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A Quantum-Deformed Navier–Stokes–Cahn–Hilliard Framework for Interfacial Multiphase Flow in Porous Disk Geometries

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Abstract: Purpose: This paper develops a generalized (q,τ)-deformed multiphase-flow framework for porous disk geometries containing multiple internal solid inclusions. The aim is to extend the single-obstacle deformation idea to a more realistic porous domain where several unsymmetric starlike obstacles interact with a moving fluid–fluid interface. Method: A coupled Navier–Stokes–Cahn–Hilliard system is formulated in the porous fluid region $\Omega_f^{(N)} = \mathbb{D} \setminus \bigcup_{k=1}^{N} \overline{\Omega_{s,k}}$. The deformation parameters $q \in (0,1]$ and $\tau \geq 0$ are introduced through a multi-obstacle boundary indicator, adaptive viscosity, deformed mobility, and localized boundary-stabilization force. The phase field ϕ is transported by the velocity field $\mathbf{u}$, while the phase gradients feed back into the momentum equation through a deformed interfacial force. Results: Theoretical results show that the porous domain is well defined under explicit containment and disjointness conditions. The localization functions are positive, bounded, and concentrated near the solid inclusions. The deformed viscosity and mobility remain positive and bounded under natural gradient assumptions. The coupled model conserves total phase mass under no-flux and impermeability boundary conditions, and a formal phase-field energy dissipation relation is obtained. Simulations demonstrate that decreasing q and increasing τ reduce spurious oscillations, smooth the interface, and localize stabilization near high-gradient regions. Conclusions: The proposed model provides a tunable extension of the classical coupled Navier–Stokes–Cahn–Hilliard formulation. When $q \to 1$, the deformation terms vanish and the classical model is recovered. The framework is suitable for computational studies of interfacial multiphase flow in porous media, microstructured domains, and bounded complex geometries.

Keywords: (q,τ)-deformation; lattice Boltzmann method; Navier–Stokes–Cahn–Hilliard system; porous disk geometry; starlike inclusions; interfacial stabilization

1. Introduction

Multiphase flow in complex and porous geometries is an important topic in computational physics, fluid mechanics, porous media modeling, and microstructured-material simulation [1–4]. Such systems arise in droplet transport, porous microfluidics, biological membranes, heterogeneous materials, filtration processes, and interfacial flow through media containing multiple internal inclusions. In these problems, the dynamics are governed not only by the interaction between two fluid phases, but also by the geometric influence of solid boundaries, pore structures, and curved internal obstacles [5–7].

The Lattice Boltzmann Method (LBM) is widely used for simulating fluid flow because of its mesoscopic structure, simple streaming–collision algorithm, and natural ability to handle complex boundaries [8–11]. In multiphase problems, LBM can be coupled with phase-field models such as the Cahn–Hilliard equation in order to describe diffuse fluid–fluid interfaces, droplet deformation, phase separation, and interfacial transport [12–14]. However, classical multiphase LBM

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and coupled Navier–Stokes–Cahn–Hilliard schemes may suffer from spurious currents, interface oscillations, and boundary-induced artifacts, especially near curved solid inclusions and high-gradient interfacial regions [15–18]. The classical LBM was recovered as $q \to 1$, while the deformation parameters q and τ provided local control of numerical damping near the fluid–fluid interface and the solid obstacle. Although this setting is useful as a first model, many physical porous systems contain several internal solid inclusions rather than a single obstacle. Motivated by this limitation, the present paper develops a new (q,τ)-deformed framework (see [19–21]) for coupled multiphase flow in a porous disk geometry containing multiple internal inclusions. The porous fluid domain is defined by

$$\Omega_f^{(N)} = \mathbb{D} \setminus \bigcup_{k=1}^{N} \overline{\Omega_{s,k}},$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk and $\Omega_{s,k}$ are internal unsymmetric starlike solid obstacles. This geometry generalizes the single-obstacle case and provides a more flexible representation of porous or microstructured domains.

The main idea of the proposed model is to couple the Navier–Stokes equations with a deformed Cahn–Hilliard phase-field equation. The phase variable ϕ identifies the two fluid phases and the diffuse interface, while the velocity field $\mathbf{u}$ describes hydrodynamic transport. The coupling is achieved through the conservative transport term $\nabla \cdot (\phi \mathbf{u})$ in the Cahn–Hilliard equation and through the interfacial force $-\kappa_0 q^\tau \phi \nabla \mu$ in the momentum equation. Thus, the velocity field transports and deforms the phase distribution, while the phase-field gradients feed back into the fluid momentum. A key contribution of this paper is the introduction of a multi-obstacle porous-boundary indicator

$$\mathcal{G}_s^{(N)}(z) = \sum_{k=1}^{N} \chi_{b,k}(z) |\nabla \Psi_{s,k}(z,\bar{z})|,$$

where $\Psi_{s,k}$ is a level-set function for the k-th solid inclusion and

$$\chi_{b,k}(z) = \exp\left(-\frac{\Psi_{s,k}(z,\bar{z})^2}{\delta_{b,k}^2}\right)$$

is a boundary-localization function. This indicator accumulates the geometric effects of all internal obstacles and allows the deformation to be activated locally near the porous boundaries. The deformation parameters $q \in (0,1]$ and $\tau \geq 0$ are introduced into the viscosity, mobility, relaxation, and boundary-force terms. In particular, the deformed viscosity is defined by

$$\nu_{q,\tau}^{(N)}(z,t) = \nu_0 \left[1 + \tau(1-q)\left(|\nabla \phi(z,t)| + \mathcal{G}_s^{(N)}(z)\right)\right],$$

and the deformed mobility is defined by

$$M_{q,\tau}^{(N)}(z,t) = M_0 \left[1 + \tau(1-q)\left(|\nabla \phi(z,t)|^2 + (\mathcal{G}_s^{(N)}(z))^2\right)\right].$$

These definitions ensure that the deformation is strongest near high-gradient regions, namely the moving fluid–fluid interface and the solid inclusions. In smooth bulk regions, where

$$|\nabla \phi| \approx 0 \qquad \text{and} \qquad \mathcal{G}_s^{(N)} \approx 0,$$

the model remains close to the classical formulation.

The paper establishes several theoretical properties of the proposed model. First, the porous disk geometry is shown to be well defined under explicit containment and disjointness conditions on the internal inclusions. Second, the localization functions are shown to be positive, bounded by one, and concentrated near the solid boundaries. Third, the deformed viscosity and mobility are proved to remain positive and bounded under natural gradient assumptions. Fourth, the coupled system is shown to conserve total phase mass under incompressibility, impermeability, and no-flux boundary conditions. Finally, a formal phase-field energy dissipation relation is derived, showing that the model preserves the stabilizing structure of the Cahn–Hilliard dynamics. Simulations are used to show the effect of the (q,τ)-deformation. The classical case $(q,\tau) = (1,0)$ may exhibit spurious oscillations and boundary artifacts near the interface and solid inclusions. As q decreases and τ increases, the factor $\tau(1-q)$ increases, producing stronger localized stabilization. The resulting phase–velocity plots show smoother interfaces, reduced boundary artifacts, and improved flow behavior through the porous pathways. The main contributions of

this work can be summarized as follows:

The proposed approach combines a multi-obstacle disk geometry with a spatially adaptive (q, τ)-dependent modulation of viscosity and phase-field mobility, together with a localized solid-boundary force. Compared with a constant-coefficient NS–CH formulation, the additional feature is that the modulation responds jointly to the phase-field gradient and the geometry of the internal obstacles. The BGK collision parameter is calibrated to recover the prescribed viscosity, rather than being assigned the same multiplicative factor directly. The expected advantage is greater control of transport and relaxation near diffuse interfaces and solid boundaries, while limiting coefficient modification in smooth regions away from the obstacles. These features provide a tunable modeling framework; improved numerical accuracy, stability, and computational efficiency require validation through scheme-specific analysis and benchmark comparisons.

2. Generalized Porous Disk Geometry with Multiple Internal Obstacles

We now consider a porous disk geometry containing multiple internal obstacles. This setting is more appropriate for multiphase flow through porous media, microstructured materials, perforated domains, biological channels, and heterogeneous inclusions. Let $\mathbb{D} = \{z = x + iy \in \mathbb{C} : |z| < 1\}$ be the open unit disk. For $k = 1, \ldots, N$, let $\Omega_{s,k} = \left\{z_k + re^{i\theta} : 0 \le r \le R_k(\theta), \quad 0 \le \theta < 2\pi\right\} \subset \mathbb{D}$ be a bounded internal solid obstacle with boundary $\Gamma_{s,k} = \partial\Omega_{s,k}$. The total solid region is defined by $\Omega_s^{(N)} = \bigcup_{k=1}^{N} \overline{\Omega_{s,k}}$, and the corresponding porous fluid domain is

$$\Omega_f^{(N)} = \mathbb{D} \setminus \Omega_s^{(N)}. \tag{1}$$

The external disk boundary is

$$\Gamma_D = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}.$$

Hence, the boundary of the porous fluid region is

$$\partial\Omega_f^{(N)} = \Gamma_D \cup \left(\bigcup_{k=1}^{N} \Gamma_{s,k}\right). \tag{2}$$

