Field theory approach to eigenstate thermalization in random quantum circuits
Supplemental Material

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I. INTRODUCTION

In this Supplementary Material, we study the correlation function $C_{nn'm'm}(\omega)$ defined as

$$C_{nn'm'm}(\omega) = \left\langle \sum_{\nu,\mu} \langle n|\nu\rangle \langle \nu|n'\rangle \langle m'|\mu\rangle \langle \mu|m\rangle \, \delta\left(\omega - E_\nu + E_\mu\right) \right\rangle. \tag{S1}$$

Here $|\nu\rangle$ represents the (quasi)energy eigenstate with (quasi)energy E_ν, and $\{|n\rangle\}$ denotes an arbitrary set of basis states of interest. The angular bracket indicates ensemble averaging. The Fourier transform of the correlation function $C_{nn'm'm}(\omega)$ acquires the form

$$C_{nn'm'm}(t) = \left\langle U_{nn'}(t) U^\dagger_{m'm}(t) \right\rangle, \tag{S2}$$

where $U_{nn'}(t)$ represents the matrix element of the time evolution operator at time t (measured in units of period for Floquet systems) in the basis $\{|n\rangle\}$, i.e., $U_{nn'}(t) = \langle n| U(t) |n'\rangle$.

In Sec. II, we provide a brief derivation of the connection between the correlation function $C_{nn'm'm}(t)$ and the spectral form factor (SFF) $K(t)$ for the circular unitary ensemble (CUE). In Sec. III, we use a sigma-model approach to prove that, for a large family of Floquet random quantum circuits (where independently Haar distributed random unitary matrices are applied to pairs of neighboring qudits of local Hilbert space dimension $q \to \infty$), the correlation function $C_{nn'm'm}(\omega)$ coincides with that of the CUE to the leading order in the Hilbert space dimension N. In Sec. IV, we investigate the higher order fluctuations of the sigma-model supermatrix field for the CUE, and compute their contribution to the CUE correlation function $C_{nn'm'm}(\omega)$. Finally, in Sec. V, we use the obtained result for the correlation function $C_{nn'm'm}(t)$ to derive some of the physical properties of the Floquet random quantum circuits under consideration, including the SFF, the partial spectral form factor (PSFF) and the dynamical correlation function of operators.

II. CORRELATION FUNCTION OF THE QUASIENERGY EIGENSTATES FOR THE CUE

Consider now an ensemble of Floquet systems (with $U(t) = U^t$) whose Floquet operators U are $N \times N$ random matrices drawn from the circular unitary ensemble (CUE). Diagonalizing the Floquet operator U by a unitary matrix V:

$$U = V \mathsf{diag}\left\{e^{-iE_1}, e^{-iE_2}, ..., e^{-iE_N}\right\} V^\dagger, \tag{S3}$$

and using the fact that the quasienergy eigenvalues ($\{E_i\}$) and eigenstates (columns of the diagonalizing matrix V) of the CUE are statistically independent, one can rewrite $C_{nn'm'm}(\omega)$ as

$$C_{nn'm'm}(\omega) = \sum_{\nu,\mu=1}^{N} \int_{\mathcal{U}(N)} dV V_{n\nu} V^\dagger_{\nu n'} V_{m'\mu} V^\dagger_{\mu m} \int \prod_{\gamma=1}^{N} dE_\gamma P(E_1, E_2, ..., E_N) \delta\left(\omega - E_\nu + E_\mu\right). \tag{S4}$$

Here dV denotes the Haar measure on the unitary group $\mathcal{U}(N)$ and is normalized by $\int_{\mathcal{U}(N)} dV = 1$. $P(E_1, E_2, ..., E_N)$ represents the joint probability distribution of the quasienergies $\{E_\gamma\}$.

The Haar integral in the equation above can be calculated with the help of the Weingarten calculus [S1–S8]:

$$\begin{aligned}&\int_{\mathcal{U}(N)} dV V_{i_1 j_1} V_{i_2 j_2} V^\dagger_{j'_1 i'_1} V^\dagger_{j'_2 i'_2}\\ =&\frac{1}{N^2-1}\left(\delta_{i_1,i'_1}\delta_{i_2,i'_2}\delta_{j_1,j'_1}\delta_{j_2,j'_2} + \delta_{i_1,i'_2}\delta_{i_2,i'_1}\delta_{j_1,j'_2}\delta_{j_2,j'_1}\right) - \frac{1}{N(N^2-1)}\left(\delta_{i_1,i'_1}\delta_{i_2,i'_2}\delta_{j_1,j'_2}\delta_{j_2,j'_1} + \delta_{i_1,i'_2}\delta_{i_2,i'_1}\delta_{j_1,j'_1}\delta_{j_2,j'_2}\right).\end{aligned} \tag{S5}$$

Inserting Eq. S13 into Eq. S4, one obtains

$$C_{nn'm'm}(\omega) = \frac{1}{N^2-1} R_2(\omega)\left(\delta_{nn'}\delta_{mm'} - \frac{1}{N}\delta_{mn}\delta_{n'm'}\right) + \left(\frac{N}{(N^2-1)}\delta_{mn}\delta_{n'm'} - \frac{1}{(N^2-1)}\delta_{nn'}\delta_{mm'}\right)\delta(\omega), \tag{S6}$$

where $R_2(\omega)$ denotes the two-level correlation function defined as

$$R_2(\omega) = \left\langle \sum_{\nu,\mu=1}^{N} \delta\left(\omega - E_\nu + E_\mu\right)\right\rangle. \tag{S7}$$

For the CUE, it is known that $R_2(\omega)$ is given by [S9, S10]:

$$R_2(\omega) = \frac{N^2}{2\pi} - \frac{1}{2\pi}\frac{\sin^2(N\omega/2)}{\sin^2(\omega/2)} + N\delta(\omega). \tag{S8}$$

In the Fourier representation, Eq. S6 becomes

$$C_{nn'm'm}(t) = \frac{1}{N^2-1} K(t)\left(\delta_{nn'}\delta_{mm'} - \frac{1}{N}\delta_{mn}\delta_{n'm'}\right) + \frac{N}{(N^2-1)}\delta_{mn}\delta_{n'm'} - \frac{1}{(N^2-1)}\delta_{nn'}\delta_{mm'}, \tag{S9}$$

where $K(t)$ indicates the spectral form factor (SFF):

$$K(t) = \left\langle \sum_{\nu,\mu=1}^{N} e^{-iE_\mu t + iE_\nu t} \right\rangle = \left\langle \operatorname{Tr} U(t) \operatorname{Tr} U^\dagger(t) \right\rangle . \tag{S10}$$

For the CUE ensemble, $K(t)$ acquires the form

$$K(t) = \min(|t|, N) + N^2 \delta_{t,0}. \tag{S11}$$

In an analogous manner, we now investigate the statistics of the matrix elements of an arbitrary Hermitian operator O in the basis of the quasienergy eigenstates of the CUE, i.e., $O_{\nu\mu} \equiv \langle \nu | O | \mu \rangle$. The ensemble average of $O_{\nu\mu}$ is given by

$$\langle O_{\nu\mu} \rangle = \sum_{n,m=1}^{N} O_{nm} \int_{\mathcal{U}(N)} dV V^\dagger_{\nu n} V_{m\mu} = \frac{1}{N} \operatorname{Tr}(O) \delta_{\nu\mu}. \tag{S12}$$

Here we have used the following Haar integral

$$\int_{\mathcal{U}(N)} dV \, V_{ij} V^\dagger_{j'i'} = \frac{1}{N} \delta_{i,i'} \delta_{j,j'}. \tag{S13}$$

Similarly, we find that the variance of the matrix element $O_{\nu\mu}$ is

$$\begin{aligned} \left\langle |O_{\nu\mu}|^2 \right\rangle &= \sum_{n,m,n',m'=1}^{N} O_{nm} O_{m'n'} \int_{\mathcal{U}(N)} dV V^\dagger_{\nu n} V_{m\mu} V^\dagger_{\mu m'} V_{n'\nu} \\ &= \left[\frac{1}{N^2-1} \operatorname{Tr}(O^2) - \frac{1}{N(N^2-1)} \operatorname{Tr}^2(O) \right] + \delta_{\nu\mu} \left[\frac{1}{N^2-1} \operatorname{Tr}^2(O) - \frac{1}{N(N^2-1)} \operatorname{Tr}(O^2) \right]. \end{aligned} \tag{S14}$$

Combining Eqs. S12 and S14, one can express $O_{\nu\mu}$ as

$$\begin{aligned} |O_{\nu\mu}| =& \frac{1}{N} \operatorname{Tr}(O) \delta_{\nu\mu} + \left\{ \left[\frac{1}{N^2-1} \operatorname{Tr}(O^2) - \frac{1}{N(N^2-1)} \operatorname{Tr}^2(O) \right] \right. \\ & \left. + \, \delta_{\nu\mu} \left[\left(\frac{1}{N^2-1} - \frac{1}{N^2} \right) \operatorname{Tr}^2(O) - \frac{1}{N(N^2-1)} \operatorname{Tr}(O^2) \right] \right\}^{1/2} R_{\nu\mu} \\ \approx & \frac{1}{N} \operatorname{Tr}(O) \delta_{\nu\mu} + \left[\frac{1}{N^2} \operatorname{Tr}(O^2) - \frac{1}{N^3} \operatorname{Tr}^2(O) \right]^{1/2} R_{\nu\mu}, \end{aligned} \tag{S15}$$

where in the last equality we kept only the leading order terms in $1/N$.

From the derivations above, it is straightforward to see that Eqs. S6, S9 and S15 are also applicable to an ensemble of Hamiltonian systems whose time-evolution operators $U(t) = e^{-iHt}$ are generated by Hermitian Hamiltonians H drawn from the Gaussian unitary ensemble (GUE), see also Ref. [S11–S13].

III. CORRELATION FUNCTION OF THE QUASIENERGY EIGENSTATES FOR THE FLOQUET RANDOM QUANTUM CIRCUITS

In this section, we first construct a supersymmetric (SUSY) sigma model for the calculation of the correlation function $C_{nn'm'm}(\omega)$, applicable to any ensemble of Floquet systems. We then employ this sigma model to a large family of Floquet random quantum circuits which consist of qudits in an arbitrary dimensional cubic lattice with either periodic or open boundary condition. Within each period, random unitary matrices drawn independent from the CUE are applied to different pairs of neighboring qudits at various substeps. We find the correlation function $C_{nn'm'm}(\omega)$ for any model in this family agrees with the RMT prediction Eq. S6 in the limit of large single-particle Hilbert space dimension $q \to \infty$, irrespective of the ordering of the local gates. This sigma model calculation is a generalization to that of the two-level correlation function $R_2(\omega)$ in Ref. [S14], see also the review [S15] and the references therein.

A. Preliminaries

Consider first an arbitrary ensemble of Floquet systems. To evaluate the correlation function $C_{nn'm'm}(\omega)$, one can make use of the generating function

$$\mathcal{Z}[J]=\left\langle \int_0^{2\pi}\frac{d\phi}{2\pi}\frac{\det\left(1-\alpha_{\mathrm{F}}e^{i\phi}U+J^{+}\right)\det\left(1-\beta_{\mathrm{F}}e^{-i\phi}U^{\dagger}+J^{-}\right)}{\det\left(1-\alpha_{\mathrm{B}}e^{i\phi}U\right)\det\left(1-\beta_{\mathrm{B}}e^{-i\phi}U^{\dagger}\right)}\right\rangle . \tag{S16}$$

Here the angular bracket represents averaging over the ensemble of the Floquet operators U. $J^{\pm}$ is a $N\times N$ source matrix, with N being the dimension of Hilbert space wherein the Floquet operator acts.

