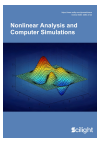
Scilight

Nonlinear Analysis and Computer Simulations
https://www.sciltp.com/journals/nacs



Article

A Deformed L1 Finite-Difference Algorithm for a (q, τ)-Fractional Heat-Conduction Model

Rabha W. Ibrahim [1,2]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences (SIMATS), Chennai 602105, Tamil Nadu, India; rabhaibrahim@yahoo.com or rabhaibrahim133@gmail.com
[2] Information and Communication Technology Research Group, Scientific Research Center, Al-Ayen University, Nasiriyah 64001, Thi-Qar, Iraq

How To Cite: Ibrahim, R.W. A Deformed L1 Finite-Difference Algorithm for a (q, τ)-Fractional Heat-Conduction Model. *Nonlinear Analysis and Computer Simulations* **2026**, *1*(4), 19. https://doi.org/10.53941/nacs.2026.100019

Received: 3 October 2026
Revised: 3 November 2026
Accepted: 24 November 2026
Published: 26 November 2026

Abstract: This paper develops a deformed L1 finite-difference algorithm for a (q, τ)-fractional heat-conduction model with a regularized Caputo-type memory kernel. The deformation enters through the effective parameter $\varepsilon_{q,\tau} = \tau(1 - q)$. We establish consistency properties of the associated deformed Gamma and Mittag–Leffler functions and obtain an L^1 kernel estimate that quantifies convergence to the classical Caputo operator at fixed fractional order. An explicit example shows that the classical Caputo and first-order limits must be taken in the specified order rather than as an unrestricted joint limit. For fixed admissible deformation parameters and compatible initial data, existence, uniqueness, and continuous dependence of strong solutions are proved using a Volterra formulation and a contraction argument. An implicit deformed L1 scheme is derived for a diffusion–reaction heat equation, and the discrete system is proved uniquely solvable at each time level. A manufactured-solution benchmark specifies an exact test problem for assessing spatial accuracy. The results establish operator-level asymptotic consistency and fixed-parameter solvability without asserting uniform convergence of the corresponding evolution solutions.

Keywords: (q, τ)-fractional calculus; Caputo fractional derivative; deformed Gamma function; fractional heat conduction; fractional differential equation; Mittage-Leffler function

2020 MSC: 26A33; 34A08; 34A55

1. Introduction

Fractional evolution equations provide an important mathematical framework for describing processes with memory, hereditary effects, anomalous diffusion, and nonlocal temporal dynamics [1–3]. In contrast to classical integer-order evolution equations, fractional models incorporate the past history of the state variable through singular convolution kernels [4–6]. This feature makes them suitable for modeling diffusion, relaxation, viscoelasticity, transport, biological processes, and other systems in which the present state depends not only on the instantaneous dynamics but also on accumulated memory effects [7–10]. Among fractional operators, the Caputo derivative is particularly useful in evolution problems [11–14] because it allows the use of classical initial conditions. For $0 < \alpha < 1$, the classical Caputo derivative is given by

$$ {}^{C}D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\, ds. $$

The singular kernel $(t - s)^{-\alpha}$ controls the memory strength, and the parameter α determines the degree of nonlocality. As $\alpha \to 1^-$, the fractional derivative approaches the standard first-order derivative, and the corresponding fractional evolution equation converges formally to a classical parabolic evolution equation. In recent years, deformed and generalized fractional operators have attracted attention because they offer additional flexibility in modeling nonlocal phenomena. In such operators, extra deformation parameters are introduced into the memory kernel or the integral structure. These parameters allow the model to describe a wider range of transient behaviors,

Copyright: © 2026 by the authors. This is an open access article under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).
Publisher's Note: Scilight stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

including enhanced damping, modified memory decay, and deviations from standard fractional dynamics. However, the introduction of deformation parameters must be supported by rigorous mathematical analysis. In particular, one must clarify whether the deformed model is consistent with the corresponding classical fractional model when the deformation vanishes. The present paper focuses on a (q,τ)-deformed fractional evolution equation. The parameter $q \in (0,1)$ measures the departure from the classical fractional regime, while $\tau > 0$ controls the deformation scale [15–17]. The main asymptotic regime is $q \to 1^-$, $\tau \to 0^+$. In this regime, the deformation disappears and the deformed memory kernel should converge to the classical Caputo kernel. Therefore, the pair (q,τ) can be viewed as a set of perturbation parameters, and the corresponding family of deformed evolution equations can be studied through asymptotic analysis.

The connection between (q,τ)-deformation and asymptotic analysis is central to this work. Here we study the deformation limit for the underlying kernels and operators. In the proposed setting, the limiting process $(q,\tau) \to (1,0)$ connects the deformed fractional model to the classical Caputo fractional model. A further limiting process, $\alpha \to 1^-$, connects the fractional model to the standard parabolic evolution equation. At the operator level, the ordered limiting procedures lead from (q,τ)-deformed evolution problem $\Rightarrow$ classical fractional evolution problem $\Rightarrow$ classical parabolic evolution problem. The objective is to establish the corresponding limits for the operators, rather than asserting convergence of solutions of the evolution problems. We introduce a Caputo-type (q,τ)-fractional derivative through a positive regularized memory kernel of the form $\big(t-s+\tau(1-q)\big)^{-\alpha}$. This kernel reduces to the standard Caputo kernel $(t-s)^{-\alpha}$ as $q \to 1^-$ and $\tau \to 0^+$. The regularization term $\tau(1-q)$ controls the strength of the deformation and avoids treating q and τ as merely empirical tuning constants. Instead, they are interpreted as asymptotic perturbation parameters governing the transition between deformed and classical dynamics. We consider an abstract evolution problem of the form

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}u(t) + \mathcal{A}_{q,\tau}u(t) = f(t,u(t)), \qquad 0 < t \le T, $$

where $\mathcal{A}_{q,\tau}$ is an elliptic operator and f is a Lipschitz nonlinear term. The main contributions of this study are the analysis of a regularized (q,τ)-Caputo-type operator and its numerical implementation for heat conduction. First, we introduce the deformed Gamma function and establish consistency properties of the corresponding Mittag–Leffler function, memory kernel, and fractional derivative. The effective parameter $\varepsilon_{q,\tau} = \tau(1-q)$ controls these operator-level limits, so the individual effects of q and τ are not separately identifiable in the present formulation. Second, for fixed deformation parameters, suitable operator assumptions, and compatible initial data, we prove existence, uniqueness, and continuous dependence of strong solutions via an equivalent Volterra equation and a contraction argument. An explicit example clarifies why the classical Caputo limit and the subsequent first-order limit must be taken in order. Third, we derive an implicit deformed L1 finite-difference scheme for a heat-conduction equation and prove unique solvability of its tridiagonal system at every time level. An engineering formulation and a manufactured-solution benchmark illustrate how the discrete method can be used. Uniform convergence of solutions with respect to the deformation parameters and a fully discrete error theorem are not claimed here.

2. The (q,τ)-Fractional Integral and Caputo Derivative

We now define the (q,τ)-fractional integral and the associated Caputo-type (q,τ)-fractional derivative used in the evolution model. Throughout this subsection, we assume $0 < \alpha < 1$, $0 < q < 1$, $\tau > 0$. Let $u : [0,T] \to X$ be a sufficiently regular function, where X is a Banach space, for example $X = L^2(\Omega)$.

Definition 1 ((q,τ)-deformed fractional kernel)**.** *The (q,τ)-deformed fractional kernel of order α is defined by*

$$ K_{\alpha}^{(q,\tau)}(t) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\big(t+\tau(1-q)\big)^{-\alpha}, \qquad t > 0, \tag{1} $$

where $\Gamma_{q,\tau}(\cdot)$ is a positive (q,τ)-deformed Gamma function satisfying the consistency condition

$$ \Gamma_{q,\tau}(1-\alpha) \longrightarrow \Gamma(1-\alpha) \qquad \text{as } q \to 1^-,\ \tau \to 0^+. \tag{2} $$

Consequently,

$$ K_{\alpha}^{(q,\tau)}(t) \longrightarrow \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \qquad \text{as } q \to 1^-,\ \tau \to 0^+. \tag{3} $$

Definition 2 ((q,τ)-fractional integral)**.** *Let $0 < \alpha < 1$. The left-sided (q,τ)-fractional integral of order α is defined by*

$$\big(I_{0+}^{\alpha;(q,\tau)}u\big)(t) = \int_0^t J_\alpha^{(q,\tau)}(t-s)u(s)\,ds, \qquad 0 < t \le T, \tag{4}$$

where

$$J_\alpha^{(q,\tau)}(t) = \frac{1}{\Gamma_{q,\tau}(\alpha)}\big(t+\tau(1-q)\big)^{\alpha-1}, \qquad t > 0. \tag{5}$$

Equivalently,

$$\big(I_{0+}^{\alpha;(q,\tau)}u\big)(t) = \frac{1}{\Gamma_{q,\tau}(\alpha)}\int_0^t \big(t-s+\tau(1-q)\big)^{\alpha-1}u(s)\,ds. \tag{6}$$

When $q \to 1^-$ and $\tau \to 0^+$, we recover the classical Riemann–Liouville fractional integral:

$$\big(I_{0+}^{\alpha;(q,\tau)}u\big)(t) \longrightarrow \big(I_{0+}^{\alpha}u\big)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds. \tag{7}$$

Definition 3 (Caputo-type (q,τ)-fractional derivative)**.** *Let $0 < \alpha < 1$ and assume that $u \in W^{1,1}(0,T;X)$. The left-sided Caputo-type (q,τ)-fractional derivative of order α is defined by*

$$\big({}^{C}D_{0+}^{\alpha;(q,\tau)}u\big)(t) = \big(I_{0+}^{1-\alpha;(q,\tau)}u'\big)(t), \qquad 0 < t \le T. \tag{8}$$

Equivalently,

$$\big({}^{C}D_{0+}^{\alpha;(q,\tau)}u\big)(t) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t \big(t-s+\tau(1-q)\big)^{-\alpha}u'(s)\,ds. \tag{9}$$

The operator ${}^{C}D_{0+}^{\alpha;(q,\tau)}$ is therefore a memory operator.