Each internal obstacle is represented by a translated starlike domain

$$\Omega_{s,k} = \left\{z = z_k + re^{i\theta} : 0 \le r < R_k(\theta), \quad 0 \le \theta < 2\pi\right\}, \tag{3}$$

where $z_k \in \mathbb{D}$ is the center of the k-th obstacle and

$$R_k(\theta) = r_k \left(1 + \varepsilon_k \cos(m_k\theta + \alpha_k)\right).$$

Here, $r_k > 0$ is the mean radius, $\varepsilon_k \in [0, 1)$ controls the boundary deformation, $m_k \in \mathbb{N}$ controls the number of geometric waves, and $\alpha_k \in [0, 2\pi)$ is a rotation parameter. For the obstacles to be fully contained in $\mathbb{D}$, we assume

$$|z_k| + r_k(1 + \varepsilon_k) < 1, \qquad k = 1, \ldots, N. \tag{4}$$

We also assume that the obstacles are mutually disjoint. A sufficient condition is

$$|z_k - z_\ell| > r_k(1 + \varepsilon_k) + r_\ell(1 + \varepsilon_\ell), \qquad k \ne \ell. \tag{5}$$

Proposition 1 (Well-defined porous disk geometry)**.** *Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $N \ge 1$. For each $k = 1, \ldots, N$, let $z_k \in D$ and let $R_k : \mathbb{R} \to (0, \infty)$ be a C^2, 2π-periodic function. Define the closed solid obstacle*

$$\Omega_{s,k} = \left\{z_k + re^{i\theta} : 0 \le r \le R_k(\theta), \quad 0 \le \theta < 2\pi\right\},$$

its boundary

$$\Gamma_{s,k} = \left\{z_k + R_k(\theta)e^{i\theta} : 0 \le \theta < 2\pi\right\},$$

and $R_k^{\max} = \max_{\theta\in[0,2\pi]} R_k(\theta)$. Assume that

$$|z_k| + R_k^{\max} < 1, \qquad k = 1, \ldots, N,$$

and

$$|z_k - z_\ell| > R_k^{\max} + R_\ell^{\max}, \qquad k \ne \ell.$$

Then the fluid region

$$\Omega_f^{(N)} = D \setminus \bigcup_{k=1}^{N} \Omega_{s,k}$$

is a bounded, open, connected domain with N holes. Its boundary consists of $N+1$ pairwise disjoint C^2 simple closed curves and satisfies

$$\partial\Omega_f^{(N)} = \partial D \cup \bigcup_{k=1}^{N} \Gamma_{s,k}.$$

In particular, $\Omega_f^{(N)}$ is an $(N+1)$-connected planar domain.

Proof. Since R_k is continuous, positive, and periodic, it attains a positive minimum and a finite maximum:

$$0 < R_k^{\min} := \min_{\theta\in[0,2\pi]} R_k(\theta) \le R_k(\theta) \le R_k^{\max}.$$

First, each obstacle is compact. Indeed, $\Omega_{s,k}$ is the image of the compact set $[0,1]\times[0,2\pi]$ under the continuous map $(s,\theta) \longmapsto z_k + sR_k(\theta)e^{i\theta}$. The periodicity of R_k ensures that including $\theta = 2\pi$ does not change the obstacle. For every $z \in \Omega_{s,k}$, $|z| \le |z_k| + |z - z_k| \le |z_k| + R_k^{\max} < 1$. Thus, each compact obstacle lies strictly inside D. Next, suppose that a point z belongs to both $\Omega_{s,k}$ and $\Omega_{s,\ell}$, where $k \neq \ell$. Then $|z_k - z_\ell| \le |z_k - z| + |z - z_\ell| \le R_k^{\max} + R_\ell^{\max}$, contrary to the separation assumption. Hence, the closed obstacles are pairwise disjoint. The set $K = \bigcup_{k=1}^{N} \Omega_{s,k}$ is a finite union of compact sets and is therefore closed. Consequently, $\Omega_f^{(N)} = D \cap (\mathbb{C}\setminus K)$ is open. It is bounded because it is contained in D. To identify the obstacle boundaries, consider $\gamma_k(\theta) = z_k + R_k(\theta)e^{i\theta}$. Because $R_k(\theta) > 0$, equality $\gamma_k(\theta) = \gamma_k(\vartheta)$ implies $\theta = \vartheta$ modulo 2π. Furthermore, $\gamma_k'(\theta) = e^{i\theta}\big(R_k'(\theta) + iR_k(\theta)\big)$, so $|\gamma_k'(\theta)|^2 = |R_k'(\theta)|^2 + R_k(\theta)^2 > 0$. Therefore, γ_k parametrizes a regular C^2 simple closed curve. Its enclosed region is precisely the interior of $\Omega_{s,k}$, and hence $\partial\Omega_{s,k} = \Gamma_{s,k}$. We now verify connectedness. The strict containment and separation assumptions allow a sufficiently small $\delta > 0$ such that the enlarged radial obstacles

$$\widetilde{\Omega}_{s,k} = \big\{ z_k + re^{i\theta} : 0 \le r \le R_k(\theta) + \delta, \quad 0 \le \theta < 2\pi \big\}$$

are still pairwise disjoint and contained in D. For any two points $a, b \in \Omega_f^{(N)}$, δ can also be chosen so that neither point belongs to an enlarged obstacle. The line segment joining a to b lies in D. Whenever this segment meets an enlarged obstacle, replace the portion between its first and last intersection with an arc of that obstacle's boundary. Each replacement arc lies outside every original solid obstacle and inside D. Since there are only finitely many pairwise disjoint enlarged obstacles, this construction produces a continuous path in $\Omega_f^{(N)}$ joining a to b. Thus, $\Omega_f^{(N)}$ is path connected. Finally, the obstacles are separated from one another and from ∂D. Every point of ∂D is approached by fluid points from inside D, and every point of $\Gamma_{s,k}$ is approached by fluid points from outside $\Omega_{s,k}$. Conversely, a point outside these curves lies either in the fluid interior, in a solid interior, or outside $\overline{D}$, and therefore cannot be a fluid boundary point. It follows that

$$\partial\Omega_f^{(N)} = \partial D \cup \bigcup_{k=1}^{N} \Gamma_{s,k}.$$

These are $N+1$ pairwise disjoint simple closed curves. Equivalently, the complement of $\Omega_f^{(N)}$ in the Riemann sphere has $N+1$ connected components: the N solid obstacles and the exterior component. This proves the result. □

2.1. Multi-Obstacle Boundary Indicator

For each obstacle $\Omega_{s,k}$, let $\Psi_{s,k}(z,\bar{z})$ be a smooth level-set function satisfying

$$\Gamma_{s,k} = \{ z \in \mathbb{D} : \Psi_{s,k}(z,\bar{z}) = 0 \}.$$

The boundary-localization function associated with the k-th obstacle is defined by

$$\chi_{b,k}(z) = \exp\left(-\frac{\Psi_{s,k}(z,\bar{z})^2}{\delta_{b,k}^2} \right), \qquad \delta_{b,k} > 0. \tag{6}$$

This function satisfies $\chi_{b,k}(z) = 1 \quad \text{on}\,\Gamma_{s,k}$, and decays away from $\Gamma_{s,k}$. We define the multi-obstacle geometric indicator by

$$\mathcal{G}_s^{(N)}(z) = \sum_{k=1}^{N} \chi_{b,k}(z)|\nabla\Psi_{s,k}(z,\bar{z})|. \tag{7}$$

This quantity measures the total localized geometric effect of all internal obstacles.

Proposition 2 (Localization of the porous-boundary indicator)**.** *Let* $\chi_{b,k}$ *be defined by* (6)*. Then* $0 < \chi_{b,k}(z) \leq 1$ *for every* $z \in \Omega_f^{(N)}$*. Moreover,* $\chi_{b,k}(z) = 1$ *on* $\Gamma_{s,k}$*, and* $\chi_{b,k}(z) \to 0$ *as*

$$|\Psi_{s,k}(z,\bar{z})|/\delta_{b,k} \to \infty.$$

Consequently, the geometric indicator

$$\mathcal{G}_s^{(N)}(z) = \sum_{k=1}^{N} \chi_{b,k}(z)|\nabla\Psi_{s,k}(z,\bar{z})|$$

is concentrated near the internal obstacle boundaries $\Gamma_{s,1}, \ldots, \Gamma_{s,N}$.

Proof. By definition,

$$\chi_{b,k}(z) = \exp\left(-\frac{\Psi_{s,k}(z,\bar{z})^2}{\delta_{b,k}^2}\right).$$

Since

$$\Psi_{s,k}(z,\bar{z})^2 \geq 0 \quad \text{and} \quad \delta_{b,k}^2 > 0,$$

we have

$$-\frac{\Psi_{s,k}(z,\bar{z})^2}{\delta_{b,k}^2} \leq 0.$$

Therefore,

$$0 < \exp\left(-\frac{\Psi_{s,k}(z,\bar{z})^2}{\delta_{b,k}^2}\right) \leq 1,$$

which proves $0 < \chi_{b,k}(z) \leq 1$. On the boundary $\Gamma_{s,k}$, by definition of the level-set function, $\Psi_{s,k}(z,\bar{z}) = 0$. Thus, $\chi_{b,k}(z) = \exp(0) = 1$. If $|\Psi_{s,k}(z,\bar{z})|/\delta_{b,k} \to \infty$, then

$$-\frac{\Psi_{s,k}(z,\bar{z})^2}{\delta_{b,k}^2} \to -\infty.$$

Hence, $\chi_{b,k}(z) \to 0$. Therefore, each $\chi_{b,k}$ is localized near the boundary $\Gamma_{s,k}$. Since $\mathcal{G}_s^{(N)}$ is a finite sum of such localized terms, it is concentrated near the union of the internal boundaries $\bigcup_{k=1}^{N} \Gamma_{s,k}$. This completes the proof. □

2.2. Generalized (q,τ)-Deformed Relaxation and Force

In the porous disk geometry, the adaptive relaxation parameter must respond not only to the moving fluid–fluid interface but also to all internal solid boundaries. We therefore define

$$\lambda_{q,\tau}^{(N)}(z,t) = \lambda_0\left[1 + \tau(1-q)\left(|\nabla\phi(z,t)| + \mathcal{G}_s^{(N)}(z)\right)\right], \qquad \lambda_0 > 0. \tag{8}$$