Taking derivatives of the generating function $\mathcal{Z}[J]$ with respect to the matrix elements of the source field $J^{\pm}$, and then setting $J=0$, $\alpha_{\mathrm{B}}=\alpha_{\mathrm{F}}$ and $\beta_{\mathrm{B}}=\beta_{\mathrm{F}}$ [S33], we have

$$\bar{C}_{nn'm'm}(\alpha_{\mathrm{F}},\beta_{\mathrm{F}})\equiv\left.\frac{\partial^2\mathcal{Z}[J]}{\partial J^{+}_{n'n}\partial J^{-}_{mm'}}\right|_{J=0,\alpha_{\mathrm{B}}=\alpha_{\mathrm{F}},\beta_{\mathrm{B}}=\beta_{\mathrm{F}}}=\sum_{k=0}^{\infty}C_{nn'm'm}(k)\left(\alpha_{\mathrm{F}}\beta_{\mathrm{F}}\right)^{k}, \tag{S17}$$

where $C_{nn'm'm}(k)$ is defined in Eq. S2 with k being the discrete time measured in units of period. We can therefore see that its Fourier transform $C_{nn'm'm}(\omega)$ (Eq. S1) can be obtained from $\bar{C}_{nn'm'm}(\alpha_{\mathrm{F}},\beta_{\mathrm{F}})$ when the product of α_{F} and β_{F} is set to $\alpha_{\mathrm{F}}\beta_{\mathrm{F}}=e^{i\omega}$:

$$C_{nn'm'm}(\omega)=\frac{1}{2\pi}\left(\bar{C}_{nn'm'm}(\alpha_{\mathrm{F}},\beta_{\mathrm{F}})+\bar{C}^{*}_{n'nmm'}(\alpha_{\mathrm{F}},\beta_{\mathrm{F}})-\delta_{nn'}\delta_{mm'}\right). \tag{S18}$$

B. Sigma model

The starting point of the sigma model derivation is the following Gaussian superintegral representation of the generating function:

$$\mathcal{Z}[J]=\left\langle\int_0^{2\pi}\frac{d\phi}{2\pi}\int\mathcal{D}(\bar\psi,\psi)\exp\left\{\begin{aligned}&-\bar\psi_i^{+,F}\left(\delta_{ij}-\alpha_{\mathrm{F}}e^{i\phi}U_{ij}+J^{+}_{ij}\right)\psi_j^{+,F}-\bar\psi_i^{-,F}\left(\delta_{ij}-\beta_{\mathrm{F}}e^{-i\phi}U^{\dagger}_{ij}+J^{-}_{ij}\right)\psi_j^{-,F}\\&-\bar\psi_i^{+,B}\left(\delta_{ij}-\alpha_{\mathrm{B}}e^{i\phi}U_{ij}\right)\psi_j^{+,B}-\bar\psi_i^{-,B}\left(\delta_{ij}-\beta_{\mathrm{B}}e^{-i\phi}U^{\dagger}_{ij}\right)\psi_j^{-,B}\end{aligned}\right\}\right\rangle. \tag{S19}$$

Here $\psi_i^{s,a}$ carries an index $s=\pm$ labeling the retarded(+)/advanced(-) space associated with the forward/backward time evolution $U/U^{\dagger}$, an index $a=\mathsf{B},\mathsf{F}$ labeling the Boson(B)/Fermion(F) space, and $i=1,2,...,N$ which labels the Hilbert ($\mathcal{H}$) space. The integration measure $\mathcal{D}(\bar\psi,\psi)$ is normalized such that the generating function $\mathcal{Z}=1$ when $J=0$ and $\alpha_{\mathrm{F}}=\alpha_{\mathrm{B}}=\beta_{\mathrm{F}}=\beta_{\mathrm{B}}=0$.

For simplicity, we introduce $2N\times2N$ matrices α_J and β_J in the direct product of Boson-Fermion and Hilbert spaces ($\mathsf{BF}\otimes\mathcal{H}$):

$$\alpha_J=\begin{bmatrix}\alpha_{\mathrm{B}}\mathbb{1}_{\mathcal{H}} & 0\\ 0 & \alpha_{\mathrm{F}}(1+J^{+})^{-1}\end{bmatrix}_{\mathsf{BF}},\qquad \beta_J=\begin{bmatrix}\beta_{\mathrm{B}}\mathbb{1}_{\mathcal{H}} & 0\\ 0 & \beta_{\mathrm{F}}(1+J^{-})^{-1}\end{bmatrix}_{\mathsf{BF}}, \tag{S20}$$

and rewrite Eq. S19 as

$$\begin{aligned}\mathcal{Z}[J]=&\det(1+J^{+})\det(1+J^{-})\\&\times\left\langle\int_0^{2\pi}\frac{d\phi}{2\pi}\int\mathcal{D}(\bar\psi,\psi)\exp\left\{-\begin{bmatrix}\bar\psi^{+} & \bar\psi^{-}\end{bmatrix}\begin{bmatrix}1-e^{i\phi}\alpha_J\mathbf{U} & 0\\ 0 & 1-e^{-i\phi}\beta_J\mathbf{U}^{\dagger}\end{bmatrix}_{\mathsf{RA}}\begin{bmatrix}\psi^{+}\\ \psi^{-}\end{bmatrix}\right\}\right\rangle.\end{aligned} \tag{S21}$$

Here the subscript in $[.]_{\mathsf{BF}}$ ($[.]_{\mathsf{RA}}$) indicates that the matrix is expressed in the Boson-Fermion (retarded-advanced) space. We use $\mathbb{1}_{\mathcal{H}}$ ($\mathbb{1}_{\mathsf{BF}}$) to denote the identity matrix in the Hilbert (Boson-Fermion) space. For simplicity, we also define $\mathbf{U}=U\otimes\mathbb{1}_{\mathsf{BF}}$.

The next step is applying the color-flavor transformation [S16–S19] to convert the integration over the center phase ϕ to an integration over $2N\times2N$ supermatrices Z and $\tilde{Z}$:

$$\begin{aligned}\mathcal{Z}[J]=&\det(1+J^{+})\det(1+J^{-})\\&\times\left\langle\int D(\tilde Z,Z)\exp\left[\mathrm{STr}\ln(1-\tilde ZZ)\right]\int\mathcal{D}(\bar\psi,\psi)\exp\left\{-\begin{bmatrix}\bar\psi^{+} & \bar\psi^{-}\end{bmatrix}\begin{bmatrix}1 & Z\beta_J\mathbf{U}^{\dagger}\\ \tilde Z\alpha_J\mathbf{U} & 1\end{bmatrix}_{\mathsf{RA}}\begin{bmatrix}\psi^{+}\\ \psi^{-}\end{bmatrix}\right\}\right\rangle.\end{aligned} \tag{S22}$$

Here Z_{ij}^{ab} carries indices $a, b = \mathsf{B}, \mathsf{F}$ that label the Boson-Fermion space and indices $i, j = 1, 2, .., N$ that label the Hilbert space. The integration runs over all $2N \times 2N$ complex supermatrices Z and $\tilde{Z}$ with the constraints: $\tilde{Z}^{\mathsf{FF}} = -Z^{\mathsf{FF}\,\dagger}$, $\tilde{Z}^{\mathsf{BB}} = Z^{\mathsf{BB}\,\dagger}$, and $|Z^{\mathsf{BB}\,\dagger} Z^{\mathsf{BB}}| < 1$. STr represents the supertrace over the $\mathsf{BF} \otimes \mathcal{H}$ space. The integration measure $\mathcal{D}(\tilde{Z}, Z)$ is also normalized by $\mathcal{Z}[J=0] = 1$ when $\alpha_{\mathrm{F}} = \alpha_{\mathrm{B}} = \beta_{\mathrm{F}} = \beta_{\mathrm{B}} = 0$.

Carrying out the integration over ψ and $\bar{\psi}$, we arrive at the SUSY sigma model representation of the generating function:

$$\begin{aligned} \mathcal{Z}[J] &= \det(1+J^+)\det(1+J^-)\left\langle \int \mathcal{D}(\tilde{Z}, Z)\exp(-S[\tilde{Z}, Z]) \right\rangle, \\ S[\tilde{Z}, Z] &= -\,\mathrm{STr}\ln(1-\tilde{Z}Z) + \mathrm{STr}\ln\left[1 - \tilde{Z}\alpha_J \mathbf{U} Z \beta_J \mathbf{U}^\dagger\right]. \end{aligned} \tag{S23}$$

We emphasize that this sigma model is exact and is applicable to an arbitrary ensemble of Floquet systems.

To compare with the sigma model for disordered Hamiltonian systems [S20], one can make the transformation

$$Q = T\Lambda T^{-1}, \qquad T = \begin{bmatrix} 1 & Z \\ \tilde{Z} & 1 \end{bmatrix}_{\mathsf{RA}}, \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{\mathsf{RA}}, \tag{S24}$$

after which the action becomes

$$S[Q] = \mathrm{STr}\ln\left[\left(1 - \begin{bmatrix} \alpha_J \mathbf{U} & 0 \\ 0 & \beta_J \mathbf{U}^\dagger \end{bmatrix}\right) Q\Lambda + \left(1 + \begin{bmatrix} \alpha_J \mathbf{U} & 0 \\ 0 & \beta_J \mathbf{U}^\dagger \end{bmatrix}\right)\right]. \tag{S25}$$

Supermatrix field Q stays on the manifold with the constraints $\mathrm{STr}Q = 0$ and $Q^2 = 1$.

C. Saddle points

Starting from the sigma model Eq. S23, we now apply it to the Floquet random quantum circuits under study. We consider the large $N \to \infty$ limit, and focus on the saddle points and quadratic fluctuations around them. In Ref. [S14], we have shown that the contribution from higher-order fluctuations for the current model is of higher order in $1/N$.

Taking variation of the action $S[\tilde{Z}, Z]$ (Eq. S23) with respect to $\tilde{Z}$ yields the saddle point equation for an arbitrary configuration of Floquet operator U:

$$Z(1-\tilde{Z}Z)^{-1} = \alpha_J \mathbf{U} Z \beta_J \mathbf{U}^\dagger \left(1 - \tilde{Z}\alpha_J \mathbf{U} Z \beta_J \mathbf{U}^\dagger\right)^{-1}. \tag{S26}$$

Note that, in Eq. S23, an averaging over the ensemble of Floquet operator U is required. We only look for solutions that solve the saddle point equation for all Floquet operators in the ensemble under consideration, and ignore all U dependent solution assuming they are not important to the ensemble averaged field theory.

It is easy to see that

$$Z^{(s)} = \tilde{Z}^{(s)} = 0, \tag{S27}$$

is a solution to the saddle point equation. The corresponding $Q^{(s)}$ matrix which is related to $Z^{(s)}, \tilde{Z}^{(s)}$ by Eq. S24 assumes the form

$$Q^{(s)} = \sigma^3_{\mathsf{RA}} \otimes \mathbb{1}_{\mathsf{BF}} \otimes \mathbb{1}_{\mathcal{H}}, \tag{S28}$$

where σ^3_{RA} is the Pauli matrix acting in the retarded-advanced space.

Another solution to the saddle point equation takes the form

$$Z^{(ns)} = \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}_{\mathsf{BF}}, \qquad \tilde{Z}^{(ns)} = \begin{bmatrix} 0 & 0 \\ 0 & -z^\dagger \end{bmatrix}_{\mathsf{BF}}, \tag{S29}$$

where z is a $N \times N$ matrix that solves the following equation

$$z(1+z^\dagger z)^{-1} = \alpha_{\mathrm{F}}\beta_{\mathrm{F}}(1+J^+)^{-1} U z (1+J^-)^{-1} U^\dagger (1 + \alpha_{\mathrm{F}}\beta_{\mathrm{F}} z^\dagger (1+J^+)^{-1} U z (1+J^-)^{-1} U^\dagger)^{-1}. \tag{S30}$$

The solution to the equation above can be expressed as $z = c_0 z'$ where $c_0' \to \infty$ and z' is an arbitrary invertible matrix of order unity. The corresponding $Q^{(ns)}$ matrix for the nonstandard saddle point is given by

$$Q^{(ns)} = \sigma^3_{\mathsf{RA}} \otimes \sigma^3_{\mathsf{BF}} \otimes \mathbb{1}_{\mathcal{H}}, \tag{S31}$$

where σ^3_{BF} represents the Pauli matrix in the Fermion-Boson space. We note that $Q^{(ns)}$ is similar to the non-trivial saddle point considered by Andreev and Altshuler [S21] in the calculation of the two-level correlation function $R_2(\omega)$ for disordered Hamiltonian systems.