Example 1 (Joint and iterated classical limits)**.** *Let $u(t) = t$ on $[0,T]$, and set*

$$\delta = 1-\alpha \in (0,1), \qquad \varepsilon = \tau(1-q) > 0.$$

For the normalization

$$\Gamma_{q,\tau}(\delta) = \frac{\Gamma(\delta+\varepsilon)}{\Gamma(1+\varepsilon)},$$

Definition 3 gives

$${}^{C}D_{0+}^{1-\delta;(q,\tau)}u(t) = \frac{(t+\varepsilon)^\delta - \varepsilon^\delta}{\delta\Gamma_{q,\tau}(\delta)}, \qquad t > 0. \tag{10}$$

The unrestricted joint limit as $\delta \to 0^+$, $q \to 1^-$, and $\tau \to 0^+$ does not exist. However, the iterated limit satisfies

$$\lim_{\delta\to 0^+}\left[\lim_{\substack{q\to 1^-\\ \tau\to 0^+}} {}^{C}D_{0+}^{1-\delta;(q,\tau)}u(t)\right] = 1 = u'(t) \tag{11}$$

for every fixed $t \in (0,T]$. Since $u'(s) = 1$, direct integration gives

$$\begin{aligned} {}^{C}D_{0+}^{1-\delta;(q,\tau)}u(t) &= \frac{1}{\Gamma_{q,\tau}(\delta)}\int_0^t (t-s+\varepsilon)^{\delta-1}\,ds \\ &= \frac{(t+\varepsilon)^\delta - \varepsilon^\delta}{\delta\Gamma_{q,\tau}(\delta)}, \end{aligned}$$

which proves (10). First, consider the simultaneous parameter path

$$\varepsilon = e^{-1/\delta}, \qquad q = 1-\sqrt{\varepsilon}, \qquad \tau = \sqrt{\varepsilon}.$$

These parameters are admissible and satisfy

$$q \to 1^-, \qquad \tau \to 0^+, \qquad \tau(1-q) = \varepsilon.$$

The Gamma recurrence formula yields

$$\begin{aligned}\delta\Gamma_{q,\tau}(\delta) &= \frac{\delta\Gamma(\delta+\varepsilon)}{\Gamma(1+\varepsilon)} \\ &= \frac{\delta}{\delta+\varepsilon}\frac{\Gamma(1+\delta+\varepsilon)}{\Gamma(1+\varepsilon)} \longrightarrow 1,\end{aligned}$$

because $\varepsilon/\delta \to 0$. *For every fixed* $t > 0$,

$$(t+\varepsilon)^{\delta} = \exp\big(\delta\log(t+\varepsilon)\big) \longrightarrow 1, \qquad \varepsilon^{\delta} = e^{-1}.$$

Consequently, we get

$$ {}^{C}D_{0+}^{1-\delta;(q,\tau)}u(t) \longrightarrow 1 - e^{-1}. \tag{12}$$

To demonstrate that the joint limit does not exist, consider a second admissible path:

$$\varepsilon = e^{-1/\delta^2}, \qquad q = 1-\sqrt{\varepsilon}, \qquad \tau = \sqrt{\varepsilon}.$$

Again, $\varepsilon/\delta \to 0$, *so the same Gamma recurrence argument gives*

$$\delta\Gamma_{q,\tau}(\delta) \longrightarrow 1.$$

In this case,

$$(t+\varepsilon)^{\delta} \longrightarrow 1, \qquad \varepsilon^{\delta} = e^{-1/\delta} \longrightarrow 0.$$

Therefore,

$$ {}^{C}D_{0+}^{1-\delta;(q,\tau)}u(t) \longrightarrow 1. \tag{13}$$

The distinct limits in (12) *and* (13) *prove that the unrestricted joint limit does not exist. Finally, fix* $\delta \in (0,1)$ *and first let* $q \to 1^{-}$ *and* $\tau \to 0^{+}$. *Then* $\varepsilon \to 0^{+}$ *and* $\Gamma_{q,\tau}(\delta) \longrightarrow \Gamma(\delta)$. *Hence, we obtain*

$$\begin{aligned}\lim_{\substack{q\to 1^{-}\\ \tau\to 0^{+}}} {}^{C}D_{0+}^{1-\delta;(q,\tau)}u(t) &= \frac{t^{\delta}}{\delta\Gamma(\delta)} \\ &= \frac{t^{\delta}}{\Gamma(1+\delta)}.\end{aligned}$$

Taking $\delta \to 0^{+}$ *now gives* $\lim_{\delta\to 0^{+}} \frac{t^{\delta}}{\Gamma(1+\delta)} = 1$, *which proves* (11).

Proposition 1 (Linearity)**.** *Let* $u, v \in W^{1,1}(0,T;X)$ *and let* $a, b \in \mathbb{R}$. *Then*

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}(au+bv)(t) = a\,{}^{C}D_{0+}^{\alpha;(q,\tau)}u(t) + b\,{}^{C}D_{0+}^{\alpha;(q,\tau)}v(t). \tag{14}$$

Proof. Using Definition 3, we have

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}(au+bv)(t) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t \big(t-s+\tau(1-q)\big)^{-\alpha}(au'(s)+bv'(s))\,ds.$$

By linearity of the Bochner integral,

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}(au+bv)(t) = a\,{}^{C}D_{0+}^{\alpha;(q,\tau)}u(t) + b\,{}^{C}D_{0+}^{\alpha;(q,\tau)}v(t).$$

□

Proposition 2 (Derivative of a constant)**.** *If* $u(t) = c$, *where* $c \in X$ *is constant, then*

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}u(t) = 0. \tag{15}$$

Proof. Since $u(t) = c$, we have $u'(t) = 0$. Therefore,

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}u(t) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t \big(t-s+\tau(1-q)\big)^{-\alpha}u'(s)\,ds = 0.$$

□

Proposition 3 (Classical consistency)**.** *Let $u \in W^{1,1}(0,T;X)$. Assume that $\Gamma_{q,\tau}(1-\alpha) \to \Gamma(1-\alpha)$ as $q \to 1^-$ and $\tau \to 0^+$. Then*

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}u(t) \longrightarrow {}^{C}D_{0+}^{\alpha}u(t) \tag{16}$$

for every $t \in (0,T]$ for which the classical Caputo derivative exists.

Proof. By Definition 3,

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}u(t) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \int_0^t \big(t-s+\tau(1-q)\big)^{-\alpha} u'(s)\, ds. $$

As $q \to 1^-$ and $\tau \to 0^+$, we have $\Gamma_{q,\tau}(1-\alpha) \to \Gamma(1-\alpha)$, and $\big(t-s+\tau(1-q)\big)^{-\alpha} \to (t-s)^{-\alpha}$. Hence,

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}u(t) \to \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\, ds = {}^{C}D_{0+}^{\alpha}u(t). $$

Therefore, the (q,τ)-Caputo derivative is asymptotically consistent with the classical Caputo derivative. □

Asymptotic analysis provides the natural mathematical setting for studying (q,τ)-deformed models, because the deformation parameters q and τ quantify the deviation from the corresponding classical operator. In the proposed framework, the regime $q \to 1^-$ and $\tau \to 0^+$ represents the vanishing-deformation limit. Under this limit, the (q,τ)-deformed memory kernel converges to the standard Caputo kernel, and the corresponding derivative converges for functions satisfying the stated regularity assumptions. Thus q and τ enter the operator through $\varepsilon_{q,\tau} = \tau(1-q)$; the results below quantify the kernel and derivative limits but do not establish convergence or stability of parameter-dependent solutions.

Proposition 4 (Regularized deformed Gamma function and kernel consistency)**.** *Let $0 < \alpha < 1$, $T > 0$, $0 < q < 1$, and $\tau > 0$. Define $\varepsilon_{q,\tau} := \tau(1-q)$, and assume that $0 < \varepsilon_{q,\tau} \le \varepsilon_0$ for some fixed constant $\varepsilon_0 > 0$. Introduce the (q,τ)-deformed Gamma function by*

$$ \Gamma_{q,\tau}(z) := \frac{\Gamma(z+\varepsilon_{q,\tau})}{\Gamma(1+\varepsilon_{q,\tau})}, \qquad z > 0. $$

Then the following statements hold.