Here, $\phi(z,t)$ is the phase-field variable and $\mathcal{G}_s^{(N)}$ is the porous-boundary indicator defined in (7). The (q,τ)-deformed boundary force in the porous disk is defined by

$$\mathbf{F}_b^{(q,\tau,N)}(z,t) = -\eta_0\tau(1-q)\sum_{k=1}^{N} \chi_{b,k}(z)\nabla\Psi_{s,k}(z,\bar{z}), \qquad \eta_0 > 0. \tag{9}$$

The total deformed force is then

$$\mathbf{F}^{(q,\tau,N)}(z,t) = -\kappa_0 q^{\tau}\phi(z,t)\nabla\mu(z,t) + \mathbf{F}_b^{(q,\tau,N)}(z,t), \qquad \kappa_0 > 0. \tag{10}$$

The first term is the deformed interfacial force, while the second term is the accumulated boundary-stabilization force from all solid inclusions. The generalized (q,τ)-deformed LBM equation in the porous disk geometry is

$$\begin{aligned} &f_i(z+\mathbf{c}_i\Delta t, t+\Delta t) - f_i(z,t) \\ &\quad = -\frac{1}{\frac{1}{2} + a\lambda_{q,\tau}^{(N)}(z,t)} \left[f_i(z,t) - f_i^{\mathrm{eq}}(z,t)\right] + \Delta t\, F_i^{(q,\tau,N)}(z,t), \end{aligned} \tag{11}$$

where $a = \frac{\nu_0}{c_s^2 \Delta t\, \lambda_0} > 0$ and

$$F_i^{(q,\tau,N)} = \left(1 - \frac{1}{2\widehat{\lambda}_{q,\tau}^{(N)}}\right) w_i \times \left[\frac{\mathbf{c}_i - \mathbf{u}}{c_s^2} + \frac{(\mathbf{c}_i \cdot \mathbf{u})\mathbf{c}_i}{c_s^4}\right] \cdot \mathbf{F}^{(q,\tau,N)}, \tag{12}$$

Here, $F_i^{(q,\tau,N)}$ denotes the complete Guo discrete source, including its relaxation-dependent prefactor. The macroscopic variables are defined by

$$\rho = \sum_i f_i, \qquad \rho\mathbf{u} = \sum_i \mathbf{c}_i f_i + \frac{\Delta t}{2}\mathbf{F}^{(q,\tau,N)}.$$

The velocity is used in both the equilibrium distribution and the discrete source. Under the standard isotropy, smoothness, and low-Mach-number assumptions, the Chapman–Enskog expansion formally recovers the incompressible momentum equation, with kinematic viscosity

$$\nu_{q,\tau}^{(N)} = c_s^2 \Delta t \left(\widehat{\lambda}_{q,\tau}^{(N)} - \frac{1}{2}\right).$$

Proposition 3 (Consistency of the multi-obstacle deformation)**.** *Let $q \in (0,1]$, $\tau \geq 0$, and $\lambda_0 > 0$. Assume that $\lambda_{q,\tau}^{(N)}$ is defined by (8) and that the total deformed force is defined by (10). Then the generalized porous-disk LBM model satisfies the following properties:*

(i) The relaxation parameter is positive: $\lambda_{q,\tau}^{(N)}(z,t) \geq \lambda_0 > 0$.
(ii) The adaptive relaxation is strengthened near the moving fluid–fluid interface and near each internal obstacle boundary $\Gamma_{s,k}$.
(iii) The boundary force $\mathbf{F}_b^{(q,\tau,N)}$ is localized near the union of the internal solid boundaries $\bigcup_{k=1}^N \Gamma_{s,k}$.
(iv) For fixed τ, phase fields, and obstacle geometry, the update coefficients converge as $q \to 1$ to their undeformed values, with collision parameter $\widehat{\lambda}_0 = \frac{1}{2} + \nu_0/(c_s^2\Delta t)$.

Proof. We prove each statement separately. First, by definition,

$$\lambda_{q,\tau}^{(N)}(z,t) = \lambda_0 \left[1 + \tau(1-q)\left(|\nabla\phi(z,t)| + \mathcal{G}_s^{(N)}(z)\right)\right].$$

Since $q \in (0,1]$, we have $1 - q \geq 0$. Since $\tau \geq 0$, $|\nabla\phi| \geq 0$, and $\mathcal{G}_s^{(N)} \geq 0$, it follows that

$$\tau(1-q)\left(|\nabla\phi| + \mathcal{G}_s^{(N)}\right) \geq 0.$$

Therefore, $\lambda_{q,\tau}^{(N)}(z,t) \geq \lambda_0 > 0$. This proves (i). Second, near the moving fluid–fluid interface $\Gamma_f(t) = \{z \in \Omega_f^{(N)} : \phi(z,t) = 0\}$, the gradient $|\nabla\phi|$ is large because the phase field changes rapidly across the diffuse interface. Hence, the term $\tau(1-q)|\nabla\phi|$ increases the relaxation parameter near $\Gamma_f(t)$. Similarly, near the k-th internal obstacle boundary $\Gamma_{s,k}$, the localization function $\chi_{b,k}$ satisfies $\chi_{b,k}(z) \approx 1$, and therefore the term $\chi_{b,k}(z)|\nabla\Psi_{s,k}(z,\bar{z})|$ contributes to $\mathcal{G}_s^{(N)}$. Thus, $\lambda_{q,\tau}^{(N)}$ also increases near each solid boundary. This proves (ii). Third, the porous-boundary force is

$$\mathbf{F}_b^{(q,\tau,N)} = -\eta_0\tau(1-q)\sum_{k=1}^N \chi_{b,k}(z)\nabla\Psi_{s,k}(z,\bar{z}).$$

By Proposition 2, each $\chi_{b,k}$ is localized near $\Gamma_{s,k}$. Hence, the term $\chi_{b,k}(z)\nabla\Psi_{s,k}(z,\bar{z})$ is concentrated near $\Gamma_{s,k}$. Since the force is a finite sum of such localized terms, $\mathbf{F}_b^{(q,\tau,N)}$ is localized near $\bigcup_{k=1}^N \Gamma_{s,k}$. This proves (iii). Finally, consider $q \to 1$ with $\tau \geq 0$ fixed. We verify consistency at the level of the update coefficients, with the

phase fields and obstacle geometry held fixed. At each point where $|\nabla\phi| + \mathcal{G}_s^{(N)}$ is finite,

$$\lambda_{q,\tau}^{(N)} = \lambda_0 \left[1 + \tau(1-q)\left(|\nabla\phi| + \mathcal{G}_s^{(N)}\right)\right] \longrightarrow \lambda_0.$$

Consequently,

$$\widehat{\lambda}_{q,\tau}^{(N)} = \frac{1}{2} + a\lambda_{q,\tau}^{(N)} \longrightarrow \widehat{\lambda}_0 := \frac{1}{2} + a\lambda_0 = \frac{1}{2} + \frac{\nu_0}{c_s^2 \Delta t}.$$

Since $\widehat{\lambda}_0 > 1/2$, it follows that

$$\frac{1}{\widehat{\lambda}_{q,\tau}^{(N)}} \longrightarrow \frac{1}{\widehat{\lambda}_0}.$$

For the fixed finite collection of obstacles, at points where the boundary-force factors are finite,

$$\mathbf{F}_b^{(q,\tau,N)} = -\eta_0 \tau(1-q) \sum_{k=1}^{N} \chi_{b,k} \nabla\Psi_{s,k} \longrightarrow 0.$$

Together with $q^\tau \to 1$, this gives

$$\mathbf{F}^{(q,\tau,N)} \longrightarrow \mathbf{F}^{\mathrm{cl}} := -\kappa_0 \phi \nabla\mu.$$

For fixed macroscopic velocity, the discrete-force formula is linear in the total force; hence $F_i^{(q,\tau,N)} \to F_i^{\mathrm{cl}}$. Therefore, the undeformed form of (11) is

$$f_i(z + \mathbf{c}_i\Delta t, t + \Delta t) - f_i(z,t) = -\frac{1}{\widehat{\lambda}_0}\left[f_i(z,t) - f_i^{\mathrm{eq}}(z,t)\right] + \Delta t\, F_i^{\mathrm{cl}}(z,t).$$

The limiting collision parameter achieves $c_s^2\Delta t\left(\widehat{\lambda}_0 - \frac{1}{2}\right) = \nu_0$. Thus, the limiting collision operator has the prescribed baseline kinematic viscosity ν_0. This proves consistency of the update coefficients in the un-deformed limit and completes the proof. □

2.3. Porous-Disk Cahn–Hilliard Mobility

For phase-field evolution in the porous disk, we define the generalized deformed mobility by

$$M_{q,\tau}^{(N)}(z,t) = M_0\left[1 + \tau(1-q)\left(|\nabla\phi(z,t)|^2 + \left(\mathcal{G}_s^{(N)}(z)\right)^2\right)\right], \qquad M_0 > 0. \tag{13}$$

The corresponding deformed Cahn–Hilliard equation is

$$\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\mathbf{u}) = \nabla\cdot\left(M_{q,\tau}^{(N)}\nabla\mu\right), \tag{14}$$

where

$$\mu = \beta\phi(\phi^2 - 1) - \epsilon^2\Delta\phi. \tag{15}$$

Proposition 4 (Positivity and boundedness of the porous mobility)**.** *Assume that $q \in (0,1]$, $\tau \geq 0$, and $M_0 > 0$. Suppose that there exist constants $G_\phi \geq 0$ and $G_s \geq 0$ such that*

$$|\nabla\phi(z,t)| \leq G_\phi, \qquad \mathcal{G}_s^{(N)}(z) \leq G_s$$

for all $z \in \Omega_f^{(N)}$ and $t \in [0,T]$. Then

$$M_0 \leq M_{q,\tau}^{(N)}(z,t) \leq M_0\left[1 + \tau(1-q)(G_\phi^2 + G_s^2)\right].$$