In the following, we call $Z^{(s)} = 0$ the standard saddle point and the non-trivial solution $Z^{(ns)}$ the nonstandard saddle point. Inserting Eq. S27 and Eq. S29 into Eq. S23, and making use of the saddle point equation, we find the actions at these saddle points

$$S^{(s)} \equiv S[\tilde{Z}^{(s)}, Z^{(s)}] = 0, \tag{S32a}$$
$$S^{(ns)} \equiv S[\tilde{Z}^{(ns)}, Z^{(ns)}] = -\operatorname{Tr}\ln\left[\alpha_{\rm F}\beta_{\rm F}(1+J^+)^{-1}(1+J^-)^{-1}\right], \tag{S32b}$$

where Tr denotes the trace over the Hilbert space only. Note that the Floquet operator U disappears from the saddle point actions.

When the source field $J^\pm$ is set to zero, the action at the nonstandard saddle point assumes the value of $S^{(ns)}|_{J=0} = -N\ln(\alpha_{\rm F}\beta_{\rm F})$. If we introduce an infinitesimal imaginary part η to $\ln(\alpha_{\rm F}\beta_{\rm F}) = i\omega - \eta$ with $\eta N \gtrsim 1$, the real part of the action at the nonstandard saddle point $\operatorname{Re} S^{(ns)} = N\eta$ is larger compared with the one at the standard saddle point $\operatorname{Re} S^{(s)} = 0$. In this case, it is enough to take into account the dominant saddle point, i.e., the standard saddle point, as well as the quadratic fluctuations around it. The correlation function $C_{nn'm'm}(\omega)$ obtained from derivatives of the generating function $\mathcal{Z}[J]$ is now smoothed over. More specifically, from Eqs. S17 and S18, one can see that introducing an infinitesimal imaginary part η to $\ln(\alpha_{\rm F}\beta_{\rm F}) = i\omega - \eta$ replace the 2π-periodic delta function $\delta(\omega)$ in the definition of $C_{nn'm'm}(\omega)$ (Eq. S1) with a 2π-periodic Lorentzian of width η, i.e., $\delta(\omega) = \sum_{n=-\infty}^{\infty} e^{in\omega}/2\pi \to \sum_{n=-\infty}^{\infty} e^{in\omega - |n|\eta}/2\pi$. $C_{nn'm'm}(\omega)$ now measures the correlation of pairs of quasienergy eigenstates with quasienergy separations $E_\nu - E_\mu$ around ω weighted by the Lorentzian. Therefore, we can see that consideration of only standard saddle point yields the smoothed correlation function $C_{nn'm'm}(\omega)$.

D. Quadratic fluctuations around the standard saddle point

The quadratic fluctuations around the standard saddle point (Eq. S27) are governed by the action

$$\delta S_2^{(s)}[\tilde{Z}, Z] = \operatorname{STr}\left[\tilde{Z} Z\right] - \operatorname{STr}\left[\tilde{Z}\alpha_J \mathbf{U} Z \beta_J \mathbf{U}^\dagger\right]. \tag{S33}$$

Using Eq. S32a and ignoring the higher order fluctuations whose contributions are of higher order in $1/N$ [S14], we approximate the contribution from the standard saddle point to the generating function $\mathcal{Z}[J]$ by

$$\begin{aligned}
&\mathcal{Z}^{(s)}[J] \approx \det(1+J^+)\det(1+J^-)\int \mathcal{D}(\tilde{Z}, Z) e^{-S^{(s)}_{\mathsf{eff}}[\tilde{Z},Z]},\\
&S^{(s)}_{\mathsf{eff}}[\tilde{Z}, Z] = \left\langle \delta S_2^{(s)}[\tilde{Z}, Z]\right\rangle = \sum_{i,j,i',j'}\sum_{a,b=\mathsf{B},\mathsf{F}} s_a \tilde{Z}^{ab}_{ij} Z^{ba}_{j'i'}\left(\delta_{ii'}\delta_{jj'} - \frac{1}{N}(\alpha_J)^{bb}_{ji}(\beta_J)^{aa}_{i'j'}\right),
\end{aligned} \tag{S34}$$

where $s_{\mathsf{B/F}} = \pm 1$. We have used the fact that, for the Floquet random quantum circuits under consideration [S14],

$$\left\langle U_{ij} U^\dagger_{j'i'}\right\rangle = \frac{1}{N}\delta_{ii'}\delta_{jj'}, \tag{S35}$$

in an arbitrary orthonormal basis. Note that this equation also holds for the CUE of dimension N. Therefore, the quadratic fluctuations around the standard saddle point for the current model is described by the same ensemble averaged effective field theory as that of the CUE.

To proceed, we divide Z into two parts which involve, respectively, its diagonal and off-diagonal components in the Hilbert space:

$$Z = X + Y, \qquad X^{ab}_{ij} = \delta_{ij} Z^{ab}_{ii}, \qquad Y^{ab}_{ij} = (1-\delta_{ij}) Z^{ab}_{ij}, \tag{S36}$$

and similarly for $\tilde{Z}$. We then perform a Fourier transformation of the diagonal components X and $\tilde{X}$:

$$X^{ab}(k) = \sum_{j=1}^{N} X_{jj}^{ab} e^{-i2\pi kj/N}, \qquad \tilde{X}^{ab}(k) = \sum_{j=1}^{N} \tilde{X}_{jj}^{ab} e^{i2\pi kj/N}. \tag{S37}$$

In terms of $X(k)$, $\tilde{X}(k)$, Y_{ij} and $\tilde{Y}_{ij}$, the ensemble averaged effective action $S_{\mathsf{eff}}^{(s)}$ in Eq. S34 can be expressed as

$$\begin{aligned} S_{\mathsf{eff}}^{(s)}[X,Y,\tilde{X},\tilde{Y}] = &\sum_{a,b=\mathsf{B},\mathsf{F}} \sum_{i\neq j} s_a \tilde{Y}_{ij}^{ab} Y_{ji}^{ba} + \frac{1}{N} \sum_{a,b=\mathsf{B},\mathsf{F}} \sum_{k=0}^{N-1} s_a \tilde{X}^{ab}(k) X^{ba}(k) \\ &- \frac{1}{N} \sum_{a,b=\mathsf{B},\mathsf{F}} s_a \alpha_b \beta_a \left(\tilde{X}^{ab}(0) - \delta_{bF} \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}^{aF}(k) \sum_i e^{-i2\pi ki/N} J_{ii}^{+} - \delta_{bF} \sum_{i\neq j} \tilde{Y}_{ij}^{aF} J_{ji}^{+} \right) \\ &\times \left(X^{ba}(0) - \delta_{aF} \frac{1}{N} \sum_{k=0}^{N-1} X^{bF}(k) \sum_{i'} e^{i2\pi ki'/N} J_{i'i'}^{-} - \delta_{aF} \sum_{i'\neq j'} Y_{j'i'}^{bF} J_{i'j'}^{-} \right). \end{aligned} \tag{S38}$$

Here we have employed the approximation $(1+J^{\pm})^{-1} \approx 1 - J^{\pm}$ for small source matrix field J.

When the source field is set to $J=0$, the action $S_{\mathsf{eff}}^{(s)}$ reduces to

$$S_{\mathsf{eff}}^{(s)}[X,Y,\tilde{X},\tilde{Y}]|_{J=0} = \sum_{a,b=\mathsf{B},\mathsf{F}} \sum_{i\neq j} s_a \tilde{Y}_{ij}^{ab} Y_{ji}^{ba} + \frac{1}{N} \sum_{a,b=\mathsf{B},\mathsf{F}} \sum_{k=0}^{N-1} s_a \tilde{X}^{ab}(k) X^{ba}(k)(1-\delta_{k,0}\alpha_b\beta_a). \tag{S39}$$

The bare propagators for X and Y arising from this action are given by

$$\begin{aligned} \left\langle X^{b'a'}(k')\tilde{X}^{ab}(k)\right\rangle_0 &= s_a s_b \left\langle \tilde{X}^{ab}(k) X^{b'a'}(k')\right\rangle_0 = s_a \frac{N}{1-\alpha_b\beta_a\delta_{k,0}} \delta_{kk'}\delta_{aa'}\delta_{bb'}, \\ \left\langle Y_{j'i'}^{b'a'} \tilde{Y}_{ij}^{ab}\right\rangle_0 &= s_a s_b \left\langle \tilde{Y}_{ij}^{ab} Y_{j'i'}^{b'a'}\right\rangle_0 = s_a \delta_{ii'}\delta_{jj'}\delta_{aa'}\delta_{bb'}. \end{aligned} \tag{S40}$$

Here the angular bracket with subscript 0 represents the functional average with the action $S_{\mathsf{eff}}^{(s)}|_{J=0}$ (Eq. S39). We find the contribution to the generating function from the standard saddle point and the quadratic fluctuations around it for zero source field $J=0$,

$$\mathcal{Z}^{(s)}[J=0] = \frac{(1-\alpha_{\mathrm{F}}\beta_{\mathrm{B}})(1-\alpha_{\mathrm{B}}\beta_{\mathrm{F}})}{(1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}})(1-\alpha_{\mathrm{B}}\beta_{\mathrm{B}})}. \tag{S41}$$

The contribution of the standard saddle point to the correlation function $\bar{C}_{nn'm'm}(\alpha_{\mathrm{F}},\beta_{\mathrm{F}})$ can be obtained from derivatives of $\mathcal{Z}^{(s)}[J]$ (Eq. S34) with respect to the source field:

$$\begin{aligned} &\bar{C}_{nn'mm'}^{(s)}(\alpha_{\mathrm{F}},\beta_{\mathrm{F}}) = \left.\frac{\partial^2 \mathcal{Z}^{(s)}[J]}{\partial J_{n'n}^{+} \partial J_{mm'}^{-}}\right|_{J=0} \\ &= \mathcal{Z}^{(s)}[J=0] \left\langle \left(\delta_{nn'}\delta_{mm'} - \frac{\partial^2 S_{\mathsf{eff}}^{(s)}}{\partial J_{n'n}^{+}\partial J_{mm'}^{-}} + \frac{\partial S_{\mathsf{eff}}^{(s)}}{\partial J_{n'n}^{+}} \frac{\partial S_{\mathsf{eff}}^{(s)}}{\partial J_{mm'}^{-}} - \delta_{nn'} \frac{\partial S_{\mathsf{eff}}^{(s)}}{\partial J_{mm'}^{-}} - \delta_{mm'} \frac{\partial S_{\mathsf{eff}}^{(s)}}{\partial J_{n'n}^{+}} \right)\Bigg|_{J=0} \right\rangle_0 . \end{aligned} \tag{S42}$$

Using the expressions for the quadratic action $S_{\mathsf{eff}}^{(s)}$ in Eq. S38 and the bare propagators in Eq. S40, we find

$$\left\langle \left.\frac{\partial S_{\mathsf{eff}}^{(s)}}{\partial J_{n'n}^{+}}\right|_{J=0} \right\rangle_0 = \delta_{nn'} \frac{1}{N^2} \sum_a s_a \alpha_{\mathrm{F}} \beta_a \left\langle \tilde{X}^{a\mathsf{F}}(0) X^{\mathsf{F}a}(0)\right\rangle_0 = \delta_{nn'} \frac{1}{N} \frac{\alpha_{\mathrm{F}}(\beta_{\mathrm{F}}-\beta_{\mathrm{B}})}{(1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}})(1-\alpha_{\mathrm{F}}\beta_{\mathrm{B}})}, \tag{S43a}$$

$$\left\langle \left.\frac{\partial S_{\mathsf{eff}}^{(s)}}{\partial J_{mm'}^{-}}\right|_{J=0} \right\rangle_0 = \delta_{mm'} \frac{1}{N^2} \sum_{b=B,F} s_F \alpha_b \beta_{\mathrm{F}} \left\langle \tilde{X}^{\mathsf{F}b}(0) X^{b\mathsf{F}}(0)\right\rangle_0 = \delta_{mm'} \frac{1}{N} \frac{\beta_{\mathrm{F}}(\alpha_{\mathrm{F}}-\alpha_{\mathrm{B}})}{(1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}})(1-\alpha_{\mathrm{B}}\beta_{\mathrm{F}})}, \tag{S43b}$$