1. *The function $\Gamma_{q,\tau}$ is positive and satisfies $\Gamma_{q,\tau}(1) = 1$ and the deformed recurrence relation $\Gamma_{q,\tau}(z+1) = \big(z+\varepsilon_{q,\tau}\big)\Gamma_{q,\tau}(z)$.*
2. *For every compact interval $I = [a,b] \subset (0,\infty)$, there exists a constant $C_I > 0$, independent of q and τ, such that*

$$ \sup_{z \in I} |\Gamma_{q,\tau}(z) - \Gamma(z)| \le C_I \varepsilon_{q,\tau}. $$

   Consequently, $\Gamma_{q,\tau}(z) \longrightarrow \Gamma(z)$ uniformly on compact subsets of $(0,\infty)$ as $q \to 1^-$, $\tau \to 0^+$.
3. *The deformed Mittag–Leffler function*

$$ E_{\alpha}^{(q,\tau)}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{q,\tau}(\alpha n + 1)} $$

   is well defined for every $z \in \mathbb{C}$. Moreover,

$$ E_{\alpha}^{(q,\tau)}(z) \longrightarrow E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} $$

   locally uniformly in $\mathbb{C}$ as $\varepsilon_{q,\tau} \to 0$.
4. *Define*

$$ K_{\alpha}^{(q,\tau)}(t) = \frac{(t+\varepsilon_{q,\tau})^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)}, \qquad t > 0, $$

   and

$$ K_{\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. $$

Then there exists a constant $C = C(\alpha, T, \varepsilon_0) > 0$ such that

$$\left\| K_\alpha^{(q,\tau)} - K_\alpha \right\|_{L^1(0,T)} \leq C \varepsilon_{q,\tau}^{1-\alpha}.$$

5. *If X is a Banach space and $v \in W^{1,2}(0,T;X)$, then*

$$\left\| {}^C D_{0+}^{\alpha;(q,\tau)} v - {}^C D_{0+}^{\alpha} v \right\|_{L^2(0,T;X)} \leq C \varepsilon_{q,\tau}^{1-\alpha} \|v'\|_{L^2(0,T;X)},$$

where

$${}^C D_{0+}^{\alpha;(q,\tau)} v(t) = \int_0^t K_\alpha^{(q,\tau)}(t-s) v'(s)\, ds$$

and

$${}^C D_{0+}^{\alpha} v(t) = \int_0^t K_\alpha(t-s) v'(s)\, ds.$$

Therefore,

$${}^C D_{0+}^{\alpha;(q,\tau)} v \longrightarrow {}^C D_{0+}^{\alpha} v \quad \text{in } L^2(0,T;X)$$

as $q \to 1^-$ and $\tau \to 0^+$.

Proof. Set $\varepsilon := \varepsilon_{q,\tau} = \tau(1-q)$. For the first assertion, positivity follows directly from the positivity of the classical Gamma function on $(0, \infty)$. In addition,

$$\Gamma_{q,\tau}(1) = \frac{\Gamma(1+\varepsilon)}{\Gamma(1+\varepsilon)} = 1.$$

Using the classical recurrence relation

$$\Gamma(z+1+\varepsilon) = (z+\varepsilon)\Gamma(z+\varepsilon),$$

we obtain

$$\Gamma_{q,\tau}(z+1) = \frac{\Gamma(z+1+\varepsilon)}{\Gamma(1+\varepsilon)} = \frac{(z+\varepsilon)\Gamma(z+\varepsilon)}{\Gamma(1+\varepsilon)} = (z+\varepsilon)\Gamma_{q,\tau}(z).$$

We next prove the uniform convergence of the deformed Gamma function. Let $I = [a,b] \subset (0,\infty)$. Since $(z,\varepsilon) \longmapsto \frac{\Gamma(z+\varepsilon)}{\Gamma(1+\varepsilon)}$ is continuously differentiable on the compact set $[a,b] \times [0,\varepsilon_0]$, its partial derivative with respect to ε is bounded there. Hence, by the mean-value theorem, there exists $C_I > 0$ such that

$$\left| \frac{\Gamma(z+\varepsilon)}{\Gamma(1+\varepsilon)} - \Gamma(z) \right| \leq C_I \varepsilon$$

for every $z \in I$. Therefore, $\sup_{z \in I} |\Gamma_{q,\tau}(z) - \Gamma(z)| \leq C_I \varepsilon$. Since $\varepsilon = \tau(1-q) \to 0$ whenever $q \to 1^-$ and $\tau \to 0^+$, the required compact uniform convergence follows. We now consider the deformed Mittag–Leffler function. From the definition of $\Gamma_{q,\tau}$,

$$\frac{1}{\Gamma_{q,\tau}(\alpha n + 1)} = \frac{\Gamma(1+\varepsilon)}{\Gamma(\alpha n + 1 + \varepsilon)}.$$

Thus, we have

$$E_\alpha^{(q,\tau)}(z) = \Gamma(1+\varepsilon) \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1 + \varepsilon)}.$$

For $0 \leq \varepsilon \leq \varepsilon_0$, the denominator $\Gamma(\alpha n + 1 + \varepsilon)$ has the same superexponential growth in n as $\Gamma(\alpha n + 1)$. Hence, for every $R > 0$, the series converges uniformly for $|z| \leq R$, uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$. Since $\Gamma(1+\varepsilon) \to 1$ and $\Gamma(\alpha n + 1 + \varepsilon) \to \Gamma(\alpha n + 1)$ for each fixed n, term-wise convergence together with uniform convergence on compact sets gives $E_\alpha^{(q,\tau)}(z) \to E_\alpha(z)$ locally uniformly in $\mathbb{C}$. It remains to estimate the difference between the kernels. We let

$$K_\alpha^{(q,\tau)}(t) - K_\alpha(t) = \left[\frac{1}{\Gamma_{q,\tau}(1-\alpha)} - \frac{1}{\Gamma(1-\alpha)} \right] (t+\varepsilon)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \left[(t+\varepsilon)^{-\alpha} - t^{-\alpha} \right].$$

Consequently,

$$\left\|K_\alpha^{(q,\tau)} - K_\alpha\right\|_{L^1(0,T)} \le \left|\frac{1}{\Gamma_{q,\tau}(1-\alpha)} - \frac{1}{\Gamma(1-\alpha)}\right| \int_0^T (t+\varepsilon)^{-\alpha}\,dt + \frac{1}{\Gamma(1-\alpha)} \int_0^T \left[t^{-\alpha} - (t+\varepsilon)^{-\alpha}\right] dt.$$

The integrated in the second integral is non-negative because $t \mapsto t^{-\alpha}$ is decreasing. By the second assertion and the positivity of $\Gamma(1-\alpha)$, there exists $C_1 > 0$ such that

$$\left|\frac{1}{\Gamma_{q,\tau}(1-\alpha)} - \frac{1}{\Gamma(1-\alpha)}\right| \le C_1\varepsilon.$$

Moreover,

$$\int_0^T (t+\varepsilon)^{-\alpha}\,dt = \frac{(T+\varepsilon)^{1-\alpha} - \varepsilon^{1-\alpha}}{1-\alpha} \le C_2.$$

Therefore, the first term is bounded by $C_3\varepsilon$. For the second term, direct integration gives

$$\int_0^T \left[t^{-\alpha} - (t+\varepsilon)^{-\alpha}\right] dt = \frac{T^{1-\alpha} - (T+\varepsilon)^{1-\alpha} + \varepsilon^{1-\alpha}}{1-\alpha}.$$

Since $T^{1-\alpha} - (T+\varepsilon)^{1-\alpha} \le 0$, we obtain

$$\int_0^T \left[t^{-\alpha} - (t+\varepsilon)^{-\alpha}\right] dt \le \frac{\varepsilon^{1-\alpha}}{1-\alpha}.$$

Hence, we have

$$\left\|K_\alpha^{(q,\tau)} - K_\alpha\right\|_{L^1(0,T)} \le C_3\varepsilon + C_4\varepsilon^{1-\alpha}.$$

For $0 < \varepsilon \le \varepsilon_0$, the right-hand side can be bounded by $C\varepsilon^{1-\alpha}$. Thus, we get

$$\left\|K_\alpha^{(q,\tau)} - K_\alpha\right\|_{L^1(0,T)} \le C\varepsilon^{1-\alpha}.$$

Finally, let $v \in W^{1,2}(0,T;X)$. Then

$${}^C D_{0+}^{\alpha;(q,\tau)} v - {}^C D_{0+}^{\alpha} v = \left(K_\alpha^{(q,\tau)} - K_\alpha\right) * v'.$$

Young's convolution inequality yields

$$\left\|{}^C D_{0+}^{\alpha;(q,\tau)} v - {}^C D_{0+}^{\alpha} v\right\|_{L^2(0,T;X)} \le \left\|K_\alpha^{(q,\tau)} - K_\alpha\right\|_{L^1(0,T)} \|v'\|_{L^2(0,T;X)}.$$

Using the kernel estimate, we conclude that

$$\left\|{}^C D_{0+}^{\alpha;(q,\tau)} v - {}^C D_{0+}^{\alpha} v\right\|_{L^2(0,T;X)} \le C\varepsilon^{1-\alpha} \|v'\|_{L^2(0,T;X)}.$$

Substituting $\varepsilon = \tau(1-q)$ completes the proof. □

3. A (q,τ)-Deformed Fractional Evolution Model

In this section, we introduce a (q,τ)-deformed fractional evolution model and establish its basic analytical properties. The purpose is to formulate a mathematically rigorous framework in which the deformation parameters (q,τ) act as tunable perturbation parameters, while the fractional order α controls the memory effect. The analysis addresses fixed-parameter well-posedness and distinguishes this property from the operator-level asymptotic consistency proved in Section 2.