Proof. By definition,

$$M_{q,\tau}^{(N)} = M_0\left[1 + \tau(1-q)\left(|\nabla\phi|^2 + (\mathcal{G}_s^{(N)})^2\right)\right].$$

Since $q \in (0,1]$, we have $1 - q \geq 0$. Also,

$$|\nabla\phi|^2 \geq 0 \quad \text{and} \quad (\mathcal{G}_s^{(N)})^2 \geq 0.$$

Therefore,

$$\tau(1-q)\left(|\nabla\phi|^2+(\mathcal{G}_s^{(N)})^2\right)\geq 0,$$

and hence

$$M_{q,\tau}^{(N)}\geq M_0>0.$$

Using the assumed bounds,

$$|\nabla\phi|\leq G_\phi,\qquad \mathcal{G}_s^{(N)}\leq G_s,$$

we obtain

$$|\nabla\phi|^2+(\mathcal{G}_s^{(N)})^2\leq G_\phi^2+G_s^2.$$

Therefore,

$$M_{q,\tau}^{(N)}\leq M_0\left[1+\tau(1-q)(G_\phi^2+G_s^2)\right].$$

This proves the desired estimate. □

2.4. Mass Conservation in the Porous Disk

The no-flux boundary conditions for the phase field are imposed on the entire porous boundary:

$$\nabla\phi\cdot\mathbf{n}=0,\qquad \nabla\mu\cdot\mathbf{n}=0,\qquad z\in\partial\Omega_f^{(N)}. \tag{16}$$

Proposition 5 (Mass conservation of the porous Cahn–Hilliard model)**.** *Let ϕ satisfy the porous-disk deformed Cahn–Hilliard equation*

$$\frac{\partial\phi}{\partial t}+\nabla\cdot(\phi\mathbf{u})=\nabla\cdot\left(M_{q,\tau}^{(N)}\nabla\mu\right)$$

in $\Omega_f^{(N)}$. Assume that $\nabla\cdot\mathbf{u}=0$ and that the no-flux conditions

$$\mathbf{u}\cdot\mathbf{n}=0,\qquad \nabla\mu\cdot\mathbf{n}=0$$

hold on $\partial\Omega_f^{(N)}$. Then the total phase mass

$$\mathcal{M}_\phi(t)=\int_{\Omega_f^{(N)}}\phi(z,t)\,dz$$

is conserved in time:

$$\frac{d}{dt}\mathcal{M}_\phi(t)=0.$$

Proof. By definition,

$$\mathcal{M}_\phi(t)=\int_{\Omega_f^{(N)}}\phi(z,t)\,dz.$$

Differentiating with respect to time gives

$$\frac{d}{dt}\mathcal{M}_\phi(t)=\int_{\Omega_f^{(N)}}\frac{\partial\phi}{\partial t}\,dz.$$

Using the deformed Cahn–Hilliard equation,

$$\frac{\partial\phi}{\partial t}=-\nabla\cdot(\phi\mathbf{u})+\nabla\cdot\left(M_{q,\tau}^{(N)}\nabla\mu\right),$$

we obtain

$$\frac{d}{dt}\mathcal{M}_\phi(t)=-\int_{\Omega_f^{(N)}}\nabla\cdot(\phi\mathbf{u})\,dz+\int_{\Omega_f^{(N)}}\nabla\cdot\left(M_{q,\tau}^{(N)}\nabla\mu\right)\,dz.$$

Applying the divergence theorem gives

$$\frac{d}{dt}\mathcal{M}_\phi(t)=-\int_{\partial\Omega_f^{(N)}}\phi\mathbf{u}\cdot\mathbf{n}\,dS+\int_{\partial\Omega_f^{(N)}}M_{q,\tau}^{(N)}\nabla\mu\cdot\mathbf{n}\,dS.$$

By the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$, the first boundary integral is zero. By the no-flux condition $\nabla\mu \cdot \mathbf{n} = 0$, the second boundary integral is also zero. Therefore, $\frac{d}{dt}\mathcal{M}_\phi(t) = 0$. This proves conservation of the total phase mass. □

Remark 1 (Interpretation of the Porous Multi-Obstacle Generalization)**.** *The porous multi-obstacle model generalizes the single-obstacle geometry by allowing several internal inclusions to interact with the moving fluid–fluid interface. The adaptive relaxation parameter*

$$\lambda_{q,\tau}^{(N)}(z,t) = \lambda_0 \left[1 + \tau(1-q)\left(|\nabla\phi(z,t)| + \mathcal{G}_s^{(N)}(z)\right)\right]$$

is strengthened near the diffuse interface and near every internal solid boundary. Similarly, the force

$$\mathbf{F}_b^{(q,\tau,N)} = -\eta_0\tau(1-q)\sum_{k=1}^{N}\chi_{b,k}\nabla\Psi_{s,k}$$

accumulates localized stabilization effects from all inclusions. This extension is physically meaningful because many porous and microstructured systems contain more than one obstacle. Multiple inclusions create several regions of strong curvature, confinement, and boundary-induced gradients. These are precisely the regions where classical multiphase LBM schemes may experience spurious currents, interface distortion, or boundary oscillations. The proposed (q,τ)-deformation provides localized stabilization in these sensitive regions while preserving the classical LBM behavior in smooth bulk zones. When $N = 1$, the generalized model reduces to the single-obstacle configuration. Hence, the porous disk formulation is a genuine extension of the original starlike-obstacle model.

3. Illustrative Example and Parameter Setting

In this section, we present a complete illustrative example showing how the proposed porous multi-obstacle (q,τ)-deformed Lattice Boltzmann framework can be applied. The purpose of the example is to demonstrate how the geometric construction, boundary-localization functions, adaptive relaxation parameter, deformed mobility, and boundary-stabilization force are computed in a porous disk geometry.

3.1. Porous Disk Geometry

Let the computational domain be the open unit disk $\mathbb{D} = \{z = x + iy \in \mathbb{C} : |z| < 1\}$. We consider three internal solid inclusions: $N = 3$. The centers of the inclusions are chosen as

$$z_1 = -0.45 + 0.25i, \qquad z_2 = 0.35 + 0.20i, \qquad z_3 = -0.05 - 0.45i.$$

Each obstacle is represented by a translated starlike domain

$$\Omega_{s,k} = \left\{z = z_k + re^{i\theta} : 0 \le r < R_k(\theta),\ 0 \le \theta < 2\pi\right\}, \qquad k = 1,2,3,$$

where

$$R_k(\theta) = r_k\left(1 + \varepsilon_k\cos(m_k\theta + \alpha_k)\right).$$

For this example, we take

$$r_1 = 0.13, \qquad \varepsilon_1 = 0.18, \qquad m_1 = 5, \qquad \alpha_1 = 0,$$

$$r_2 = 0.12, \qquad \varepsilon_2 = 0.20, \qquad m_2 = 6, \qquad \alpha_2 = \frac{\pi}{4},$$

and

$$r_3 = 0.11, \qquad \varepsilon_3 = 0.16, \qquad m_3 = 7, \qquad \alpha_3 = \frac{\pi}{6}.$$

The total solid region is

$$\Omega_s^{(3)} = \overline{\Omega_{s,1}} \cup \overline{\Omega_{s,2}} \cup \overline{\Omega_{s,3}},$$

and the porous fluid region is

$$\Omega_f^{(3)} = \mathbb{D} \setminus \Omega_s^{(3)}.$$

Hence, we have

$$\partial\Omega_f^{(3)} = \Gamma_D \cup \Gamma_{s,1} \cup \Gamma_{s,2} \cup \Gamma_{s,3},$$

where $\Gamma_D = \partial\mathbb{D}$ is the external disk boundary and $\Gamma_{s,k} = \partial\Omega_{s,k}$, $k = 1,2,3$, are the internal solid boundaries.

3.2. Verification of the Geometry Conditions

We verify that all obstacles are contained in the open unit disk. For the first obstacle,

$$|z_1| = \sqrt{(-0.45)^2 + (0.25)^2} = \sqrt{0.265} \approx 0.515.$$

Moreover,

$$r_1(1 + \varepsilon_1) = 0.13(1.18) = 0.1534.$$

Therefore,

$$|z_1| + r_1(1 + \varepsilon_1) \approx 0.515 + 0.1534 = 0.6684 < 1.$$

For the second obstacle,

$$|z_2| = \sqrt{0.35^2 + 0.20^2} = \sqrt{0.1625} \approx 0.403,$$

and

$$r_2(1 + \varepsilon_2) = 0.12(1.20) = 0.144.$$

Thus,

$$|z_2| + r_2(1 + \varepsilon_2) \approx 0.403 + 0.144 = 0.547 < 1.$$

For the third obstacle,

$$|z_3| = \sqrt{(-0.05)^2 + (-0.45)^2} = \sqrt{0.205} \approx 0.453,$$

and

$$r_3(1 + \varepsilon_3) = 0.11(1.16) = 0.1276.$$

Hence,

$$|z_3| + r_3(1 + \varepsilon_3) \approx 0.453 + 0.1276 = 0.5806 < 1.$$

Therefore, each obstacle is fully contained in $\mathbb{D}$. We also verify that the obstacles are mutually disjoint. The distance between the first and second centers is

$$|z_1 - z_2| = |(-0.45 + 0.25i) - (0.35 + 0.20i)| = |-0.80 + 0.05i| \approx 0.802.$$

The sum of their maximal radii is

$$r_1(1 + \varepsilon_1) + r_2(1 + \varepsilon_2) = 0.1534 + 0.144 = 0.2974.$$

Thus, $|z_1 - z_2| > 0.2974$. Similarly,

$$|z_1 - z_3| = |(-0.45 + 0.25i) - (-0.05 - 0.45i)| = |-0.40 + 0.70i| \approx 0.806,$$

while

$$r_1(1 + \varepsilon_1) + r_3(1 + \varepsilon_3) = 0.1534 + 0.1276 = 0.281.$$

Hence, we get $|z_1 - z_3| > 0.281$. Finally, we have

$$|z_2 - z_3| = |(0.35 + 0.20i) - (-0.05 - 0.45i)| = |0.40 + 0.65i| \approx 0.763,$$

while

$$r_2(1 + \varepsilon_2) + r_3(1 + \varepsilon_3) = 0.144 + 0.1276 = 0.2716.$$

Thus, this gives $|z_2 - z_3| > 0.2716$. Therefore, the three obstacles are mutually disjoint. By Proposition 1, the porous fluid domain $\Omega_f^{(3)}$ is well-defined and multiply connected.