$$
\begin{aligned}
\left\langle \frac{\partial^2 S^{(s)}_{\mathsf{eff}}}{\partial J^+_{n'n}\partial J^-_{mm'}}\bigg|_{J=0}\right\rangle_0 =& -\delta_{nn'}\delta_{mm'}\frac{1}{N^3}\sum_{k=0}^{N-1} s_F\alpha_{\mathrm{F}}\beta_{\mathrm{F}}\left\langle \tilde{X}^{\mathsf{FF}}(k)X^{\mathsf{FF}}(k)\right\rangle_0 e^{-i2\pi k(n-m)/N} \\
& -(1-\delta_{nn'})(1-\delta_{mm'})\frac{1}{N}s_F\alpha_{\mathrm{F}}\beta_{\mathrm{F}}\left\langle \tilde{Y}^{\mathsf{FF}}_{nn'}Y^{\mathsf{FF}}_{m'm}\right\rangle_0 \\
=& -\delta_{nn'}\delta_{mm'}\frac{1}{N^2}\frac{(\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^2}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}} - \delta_{nm}\delta_{n'm'}\frac{1}{N}\alpha_{\mathrm{F}}\beta_{\mathrm{F}},
\end{aligned} \tag{S43c}
$$

$$
\begin{aligned}
\left\langle \frac{\partial S^{(s)}_{\mathsf{eff}}}{\partial J^+_{n'n}}\frac{\partial S^{(s)}_{\mathsf{eff}}}{\partial J^-_{mm'}}\bigg|_{J=0}\right\rangle_0 =& \frac{1}{N^2}\sum_{ab} s_a s_F\alpha_{\mathrm{F}}\beta_{\mathrm{F}}\alpha_b\beta_a\left\langle\left(\delta_{nn'}\sum_{k=0}^{N-1}\tilde{X}^{a\mathsf{F}}(k)\frac{1}{N}e^{-i2\pi kn/N}+(1-\delta_{nn'})\tilde{Y}^{a\mathsf{F}}_{nn'}\right)X^{\mathsf{F}a}(0)\right. \\
& \left.\times\,\tilde{X}^{\mathsf{F}b}(0)\left(\delta_{mm'}\sum_{k=0}^{N-1}X^{b\mathsf{F}}(k)\frac{1}{N}e^{i2\pi km/N}+(1-\delta_{mm'})Y^{b\mathsf{F}}_{m'm}\right)\right\rangle_0 \\
=&\delta_{nn'}\delta_{mm'}\frac{1}{N^2}\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}(\beta_{\mathrm{F}}-\beta_{\mathrm{B}})(\alpha_{\mathrm{F}}-\alpha_{\mathrm{B}})}{(1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^2(1-\alpha_{\mathrm{F}}\beta_{\mathrm{B}})(1-\alpha_{\mathrm{B}}\beta_{\mathrm{F}})} \\
&+\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}\left(\delta_{nn'}\delta_{mm'}\frac{1}{N^2}\frac{(\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^2}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}+\delta_{nm}\delta_{n'm'}\frac{1}{N}\alpha_{\mathrm{F}}\beta_{\mathrm{F}}\right).
\end{aligned} \tag{S43d}
$$

Inserting these equations into Eq. S42, and setting $\alpha_{\mathrm{B}} = \alpha_{\mathrm{F}}$ and $\beta_{\mathrm{B}} = \beta_{\mathrm{F}}$, we obtain the standard saddle point's contribution to the correlation function $\bar{C}_{nn'm'm}(\alpha_{\mathrm{F}}, \beta_{\mathrm{F}})$:

$$
\bar{C}^{(s)}_{nn'm'm}(\alpha_{\mathrm{F}},\beta_{\mathrm{F}}) = \delta_{nn'}\delta_{mm'} + \delta_{nn'}\delta_{mm'}\frac{1}{N^2}\left(\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}\right)^2 + \delta_{nm}\delta_{n'm'}\frac{1}{N}\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}. \tag{S44}
$$

This leads to

$$
C^{(s)}_{nn'm'm}(\omega) = \delta_{nn'}\delta_{mm'}\frac{1}{2\pi}\left[1+\frac{1}{N^2}\left(1-\frac{1}{2}\frac{1}{\sin^2(\omega/2)}\right)\right] - \delta_{nm}\delta_{n'm'}\frac{1}{N}\frac{1}{2\pi}, \tag{S45}
$$

where we have used Eq. S18 and set $\alpha_{\mathrm{F}}\beta_{\mathrm{F}} = e^{i\omega}$.

As mentioned earlier, consideration of only the standard saddle point gives rise to the smoothed correlation function. We can see that Eq. S45 is consistent with the RMT prediction Eq. S6, to the leading order in $1/N$, after replacing the two-level correlation function $R_2(\omega)$ with its smoothed version:

$$
R^{(s)}_2(\omega) = \frac{N^2}{2\pi}\left(1-\frac{1}{N^2}\frac{1}{2\sin^2(\omega/2)}\right). \tag{S46}
$$

For the comparison of higher order terms in Eq. S45 and Eq. S6, see the discussion at the end of this section.

E. Quadratic fluctuations around the nonstandard saddle point

To study the quadratic fluctuations around the nonstandard saddle point, we insert

$$
Z = Z^{(ns)} + \delta Z, \qquad \tilde{Z} = \tilde{Z}^{(ns)} + \delta\tilde{Z}, \tag{S47}
$$

into the action $S[\tilde{Z}, Z]$ in Eq. S23 and expand it up to the quadratic order in δZ and $\delta\tilde{Z}$. This gives rise to

$$
\begin{aligned}
\delta S^{(ns)}_2[\delta\tilde{Z},\delta Z] =& \mathrm{STr}\left[(1-\tilde{Z}^{(ns)}Z^{(ns)})^{-1}\delta\tilde{Z}\delta Z\right] - \mathrm{STr}\left[(1-\tilde{Z}^{(ns)}\alpha_J\mathbf{U}Z^{(ns)}\beta_J\mathbf{U}^\dagger)^{-1}\delta\tilde{Z}\alpha_J\mathbf{U}\delta Z\beta_J\mathbf{U}^\dagger\right] \\
&+\frac{1}{2}\mathrm{STr}\left[\left((1-\tilde{Z}^{(ns)}Z^{(ns)})^{-1}(\delta\tilde{Z}Z^{(ns)}+\tilde{Z}^{(ns)}\delta Z)\right)^2\right] \\
&-\frac{1}{2}\mathrm{STr}\left[\left((1-\tilde{Z}^{(ns)}\alpha_J\mathbf{U}Z^{(ns)}\beta_J\mathbf{U}^\dagger)^{-1}(\delta\tilde{Z}\alpha_J\mathbf{U}Z^{(ns)}\beta_J\mathbf{U}^\dagger+\tilde{Z}^{(ns)}\alpha_J\mathbf{U}\delta Z\beta_J\mathbf{U}^\dagger)\right)^2\right].
\end{aligned} \tag{S48}
$$

Making use of the expression for the nonstandard saddle point $Z^{(ns)}$ (Eq. S29) and the saddle point equation (Eq. S30), we obtain

$$\begin{aligned}
&\delta S_2^{(ns)}[\delta\tilde{Z},\delta Z] = +\,\mathrm{Tr}\left[\delta\tilde{Z}^{\mathsf{BB}}\delta Z^{\mathsf{BB}}\right] - \mathrm{Tr}\left[\delta\tilde{Z}^{\mathsf{BB}}\alpha_{\mathrm{B}}U\delta Z^{\mathsf{BB}}\beta_{\mathrm{B}}U^\dagger\right]\\
&-\mathrm{Tr}\left[\delta\tilde{Z}^{\mathsf{FF}}(1+zz^\dagger)^{-1}\delta Z^{\mathsf{FF}}(1+z^\dagger z)^{-1}\right] + \mathrm{Tr}\left[\delta\tilde{Z}^{\mathsf{FF}}(z^\dagger)^{-1}U(\beta_J^{\mathsf{FF}})^{-1}z^\dagger(1+zz^\dagger)^{-1}\delta Z^{\mathsf{FF}}z^{-1}U^\dagger(\alpha_J^{\mathsf{FF}})^{-1}z(1+z^\dagger z)^{-1}\right]\\
&+\mathrm{Tr}\left[\delta\tilde{Z}^{\mathsf{BF}}(1+zz^\dagger)^{-1}\delta Z^{\mathsf{FB}}\right] - \mathrm{Tr}\left[\delta\tilde{Z}^{\mathsf{BF}}(z^\dagger)^{-1}U(\beta_J^{\mathsf{FF}})^{-1}z^\dagger(1+zz^\dagger)^{-1}\delta Z^{\mathsf{FB}}\beta_{\mathrm{B}}U^\dagger\right]\\
&-\mathrm{Tr}\left[\delta\tilde{Z}^{\mathsf{FB}}\delta Z^{\mathsf{BF}}(1+z^\dagger z)^{-1}\right] + \mathrm{Tr}\left[\delta\tilde{Z}^{\mathsf{FB}}\alpha_{\mathrm{B}}U\delta Z^{\mathsf{BF}}z^{-1}U^\dagger(\alpha_J^{\mathsf{FF}})^{-1}z(1+z^\dagger z)^{-1}\right].
\end{aligned} \tag{S49}$$

Applying the transformation:

$$\begin{aligned}
&\delta Z^{\mathsf{BB}}\to\delta Z^{\mathsf{BB}},\quad \delta Z^{\mathsf{FF}}\to(1+zz^\dagger)(z^\dagger)^{-1}\delta Z^{\mathsf{FF}}z,\quad \delta Z^{\mathsf{FB}}\to(1+zz^\dagger)(z^\dagger)^{-1}\delta Z^{\mathsf{FB}},\quad \delta Z^{\mathsf{BF}}\to\delta Z^{\mathsf{BF}}z,\\
&\delta\tilde{Z}^{\mathsf{BB}}\to\delta\tilde{Z}^{\mathsf{BB}},\quad \delta\tilde{Z}^{\mathsf{FF}}\to(1+z^\dagger z)z^{-1}\delta\tilde{Z}^{\mathsf{FF}}z^\dagger,\quad \delta\tilde{Z}^{\mathsf{BF}}\to\delta\tilde{Z}^{\mathsf{BF}}z^\dagger,\quad \delta\tilde{Z}^{\mathsf{FB}}\to(1+z^\dagger z)z^{-1}\delta\tilde{Z}^{\mathsf{FB}},
\end{aligned} \tag{S50}$$

one finds that Eq. S49 can be further simplified to

$$\delta S_2^{(ns)}[\delta\tilde{Z},\delta Z] = \mathrm{STr}\left[\delta\tilde{Z}\delta Z\right] - \mathrm{STr}\left[\delta\tilde{Z}\mathbf{U}\tilde{\alpha}_J\delta Z\mathbf{U}^\dagger\tilde{\beta}_J\right]. \tag{S51}$$

Here $\tilde{\alpha}_J$ and $\tilde{\beta}_J$ are $2N\times 2N$ matrices defined as

$$\tilde{\alpha}_J = \begin{bmatrix}\alpha_{\mathrm{B}}\mathbb{1}_{\mathcal{H}} & 0\\ 0 & \beta_{\mathrm{F}}^{-1}(1+J^-)\end{bmatrix}_{\mathsf{BF}},\qquad \tilde{\beta}_J = \begin{bmatrix}\beta_{\mathrm{B}}\mathbb{1}_{\mathcal{H}} & 0\\ 0 & \alpha_{\mathrm{F}}^{-1}(1+J^+)\end{bmatrix}_{\mathsf{BF}}. \tag{S52}$$

Note that although $\mathbf{U}$ and $\tilde{\alpha}_J$ ($\mathbf{U}^\dagger$ and $\tilde{\beta}_J$) do not always commute, one can still permute them in the action. In particular, if one apply the transformation

$$\delta Z\to\mathbf{U}\delta Z\mathbf{U},\qquad \delta\tilde{Z}\to\mathbf{U}^\dagger\delta Z\mathbf{U}^\dagger, \tag{S53}$$

the action becomes

$$\delta S_2^{(ns)}[\delta Z,\delta\tilde{Z}] = \mathrm{STr}\left[\delta\tilde{Z}\delta Z\right] - \mathrm{STr}\left[\delta\tilde{Z}\tilde{\alpha}_J\mathbf{U}\delta Z\tilde{\beta}_J\mathbf{U}^\dagger\right]. \tag{S54}$$

which is identical to that of the standard saddle point (Eq. S38) after the replacement $\alpha_J\to\tilde{\alpha}_J$ and $\beta_J\to\tilde{\beta}_J$.