Let $\Omega \subset \mathbb{R}^d$, $d \ge 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $T > 0$. We denote $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, with the standard embeddings $V \hookrightarrow H \hookrightarrow V'$. For $0 < \alpha \le 1$, $0 < q < 1$, and $\tau > 0$, we consider the following (q,τ)-deformed fractional evolution problem:

$${}^C D_t^{\alpha;(q,\tau)} u(t,x) + \mathcal{A}_{q,\tau} u(t,x) = f(t,x,u(t,x)), \qquad (t,x) \in (0,T] \times \Omega, \tag{17}$$

subject to the initial and boundary conditions

$$u(0,x) = u_0(x), \qquad x \in \Omega, \qquad u(t,x) = 0, \qquad (t,x) \in (0,T] \times \partial\Omega. \tag{18}$$

Here, ${}^{C}D_t^{\alpha;(q,\tau)}$ denotes a Caputo-type (q,τ)-deformed fractional derivative and $\mathcal{A}_{q,\tau}$ is a second-order elliptic operator defined by

$$\mathcal{A}_{q,\tau}u = -\nabla \cdot \big(a_{q,\tau}(x)\nabla u\big) + V_{q,\tau}(x)u. \tag{19}$$

The coefficient $a_{q,\tau}$ represents the deformed diffusion rate, while $V_{q,\tau}$ is a deformed potential or reaction coefficient. For analytical clarity, we assume that the deformed Caputo derivative can be represented in the form

$${}^{C}D_t^{\alpha;(q,\tau)}u(t) = \int_0^t K_\alpha^{(q,\tau)}(t-s)u'(s)\,ds, \tag{20}$$

where the memory kernel $K_\alpha^{(q,\tau)}$ is non-negative, locally integrable, and satisfies

$$K_\alpha^{(q,\tau)}(t) \longrightarrow \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \quad \text{as } q \to 1^-,\ \tau \to 0^+. \tag{21}$$

Thus, in the limiting case, ${}^{C}D_t^{\alpha;(q,\tau)}$ reduces to the classical Caputo derivative. If in addition $\alpha \to 1^-$, then the model formally approaches the classical parabolic evolution equation

$$\partial_t u(t,x) + \mathcal{A}u(t,x) = f(t,x,u(t,x)), \qquad (t,x) \in (0,T] \times \Omega, \tag{22}$$

where $\mathcal{A}u = -\nabla \cdot (a(x)\nabla u) + V(x)u$.

3.1. Assumptions

We impose the following assumptions.

Assumption 1. *The coefficients $a_{q,\tau}$ and $V_{q,\tau}$ satisfy $a_{q,\tau} \in L^\infty(\Omega)$, $V_{q,\tau} \in L^\infty(\Omega)$, and there exist constants $a_0 > 0$ and $V_0 \geq 0$, independent of q and τ, such that*

$$a_{q,\tau}(x) \geq a_0, \qquad V_{q,\tau}(x) \geq -V_0, \qquad \textit{for a.e. } x \in \Omega.$$

Moreover, there exist functions $a, V \in L^\infty(\Omega)$ and a constant $C > 0$ such that

$$\|a_{q,\tau} - a\|_{L^\infty(\Omega)} + \|V_{q,\tau} - V\|_{L^\infty(\Omega)} \leq C\left([\tau(1-q)]^{1-\alpha} + (1-q) + \tau\right).$$

Assumption 2. *The nonlinear term $f : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in (t,x) and globally Lipschitz continuous in the last variable; that is, there exists $L_f > 0$ such that*

$$|f(t,x,r) - f(t,x,s)| \leq L_f|r-s|$$

for all $r, s \in \mathbb{R}$ and for a.e. $(t,x) \in (0,T) \times \Omega$. In addition, $f(t,x,0) \in L^2(0,T;H)$.

Assumption 3. *For fixed admissible parameters, the kernel $K_\alpha^{(q,\tau)}$ belongs to $W^{1,1}(0,T)$ and satisfies*

$$K_\alpha^{(q,\tau)}(t) \geq 0, \qquad (K_\alpha^{(q,\tau)})'(t) \leq 0 \quad \textit{for a.e. } t \in (0,T).$$

3.2. Weak Formulation

Define the bilinear form

$$B_{q,\tau}(u,v) = \int_\Omega a_{q,\tau}(x)\nabla u \cdot \nabla v\,dx + \int_\Omega V_{q,\tau}(x)uv\,dx, \qquad u, v \in V. \tag{23}$$

By Assumption 1, there exist constants $M > 0$ and $\lambda > 0$ such that

$$|B_{q,\tau}(u,v)| \leq M\|u\|_V\|v\|_V, \tag{24}$$

and

$$B_{q,\tau}(u,u) + \lambda\|u\|_H^2 \geq c_0\|u\|_V^2, \tag{25}$$

where $c_0 > 0$ is independent of q and τ provided the coefficient bounds are uniform. For the following variational identity, we restrict attention to solutions with the additional time regularity

$$u_{q,\tau,\alpha} \in H^1(0,T;H) \cap L^2(0,T;V).$$

This regularity ensures $u_{q,\tau,\alpha} \in C([0,T];H)$, makes the initial trace $u_{q,\tau,\alpha}(0) = u_0$ meaningful in H, and defines the regularized Caputo-type derivative as an element of $L^2(0,T;H)$ for each fixed q,τ. Such a solution satisfies the variational identity

$$\left\langle {}^C D_t^{\alpha;(q,\tau)} u_{q,\tau,\alpha}(t), v \right\rangle_{V',V} + B_{q,\tau}\big(u_{q,\tau,\alpha}(t), v\big) = \big(f(t,\cdot,u_{q,\tau,\alpha}(t)), v\big)_H \tag{26}$$

for every $v \in V$ and for a.e. $t \in (0,T)$.

3.3. Existence and Uniqueness

Because the regularized kernel is bounded at the origin, the well-posedness statement below is formulated for compatible initial data and a more regular source term. We consider $f(t,\cdot,u) = g(t) + F(u)$ under the hypotheses stated below. The resulting estimates hold for fixed deformation parameters; uniformity in the vanishing-regularization limit is not asserted. For $u \in C([0,T];H)$, the extended (q,τ)-deformed Caputo-type derivative is defined by

$${}^C D_{0+}^{\alpha;(q,\tau)} u(t) = K_\alpha^{(q,\tau)}(0)u(t) - K_\alpha^{(q,\tau)}(t)u(0) + \int_0^t (K_\alpha^{(q,\tau)})'(t-s)u(s)\,ds. \tag{27}$$

For $u \in W^{1,1}(0,T;H)$, integration by parts shows that this expression coincides with ${}^C D_{0+}^{\alpha;(q,\tau)} u$ as originally defined in Definition 3.

Proposition 5 (Well-posedness of the deformed evolution problem)**.** *Let H be a real Hilbert space, let $T > 0$, and fix $0 < \alpha < 1, 0 < q < 1, \tau > 0$. Set*

$$\varepsilon = \tau(1-q), \qquad K(t) = K_\alpha^{(q,\tau)}(t) = \frac{(t+\varepsilon)^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)}, \qquad 0 \leq t \leq T,$$

where $\Gamma_{q,\tau}(1-\alpha) > 0$, and write

$$k_0 = K(0) = \frac{\varepsilon^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)}.$$

Assume the following:

1. *The operator $A = \mathcal{A}_{q,\tau}$ is densely defined, self-adjoint, and satisfies $(Aw,w)_H \geq a_*\|w\|_H^2$, $w \in D(A)$, for some $a_* \geq 0$.*
2. *The nonlinear mapping $F : H \to H$ is continuously Fréchet differentiable and $\sup_{w\in H}\|F'(w)\|_{\mathcal{L}(H)} \leq L$, $d := k_0 + a_* - L > 0$.*
3. *The source satisfies $g \in C^1([0,T];H)$.*
4. *The initial value $u_0 \in D(A)$ satisfies the compatibility condition*

$$Au_0 = g(0) + F(u_0). \tag{28}$$

Then the problem

$$\int_0^t K(t-s)u'(s)\,ds + Au(t) = g(t) + F(u(t)), \qquad 0 < t \leq T, \qquad u(0) = u_0, \tag{29}$$

admits a unique solution $u \in C^1([0,T];H) \cap C([0,T];D(A))$, where $D(A)$ is equipped with its graph norm. Define $b = \max_{0\leq t\leq T}|K'(t)|$. The solution satisfies

$$\|u'\|_{C([0,T];H)} \leq \frac{\|g'\|_{C([0,T];H)}}{d}\exp\left(\frac{bT}{d}\right) \tag{30}$$

and

$$\|u\|_{C([0,T];H)} \le \|u_0\|_H + \frac{T\|g'\|_{C([0,T];H)}}{d} \exp\left(\frac{bT}{d}\right). \tag{31}$$

Moreover, if u and $\widetilde{u}$ correspond to compatible data (u_0, g) and $(\widetilde{u}_0, \widetilde{g})$, with the same A, F, and K, then

$$\|u - \widetilde{u}\|_{C([0,T];H)} \le \frac{\|g - \widetilde{g}\|_{C([0,T];H)} + k_0\|u_0 - \widetilde{u}_0\|_H}{d} \exp\left(\frac{bT}{d}\right). \tag{32}$$

The constants in these estimates may depend on α, q, and τ.