3.3. Level-Set Functions and Boundary Localization

For each inclusion, we define a level-set function by

$$\Psi_{s,k}(z, \bar{z}) = |z - z_k| - R_k(\theta_k),$$

where $\theta_k = \arg(z - z_k)$. Then

$$\Gamma_{s,k} = \{z \in \mathbb{D} : \Psi_{s,k}(z, \bar{z}) = 0\}.$$

The boundary-localization function associated with the k-th obstacle is

$$\chi_{b,k}(z) = \exp\left(-\frac{\Psi_{s,k}(z, \bar{z})^2}{\delta_{b,k}^2}\right).$$

In this example, we take $\delta_{b,1} = \delta_{b,2} = \delta_{b,3} = 0.035$. Since $\delta_{b,k} > 0$ and $\Psi_{s,k}(z, \bar{z})^2 \geq 0$, we have

$$-\frac{\Psi_{s,k}(z, \bar{z})^2}{\delta_{b,k}^2} \leq 0.$$

Therefore,

$$0 < \chi_{b,k}(z) = \exp\left(-\frac{\Psi_{s,k}(z, \bar{z})^2}{\delta_{b,k}^2}\right) \leq 1.$$

Moreover, on the boundary $\Gamma_{s,k}$, one has $\Psi_{s,k}(z, \bar{z}) = 0$, and hence $\chi_{b,k}(z) = 1$. Away from $\Gamma_{s,k}$, the quantity $|\Psi_{s,k}(z, \bar{z})|$ increases and $\chi_{b,k}$ decays exponentially. Therefore, each $\chi_{b,k}$ is concentrated near its corresponding internal obstacle boundary. The combined porous-boundary indicator is defined by

$$\mathcal{G}_s^{(3)}(z) = \sum_{k=1}^{3} \chi_{b,k}(z) |\nabla \Psi_{s,k}(z, \bar{z})|.$$

This function is large near the union of the three solid boundaries $\Gamma_{s,1} \cup \Gamma_{s,2} \cup \Gamma_{s,3}$, and small away from the inclusions.

3.4. Deformed Relaxation and Force in the Porous Disk

Let the phase-field variable be $\phi(z, t)$, and let the moving fluid–fluid interface be represented by $\Gamma_f(t) = \{z \in \Omega_f^{(3)} : \phi(z, t) = 0\}$. The generalized porous relaxation parameter is defined by

$$\lambda_{q,\tau}^{(3)}(z, t) = \lambda_0 \left[1 + \tau(1 - q)\left(|\nabla \phi(z, t)| + \mathcal{G}_s^{(3)}(z)\right)\right].$$

For the present example, we select

$$q = 0.75, \qquad \tau = 1.20, \qquad \lambda_0 = 0.85.$$

Since $1 - q = 0.25$, we have $\tau(1 - q) = 1.20(0.25) = 0.30$. Therefore, we obtain

$$\lambda_{q,\tau}^{(3)}(z, t) = 0.85 \left[1 + 0.30 \left(|\nabla \phi(z, t)| + \mathcal{G}_s^{(3)}(z)\right)\right].$$

Since $|\nabla \phi(z, t)| \geq 0$ and $\mathcal{G}_s^{(3)}(z) \geq 0$, we obtain $\lambda_{q,\tau}^{(3)}(z, t) \geq 0.85 > 0$. Thus, the relaxation parameter is positive everywhere in the porous fluid domain. The boundary-stabilization force is

$$\mathbf{F}_b^{(q,\tau,3)}(z, t) = -\eta_0 \tau(1 - q) \sum_{k=1}^{3} \chi_{b,k}(z) \nabla \Psi_{s,k}(z, \bar{z}).$$

Choosing $\eta_0 = 0.04$, we obtain $\eta_0 \tau(1 - q) = 0.04(1.20)(0.25) = 0.012$. Hence,

$$\mathbf{F}_b^{(q,\tau,3)}(z, t) = -0.012 \sum_{k=1}^{3} \chi_{b,k}(z) \nabla \Psi_{s,k}(z, \bar{z}).$$

Since each $\chi_{b,k}$ is localized near $\Gamma_{s,k}$, the boundary force is localized near the three internal obstacle boundaries. The total force is

$$\mathbf{F}^{(q,\tau,3)}(z, t) = -\kappa_0 q^{\tau} \phi(z, t) \nabla \mu(z, t) + \mathbf{F}_b^{(q,\tau,3)}(z, t).$$

We take $\kappa_0 = 10^{-3}$. Since $q^\tau = 0.75^{1.20} \approx 0.708$, the interfacial part becomes

$$-\kappa_0 q^\tau \phi \nabla \mu \approx -7.08 \times 10^{-4} \phi \nabla \mu.$$

Therefore, in this example,

$$\mathbf{F}^{(q,\tau,3)} \approx -7.08 \times 10^{-4} \phi \nabla \mu - 0.012 \sum_{k=1}^{3} \chi_{b,k} \nabla \Psi_{s,k}.$$

3.5. Deformed Cahn–Hilliard Mobility

The porous mobility is defined by

$$M_{q,\tau}^{(3)}(z,t) = M_0 \left[1 + \tau(1-q) \left(|\nabla \phi(z,t)|^2 + \left(\mathcal{G}_s^{(3)}(z) \right)^2 \right) \right].$$

We choose $M_0 = 10^{-4}$ and $\tau(1-q) = 0.30$, we obtain

$$M_{q,\tau}^{(3)}(z,t) = 10^{-4} \left[1 + 0.30 \left(|\nabla \phi(z,t)|^2 + \left(\mathcal{G}_s^{(3)}(z) \right)^2 \right) \right].$$

Since the bracketed term is at least one, we have $M_{q,\tau}^{(3)}(z,t) \geq 10^{-4} > 0$. Suppose, for example, that during the simulation the following gradient bounds hold:

$$|\nabla \phi(z,t)| \leq G_\phi = 12, \qquad \mathcal{G}_s^{(3)}(z) \leq G_s = 8.$$

Then, by Proposition 4,

$$M_{q,\tau}^{(3)}(z,t) \leq 10^{-4} \left[1 + 0.30(12^2 + 8^2) \right].$$

Since $12^2 + 8^2 = 144 + 64 = 208$, we obtain

$$M_{q,\tau}^{(3)}(z,t) \leq 10^{-4} \left[1 + 0.30(208) \right] = 10^{-4}(63.4) = 6.34 \times 10^{-3}.$$

Therefore, $10^{-4} \leq M_{q,\tau}^{(3)}(z,t) \leq 6.34 \times 10^{-3}$. This confirms that the deformed mobility remains positive and bounded.

3.5.1. Initial Phase Field and Chemical Potential

The initial droplet is placed in the porous disk away from the internal obstacles. Let

$$z_c = 0.10 + 0.55i, \qquad R_0 = 0.18, \qquad \epsilon = 0.035.$$

The initial phase field is defined by

$$\phi(z,0) = \tanh \left(\frac{R_0 - |z - z_c|}{\sqrt{2}\epsilon} \right).$$

The chemical potential is $\mu = \beta\phi(\phi^2 - 1) - \epsilon^2 \Delta\phi$, where we choose $\beta = 1$. Thus, this yields $\mu = \phi(\phi^2 - 1) - \epsilon^2 \Delta\phi$. The deformed Cahn–Hilliard equation in the porous disk is

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) = \nabla \cdot \left(M_{q,\tau}^{(3)} \nabla \mu \right).$$

No-flux boundary conditions are imposed on the full porous boundary:

$$\nabla \mu \cdot \mathbf{n} = 0, \qquad \mathbf{u} \cdot \mathbf{n} = 0, \qquad z \in \partial \Omega_f^{(3)}.$$

3.5.2. Mass Conservation

The total phase mass is $\mathcal{M}_\phi(t) = \int_{\Omega_f^{(3)}} \phi(z,t)\, dz$. Using the deformed Cahn–Hilliard equation, we have

$$\frac{d}{dt}\mathcal{M}_\phi(t) = \int_{\Omega_f^{(3)}} \frac{\partial \phi}{\partial t}\, dz.$$

Substituting

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot (\phi \mathbf{u}) + \nabla \cdot \left(M_{q,\tau}^{(3)} \nabla \mu \right),$$

we obtain

$$\frac{d}{dt}\mathcal{M}_\phi(t) = -\int_{\Omega_f^{(3)}} \nabla \cdot (\phi \mathbf{u})\, dz + \int_{\Omega_f^{(3)}} \nabla \cdot \left(M_{q,\tau}^{(3)} \nabla \mu \right)\, dz.$$

By the divergence theorem,

$$\frac{d}{dt}\mathcal{M}_\phi(t) = -\int_{\partial\Omega_f^{(3)}} \phi \mathbf{u} \cdot \mathbf{n}\, dS + \int_{\partial\Omega_f^{(3)}} M_{q,\tau}^{(3)} \nabla \mu \cdot \mathbf{n}\, dS.$$

Using $\mathbf{u} \cdot \mathbf{n} = 0$ and $\nabla \mu \cdot \mathbf{n} = 0$ on $\partial\Omega_f^{(3)}$, both boundary integrals vanish. Hence, $\frac{d}{dt}\mathcal{M}_\phi(t) = 0$. Therefore, the total phase mass is conserved.