The contribution of the nonstandard saddle point to the generating function can be approximated as

$$\mathcal{Z}^{(ns)}[J] = (\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^N\int\mathcal{D}(\tilde{Z},Z)e^{-S_{\mathsf{eff}}^{(ns)}},\qquad S_{\mathsf{eff}}^{(ns)} = \left\langle\delta S_2^{(ns)}[\tilde{Z},Z]\right\rangle, \tag{S55}$$

where we have used the expression for the action at the nonstandard saddle point Eq. S32b. Given the close similarity between their actions (Eq. S33 and Eq. S54), the contribution from the quadratic fluctuations around the nonstandard saddle point can be deduced from that of the standard saddle point by making the replacement.

$$\alpha_{\mathrm{F}}\to\beta_{\mathrm{F}}^{-1},\quad \beta_{\mathrm{F}}\to\alpha_{\mathrm{F}}^{-1},\quad J^+\to(1+J^-)^{-1}-1\approx -J^-,\quad J^-\to(1+J^+)^{-1}-1\approx -J^+. \tag{S56}$$

From Eq. S55, we find that the contribution from the nonstandard saddle point and its quadratic fluctuations to the correlation function $\bar{C}_{nn'm'm}(\omega)$ can be expressed as

$$\bar{C}^{(ns)}_{nn'mm'}(\omega) = \left.\frac{\partial^2\mathcal{Z}^{(ns)}[J]}{\partial J^+_{n'n}\partial J^-_{mm'}}\right|_{J=0} = \mathcal{Z}^{(ns)}[J=0]\left\langle\left.\left(-\frac{\partial^2 S_{\mathsf{eff}}^{(ns)}}{\partial J^+_{n'n}\partial J^-_{mm'}} + \frac{\partial S_{\mathsf{eff}}^{(ns)}}{\partial J^+_{n'n}}\frac{\partial S_{\mathsf{eff}}^{(ns)}}{\partial J^-_{mm'}}\right)\right|_{J=0}\right\rangle_0. \tag{S57}$$

Here $Z^{(ns)}[J=0]/(\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^N$ and the derivatives of the ensemble averaged effective action $S_{\mathsf{eff}}^{(ns)}$ with respect to the source field $J^\pm$ can be deduced from their standard saddle point counterparts (Eqs. S41 and S43) by applying the replacement Eq. S56:

$$\mathcal{Z}^{(ns)}[J=0] = -(\alpha_{\rm F}\beta_{\rm F})^N \frac{(\alpha_{\rm F}-\alpha_{\rm B})(\beta_{\rm F}-\beta_{\rm B})}{(1-\alpha_{\rm B}\beta_{\rm B})(1-\alpha_{\rm F}\beta_{\rm F})}, \tag{S58a}$$

$$\left\langle \frac{\partial^2 S_{\sf eff}^{(ns)}}{\partial J^+_{n'n}\partial J^-_{mm'}}\bigg|_{J=0}\right\rangle_0 = \delta_{nn'}\delta_{mm'}\frac{1}{N^2}\frac{1}{(1-\alpha_{\rm F}\beta_{\rm F})\alpha_{\rm F}\beta_{\rm F}} - \delta_{nm}\delta_{n'm'}\frac{1}{N}\frac{1}{\alpha_{\rm F}\beta_{\rm F}}, \tag{S58b}$$

$$\begin{aligned}\left\langle \frac{\partial S_{\sf eff}^{(ns)}}{\partial J^+_{n'n}}\frac{\partial S_{\sf eff}^{(ns)}}{\partial J^-_{mm'}}\bigg|_{J=0}\right\rangle_0 =& \delta_{nn'}\delta_{mm'}\frac{1}{N^2}\frac{(1-\alpha_{\rm F}\beta_{\rm B})(1-\alpha_{\rm B}\beta_{\rm F})\alpha_{\rm F}\beta_{\rm F}}{(1-\alpha_{\rm F}\beta_{\rm F})^2(\alpha_{\rm F}-\alpha_{\rm B})(\beta_{\rm F}-\beta_{\rm B})} \\ &- \frac{1}{1-\alpha_{\rm F}\beta_{\rm F}}\left(-\delta_{nn'}\delta_{mm'}\frac{1}{N^2}\frac{1}{(1-\alpha_{\rm F}\beta_{\rm F})\alpha_{\rm F}\beta_{\rm F}} + \delta_{nm}\delta_{n'm'}\frac{1}{N}\frac{1}{\alpha_{\rm F}\beta_{\rm F}}\right).\end{aligned} \tag{S58c}$$

Inserting these equations into Eq. S57 and setting $\alpha_{\rm B}=\alpha_{\rm F}$ and $\beta_{\rm B}=\beta_{\rm F}$, we obtain

$$\bar{C}^{(ns)}_{nn'mm'}(\alpha_{\rm F},\beta_{\rm F}) = -\delta_{nn'}\delta_{mm'}\frac{1}{N^2}(\alpha_{\rm F}\beta_{\rm F})^N\frac{\alpha_{\rm F}\beta_{\rm F}}{(1-\alpha_{\rm F}\beta_{\rm F})^2}. \tag{S59}$$

Adding this result with the contribution of the standard saddle point (Eq. S44), we find the correlation function $\bar{C}_{nn'm'm}(\alpha_{\rm F},\beta_{\rm F})$:

$$\bar{C}_{nn'm'm}(\alpha_{\rm F},\beta_{\rm F}) = \delta_{nn'}\delta_{mm'}\left[1+\frac{1}{N^2}\left(1-(\alpha_{\rm F}\beta_{\rm F})^N\right)\frac{\alpha_{\rm F}\beta_{\rm F}}{(1-\alpha_{\rm F}\beta_{\rm F})^2} - \frac{1}{N^2}\frac{\alpha_{\rm F}\beta_{\rm F}}{1-\alpha_{\rm F}\beta_{\rm F}}\right] + \delta_{nm}\delta_{n'm'}\frac{1}{N}\frac{\alpha_{\rm F}\beta_{\rm F}}{1-\alpha_{\rm F}\beta_{\rm F}}. \tag{S60}$$

which leads to

$$\begin{aligned}C_{nn'm'm}(\omega) =& \left[\frac{1}{2\pi} - \frac{1}{N^2}\frac{1}{2\pi}\frac{\sin^2(N\omega/2)}{\sin^2(\omega/2)} + \frac{1}{N}\delta(\omega) + \frac{1}{N^2}\frac{1}{2\pi}\right]\delta_{nn'}\delta_{mm'} - \frac{1}{N}\frac{1}{2\pi}\delta_{nm}\delta_{n'm'} + \\ &- \frac{1}{N^2}\delta(\omega)\delta_{nn'}\delta_{mm'} + \frac{1}{N}\delta(\omega)\delta_{nm}\delta_{n'm'},\end{aligned} \tag{S61}$$

Here we have made use of Eq. S18 and the following identity:

$$2\,\mathrm{Re}\,\frac{e^{i\omega}}{1-e^{i\omega}} + 1 = 2\pi\delta(\omega). \tag{S62}$$

Eq. S61 is consistent with the RMT prediction Eq. S6 to the leading order in $1/N$. In particular, we have obtained the additional oscillatory term $\frac{1}{4\pi N^2}\frac{\cos(N\omega)}{\sin^2(\omega/2)}\delta_{nn'}\delta_{mm'}$ by taking into account the contribution from the nonstandard saddle point. Note that the higher order terms in Eq. S61 do not agree with the corresponding terms in Eq. S6. In particular, instead of $\delta_{nn'}\delta_{mm'}R_2(\omega)/N^4$ and $-\delta_{nm}\delta_{n'm'}R_2(\omega)/N^3$ in the CUE result Eq. S6, we find $\delta_{nn'}\delta_{mm'}R_2^{(dis)}(\omega)/N^4$ and $-\delta_{nm}\delta_{n'm'}R_2^{(dis)}(\omega)/N^3$, where the disconnected part of the two-level correlation function is $R_2^{(dis)}(\omega) = N^2/2\pi$. However, the higher order fluctuations are needed to find the exact result for $C_{nn'm'm}(\omega)$ at these higher orders and to compare with the RMT prediction. The calculation of the contribution from the higher order fluctuations is rather cumbersome. We instead consider for simplicity the CUE in Sec. IV, and investigate the higher order fluctuations of the supermatrix field Z in its sigma model. We demonstrate that the next leading higher order term $-\delta_{mn}\delta_{n'm'}R_2(\omega)/N^3$ in the CUE result can indeed be recovered within the sigma model approach by taking into account the higher order fluctuations.

IV. HIGHER ORDER FLUCTUATIONS FOR THE CUE

In this section, we use the SUSY sigma model to rederive the correlation function $C_{nn'm'm}(\omega)$ for the CUE, i.e., Eq. S6. The goal of this section is to demonstrate that the second term in Eq. S6, which is of higher order in the large N expansion, arises from the higher order fluctuations. Unlike the earlier derivation in Sec. II which only shows the relation between $C_{nn'm'm}(\omega)$ and $R_2(\omega)$, this sigma model calculation also gives the explicit expression for $R_2(\omega)$ (Eq. S7). Moreover, there is no need of the Weingarten calculus which can also be derived in the sigma model framework [S14]. Note that the sigma model calculation performed in Sec. III also applies directly to the CUE whose second moment of the Floquet operator is also given by Eq. S35. In this section, we employ a slightly different approach which makes use of the color-flavor transformation to perform the ensemble averaging. This approach greatly simplifies the structure of the sigma model (see details below), but can not be directly applied to the Floquet random quantum circuits we considered.

A. Sigma model for the CUE

We start from the superintegral representation of the generating function $\mathcal{Z}[J]$ Eq. S22, which is applicable to an arbitrary ensemble of Floquet systems. Instead of converting the integration over the center phase ϕ into an integration over supermatrix matrices, we first average over the ensemble of the CUE matrices U. This ensemble averaging can be performed using the color-flavor transformation [S16–S19]:

$$\begin{aligned}\mathcal{Z}[J] =& \det(1+J^+)\det(1+J^-)\int_0^{2\pi}\frac{d\phi}{2\pi}\int D(\tilde{Z},Z)\exp\left[N\mathrm{Str}\ln(1-\tilde{Z}Z)\right] \\ &\times \int \mathcal{D}(\bar{\psi},\psi)\exp\left\{-\begin{bmatrix}\bar{\psi}^+ & \bar{\psi}^-\end{bmatrix}\begin{bmatrix}1 & e^{i\phi}\alpha_J Z \\ e^{-i\phi}\beta_J\tilde{Z} & 1\end{bmatrix}_{\mathsf{RA}}\begin{bmatrix}\psi^+ \\ \psi^-\end{bmatrix}\right\}.\end{aligned} \tag{S63}$$

Here Str traces over the BF space only, in comparison with STr which is introduced earlier and traces over the product space $\mathsf{BF}\otimes\mathcal{H}$. We emphasize that, unlike in Sec. III, here Z and $\tilde{Z}$ are 2×2 matrices defined in the BF space only. They are subject to the constraints $\tilde{Z}^{\mathsf{FF}} = -Z^{\mathsf{FF}\,*}$, $\tilde{Z}^{\mathsf{BB}} = Z^{\mathsf{BB}\,*}$, and $|Z^{\mathsf{BB}}| < 1$.

Integrating out the supervectors $\bar{\psi}$ and ψ, we arrive at the zero-dimensional SUSY sigma model for the CUE [S10, S16]:

$$\begin{aligned}\mathcal{Z}[J] =& \det(1+J^+)\det(1+J^-)\int D(\tilde{Z},Z)\exp(-S[\tilde{Z},Z]), \\ S[\tilde{Z},Z] =& -N\mathrm{Str}\ln(1-\tilde{Z}Z) + \mathrm{STr}\ln(1-(\tilde{Z}\otimes\mathbb{1}_{\mathcal{H}})\alpha_J(Z\otimes\mathbb{1}_{\mathcal{H}})\beta_J).\end{aligned} \tag{S64}$$

Compared with the sigma model in Eq. S23 for an arbitrary ensemble of Floquet systems, here we have converted the averaging over the CUE matrices U (instead of the averaging over the center phase ϕ) into an integration over the supermatrices Z and $\tilde{Z}$ which have no structure in the Hilbert space. It is therefore much easier to study the role played by the higher order fluctuations in the case of the CUE.