Proof. For the fixed parameters under consideration, $K \in C^\infty([0,T])$, and

$$K(t) > 0, \qquad K'(t) = -\frac{\alpha(t+\varepsilon)^{-\alpha-1}}{\Gamma_{q,\tau}(1-\alpha)} < 0.$$

Step 1: Invertibility of the instantaneous operator. Define $B : D(A) \to H$, $Bw = (k_0 I + A)w - F(w)$. Since A is self-adjoint and bounded below by $a_* I$, the operator $k_0 I + A$ is invertible and $\|(k_0 I + A)^{-1}\|_{\mathcal{L}(H)} \le \frac{1}{k_0 + a_*}$. For each $h \in H$, the equation $Bw = h$ is equivalent to $w = (k_0 I + A)^{-1}\big(h + F(w)\big)$. The right-hand side is a contraction on H, because $\frac{L}{k_0 + a_*} < 1$. Consequently, B is objective. Denote its inverse by $Q : H \to D(A)$. For $x, y \in D(A)$,

$$(Bx - By, x - y)_H = k_0\|x - y\|_H^2 + (A(x - y), x - y)_H - (F(x) - F(y), x - y)_H \ge d\|x - y\|_H^2.$$

Hence, we have

$$\|Q(h_1) - Q(h_2)\|_H \le d^{-1}\|h_1 - h_2\|_H. \tag{33}$$

The identity $AQ(h) = h + F(Q(h)) - k_0 Q(h)$ also shows that Q is continuous into $D(A)$ with its graph norm. Furthermore, $B'(w) = k_0 I + A - F'(w)$ is invertible by the same resolvent contraction argument, and its inverse has norm at most d^{-1} on H. The inverse mapping theorem, applied with the graph norm on $D(A)$, therefore gives

$$Q \in C^1(H; D(A)), \qquad \|DQ(h)\|_{\mathcal{L}(H)} \le d^{-1}.$$

Step 2: Equivalent Volterra equation. For $u \in C^1([0,T]; H)$, integration by parts gives

$$(K * u')(t) = k_0 u(t) - K(t)u_0 + \int_0^t K'(t - s)u(s)\, ds. \tag{34}$$

Thus, the evolution equation is equivalent to

$$u(t) = Q\left(g(t) + K(t)u_0 - \int_0^t K'(t - s)u(s)\, ds\right). \tag{35}$$

Define the right-hand side of (35) as Φu. For $\eta > 0$, equip $C([0,T]; H)$ with the equivalent norm $\|u\|_\eta = \sup_{0 \le t \le T} e^{-\eta t}\|u(t)\|_H$. Using (33), we obtain

$$\begin{aligned} e^{-\eta t}\|(\Phi u)(t) - (\Phi v)(t)\|_H &\le \frac{b}{d}\int_0^t e^{-\eta(t-s)} e^{-\eta s}\|u(s) - v(s)\|_H\, ds \\ &\le \frac{b}{d\eta}\|u - v\|_\eta. \end{aligned}$$

Choosing $\eta > b/d$ makes Φ a contraction. The Banach fixed-point theorem yields a unique $u \in C([0,T]; H)$ satisfying (35). At $t = 0$, $u(0) = Q(g(0) + k_0 u_0) = u_0$, because (28) implies $Bu_0 = g(0) + k_0 u_0$.

Step 3: Regularity and recovery of the equation. Set $h(t) = g(t) + K(t)u_0 - \int_0^t K'(t - s)u(s)\, ds$. Since $K \in C^2([0,T])$ and u is continuous, the convolution in this expression is continuously differentiable, with

$$\frac{d}{dt}\int_0^t K'(t - s)u(s)\, ds = K'(0)u(t) + \int_0^t K''(t - s)u(s)\, ds.$$

Therefore, $h \in C^1([0,T]; H)$. Since $u = Q \circ h$, the regularity of Q gives $u \in C^1([0,T]; H) \cap C([0,T]; D(A))$. Equation (34) now applies, proving that u satisfies (29). Conversely, every solution with this regularity satisfies

(35); uniqueness follows.

Step 4: A priori estimates. Using $u(0) = u_0$, the derivative of h can also be written as $h'(t) = g'(t) - \int_0^t K'(t-s)u'(s)\,ds$. Indeed, $\frac{d}{dt}(K' * u)(t) = K'(t)u_0 + (K' * u')(t)$. The chain rule and the bound on DQ yield

$$\|u'(t)\|_H \le \frac{\|g'(t)\|_H}{d} + \frac{b}{d}\int_0^t \|u'(s)\|_H\,ds.$$

The ordinary Gronwall inequality proves (30). Integrating u' from 0 to t gives (31). For two compatible solutions, subtracting their Volterra equations gives

$$\|u(t) - \widetilde{u}(t)\|_H \le \frac{\|g - \widetilde{g}\|_{C([0,T];H)} + k_0\|u_0 - \widetilde{u}_0\|_H}{d} + \frac{b}{d}\int_0^t \|u(s) - \widetilde{u}(s)\|_H\,ds.$$

again application of the ordinary Gronwall inequality proves (32). □

4. Engineering Application

We consider transient heat conduction in a thin homogeneous engineering component, such as a metallic strip, a composite layer, or a thermal barrier coating. The component occupies the one-dimensional spatial domain $0 < x < L$, where $L > 0$ denotes its length. The temperature field is denoted by $T(t, x)$, and the surrounding environment is maintained at a constant reference temperature T_{a}. To remove the ambient temperature from the governing equation, we introduce the excess temperature $\theta(t, x) = T(t, x) - T_{\mathrm{a}}$. The quantity $\theta(t, x)$ therefore represents the temperature difference between the component and its environment. In materials with thermal memory, the present heat-transfer response may depend on the previous thermal history. Such behavior appears in heterogeneous composites, porous thermal media, polymeric materials, and layered structures. To describe this effect, we use the (q, τ)-deformed fractional diffusion–reaction model

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}\theta(t, x) - D_{q,\tau}\frac{\partial^2\theta(t, x)}{\partial x^2} + \mu_{q,\tau}\theta(t, x) = Q(t, x), \tag{36}$$

for $(t, x) \in (0, T_{\mathrm{f}}] \times (0, L)$. Here, $D_{q,\tau} > 0$ is the effective deformed thermal diffusivity; $\mu_{q,\tau} \ge 0$ is the effective heat-loss coefficient; $Q(t, x)$ is a distributed thermal source; $\alpha \in (0, 1)$ controls the strength of thermal memory; $q \in (0, 1)$ measures the departure from the classical fractional regime; $\tau > 0$ controls the deformation scale. The Caputo-type deformed derivative is defined by

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}\theta(t, x) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t \big(t - s + \tau(1-q)\big)^{-\alpha}\frac{\partial\theta(s, x)}{\partial s}\,ds. \tag{37}$$

The parameters q and τ jointly control the deformation through the effective parameter $\varepsilon_{q,\tau} = \tau(1-q)$. When q is close to 1 and τ is small, the thermal response approaches the classical Caputo fractional heat-conduction model. Smaller q or larger τ introduce stronger deformation and modify the thermal relaxation rate. We assume that both ends of the component are maintained at the ambient temperature. Hence, $\theta(t, 0) = \theta(t, L) = 0$, $0 < t \le T_{\mathrm{f}}$. The initial temperature distribution is chosen as

$$\theta(0, x) = \theta_{\max}\sin\left(\frac{\pi x}{L}\right), \qquad 0 \le x \le L, \tag{38}$$

where $\theta_{\max} > 0$ is the maximum initial excess temperature. This profile is physically meaningful because it vanishes at both boundaries and reaches its maximum at the midpoint of the component. For the homogeneous cooling problem, we formally set $Q(t, x) = 0$. Therefore, Equation (36) becomes

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}\theta(t, x) - D_{q,\tau}\theta_{xx}(t, x) + \mu_{q,\tau}\theta(t, x) = 0. \tag{39}$$

4.1. Compatibility Caveat

For each fixed q, τ, the memory derivative of a continuously differentiable function vanishes at $t = 0$. Thus a strong solution covered by Proposition 5 must satisfy $\mathcal{A}_{q,\tau}\theta(0) = Q(0)$, whereas the nonzero sinusoidal initial condition above does not satisfy this identity when $Q = 0$ and $D_{q,\tau} > 0$. Accordingly, this homogeneous cooling setup is a formal engineering model, not an example covered by the strong-solution theorem. The following Mittag–Leffler expression is only a heuristic comparison function; it is not an exact solution formula for the shifted

kernel. The manufactured-source benchmark below provides a compatible exact test solution.