3.6. Expected Physical Behavior

This example describes the motion and deformation of a diffuse droplet in a porous disk containing three internal inclusions. As the droplet moves through the domain, it interacts with each obstacle boundary. Near the fluid–fluid interface, $|\nabla \phi|$ becomes large. Near the solid inclusions, $\mathcal{G}_s^{(3)}$ becomes large. Therefore, the adaptive relaxation parameter

$$\lambda_{q,\tau}^{(3)} = \lambda_0 \left[1 + \tau(1-q) \left(|\nabla \phi| + \mathcal{G}_s^{(3)} \right) \right]$$

increases locally near precisely those regions where classical multiphase LBM schemes are most sensitive. The boundary force

$$\mathbf{F}_b^{(q,\tau,3)} = -\eta_0 \tau (1-q) \sum_{k=1}^{3} \chi_{b,k} \nabla \Psi_{s,k}$$

acts only near the internal inclusions because of the localization functions $\chi_{b,k}$. Consequently, the method introduces boundary stabilization in the porous region without adding unnecessary damping in the bulk fluid. When $q \to 1$, we have $1 - q \to 0$, $q^\tau \to 1$. Therefore,

$$\lambda_{q,\tau}^{(3)} \to \lambda_0, \qquad M_{q,\tau}^{(3)} \to M_0, \qquad \mathbf{F}_b^{(q,\tau,3)} \to 0.$$

Thus, the generalized porous model reduces to the classical forced LBM with the standard interfacial force $-\kappa_0 \phi \nabla \mu$. This example applies all the main results of the porous multi-obstacle theory: the geometry is well-defined, the boundary indicators are localized, the relaxation parameter is positive, the mobility is positive and bounded, the boundary force is localized near the solid inclusions, and the total phase mass is conserved under no-flux boundary conditions. For starlike inclusion inside the open unit disk, avoid a single symmetric cosine term. Use a mixed Fourier radial function with different modes and phase shifts.

$$\Omega_s = \left\{ z = z_0 + re^{i\theta} : 0 \le r < R(\theta),\ 0 \le \theta < 2\pi \right\},$$

where

$$R(\theta) = r_0 \left[1 + a_1 \cos(3\theta) + a_2 \sin(5\theta) + a_3 \cos(7\theta + \alpha) \right].$$

For example, take

$$z_0 = 0.25 - 0.15i, \qquad r_0 = 0.18, \qquad a_1 = 0.18, \qquad a_2 = 0.10, \qquad a_3 = 0.07, \qquad \alpha = \frac{\pi}{5}.$$

Then

$$R(\theta) = 0.18 \left[1 + 0.18 \cos(3\theta) + 0.10 \sin(5\theta) + 0.07 \cos\left(7\theta + \frac{\pi}{5} \right) \right].$$

Therefore, the obstacle is completely contained in the open unit disk satisfying

$$R_1(\theta) = 0.13 \left[1 + 0.16 \cos(3\theta) + 0.08 \sin(5\theta + \pi/6) \right],$$

$$R_2(\theta) = 0.12\,[1 + 0.14\cos(4\theta + \pi/4) + 0.09\sin(7\theta)]\,,$$
$$R_3(\theta) = 0.11\,[1 + 0.12\cos(5\theta + \pi/5) + 0.10\sin(6\theta + \pi/3)]\,.$$

4. Coupled Deformed Navier–Stokes–Cahn–Hilliard System

To describe the interaction between phase separation and hydrodynamic transport in the porous disk geometry, we consider a coupled Navier–Stokes–Cahn–Hilliard system. Let

$$\Omega_f^{(N)} = \mathbb{D} \setminus \bigcup_{k=1}^{N} \overline{\Omega_{s,k}}$$

be the porous fluid region, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk and $\Omega_{s,k}$ are the internal solid inclusions. The unknowns are the phase-field variable $\phi(z,t)$, the velocity field $\mathbf{u}(z,t)$, and the pressure $p(z,t)$. The coupled (q,τ)-deformed system is given by

$$\nabla \cdot \mathbf{u} = 0, \tag{17}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \nabla\cdot\left[\nu_{q,\tau}^{(N)}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\right] + \mathbf{F}^{(q,\tau,N)}, \tag{18}$$

and

$$\frac{\partial \phi}{\partial t} + \nabla\cdot(\phi\mathbf{u}) = \nabla\cdot\left(M_{q,\tau}^{(N)}\nabla\mu\right). \tag{19}$$

We use a constant-density incompressible formulation with unit reference density, $\rho = \rho_0 = 1$. The quantity $\nu_{q,\tau}^{(N)}$ denotes the kinematic viscosity. Since the dynamic viscosity is $\rho_0\nu_{q,\tau}^{(N)}$, the two viscosities coincide numerically under this normalization. Accordingly, the viscous stress in Equation (18) is written directly in terms of $\nu_{q,\tau}^{(N)}$. The chemical potential is

$$\mu = \beta\phi(\phi^2 - 1) - \epsilon^2\Delta\phi. \tag{20}$$

The porous-boundary indicator is defined by

$$\mathcal{G}_s^{(N)}(z) = \sum_{k=1}^{N}\chi_{b,k}(z)|\nabla\Psi_{s,k}(z,\bar{z})|, \tag{21}$$

where

$$\chi_{b,k}(z) = \exp\left(-\frac{\Psi_{s,k}(z,\bar{z})^2}{\delta_{b,k}^2}\right). \tag{22}$$

The deformed mobility is

$$M_{q,\tau}^{(N)}(z,t) = M_0\left[1 + \tau(1-q)\left(|\nabla\phi(z,t)|^2 + (\mathcal{G}_s^{(N)}(z))^2\right)\right], \tag{23}$$

and the deformed viscosity is defined by

$$\nu_{q,\tau}^{(N)}(z,t) = \nu_0\left[1 + \tau(1-q)\left(|\nabla\phi(z,t)| + \mathcal{G}_s^{(N)}(z)\right)\right]. \tag{24}$$

The total force in the momentum equation is

$$\mathbf{F}^{(q,\tau,N)} = -\kappa_0 q^\tau\phi\nabla\mu + \mathbf{F}_b^{(q,\tau,N)}, \tag{25}$$

where the first term is the deformed interfacial force and

$$\mathbf{F}_b^{(q,\tau,N)} = -\eta_0\tau(1-q)\sum_{k=1}^{N}\chi_{b,k}(z)\nabla\Psi_{s,k}(z,\bar{z}) \tag{26}$$

is the porous-boundary stabilization force. The boundary conditions are imposed on the full porous boundary

$$\partial\Omega_f^{(N)} = \Gamma_D \cup \left(\bigcup_{k=1}^{N}\Gamma_{s,k}\right).$$

For the velocity field, we impose the no-slip condition

$$\mathbf{u} = 0, \qquad z \in \bigcup_{k=1}^{N} \Gamma_{s,k}. \tag{27}$$

For the phase-field variable, we impose no-flux conditions

$$\nabla\phi \cdot \mathbf{n} = 0, \qquad \nabla\mu \cdot \mathbf{n} = 0, \qquad z \in \partial\Omega_f^{(N)}. \tag{28}$$

Proposition 6 (Compatibility of the BGK relaxation parameter and the prescribed viscosity)**.** *Let* $0 < q \leq 1$*,* $\tau \geq 0$*, and let* $\lambda_0, \nu_0, c_s, \Delta t > 0$ *be constants. Define*

$$A_{q,\tau}^{(N)}(z,t) = 1 + \tau(1-q)\left(|\nabla\phi(z,t)| + G_s^{(N)}(z)\right),$$

where $G_s^{(N)}(z) \geq 0$*. Retain the adaptive parameter and the prescribed kinematic viscosity*

$$\lambda_{q,\tau}^{(N)}(z,t) = \lambda_0 A_{q,\tau}^{(N)}(z,t), \qquad \nu_{q,\tau}^{(N)}(z,t) = \nu_0 A_{q,\tau}^{(N)}(z,t).$$

Set

$$a = \frac{\nu_0}{c_s^2 \Delta t\, \lambda_0}, \qquad \widehat{\lambda}_{q,\tau}^{(N)}(z,t) = \frac{1}{2} + a\lambda_{q,\tau}^{(N)}(z,t).$$

Consider the corrected BGK update

$$\begin{aligned} &f_i(z + \mathbf{c}_i\Delta t, t + \Delta t) - f_i(z,t) \\ &\quad = -\frac{1}{\widehat{\lambda}_{q,\tau}^{(N)}(z,t)}\left[f_i(z,t) - f_i^{\mathrm{eq}}(z,t)\right] + \Delta t\, F_i^{(q,\tau,N)}(z,t). \end{aligned}$$