B. Saddle points

From Eq. S64, we obtain the saddle point equation:

$$\left(Z(1-\tilde{Z}Z)^{-1}\right)\otimes\mathbb{1}_{\mathcal{H}} = \alpha_J(Z\otimes\mathbb{1}_{\mathcal{H}})\beta_J\left[1-(\tilde{Z}\otimes\mathbb{1}_{\mathcal{H}})\alpha_J(Z\otimes\mathbb{1}_{\mathcal{H}})\beta_J\right]^{-1}. \tag{S65}$$

A trivial solution to this equation is the standard saddle point given by Eq. S27. We emphasize again that $Z^{(s)}$ and $\tilde{Z}^{(s)}$ are now 2×2 supermatrices acting in the BF space only. As before, the saddle point equation has another solution - the nonstandard saddle point. It acquires the form of Eq. S29 where z is no longer a matrix in the Hilbert space but rather a complex number with$|z| = \infty$. The actions at the standard and nonstandard saddle points are given by Eq. S32a and Eq. S32b, respectively.

For simplicity, we will focus on the fluctuations around the standard saddle point, and derive the smoothed correlation function $C^{(s)}_{nn'm'm}(\omega)$. We will show that the higher order term $-\delta_{mn}\delta_{n'm'}R_2^{(s)}(\omega)/N(N^2-1)$ in the smoothed correlation function $C^{(s)}_{nn'm'm}(\omega)$ for the CUE, missing from the earlier sigma model calculation, comes from higher order fluctuations around the standard saddle point.

C. Quadratic fluctuations around the standard saddle point

We consider first the quadratic fluctuations around the standard saddle point $Z^{(s)} = 0$ which are governed by the action:

$$\delta S_2^{(s)}[\tilde{Z}, Z] = N \sum_{ab} s_a \tilde{Z}^{ab} Z^{ba} \left[1 - \frac{1}{N} \operatorname{Tr}(\alpha_J^{bb} \beta_J^{aa})\right]. \tag{S66}$$

From this equation, we find the bare propagator for Z arising from the quadratic action at $J = 0$:

$$\left\langle Z^{ba} \tilde{Z}^{a'b'} \right\rangle_0 = s_a s_b \left\langle \tilde{Z}^{a'b'} Z^{ba} \right\rangle_0 = \delta_{aa'} \delta_{bb'} \frac{1}{N} \frac{s_a}{1 - \alpha_b \beta_a}. \tag{S67}$$

As before, we use the angular bracket with subscript 0 to denote the functional averaging with the quadratic action $\delta S_2^{(s)}$ at $J = 0$.

Performing the Gaussian integration governed by the action $\delta S_2^{(s)}$ in Eq. S66, and making use of Eq. S32a, we find the contribution to the generating function from the quadratic fluctuations around the standard saddle point

$$\mathcal{Z}^{(s)}[J] = \det(1 + J^+) \det(1 + J^-) \frac{\left[1 - \frac{1}{N} \alpha_{\rm B} \beta_{\rm F} \operatorname{Tr}\left((1 + J^-)^{-1}\right)\right] \left[1 - \frac{1}{N} \alpha_{\rm F} \beta_{\rm B} \operatorname{Tr}\left((1 + J^+)^{-1}\right)\right]}{(1 - \alpha_{\rm B} \beta_{\rm B}) \left[1 - \frac{1}{N} \alpha_{\rm F} \beta_{\rm F} \operatorname{Tr}\left((1 + J^+)^{-1} (1 + J^-)^{-1}\right)\right]}. \tag{S68}$$

Taking derivatives of the generating function $\mathcal{Z}^{(s)}[J]$ in the equation above with respect to the source field $J^{\pm}$ and setting $\alpha_{\rm B} = \alpha_{\rm F}$ and $\beta_{\rm B} = \beta_{\rm F}$, we find that the quadratic fluctuations' contribution to $\bar{C}^{(s)}_{nn'm'm}(\alpha_{\rm F}, \beta_{\rm F})$ is given by an expression identical to the earlier result Eq. S44, as expected. This also gives rise to the leading order smoothed correlation function $C^{(s)}_{nn'm'm}(\omega)$ in Eq. S45.

D. Higher order fluctuations around the standard saddle point

We now consider the higher order fluctuations and expand the action $S[\tilde{Z}, Z]$ around the standard saddle point up to sixth order in Z and $\tilde{Z}$:

$$\delta S_6[\tilde{Z}, Z] = \sum_{k=1}^{3} \frac{1}{k} \left\{ N \operatorname{Str}\left[\left(\tilde{Z} Z\right)^k\right] - \operatorname{STr}\left[\left((\tilde{Z} \otimes \mathbb{1}_{\mathcal{H}}) \alpha_J (Z \otimes \mathbb{1}_{\mathcal{H}}) \beta_J\right)^k\right]\right\}. \tag{S69}$$

From this equation, it is easier to see that the correlation function $\bar{C}_{nn'm'm}$ given by the derivatives of the generating function $\mathcal{Z}[J]$ with respect $J^+_{n'n}$ and $J^-_{mm'}$ (Eq. S17) contains two parts $\bar{C}^{(1)}_{nn'm'm}$ and $\bar{C}^{(2)}_{nn'm'm}$ which are proportional to $\delta_{nn'} \delta_{mm'}$ and $\delta_{nm} \delta_{n'm'}$, respectively. In this section, we will focus on the part $\bar{C}^{(2)}_{nn'm'm}$ and recover the higher order term $-\delta_{mn} \delta_{n'm'} R_2(\omega)/N^3$ in the CUE result Eq. S6.

From Eq. S69, one can see that $\bar{C}^{(2)}_{nn'm'm}$ arises from

$$\bar{C}^{(2)}_{nn'm'm} = \sum_{k=1}^{3} \frac{1}{k} \left\langle \frac{\partial^2 \operatorname{STr}\left[\left((\tilde{Z} \otimes \mathbb{1}_{\mathcal{H}}) \alpha_J (Z \otimes \mathbb{1}_{\mathcal{H}}) \beta_J\right)^k\right]}{\partial J^+_{n'n} \partial J^-_{mm'}} \bigg|_{J=0} \right\rangle. \tag{S70}$$

Here and throughout this section we use the angular bracket without any subscript (with subscript 0) to denote the functional averaging with the action $\delta S_6[\tilde{Z}, Z]$ in Eq. S69 (quadratic action $\delta S_2[\tilde{Z}, Z]$ in Eq. S66) at $J = 0$.

The $k = 1$ term in Eq. S70 is given by

$$\begin{aligned}
&\left\langle \frac{\partial^2 \operatorname{STr}\left[\left((\tilde{Z} \otimes \mathbb{1}_{\mathcal{H}}) \alpha_J (Z \otimes \mathbb{1}_{\mathcal{H}}) \beta_J\right)\right]}{\partial J^+_{n'n} \partial J^-_{mm'}} \bigg|_{J=0} \right\rangle = \delta_{nm} \delta_{n'm'} s_F \alpha_{\rm F} \beta_{\rm F} \left\langle \tilde{Z}^{\sf FF} Z^{\sf FF} \right\rangle \\
&= \delta_{nm} \delta_{n'm'} s_F \alpha_{\rm F} \beta_{\rm F} \left\langle \tilde{Z}^{\sf FF} Z^{\sf FF} \left(1 - S^{(4)}_{\sf int} - S^{(6)}_{\sf int} + \frac{1}{2} \left(S^{(4)}_{\sf int}\right)^2\right) \right\rangle_0 + O(1/N^4) \\
&= \delta_{nm} \delta_{n'm'} \left\{ \frac{1}{N} \frac{\alpha_{\rm F} \beta_{\rm F}}{1 - \alpha_{\rm F} \beta_{\rm F}} - \frac{1}{N^3} \frac{\alpha_{\rm F} \beta_{\rm F} \left[1 - (\alpha_{\rm F} \beta_{\rm F})^3\right]}{(1 - \alpha_{\rm F} \beta_{\rm F})^4} + \frac{1}{N^3} \frac{\alpha_{\rm F} \beta_{\rm F} \left[1 - (\alpha_{\rm F} \beta_{\rm F})^2\right]^2}{(1 - \alpha_{\rm F} \beta_{\rm F})^5} \right\} + O(1/N^4).
\end{aligned} \tag{S71}$$

Here $S_{\mathsf{int}}^{(4)}$ and $S_{\mathsf{int}}^{(6)}$ are, respectively, the quartic order ($k=2$) and sixth order ($k=3$) term in $\delta S_6[\tilde{Z}, Z]$ (Eq. S69) at $J=0$:

$$\begin{aligned}
S_{\mathsf{int}}^{(4)} &= \frac{N}{2} \sum_{a_1,b_1,a_2,b_2} s_{a_1} \tilde{Z}^{a_1 b_1} Z^{b_1 a_2} \tilde{Z}^{a_2 b_2} Z^{b_2 a_1} \left(1 - \alpha_{b_1}\beta_{a_2}\alpha_{b_2}\beta_{a_1}\right), \\
S_{\mathsf{int}}^{(6)} &= \frac{N}{3} \sum_{a_1,b_1,a_2,b_2,a_3,b_3} s_{a_1} \tilde{Z}^{a_1 b_1} Z^{b_1 a_2} \tilde{Z}^{a_2 b_2} Z^{b_2 a_3} \tilde{Z}^{a_3 b_3} Z^{b_3 a_1} \left(1 - \alpha_{b_1}\beta_{a_2}\alpha_{b_2}\beta_{a_3}\alpha_{b_3}\beta_{a_1}\right).
\end{aligned} \tag{S72}$$

In the last equality of Eq. S71, we have used the expression for the bare Z propagator at $J=0$ (Eq. S67) and set $\alpha_{\mathrm{B}} = \alpha_{\mathrm{F}}$ and $\beta_{\mathrm{B}} = \beta_{\mathrm{F}}$.

Similarly, we find the $k=2$ term in Eq. S70 :

$$\begin{aligned}
&\frac{1}{2} \left\langle \frac{\partial^2 \mathrm{STr}\left[\left((\tilde{Z}\otimes \mathbb{1}_{\mathcal{H}})\alpha_J (Z\otimes \mathbb{1}_{\mathcal{H}})\beta_J\right)^2\right]}{\partial J^+_{n'n}\partial J^-_{mm'}}\Bigg|_{J=0} \right\rangle \\
&= \delta_{nm}\delta_{n'm'} \sum_{ab} \left\langle s_a \tilde{Z}^{aF}\alpha_{\mathrm{F}} Z^{\mathsf{FF}}\beta_{\mathrm{F}}\tilde{Z}^{Fb}\alpha_b Z^{ba}\beta_a + s_F \tilde{Z}^{\mathsf{FF}}\alpha_{\mathrm{F}} Z^{Fa}\beta_a \tilde{Z}^{ab}\alpha_b Z^{bF}\beta_{\mathrm{F}} \right\rangle \\
&= \delta_{nm}\delta_{n'm'} \left\langle \sum_{ab} \left(s_a \tilde{Z}^{aF}\alpha_{\mathrm{F}} Z^{\mathsf{FF}}\beta_{\mathrm{F}}\tilde{Z}^{Fb}\alpha_b Z^{ba}\beta_a + s_F \tilde{Z}^{\mathsf{FF}}\alpha_{\mathrm{F}} Z^{Fa}\beta_a \tilde{Z}^{ab}\alpha_b Z^{bF}\beta_{\mathrm{F}} \right)\left(1 - S_{\mathsf{int}}^{(4)}\right) \right\rangle_0 + O(1/N^4) \\
&= -\delta_{nm}\delta_{n'm'} \frac{2}{N^3} \frac{(\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^2\left[1-(\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^2\right]}{(1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^4} + O(1/N^4),
\end{aligned} \tag{S73}$$

as well as the $k=3$ term:

$$\begin{aligned}
&\frac{1}{3} \left\langle \frac{\partial^2 \mathrm{STr}\left[\left((\tilde{Z}\otimes \mathbb{1}_{\mathcal{H}})\alpha_J (Z\otimes \mathbb{1}_{\mathcal{H}})\beta_J\right)^3\right]}{\partial J^+_{n'n}\partial J^-_{mm'}}\Bigg|_{J=0} \right\rangle \\
&= \delta_{nm}\delta_{n'm} \sum_{aba'b'} \left[s_a \left\langle \tilde{Z}^{aF}\alpha_{\mathrm{F}} Z^{\mathsf{FF}}\beta_{\mathrm{F}}\tilde{Z}^{Fb}\alpha_b Z^{ba'}\beta_{a'}\tilde{Z}^{a'b'}\alpha_{b'} Z^{b'a}\beta_a \right\rangle_0 \right. \\
&\left. + s_a \left\langle \tilde{Z}^{aF}\alpha_{\mathrm{F}} Z^{Fb}\beta_b \tilde{Z}^{ba'}\alpha_{a'} Z^{a'F}\beta_{\mathrm{F}}\tilde{Z}^{Fb'}\alpha_{b'} Z^{b'a}\beta_a \right\rangle_0 + s_F \left\langle \tilde{Z}^{\mathsf{FF}}\alpha_{\mathrm{F}} Z^{Fa}\beta_a \tilde{Z}^{ab}\alpha_b Z^{ba'}\beta_{a'}\tilde{Z}^{a'b'}\alpha_{b'} Z^{b'F}\beta_{\mathrm{F}} \right\rangle_0 \right] + O(1/N^4) \\
&= \delta_{nm}\delta_{n'm} \frac{3}{N^3} \left(\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}} \right)^3 + O(1/N^4).
\end{aligned} \tag{S74}$$