Following the asymptotic structure of the proposed model, we define

$$D_{q,\tau} = D_0\big[1+\tau(1-q)\big], \qquad \mu_{q,\tau} = \mu_0\big[1+\tau(1-q)\big], \tag{40}$$

where $D_0 > 0$ and $\mu_0 \geq 0$ are the classical thermal diffusivity and heat-loss coefficient, respectively. As $q \to 1^-$, $\tau \to 0^+$, we have $D_{q,\tau} \to D_0$, $\mu_{q,\tau} \to \mu_0$. Thus, the deformed engineering model approaches the standard fractional heat-conduction equation. Since

$$-\frac{d^2}{dx^2}\sin\left(\frac{\pi x}{L}\right) = \left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right),$$

we seek a separated representation of the form

$$\theta_{q,\tau,\alpha}(t,x) = A_{q,\tau,\alpha}(t)\sin\left(\frac{\pi x}{L}\right). \tag{41}$$

Substitution into Equation (39) gives

$${}^{C}D_{0+}^{\alpha;(q,\tau)} A_{q,\tau,\alpha}(t) + \lambda_{q,\tau} A_{q,\tau,\alpha}(t) = 0, \qquad A_{q,\tau,\alpha}(0) = \theta_{\max}, \tag{42}$$

where

$$\lambda_{q,\tau} = D_{q,\tau}\left(\frac{\pi}{L}\right)^2 + \mu_{q,\tau}. \tag{43}$$

For a heuristic comparison only (not an exact representation of the shifted-kernel equation), consider the deformed Mittag–Leffler expression

$$A_{q,\tau,\alpha}(t) \approx \theta_{\max} E_{\alpha}^{(q,\tau)}\left(-\lambda_{q,\tau} t^{\alpha}\right), \tag{44}$$

and consequently

$$\theta_{q,\tau,\alpha}(t,x) \approx \theta_{\max} E_{\alpha}^{(q,\tau)}\left(-\lambda_{q,\tau} t^{\alpha}\right)\sin\left(\frac{\pi x}{L}\right). \tag{45}$$

The corresponding physical temperature is recovered from

$$T_{q,\tau,\alpha}(t,x) = T_{\mathrm{a}} + \theta_{q,\tau,\alpha}(t,x). \tag{46}$$

4.2. Engineering Parameter Values

For an illustrative thermal experiment, we take

$$L = 1\ \mathrm{m}, \qquad T_{\mathrm{a}} = 293.15\ \mathrm{K}, \qquad \theta_{\max} = 50\ \mathrm{K},$$

together with

$$D_0 = 1, \qquad \mu_0 = 0.5, \qquad \alpha = 0.85.$$

The four deformation regimes are

$$(q,\tau) \in \{(0.99,0.05),(0.90,0.20),(0.80,0.50),(0.70,1.00)\}.$$

These cases represent near-classical, mild, moderate, and strong deformation, respectively. The initial temperature field is identical for all cases: $T(0,x) = T_{\mathrm{a}} + 50\sin(\pi x)$. Therefore, the deformation parameters affect only the subsequent thermal relaxation and must not change the initial temperature amplitude. To quantify the thermal response, we introduce the following engineering measures.

4.2.1. Maximum Excess Temperature

$$\Theta_{\max}(t;q,\tau) = \max_{0\leq x\leq L} \left|\theta_{q,\tau,\alpha}(t,x)\right|. \tag{47}$$

This quantity measures the highest temperature difference remaining inside the component at time t.

4.2.2. Average Thermal Energy

$$\mathcal{H}_{q,\tau}(t) = \frac{1}{2}\int_0^L |\theta_{q,\tau,\alpha}(t,x)|^2\, dx. \tag{48}$$

A decreasing value of $\mathcal{H}_{q,\tau}(t)$, when observed numerically, indicates decay of the excess-temperature norm; this does not itself establish convergence.

4.2.3. Cooling Time

For a prescribed tolerance $\delta > 0$, define

$$t_{\mathrm{cool}}(q,\tau) = \inf\left\{t > 0 : \Theta_{\max}(t;q,\tau) \le \delta\right\}. \tag{49}$$

This quantity is directly relevant to engineering design because it measures the time required for the component to cool to an acceptable temperature level.

4.2.4. Classical-Model Deviation

$$\mathcal{E}_{\mathrm{th}}(q,\tau) = \left(\int_0^{T_{\mathrm{f}}} \|\theta_{q,\tau,\alpha}(t,\cdot) - \theta_{\alpha}(t,\cdot)\|^2_{L^2(0,L)}\, dt\right)^{1/2}, \tag{50}$$

where θ_α denotes the solution of the corresponding classical Caputo fractional heat equation. The indicator $\mathcal{E}_{\mathrm{th}}(q,\tau)$ measures deviation from the classical fractional solution when both solutions are available. No uniform bound for this solution-level quantity is asserted: Proposition 4 concerns the memory kernels and derivatives applied to a fixed sufficiently regular function, not the parameter-dependent heat-equation solutions.

4.2.5. Engineering Interpretation

The parameter q controls the departure from the classical fractional heat-transfer regime, while τ determines the scale of the kernel regularization. Different effective deformations can yield different computed cooling histories. Any directional change in the cooling rate must be assessed from the numerical results; monotonic enhancement of damping has not been established analytically. From an engineering perspective, the model can represent components whose apparent thermal diffusivity and heat-loss behavior depend on microstructural memory, interface effects, or delayed thermal relaxation. The quantities $\Theta_{\max}$, $\mathcal{H}_{q,\tau}$, and t_{cool} provide practical measures for comparing materials and selecting deformation parameters for desired thermal performance. As $(q,\tau) \to (1,0)$, the memory kernel and the discrete L1 weights approach their classical Caputo counterparts; convergence of the corresponding heat-equation solutions is not established here. If additionally $\alpha \to 1^-$, the model approaches the standard Fourier diffusion–reaction equation. Thus, the proposed formulation provides a consistent bridge between memory-dependent thermal transport and classical engineering heat conduction.

4.3. A Deformed L1 Finite-Difference Algorithm

We construct a fully discrete scheme for the homogeneous (q,τ)-deformed diffusion–reaction problem

$${}^{C}D_{0+}^{\alpha;(q,\tau)}u(t,x) - D_{q,\tau}u_{xx}(t,x) + \mu_{q,\tau}u(t,x) = g(t,x), \tag{51}$$

for $(t,x) \in (0,T] \times (0,L)$, subject to $u(0,x) = u_0(x)$, $u(t,0) = u(t,L) = 0$. The deformed Caputo derivative is

$${}^{C}D_{0+}^{\alpha;(q,\tau)}u(t,x) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t \left(t - s + \varepsilon_{q,\tau}\right)^{-\alpha}\frac{\partial u(s,x)}{\partial s}\, ds, \tag{52}$$

where $\varepsilon_{q,\tau} = \tau(1-q) > 0$. Let

$$x_i = ih, \qquad i = 0,1,\ldots,M, \qquad h = \frac{L}{M},$$

and

$$t_n = n\Delta t, \qquad n = 0,1,\ldots,N, \qquad \Delta t = \frac{T}{N}.$$

We denote the numerical approximation of $u(t_n,x_i)$ by $U_i^n \approx u(t_n,x_i)$. The initial and boundary values are

prescribed by

$$U_i^0 = u_0(x_i), \qquad i = 0, 1, \ldots, M, \tag{53}$$

and

$$U_0^n = U_M^n = 0, \qquad n = 1, 2, \ldots, N. \tag{54}$$

The Deformed L1 approximation can be determined on each interval $[t_k, t_{k+1}]$, where the time derivative is approximated by the backward difference

$$\frac{\partial u(s, x_i)}{\partial s} \approx \frac{U_i^{k+1} - U_i^k}{\Delta t}, \qquad s \in [t_k, t_{k+1}].$$

Therefore, at $t = t_n$, Equation (52) is approximated by

$${}^{C}D_{0+}^{\alpha;(q,\tau)} u(t_n, x_i) \approx \sum_{k=0}^{n-1} b_{n-k}^{(q,\tau)} \left(U_i^{k+1} - U_i^k \right), \tag{55}$$

where the deformed L1 weights are

$$b_j^{(q,\tau)} = \frac{(j\Delta t + \varepsilon_{q,\tau})^{1-\alpha} - ((j-1)\Delta t + \varepsilon_{q,\tau})^{1-\alpha}}{\Gamma_{q,\tau}(1-\alpha)(1-\alpha)\Delta t}, \qquad j \geq 1. \tag{56}$$

Indeed,

$$\begin{aligned} &\frac{1}{\Gamma_{q,\tau}(1-\alpha)} \int_{t_k}^{t_{k+1}} \left(t_n - s + \varepsilon_{q,\tau}\right)^{-\alpha} \frac{U_i^{k+1} - U_i^k}{\Delta t}\, ds \\ &\quad = b_{n-k}^{(q,\tau)} \left(U_i^{k+1} - U_i^k \right). \end{aligned}$$

The weights satisfy

$$b_j^{(q,\tau)} > 0, \qquad b_{j+1}^{(q,\tau)} \leq b_j^{(q,\tau)}, \qquad j \geq 1, \tag{57}$$

because the function $r \longmapsto (r\Delta t + \varepsilon_{q,\tau})^{1-\alpha}$ is increasing and concave for $0 < \alpha < 1$. When $q \to 1^-$, $\tau \to 0^+$, we have $\varepsilon_{q,\tau} \to 0$, and the weights reduce to the classical L1 weights

$$b_j^{(\alpha)} = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \left[j^{1-\alpha} - (j-1)^{1-\alpha} \right]. \tag{58}$$