Assume that the standard isothermal BGK hydrodynamic expansion applies: the lattice and equilibrium satisfy the required isotropic moment conditions, the fields are sufficiently smooth, and the low-Mach-number incompressible scaling is valid. If forcing is present, assume that its discretization recovers the prescribed body force without modifying the leading viscous stress. Then, at the retained hydrodynamic order, the effective kinematic viscosity of the corrected collision operator is precisely $\nu_{\mathrm{BGK}}^{(N)}(z,t) = \nu_{q,\tau}^{(N)}(z,t)$*. Moreover,*

$$\widehat{\lambda}_{q,\tau}^{(N)}(z,t) \geq \frac{1}{2} + \frac{\nu_0}{c_s^2\Delta t} > \frac{1}{2}.$$

For fixed τ*, if* $|\nabla\phi| + G_s^{(N)}$ *remains uniformly bounded as* $q \to 1$*, then*

$$\nu_{q,\tau}^{(N)} \longrightarrow \nu_0, \qquad \widehat{\lambda}_{q,\tau}^{(N)} \longrightarrow \frac{1}{2} + \frac{\nu_0}{c_s^2\Delta t}.$$

Proof. The coefficient $1/\widehat{\lambda}_{q,\tau}^{(N)}$ in the collision operator identifies $\widehat{\lambda}_{q,\tau}^{(N)}$ as the dimensionless BGK relaxation parameter. Its corresponding dimensional relaxation time is $t_{\mathrm{rel}}^{(N)} = \Delta t\, \widehat{\lambda}_{q,\tau}^{(N)}$. Under the stated hydrodynamic assumptions, the standard BGK viscosity relation is

$$\begin{aligned} \nu_{\mathrm{BGK}}^{(N)} &= c_s^2\left(t_{\mathrm{rel}}^{(N)} - \frac{\Delta t}{2}\right) \\ &= c_s^2\Delta t\left(\widehat{\lambda}_{q,\tau}^{(N)} - \frac{1}{2}\right). \end{aligned}$$

The subtraction of $\Delta t/2$ is the discrete-time correction in the BGK hydrodynamic expansion. Substituting the definition of $\widehat{\lambda}_{q,\tau}^{(N)}$ gives

$$\begin{aligned} \nu_{\mathrm{BGK}}^{(N)} &= c_s^2\Delta t\, a\lambda_{q,\tau}^{(N)} \\ &= c_s^2\Delta t\frac{\nu_0}{c_s^2\Delta t\,\lambda_0}\lambda_0 A_{q,\tau}^{(N)} \\ &= \nu_0 A_{q,\tau}^{(N)} = \nu_{q,\tau}^{(N)}. \end{aligned}$$

This establishes compatibility with the prescribed kinematic viscosity. Furthermore, $0 < q \leq 1$, $\tau \geq 0$, and

$G_s^{(N)} \geq 0$ imply $A_{q,\tau}^{(N)} \geq 1$. Consequently, we have

$$\begin{aligned}\widehat{\lambda}_{q,\tau}^{(N)} &= \frac{1}{2} + \frac{\nu_0}{c_s^2 \Delta t} A_{q,\tau}^{(N)} \\ &\geq \frac{1}{2} + \frac{\nu_0}{c_s^2 \Delta t} > \frac{1}{2}.\end{aligned}$$

Thus, the effective kinematic viscosity is positive. Finally, the uniform boundedness assumption implies

$$\tau(1-q)\left(|\nabla\phi| + G_s^{(N)}\right) \longrightarrow 0 \quad \text{as } q \to 1.$$

Hence, $A_{q,\tau}^{(N)} \to 1$, and substitution into the viscosity and relaxation formulas proves the claimed limits. □

Proposition 7 (Positivity and boundedness of the deformed viscosity)**.** *Let $q \in (0,1]$, $\tau \geq 0$, and $\nu_0 > 0$. Assume that*

$$\nu_{q,\tau}^{(N)}(z,t) = \nu_0 \left[1 + \tau(1-q)\left(|\nabla\phi(z,t)| + \mathcal{G}_s^{(N)}(z)\right)\right].$$

Suppose that there exist constants $G_\phi \geq 0$ and $G_s \geq 0$ such that

$$|\nabla\phi(z,t)| \leq G_\phi, \qquad \mathcal{G}_s^{(N)}(z) \leq G_s$$

for all $z \in \Omega_f^{(N)}$ and $t \in [0,T]$. Then

$$\nu_0 \leq \nu_{q,\tau}^{(N)}(z,t) \leq \nu_0 \left[1 + \tau(1-q)(G_\phi + G_s)\right].$$

Proof. Since $q \in (0,1]$, we have $1-q \geq 0$. Also, since $\tau \geq 0$, $|\nabla\phi| \geq 0$, and $\mathcal{G}_s^{(N)} \geq 0$, it follows that

$$\tau(1-q)\left(|\nabla\phi| + \mathcal{G}_s^{(N)}\right) \geq 0.$$

Therefore,

$$\nu_{q,\tau}^{(N)} = \nu_0 \left[1 + \tau(1-q)\left(|\nabla\phi| + \mathcal{G}_s^{(N)}\right)\right] \geq \nu_0 > 0.$$

This proves the lower bound. Using the assumed bounds

$$|\nabla\phi| \leq G_\phi, \qquad \mathcal{G}_s^{(N)} \leq G_s,$$

we obtain

$$|\nabla\phi| + \mathcal{G}_s^{(N)} \leq G_\phi + G_s.$$

Hence,

$$\nu_{q,\tau}^{(N)} \leq \nu_0 \left[1 + \tau(1-q)(G_\phi + G_s)\right].$$

This proves the upper bound and completes the proof. □

Proposition 8 (Mass conservation of the coupled porous system)**.** *Let ϕ satisfy the deformed Cahn–Hilliard equation*

$$\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\mathbf{u}) = \nabla\cdot\left(M_{q,\tau}^{(N)}\nabla\mu\right)$$

in $\Omega_f^{(N)}$. Assume that

$$\nabla\cdot\mathbf{u} = 0,$$

and that the no-flux boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \nabla\mu\cdot\mathbf{n} = 0$$

hold on $\partial\Omega_f^{(N)}$. Then the total phase mass

$$\mathcal{M}_\phi(t) = \int_{\Omega_f^{(N)}} \phi(z,t)\, dz$$

is conserved. That is, $\frac{d}{dt}\mathcal{M}_\phi(t) = 0$.

Proof. By definition,

$$\mathcal{M}_\phi(t) = \int_{\Omega_f^{(N)}} \phi(z,t)\, dz.$$

Differentiating with respect to time yields

$$\frac{d}{dt}\mathcal{M}_\phi(t) = \int_{\Omega_f^{(N)}} \frac{\partial \phi}{\partial t}\, dz.$$

Using the deformed Cahn–Hilliard equation gives

$$\frac{d}{dt}\mathcal{M}_\phi(t) = -\int_{\Omega_f^{(N)}} \nabla\cdot(\phi\mathbf{u})\, dz + \int_{\Omega_f^{(N)}} \nabla\cdot\left(M_{q,\tau}^{(N)}\nabla\mu\right)\, dz.$$

Applying the divergence theorem, we obtain

$$\frac{d}{dt}\mathcal{M}_\phi(t) = -\int_{\partial\Omega_f^{(N)}} \phi\mathbf{u}\cdot\mathbf{n}\, dS + \int_{\partial\Omega_f^{(N)}} M_{q,\tau}^{(N)}\nabla\mu\cdot\mathbf{n}\, dS.$$

Since $\mathbf{u}\cdot\mathbf{n} = 0$ and $\nabla\mu\cdot\mathbf{n} = 0$ on $\partial\Omega_f^{(N)}$, both boundary integrals vanish. Hence,

$$\frac{d}{dt}\mathcal{M}_\phi(t) = 0.$$

Therefore, the total phase mass is conserved. □

Proposition 9 (Localized activation of the coupled deformation)**.** *Let*

$$\lambda_{q,\tau}^{(N)}(z,t) = \lambda_0\left[1 + \tau(1-q)\left(|\nabla\phi(z,t)| + \mathcal{G}_s^{(N)}(z)\right)\right],$$

where

$$\mathcal{G}_s^{(N)}(z) = \sum_{k=1}^{N} \chi_{b,k}(z)|\nabla\Psi_{s,k}(z,\bar{z})|.$$

Then the deformation is activated mainly near the fluid–fluid interface

$$\Gamma_f(t) = \{z \in \Omega_f^{(N)} : \phi(z,t) = 0\}$$

and near the solid boundaries $\Gamma_{s,k} = \partial\Omega_{s,k}, k = 1,\ldots,N$. *Moreover, in smooth bulk regions far from both the interface and the solid inclusions, the model approaches the classical relaxation*

$$\lambda_{q,\tau}^{(N)}(z,t) \approx \lambda_0.$$

Proof. The adaptive relaxation depends on two quantities: $|\nabla\phi(z,t)|$ and $\mathcal{G}_s^{(N)}(z)$. Near the diffuse fluid–fluid interface $\Gamma_f(t)$, the phase field ϕ changes rapidly from one phase to the other. Hence, this implies $|\nabla\phi(z,t)|$ is large near $\Gamma_f(t)$. Therefore, the term $\tau(1-q)|\nabla\phi|$ increases the relaxation parameter near the moving interface. Next, for each internal obstacle, the localization function

$$\chi_{b,k}(z) = \exp\left(-\frac{\Psi_{s,k}(z,\bar{z})^2}{\delta_{b,k}^2}\right)$$

satisfies $\chi_{b,k}(z) = 1$ on $\Gamma_{s,k}$, and decays away from $\Gamma_{s,k}$. Hence, $\chi_{b,k}(z)|\nabla\Psi_{s,k}(z,\bar{z})|$ is concentrated near the k-th solid boundary. Since

$$\mathcal{G}_s^{(N)}(z) = \sum_{k=1}^{N} \chi_{b,k}(z)|\nabla\Psi_{s,k}(z,\bar{z})|,$$

the porous-boundary indicator is concentrated near the union $\bigcup_{k=1}^{N}\Gamma_{s,k}$. Therefore, $\lambda_{q,\tau}^{(N)}$ is strengthened near both $\Gamma_f(t)$ and the internal solid boundaries. In smooth bulk regions far from the fluid–fluid interface and far from

the solid inclusions, one has $|\nabla\phi(z,t)| \approx 0$ and $\mathcal{G}_s^{(N)}(z) \approx 0$. Therefore, $\lambda_{q,\tau}^{(N)}(z,t) \approx \lambda_0$. This shows that the deformation is local rather than global. □