Combining everything, we arrive at the contribution from the first few leading order fluctuations around the standard saddle point to $\bar{C}^{(2)}_{nn'm'm}$ (i.e., the $\delta_{nm}\delta_{n'm'}$ proportional term in $\bar{C}^{(s)}_{nn'm'm}$):

$$\bar{C}^{(2)}_{nn'm'm}(\alpha_{\mathrm{F}}, \beta_{\mathrm{F}}) = \delta_{nm}\delta_{n'm'} \left[\frac{1}{N}\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}} - \frac{1}{N^3}\left(\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}} \right)^2 \right]. \tag{S75}$$

Note that the leading order first term in the equation above originates from the quadratic fluctuations (equivalent to the last term in Eq. S44), while the higher order second term arises from the higher order fluctuations. It is straightforward to see that the corresponding contribution to smoothed correlation function $C^{(s)}_{nn'm'm}$ is given by

$$C^{(s)}_{nn'm'm}(\omega) = -\,\delta_{nm}\delta_{n'm'} \left[\frac{1}{N}\frac{1}{2\pi} + \frac{1}{N^3}\frac{1}{2\pi}\left(1 - \frac{1}{2\sin^2(\omega/2)}\right) \right], \tag{S76}$$

We have therefore reproduced the higher order term $\delta_{mn}\delta_{n'm'}R_2^{(s)}(\omega)/N^3$ in the smoothed correlation function $C^{(s)}_{nn'm'm}(\omega)$ for the CUE by inclusion of higher order fluctuations.

V. CONNECTION TO PHYSICAL PROPERTIES

Once the correlation function $C_{nn'm'm}(\omega)$ defined in Eq. S1 is known, one can immediately obtain many important physical properties of the system. In this section, we use our result for the correlation function $C_{nn'm'm}(\omega)$ in Eq. S61 to evaluate the SFF, the PSFF, the dynamical correlation function of operators and the time evolution of the density matrix for the Floquet random quantum circuits under consideration.

A. Spectral form factor

From their definitions Eqs. S2 and S10, one can see that the SFF $K(t)$ is related to $C_{nn'm'm}(t)$ by

$$K(t) = \sum_{nm} C_{nnmm}(t). \tag{S77}$$

This is consistent with the RMT prediction Eq. S9. In particular, if we substitute Eq. S9 into the right-hand-side (RHS) of the equation above, one can immediate see that it reduces to

$$\sum_{nm} C_{nnmm}(t) = \sum_{nm} \left(\frac{K(t)-1}{N^2-1} + \frac{-K(t)/N + N}{N^2-1}\delta_{mn} \right) = K(t), \tag{S78}$$

identical to the left-hand-side (LHS) of Eq. S77.

Using our leading order in $1/N$ result for $C_{nn'm'm}$ in Eq. S61, we find

$$\sum_{nm} C_{nnmm}(t) \approx \sum_{nm} \left(\frac{K(t)-1+\delta_{t,0}}{N^2} + \frac{-\delta_{t,0}+1}{N}\delta_{mn} \right) = K(t). \tag{S79}$$

Here $K(t)$ on the RHS is the CUE SFF in Eq. S11. Eq. S79 shows that the SFF for the Floquet random quantum circuits under study agrees with the RMT result Eq. S11. Note that the difference between the first equalities of Eq. S78 and Eq. S79 arises from the higher order terms in $C_{nnmm}(t)$, which are not important for the leading order SFF.

B. Partial spectral form factor

The partial spectral form factor (PSFF) is defined as [S22, S23]

$$K_{\mathcal{A}}(t) = \left\langle \mathrm{Tr}_{\bar{\mathcal{A}}} \left[\mathrm{Tr}_{\mathcal{A}}\, U(t)\, \mathrm{Tr}_{\mathcal{A}}\, U^\dagger(t) \right] \right\rangle, \tag{S80}$$

where $\mathrm{Tr}_{\mathcal{A}}$ and $\mathrm{Tr}_{\bar{\mathcal{A}}}$ represent the trace operators over the Hilbert spaces of the subsystem $\mathcal{A}$ and its complement $\bar{\mathcal{A}}$, respectively. In terms of $C_{nn'm'm}(t)$, PSFF can be expressed as

$$K_A(t) = \sum_{i_1^{\mathcal{A}}, i_2^{\mathcal{A}}=1}^{N_{\mathcal{A}}} \sum_{j_1^{\bar{\mathcal{A}}}, j_2^{\bar{\mathcal{A}}}=1}^{N/N_{\mathcal{A}}} C_{(i_1^{\mathcal{A}}, j_1^{\bar{\mathcal{A}}}),(i_1^{\mathcal{A}}, j_2^{\bar{\mathcal{A}}}),(i_2^{\mathcal{A}}, j_2^{\bar{\mathcal{A}}}),(i_2^{\mathcal{A}}, j_1^{\bar{\mathcal{A}}})}(t), \tag{S81}$$

with $N_{\mathcal{A}}$ being the dimension of the Hilbert space of subsystem $\mathcal{A}$. Here we have used a two-component vector $(i^A, j^{\bar{A}})$ to label the Hilbert space of the entire system, whose first and second components label the subsystem $\mathcal{A}$ and its complement $\bar{\mathcal{A}}$, respectively.

Inserting the expression for the CUE correlation function $C_{nn'm'm}(t)$ Eq. S9 into the equation above, we find the RMT prediction for PSFF:

$$\begin{aligned} K_{\mathcal{A}}(t) = & \sum_{i_1^{\mathcal{A}}, i_2^{\mathcal{A}}=1}^{N_{\mathcal{A}}} \sum_{j_1^{\bar{\mathcal{A}}}, j_2^{\bar{\mathcal{A}}}=1}^{N/N_{\mathcal{A}}} \left(\frac{K(t)-1}{N^2-1}\delta_{j_1^{\bar{\mathcal{A}}}, j_2^{\bar{\mathcal{A}}}} + \frac{-K(t)/N+N}{N^2-1}\delta_{i_1^{\mathcal{A}}, i_2^{\mathcal{A}}} \right) = \frac{K(t)-1}{N^2-1} N N_{\mathcal{A}} + \frac{-K(t)/N+N}{N^2-1}\frac{N^2}{N_{\mathcal{A}}} \\ = & \frac{N N_{\mathcal{A}} - N/N_{\mathcal{A}}}{N^2-1} K(t) + \frac{N^3/N_{\mathcal{A}} - N N_{\mathcal{A}}}{N^2-1}, \end{aligned} \tag{S82}$$

This equation is equivalent to the result presented in Ref. [S22] as expected.

On the other hand, our leading order result for $C_{nn'm'm}$ Eq. S61 leads to

$$\begin{aligned}K_{\mathcal{A}}(t) \approx &\sum_{i_1^{\mathcal{A}},i_2^{\mathcal{A}}=1}^{N_{\mathcal{A}}} \sum_{j_1^{\bar{\mathcal{A}}},j_2^{\bar{\mathcal{A}}}=1}^{N/N_{\mathcal{A}}} \left(\frac{K(t)-1+\delta_{t,0}}{N^2} \delta_{j_1^{\bar{\mathcal{A}}},j_2^{\bar{\mathcal{A}}}} + \frac{-\delta_{t,0}+1}{N} \delta_{i_1^{\mathcal{A}},i_2^{\mathcal{A}}} \right) = \frac{K(t)-1+\delta_{t,0}}{N^2} N N_{\mathcal{A}} + \frac{-\delta_{t,0}+1}{N} \frac{N^2}{N_{\mathcal{A}}} \\ =&\frac{N_{\mathcal{A}}}{N} K(t) + \left(\frac{N}{N_{\mathcal{A}}} - \frac{N_{\mathcal{A}}}{N} \right) (1-\delta_{t,0}).\end{aligned} \tag{S83}$$

This is consistent with the RMT result Eq. S82 when $N_{\mathcal{A}}$ the Hilbert space dimension of the subsystem $\mathcal{A}$ satisfies $N_{\mathcal{A}} \gg 1$.

We emphasize that, for the PSFF, the contribution from the higher order in $1/N$ terms in $C_{\mathbf{nn'm'm}}$ is non-negligible when $N_{\mathcal{A}} \sim O(1)$ (which is of no relevance to the current model with local Hilbert space dimension $q \to \infty$). The number of terms of the form $C_{\mathbf{nn'm'm}}$ contributing to the summation in Eq. S81 is

$$\bar{N} = \begin{cases} N N_{\mathcal{A}}, & (1)\, \mathbf{n}=\mathbf{n'}, \mathbf{m'}=\mathbf{m}, \\ N^2/N_{\mathcal{A}} - N, & (2)\, \mathbf{n}=\mathbf{m}, \mathbf{n'}=\mathbf{m'}, \mathbf{n}\neq\mathbf{n'}, \\ N^2 - N N_{\mathcal{A}} - N^2/N_{\mathcal{A}} + N, & (3)\, \text{otherwise}. \end{cases} \tag{S84}$$

When $N_{\mathcal{A}} \gg 1$, the numbers of total contributing terms $\bar{N}$ for categories (2) and (3) are, respectively, a factor of $O(N/N_{\mathcal{A}}^2)$ and $O(N/N_{\mathcal{A}})$ larger compared with that of categroy (1). From the earlier result, $C_{\mathbf{nn'm'm}}(\omega)$ with indices lie in the categories (2) and (3) are of higher order in $1/N$ compared with that of category (1). This means that we can ignore the contribution from the higher order terms in $C_{\mathbf{nn'm'm}}$. On the other hand, when $N_{\mathcal{A}} \sim O(1)$, the higher order in $1/N$ terms in $C_{\mathbf{nn'm'm}}$ for categories (2) and (3) are required to obtain the leading order result for the PSFF.

C. Correlation function of operators

We now consider the dynamical correlation function of operators O and O':

$$C_{OO'}(t) = \frac{1}{N} \langle \mathrm{Tr}\,(O(t)O') \rangle = \frac{1}{N} \left\langle \mathrm{Tr}\left(U^\dagger(t) O U(t) O'\right) \right\rangle, \tag{S85}$$

whose Fourier transform is

$$C_{OO'}(\omega) = \frac{1}{N} \left\langle \sum_{\nu,\mu=1}^{N} O_{\mu\nu} O'_{\nu\mu} \delta(\omega - E_\nu + E_\mu) \right\rangle. \tag{S86}$$

It is not difficult to see that the operator correlation function $C_{OO'}(\omega)$ is related to $C_{nn'm'm}(\omega)$ by

$$C_{OO'}(\omega) = \frac{1}{N} \sum_{nmn'm'} O_{mn} O'_{n'm'} C_{nn'm'm}(\omega). \tag{S87}$$

For any system whose correlation function $C_{nn'm'm}(\omega)$ follows the RMT prediction Eq. S6, we have

$$C_{OO'}(\omega) = \frac{R_2(\omega)-\delta(\omega)}{N^2-1} \left(\frac{1}{N} \mathrm{Tr}(OO') - \frac{1}{N^2} \mathrm{Tr}(O)\,\mathrm{Tr}(O') \right) + \frac{\delta(\omega)}{N^2} \mathrm{Tr}(O)\,\mathrm{Tr}(O'). \tag{S88}$$

For the current model, if we insert our leading order in $1/N$ result for the correlation function $C_{nn'm'm}(\omega)$ in Eq. S61, we arrive at

$$C_{OO'}(\omega) \approx (R_2(\omega) - \delta(\omega) + \frac{1}{2\pi}) \frac{1}{N^3} \mathrm{Tr}(OO') + \left(\delta(\omega) - \frac{1}{2\pi} \right) \frac{1}{N^2} \mathrm{Tr}(O)\,\mathrm{Tr}(O'), \tag{S89}$$

which, for traceless operators ($\mathrm{Tr}\,O = 0$, $\mathrm{Tr}\,O' = 0$), reduces to

$$C_{OO'}(\omega) \approx \frac{R_2(\omega)}{N^3} \mathrm{Tr}(OO'). \tag{S90}$$

We note that the difference between the Eq. S88 and Eq. S89 is due to the higher order terms in the correlation function $C_{nn'm'm}(\omega)$. These higher order terms may be non-negligible for the leading order dynamical correlation function of some operators, depending on the specific operators under consideration.