Thus, the proposed algorithm is asymptotically consistent with the standard L1 approximation of the classical Caputo derivative. The second spatial derivative is approximated by the central difference formula

$$u_{xx}(t_n, x_i) \approx \delta_{xx} U_i^n := \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}, \qquad i = 1, \ldots, M-1. \tag{59}$$

Substituting Equations (55) and (59) into Equation (51) gives

$$\sum_{k=0}^{n-1} b_{n-k}^{(q,\tau)} \left(U_i^{k+1} - U_i^k \right) - D_{q,\tau} \delta_{xx} U_i^n + \mu_{q,\tau} U_i^n = g_i^n, \tag{60}$$

where $g_i^n = g(t_n, x_i)$. Separating the contribution containing the unknown value U_i^n, we obtain

$$b_1^{(q,\tau)} \left(U_i^n - U_i^{n-1} \right) + \sum_{k=0}^{n-2} b_{n-k}^{(q,\tau)} \left(U_i^{k+1} - U_i^k \right) - D_{q,\tau} \delta_{xx} U_i^n + \mu_{q,\tau} U_i^n = g_i^n. \tag{61}$$

Equivalently,

$$-r_{q,\tau} U_{i-1}^n + \left(b_1^{(q,\tau)} + 2r_{q,\tau} + \mu_{q,\tau} \right) U_i^n - r_{q,\tau} U_{i+1}^n = F_i^n, \tag{62}$$

where $r_{q,\tau} = \frac{D_{q,\tau}}{h^2}$ and

$$F_i^n = g_i^n + b_1^{(q,\tau)} U_i^{n-1} - \sum_{k=0}^{n-2} b_{n-k}^{(q,\tau)} \left(U_i^{k+1} - U_i^k \right). \tag{63}$$

For $n = 1$, the history sum is empty, and hence

$$F_i^1 = g_i^1 + b_1^{(q,\tau)} U_i^0. \tag{64}$$

Define $\mathbf{U}^n = \begin{bmatrix} U_1^n & U_2^n & \cdots & U_{M-1}^n \end{bmatrix}^{\mathsf{T}}$. Then the fully discrete problem can be written as

$$\mathbf{A}_{q,\tau}\mathbf{U}^n = \mathbf{F}^n, \qquad n = 1, 2, \ldots, N, \tag{65}$$

where

$$\mathbf{A}_{q,\tau} = \begin{bmatrix} d_{q,\tau} & -r_{q,\tau} & 0 & \cdots & 0 \\ -r_{q,\tau} & d_{q,\tau} & -r_{q,\tau} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -r_{q,\tau} & d_{q,\tau} & -r_{q,\tau} \\ 0 & \cdots & 0 & -r_{q,\tau} & d_{q,\tau} \end{bmatrix}, \tag{66}$$

with

$$d_{q,\tau} = b_1^{(q,\tau)} + 2r_{q,\tau} + \mu_{q,\tau}. \tag{67}$$

The vector $\mathbf{F}^n$ contains the source term, the previous solution value, and the complete memory contribution from all earlier time levels.

Proposition 6 (Unique solvability of the deformed L1 scheme)**.** *Let* $0 < \alpha < 1, 0 < q < 1, \tau > 0, D_{q,\tau} > 0, \mu_{q,\tau} \geq 0$. *Then, at every time level* t_n*, the linear system* (65) *has a unique solution.*

Proof. From Equation (56), $b_1^{(q,\tau)} > 0$. Moreover, $r_{q,\tau} = \frac{D_{q,\tau}}{h^2} > 0$. The diagonal entries of $\mathbf{A}_{q,\tau}$ satisfy

$$d_{q,\tau} = b_1^{(q,\tau)} + 2r_{q,\tau} + \mu_{q,\tau},$$

while the sum of the absolute values of the off-diagonal entries in each interior row is $2r_{q,\tau}$. Therefore, $d_{q,\tau} - 2r_{q,\tau} = b_1^{(q,\tau)} + \mu_{q,\tau} > 0$. Hence, $\mathbf{A}_{q,\tau}$ is strictly diagonally dominant. It is also symmetric with positive diagonal entries and nonpositive off-diagonal entries. Consequently, $\mathbf{A}_{q,\tau}$ is nonsingular, and the system admits a unique solution at every time level. □

4.4. Engineering Computational Benchmark

To define a reproducible manufactured-solution check for Algorithm 1, we consider transient heat conduction in a thin homogeneous engineering strip with memory-dependent heat transfer. The strip occupies the dimensionless spatial interval $0 < x < 1$, and the computational time interval is $0 < t \leq 1$. Let $T(t, x)$ denote the temperature of the strip and let T_{a} be the constant ambient temperature. We introduce the dimensionless excess temperature $u(t, x) = \frac{T(t,x) - T_{\mathrm{a}}}{T_{\mathrm{r}}}$, where $T_{\mathrm{r}} > 0$ is a reference temperature difference. The governing deformed fractional heat-conduction equation is

Algorithm 1 Implicit deformed L1 finite-difference algorithm

1: Choose h, Δt, q, τ, and α.
2: Set $x_i = ih$, $t_n = n\Delta t$, and $\varepsilon_{q,\tau} = \tau(1 - q)$.
3: Initialize $U_i^0 = u_0(x_i)$.
4: Compute the deformed convolution weights

$$\omega_j^{(q,\tau)} = \frac{(j\Delta t + \varepsilon_{q,\tau})^{1-\alpha} - ((j-1)\Delta t + \varepsilon_{q,\tau})^{1-\alpha}}{\Gamma_{q,\tau}(1-\alpha)(1-\alpha)}.$$

5: **for** $n = 1, 2, \ldots, N$ **do**
6: Assemble the memory vector from $U^0, U^1, \ldots, U^{n-1}$.
7: Assemble the tridiagonal diffusion–reaction matrix.
8: Solve the linear system for U^n.
9: Impose $U_0^n = U_M^n = 0$.
10: **end for**
11: If an exact solution is available, compute its numerical error; otherwise report selected solution diagnostics.

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}u(t,x) - D_{q,\tau}u_{xx}(t,x) + \mu_{q,\tau}u(t,x) = g(t,x), \tag{68}$$

subject to

$$ u(t,0) = u(t,1) = 0, \qquad 0 < t \le 1, \tag{69}$$

and

$$ u(0,x) = \sin(\pi x), \qquad 0 \le x \le 1. \tag{70}$$

The coefficients are defined by

$$ D_{q,\tau} = D_0\left[1 + \tau(1-q)\right], \qquad \mu_{q,\tau} = \mu_0\left[1 + \tau(1-q)\right], \tag{71}$$

where $D_0 = 1$, $\mu_0 = 0.5$. The regularization parameter is

$$ \varepsilon_{q,\tau} = \tau(1-q). \tag{72}$$

The deformed Caputo derivative is

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}u(t,x) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t (t-s+\varepsilon_{q,\tau})^{-\alpha}\, u_s(s,x)\, ds. \tag{73}$$

For numerical verification, we prescribe the exact dimensionless temperature

$$ u_{\mathrm{ex}}(t,x) = (1+t)\sin(\pi x). \tag{74}$$

This function satisfies the boundary conditions because $u_{\mathrm{ex}}(t,0) = u_{\mathrm{ex}}(t,1) = 0$, and it satisfies the initial condition $u_{\mathrm{ex}}(0,x) = \sin(\pi x)$. Its temporal derivative is $\frac{\partial u_{\mathrm{ex}}}{\partial t} = \sin(\pi x)$. Therefore, we get

$$\begin{aligned} {}^{C}D_{0+}^{\alpha;(q,\tau)}u_{\mathrm{ex}}(t,x) &= \frac{\sin(\pi x)}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t (t-s+\varepsilon_{q,\tau})^{-\alpha}\, ds \\ &= \frac{(t+\varepsilon_{q,\tau})^{1-\alpha} - \varepsilon_{q,\tau}^{1-\alpha}}{\Gamma_{q,\tau}(1-\alpha)(1-\alpha)}\sin(\pi x). \end{aligned}$$

Define

$$ \Phi_{q,\tau,\alpha}(t) = \frac{(t+\varepsilon_{q,\tau})^{1-\alpha} - \varepsilon_{q,\tau}^{1-\alpha}}{\Gamma_{q,\tau}(1-\alpha)(1-\alpha)}. \tag{75}$$

Hence,

$$ {}^{C}D_{0+}^{\alpha;(q,\tau)}u_{\mathrm{ex}}(t,x) = \Phi_{q,\tau,\alpha}(t)\sin(\pi x). \tag{76}$$

Moreover, $\frac{\partial^2 u_{\mathrm{ex}}}{\partial x^2} = -\pi^2(1+t)\sin(\pi x)$. Substitution into Equation (68) gives the manufactured thermal source

$$ g(t,x) = \left[\Phi_{q,\tau,\alpha}(t) + \left(D_{q,\tau}\pi^2 + \mu_{q,\tau}\right)(1+t)\right]\sin(\pi x). \tag{77}$$

Consequently, Equation (74) is an exact solution of the engineering benchmark. The following parameter values are used:

$$ \alpha = 0.85, \qquad D_0 = 1, \qquad \mu_0 = 0.5, \qquad T = 1. \tag{78}$$

Four deformation regimes are considered:

$$ (q,\tau) \in \{(0.99, 0.05), (0.90, 0.20), (0.80, 0.50), (0.70, 1.00)\}. \tag{79}$$

They represent near-classical, mild, moderate, and strong deformation, respectively.

Application of Algorithm 1

Let $x_i = ih, h = \frac{1}{M}, i = 0, \ldots, M$, and $t_n = n\Delta t, \Delta t = \frac{1}{N}, n = 0, \ldots, N$. The numerical solution is denoted by $U_i^n \approx u(t_n, x_i)$. The initial and boundary values are

$$ U_i^0 = \sin(\pi x_i), \qquad U_0^n = U_M^n = 0. \tag{80}$$

For $j \geq 1$, the deformed L1 weights are

$$b_j^{(q,\tau)} = \frac{(j\Delta t + \varepsilon_{q,\tau})^{1-\alpha} - ((j-1)\Delta t + \varepsilon_{q,\tau})^{1-\alpha}}{\Gamma_{q,\tau}(1-\alpha)(1-\alpha)\Delta t}. \tag{81}$$

At each time level t_n, Algorithm 1 solves

$$\sum_{k=0}^{n-1} b_{n-k}^{(q,\tau)} \left(U_i^{k+1} - U_i^k\right) - D_{q,\tau} \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} + \mu_{q,\tau} U_i^n = g(t_n, x_i), \tag{82}$$

for $i = 1, \ldots, M-1$. Define $r_{q,\tau} = \frac{D_{q,\tau}}{h^2}$, and $d_{q,\tau} = b_1^{(q,\tau)} + 2r_{q,\tau} + \mu_{q,\tau}$. The tridiagonal equation becomes

$$-r_{q,\tau} U_{i-1}^n + d_{q,\tau} U_i^n - r_{q,\tau} U_{i+1}^n = F_i^n, \tag{83}$$

where

$$F_i^n = g(t_n, x_i) + b_1^{(q,\tau)} U_i^{n-1} - H_i^n, \tag{84}$$

and the history term is

$$H_i^n = \sum_{k=0}^{n-2} b_{n-k}^{(q,\tau)} \left(U_i^{k+1} - U_i^k\right). \tag{85}$$

For $n = 1$, the history term is empty: $H_i^1 = 0$. Thus, Algorithm 1 produces the temperature vector

$$\mathbf{U}^n = \begin{bmatrix} U_1^n & U_2^n & \cdots & U_{M-1}^n \end{bmatrix}^{\mathsf{T}}$$

by solving the tridiagonal system

$$\mathbf{A}_{q,\tau} \mathbf{U}^n = \mathbf{F}^n. \tag{86}$$

Table 1 presented the four cases with different values of the pair (q, τ).

Table 1. Deformation settings for the engineering benchmark.

Case	q	τ	$\varepsilon_{q,\tau}$	Interpretation
1	0.99	0.05	0.0005	Near classical
2	0.90	0.20	0.0200	Mild deformation
3	0.80	0.50	0.1000	Moderate deformation
4	0.70	1.00	0.3000	Strong deformation

5. Conclusions

This work introduced a regularized (q, τ)-deformed Caputo-type formulation and an implicit L1 finite-difference method for fractional heat conduction. For the associated deformed Gamma and Mittag–Leffler functions and the memory kernel, we established classical consistency properties, including a quantitative L^1 estimate for the difference between the regularized and classical kernels. The classical Caputo and first-order limits were analyzed in the prescribed order, and an explicit example showed why an unrestricted joint limit cannot generally be asserted. For fixed admissible deformation parameters, strong-solution existence, uniqueness, and continuous dependence were proved under operator, source-regularity, and initial compatibility assumptions. The deformed L1 scheme was then derived for a diffusion–reaction heat equation, and its discrete linear system was shown to be uniquely solvable at each time step. A manufactured-source example supplies an exact solution for numerical verification. Because q and τ enter through $\varepsilon_{q,\tau} = \tau(1-q)$, their separate effects cannot be identified within this model. Uniform convergence of parameter-dependent solutions and full space–time numerical error bounds remain open directions for future study.

Funding

This research received no external funding.

Institutional Review Board Statement

Not applicable.

Informed Consent Statement

Not applicable.

Data Availability Statement

Not applicable.

Conflicts of Interest

The author declares no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

References

1. Rahman, R.U.; Qousini, M.; Alshehri, A.M.; et al. Evaluation of the performance of fractional evolution equations based on fractional operators and sensitivity assessment. *Results Phys.* **2023**, *49*, 106537. https://doi.org/10.1016/j.rinp.2023.106537.
2. Smadiyeva, A.G.; Torebek, B.T. Decay estimates for the time-fractional evolution equations with time-dependent coefficients. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2023**, *479*, 20230103. https://doi.org/10.1098/rspa.2023.0103.
3. Liu, Y.; Liu, Z.; Peng, S.; et al. Optimal feedback control for a class of fractional evolution equations with history-dependent operators. *Fract. Calc. Appl. Anal.* **2022**, *25*, 1108–1130. https://doi.org/10.1007/s13540-022-00054-y.
4. Berrighi, F.; Medjadj, I.; Karapınar, E. Mild solutions for conformable fractional order functional evolution equations via Meir-Keeler type fixed point theorem. *Filomat* **2025**, *39*, 1989–2002. https://doi.org/10.2298/fil2506989b.
5. Younas, T.; Ahmad, J. Novel soliton insights into generalized fractional Tzitzéica-type evolution equations using the modified Khater method. *Mod. Phys. Lett. B* **2025**, *39*, 2450441. https://doi.org/10.1142/s0217984924504414.
6. Beroudj, M.E.; Kirane, M.; Mennouni, A. An inverse problem for an evolution equation with a time fractional derivative of order $\alpha \in (1,\, 2)$ involving Legendre polynomials. *Discret. Contin. Dyn. Syst. S* **2026**, *23*, 81–98. https://doi.org/10.3934/dcdss.2025018.
7. Lachachi-Merad, N.; Baghli-Bendimerad, S.; Benchohra, M.; et al. Unique mild solution for fractional partial and neutral evolution equations with state-dependent delay. *Rocky Mt. J. Math.* **2025**, *55*, 1351–1359. https://doi.org/10.1216/rmj.2025.55.1351.
8. Rachad, H.; Bahloul, R. A mild solution approach to general fractional evolution equations with nonlocal constraints. *Gulf J. Math.* **2026**, *22*, 1–11. https://doi.org/10.56947/gjom.v22i1.3834.
9. Lemnaouar, M.R.; El Hakki, I. Controllability and Ulam stabilities for a fractional damped evolution equation using the ρ-Caputo derivative. *Int. J. Syst. Sci.* **2026**, 1–19. https://doi.org/10.1080/00207721.2026.2625787.
10. Wang, K. New computational approaches to the fractional coupled nonlinear Helmholtz equation. *Eng. Comput.* **2024**, *41*, 1285–1300. https://doi.org/10.1108/ec-08-2023-0501.
11. Meknani, B.; Alrebdi, R.; Bchatnia, A. Pseudo-Almost Automorphic $\mathcal{C}0$-Solutions: Well-Posedness and Asymptotic Behaviour in Evolution Equations with Nonlocal Constraints. *Mathematics* **2026**, *14*, 459. https://doi.org/10.3390/math14030459.
12. Zerbib, S.; Hilal, K.; Kajouni, A.; et al. Analytical results for semilinear fractional evolution systems with variable-order Φ-Caputo derivatives using generalized Laplace transforms. *Comput. Appl. Math.* **2026**, *45*, 188. https://doi.org/10.1007/s40314-025-03470-x.
13. Iqbal, M.; Liu, J.; Ali Faridi, W.; et al. Analysis of optical soliton solutions in weakly nonlocal media to the fractional nonlinear Schrödinger equation in (2+1)–dimension via efficient computational technique. *Int. J. Comput. Math.* **2026**, *103*, 1429–1453. https://doi.org/10.1080/00207160.2026.2615160.
14. My, B.K. On Fractional Pseudo-Parabolic Equations Involving Nonlocal Nonlinearities. *Asymptot. Anal.* **2026**, *148*, 863–896. https://doi.org/10.1177/09217134251386206.
15. Ibrahim, R.W. Image edge detections based on fractional deformed harmonic univalent functions. *Comput. Math. Model.* **2026**, *37*, 780–829. https://doi.org/10.1007/s10598-026-09714-7.
16. Ibrahim, R.W.; AlSheikh, M.H. Model Validation Pipeline against Longitudinal Alzheimer's Biomarker Data. *Neuroinformatics* **2026**, *24*, 33. https://doi.org/10.1007/s12021-026-09790-5.
17. Ibrahim, R.W. Quantum-deformed growth geometry in the open unit disk, with applications in subordination, extremal functions, and phase transitions. *Arab. J. Math. Sci.* **2026**, *32*, 237–253. https://doi.org/10.1108/ajms-12-2025-0204.