We now consider the case where the operators O and O' act on the same qudit. For simplicity, we use $\mathbf{n} = (n_0, \mathbf{n}^{(L-1)})$ to label the many-body state of the entire system which consists of L qudits. Here n_0 labels the single-particle state of the qudit on which O and O' operator, and the $(L-1)$-dim vector $\mathbf{n}^{(L-1)}$ labels the many-body state of the remaining $L-1$ qudits. Using this notation, we can see that $O_{\mathbf{nm}}$ is nonzero only when $\mathbf{n}^{(L-1)} = \mathbf{m}^{(L-1)}$. Therefore, the number of terms of the form $C_{\mathbf{nn'm'm}}$ contributing to the summation in Eq. S87 is of the order of

$$\bar{N} = \begin{cases} O(Nq), & (1)\, \mathbf{n} = \mathbf{n'}, \mathbf{m'} = \mathbf{m}, \\ O(N^2), & (2)\, \mathbf{n} = \mathbf{m}, \mathbf{n'} = \mathbf{m'}, \mathbf{n} \neq \mathbf{n'}, \\ O(N^2 q^2), & (3)\, \text{otherwise}. \end{cases} \tag{S91}$$

Similarly, if O and O' act on different qudits, we instead have

$$\bar{N} = \begin{cases} O(N), & (1)\, \mathbf{n} = \mathbf{n'}, \mathbf{m'} = \mathbf{m}, \\ O(N^2), & (2)\, \mathbf{n} = \mathbf{m}, \mathbf{n'} = \mathbf{m'}, \mathbf{n} \neq \mathbf{n'}, \\ O(N^2 q^2), & (3)\, \text{otherwise}. \end{cases} \tag{S92}$$

We note that, to the second leading order, $C_{nn'm'm}$ can be expressed as

$$C_{nn'm'm}(\omega) = c_1(\omega)\delta_{nn'}\delta_{mm'} + c_2(\omega)\delta_{nm}\delta_{n'm'}. \tag{S93}$$

Here $c_{1,2}(\omega)$ are functions of ω (and N), and are independent of the specific indices n, n', m, m'. This equation can be proven making use of the sigma model action $S[\tilde{Z}, Z]$ (Eq. S23) and the fact that the first few moments of the many-body Floquet operator U are nonvanishing to the leading order only if the row/column indices of all the unitary matrices U in the moment are given by the column/row indices of all the $U^\dagger$ matrices after a permutation [S14]. Specifically, one can apply the Wick's theorem to show that, up to the second leading order, $\langle \partial^2 S/\partial J^+_{n'n}\partial J^-_{mm'}\rangle$ and $\langle (\partial S/\partial J^+_{n'n})(\partial S/\partial J^-_{mm'})\rangle$ are nonzero only when the external Hilbert space row indices n, m' are related to the column indices n', m by a permutation, and they take values independent of the specific external indices. Eq. S93 then allows us to ignore the contribution from $C_{nn'm'm}(\omega)$ whose indices fall into categories (2) and (3) in Eq. S91 or Eq. S92, for traceless local operators O and O'. Therefore, one can use the leading order result for $C_{nn'm'm}(\omega)$ to determine the dynamical correlation function $C_{OO'}(\omega)$ for the traceless local operators O and O' acting on the same qudit or different qudits, and the result is given by Eq. S90.

D. Time evolution of the density matrix

Using the correlation function of the quasienergy eigenstates $C_{nn'm'm}(t)$, one can also study the time evolution of the density matrix $\rho(t) = U(t)\rho(0)U^\dagger(t)$:

$$\langle \rho_{nm}(t)\rangle = \sum_{n',m'=1}^{N} \rho_{n'm'}(t=0) C_{nn'm'm}(t), \tag{S94}$$

where $\rho(0)$ denotes the density matrix at time $t = 0$. For the CUE, one can insert Eq. S9 into the equation above and find

$$\langle \rho_{nm}(t)\rangle = (\rho_{th})_{nm}\frac{N^2 - K(t)}{N^2 - 1} + \rho_{nm}(0)\frac{K(t) - 1}{N^2 - 1}, \tag{S95}$$

where $\rho_{th} \equiv \mathbb{1}_{\mathcal{H}}/N$ is just the thermal density matrix and we have used $\mathrm{Tr}(\rho) = 1$. At large time $t \to \infty$, the SFF acquires the value $K(t \to \infty) = N$ and the equation above becomes

$$\langle \rho_{nm}(t \to \infty)\rangle = (\rho_{th})_{nm}\frac{N}{N+1} + \rho_{nm}(0)\frac{1}{N+1} \approx (\rho_{th})_{nm} + \rho_{nm}(0)\frac{1}{N}. \tag{S96}$$

Using instead the leading order result for $C_{nn'm'm}$ derived earlier for the Floquet random quantum circuits under study, we have

$$\langle \rho_{nm}(t)\rangle = (\rho_{th})_{nm}(1-\delta_{t,0}) + \rho_{nm}(0)\frac{K(t)-1+\delta_{t,0}}{N^2}, \tag{S97}$$

which at large time reduces to Eq. S96. We emphasize that, the contribution from the higher order terms in $C_{nn'm'm}(t)$ to $\langle \rho_{nm}(t)\rangle$ could be as important as that from the leading order term. In this case, consideration of the higher order fluctuations in the sigma model is required to study the time evolution of the density matrix. We leave this for the future investigation.

Note that the second term in Eq. S96 can be neglected when consider its contribution to the expectation value of a physical observable O. Within RMT, it is straightforward to see from Eq. S95 that the expectation value of O at time t is

$$\langle \mathrm{Tr}\left(O\rho(t)\right)\rangle = \frac{1}{N}\,\mathrm{Tr}(O)\frac{N^2-K(t)}{N^2-1} + \mathrm{Tr}(O\rho(0))\frac{K(t)-1}{N^2-1}, \tag{S98}$$

which is identical to the expression in Ref. [S24]. At large time $t\to\infty$, this equation reduces to

$$\langle \mathrm{Tr}\left(O\rho(t\to\infty)\right)\rangle \approx \frac{1}{N}\,\mathrm{Tr}(O) + \frac{1}{N}\,\mathrm{Tr}(\rho(0)O). \tag{S99}$$

The second term in the equation above arises from $\rho_{nm}(0)/N$ in $\langle \rho_{nm}(t\to\infty)\rangle$ (Eq. S96), and is of the order of $1/N$ smaller than the first term above, which is just the (infinite temperature) thermal average of operator O.

The discussion above shows that, at times larger than the plateau time (i.e., the time at which the SFF reaches the plateau $K(t)=N$), the expectation value of O is given approximately by its thermal equilibrium value $\mathrm{Tr}(O)/N$. The plateau time is of the order of the inverse level spacing and thus is exponentially large in system size. We emphasize that the relaxation time - the time at which the expectation value of observable O reaches the thermal equilibrium value - is actually much smaller than the plateau time [S13]. Let us now assume that the density matrix evolution is indeed described by the RMT prediction Eq. S95, and is therefore determined by the behavior of the SFF. Note that the SFF for generic chaotic systems has a slope-ramp-plateau structure. It drops nonuniversally from N^2 at $t=0$, reaches the dip, and then grows linearly (for the unitary symmetry class) until it hits the plateau at $K(t\to\infty)=N$ [S9, S10]. In this case, when the SFF takes a value $K(t)\lesssim N$, which happens at a time much earlier than the plateau time in the slope regime, the density matrix becomes approximately the thermal one $\rho_{th}=\mathbb{I}_{\mathcal{H}}/N$ in the sense that the remaining term's contribution to the expectation value of any physical observable is negligible. Moreover, $\langle\rho(t)\rangle$ remains around ρ_{th} as $K(t)\lesssim N$ for all remaining time. This also suggests that the relaxation towards thermalization process is related to the early-time non-universal slope in the SFF [S24]. On the other hand, the RMT prediction Eq. S95 may only be valid for generic ergodic systems at late times. It would be an interesting future direction to investigate the connection between the relaxation process and SFF, which may require consideration of higher order fluctuations as well as massive-mode fluctuations essential for the slope regime of the SFF [S25–S27]. Moreover, one can also estimate the statistical fluctuation of the density matrix $\rho(t)$ around the ensemble averaged result $\langle\rho(t)\rangle$ by evaluating the higher-order correlation functions of quasienergy eigenstates in the current sigma model framework.

VI. DIAGONAL MATRIX ELEMENT

In this section, we investigate the ensemble average of the diagonal matrix elements of an arbitrary physical observable O in the quasienergy eigenbasis for the Floquet random quantum circuits we studied. Specifically, we consider the quantity $\langle\sum_\nu O_{\nu\nu}\delta(\omega-E_\nu)\rangle$ which represents the ensemble average of $O_{\nu\nu}$ for the quasienergy eigenstate $|\nu\rangle$ with the eigenvalue $E_\nu=\omega$. Its Fourier transform can be expressed as

$$\left\langle \sum_\nu O_{\nu\nu}e^{-iE_\nu t}\right\rangle = \sum_{nm} O_{nm}\left\langle \sum_\nu \langle\nu|n\rangle\,\langle m|\nu\rangle\, e^{-iE_\nu t}\right\rangle = \sum_{nm} O_{nm}\left\langle\langle m|U^t|n\rangle\right\rangle. \tag{S100}$$

As before, $|\nu\rangle$ denotes the eigenstate of the Floquet operator U, while $|n\rangle\,,|m\rangle$ belong to a fixed set of basis states.

For the current model, it is obvious that

$$\left\langle\langle m|U^t|n\rangle\right\rangle = \delta_{t,0}\delta_{mn}. \tag{S101}$$

Inserting this expression into Eq. S100, we find

$$\left\langle \sum_{\nu} O_{\nu\nu} e^{-iE_\nu t} \right\rangle = \delta_{t,0}\, \mathrm{Tr}\, O, \tag{S102}$$

which after Fourier transform leads to

$$\left\langle \sum_{\nu} O_{\nu\nu} \delta(\omega - E_\nu) \right\rangle = \frac{1}{2\pi}\, \mathrm{Tr}\, O. \tag{S103}$$

This result is consistent with the eigenstate thermalization hypothesis (ETH) [S28–S30]. For Floquet systems, ETH postulates that the diagonal matrix element of the physical observable O in the quasienergy eigenbasis acquires the form $O_{\nu\nu} = \frac{1}{N}\,\mathrm{Tr}\, O + O(N^{-1/2})$ [S31, S32]. In particular, if we insert the ETH ansatz into the left hand side of Eq. S103, we arrive at

$$\left\langle \sum_{\nu} O_{\nu\nu} \delta(\omega - E_\nu) \right\rangle = \frac{R_1(\omega)}{N}\, \mathrm{Tr}\, O. \tag{S104}$$

Here $R_1(\omega)$ is the mean level density defined as

$$R_1(\omega) = \left\langle \sum_{\nu} \delta(\omega - E_\nu) \right\rangle. \tag{S105}$$

By setting O to an identical matrix in Eq. S103, one can prove that it takes the value of

$$R_1(\omega) = \frac{N}{2\pi}. \tag{S106}$$

Inserting Eq. S106 into Eq. S104 gives rise to Eq. S103.

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[S33] In principal, one can directly set $\alpha_{\mathrm{B}} = \alpha_{\mathrm{F}}$ and $\beta_{\mathrm{B}} = \beta_{\mathrm{F}}$ at the beginning of the calculation. However, as will become apparent later, it is more convenient to apply this procedure at the end.