Proposition 10 (Formal phase-field energy balance)**.** *Let $\Omega = \Omega_f^{(N)}$ be a fixed bounded domain with sufficiently smooth boundary. Suppose that $\phi, \mu, \mathbf{u}$ are sufficiently smooth solutions of*

$$\partial_t\phi + \nabla\cdot(\phi\mathbf{u}) = \nabla\cdot(M_{q,\tau}^{(N)}\nabla\mu), \qquad \mu = \beta\phi(\phi^2-1) - \varepsilon^2\Delta\phi,$$

where $M_{q,\tau}^{(N)} > 0$. Assume

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \qquad \mathbf{u}\cdot\mathbf{n} = 0, \quad \partial_{\mathbf{n}}\phi = 0, \quad \partial_{\mathbf{n}}\mu = 0 \quad \text{on } \partial\Omega.$$

Define

$$E(\phi) = \int_\Omega \left[\frac{\beta}{4}(\phi^2-1)^2 + \frac{\varepsilon^2}{2}|\nabla\phi|^2\right] dz.$$

Then

$$\frac{dE}{dt} = \int_\Omega \phi\,\mathbf{u}\cdot\nabla\mu\,dz - \int_\Omega M_{q,\tau}^{(N)}|\nabla\mu|^2\,dz.$$

In particular, if

$$\int_\Omega \phi(z,t)\,\mathbf{u}(z,t)\cdot\nabla\mu(z,t)\,dz \le 0$$

for every $t \in (0,T)$, then $E(\phi(t))$ is nonincreasing. If this integral vanishes for every $t \in (0,T)$, then

$$\frac{dE}{dt} = -\int_\Omega M_{q,\tau}^{(N)}|\nabla\mu|^2\,dz \le 0.$$

Proof. Differentiation and integration by parts, using $\partial_{\mathbf{n}}\phi = 0$, give

$$\frac{dE}{dt} = \int_\Omega \left[\beta\phi(\phi^2-1) - \varepsilon^2\Delta\phi\right]\partial_t\phi\,dz = \int_\Omega \mu\,\partial_t\phi\,dz.$$

Substituting the phase-field equation yields

$$\frac{dE}{dt} = -\int_\Omega \mu\,\nabla\cdot(\phi\mathbf{u})\,dz + \int_\Omega \mu\,\nabla\cdot(M_{q,\tau}^{(N)}\nabla\mu)\,dz.$$

The boundary conditions imply

$$-\int_\Omega \mu\,\nabla\cdot(\phi\mathbf{u})\,dz = \int_\Omega \phi\,\mathbf{u}\cdot\nabla\mu\,dz$$

and

$$\int_\Omega \mu\,\nabla\cdot(M_{q,\tau}^{(N)}\nabla\mu)\,dz = -\int_\Omega M_{q,\tau}^{(N)}|\nabla\mu|^2\,dz.$$

Combining these identities proves the result. □

5. Conclusions

This paper introduced a generalized (q,τ)-deformed framework for coupled multiphase flow in porous disk geometries containing multiple internal inclusions. In contrast to the single-obstacle configuration, the present model considers a porous fluid region of the form

$$\Omega_f^{(N)} = \mathbb{D} \setminus \bigcup_{k=1}^{N} \overline{\Omega_{s,k}},$$

where the internal solid inclusions may be represented by unsymmetric starlike boundaries. This construction provides a more realistic and flexible setting for studying interfacial flow in complex bounded geometries, porous media, and microstructured domains. The proposed formulation couples the Navier–Stokes equations with a deformed Cahn–Hilliard phase-field equation. The phase variable ϕ describes the fluid–fluid interface, while the velocity field $\mathbf{u}$ transports and deforms the phase distribution. The coupling is achieved through the conservative

transport term $\nabla \cdot (\phi \mathbf{u})$ and the deformed interfacial force $-\kappa_0 q^\tau \phi \nabla \mu$. Thus, the velocity field does not destroy the phase variable; instead, it moves the interface in a mass-conserving manner while the phase-field gradients feed back into the momentum equation. A main contribution of the paper is the introduction of a multi-obstacle porous-boundary indicator

$$\mathcal{G}_s^{(N)}(z) = \sum_{k=1}^{N} \chi_{b,k}(z) |\nabla \Psi_{s,k}(z, \bar{z})|,$$

which accumulates the localized geometric effects of all internal solid boundaries. This indicator was used to define the deformed relaxation, viscosity, mobility, and boundary-stabilization force. As a result, the (q, τ)-deformation is activated mainly near high-gradient regions, namely the moving fluid–fluid interface and the solid inclusions. In smooth bulk regions, where

$$|\nabla \phi| \approx 0 \qquad \text{and} \qquad \mathcal{G}_s^{(N)} \approx 0,$$

the model remains close to the classical formulation. Several theoretical properties were established. The porous disk geometry was shown to be well-defined under explicit containment and disjointness conditions on the inclusions. The boundary-localization functions were proved to be positive, bounded by one, and concentrated near the internal solid boundaries. The deformed viscosity and mobility were shown to remain positive and bounded under natural gradient assumptions. The coupled system was also shown to conserve the total phase mass under incompressibility, impermeability, and no-flux boundary conditions. In addition, a formal phase-field energy dissipation relation was derived, showing that the deformed mobility contributes to stable interfacial evolution. The illustrative phase–velocity results demonstrate the benefit of the deformation parameters. The classical case $(q, \tau) = (1, 0)$ may exhibit spurious oscillations and boundary artifacts near interfaces and porous inclusions. By decreasing q and increasing τ, the deformation strength $\tau(1 - q)$ increases, producing smoother interfaces, reduced boundary artifacts, and more localized stabilization around the unsymmetric starlike obstacles. These observations support the interpretation of q as a deformation parameter and τ as a regulator of local stabilization strength. When $q \to 1$, the deformation terms vanish, the porous-boundary force disappears, and the model reduces to the classical coupled Navier–Stokes–Cahn–Hilliard formulation. Therefore, the proposed system can be regarded as a tunable extension of the classical multiphase flow model. The framework preserves the physical structure of phase-field hydrodynamics while adding localized control near interfaces and complex solid boundaries.

Future work will focus on full numerical benchmark validation, comparison with classical LBM and standard Navier–Stokes–Cahn–Hilliard solvers, quantitative measurement of spurious-current reduction, convergence analysis, and extension to three-dimensional porous domains. Further developments may also include adaptive selection of (q, τ), anisotropic porous mobility, multi-component phase fields, and applications to droplet transport, porous microfluidics, biological membranes, and heterogeneous materials.

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References

1. Shokrpour Roudbari, M.; van Brummelen, E.H.; Verhoosel, C.V. A multiscale diffuse-interface model for two-phase flow in porous media. *Comput. Fluids* **2016**, *141*, 212–222. https://doi.org/10.1016/j.compfluid.2016.07.006.
2. Soltanmohammadi, R.; Iraji, S.; Rodrigues de Almeida, T.; et al. Investigation of pore geometry influence on fluid flow in heterogeneous porous media: A pore-scale study. *Energy Geosci.* **2024**, *5*, 100222. https://doi.org/10.1016/j.engeos.2023.100222.
3. Iraji, S.; De Almeida, T.R.; Munoz, E.R.; et al. The impact of heterogeneity and pore network characteristics on single and multi-phase fluid propagation in complex porous media: An X-ray computed tomography study. *Pet. Sci.* **2024**, *21*, 1719–1738. https://doi.org/10.1016/j.petsci.2024.01.015.
4. Nimvari, M.E.; Persoons, T.; Gibbons, M. Pore-scale simulation of two-phase flow in biporous media. *Phys. Fluids* **2024**, *36*, 107157. https://doi.org/10.1063/5.0232694.
5. Ebadi, M.; Meisenheimer, D.; Wildenschild, D.; et al. Topological evolution: An unexplored aspect of hysteresis for multiphase flow in porous media. *Phys. Fluids* **2026**, *38*, 026606. https://doi.org/10.1063/5.0310578.
6. Yu, T.; Zhao, J. Resolved CFD-DEM for high-fidelity multiphase flow modeling in porous media of arbitrary geometry. *Comput. Methods Appl. Mech. Eng.* **2026**, *450*, 118676. https://doi.org/10.1016/j.cma.2025.118676.
7. Mesbah, F.; Siavashi, J.; Sharifi, M. 3D pore-scale digital twin for assessing thermal effects on two-phase flow and relative permeability in Bentheimer sandstone. *Int. J. Heat Mass Transf.* **2026**, *260*, 128340. https://doi.org/10.1016/j.ijheatmasstransfer.2026.128340.
8. Dhaouadi, F.; Dumbser, M.; Gavrilyuk, S. A first-order hyperbolic reformulation of the Cahn–Hilliard equation. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2025**, *481*, 20240606. https://doi.org/10.1098/rspa.2024.0606.
9. Mandolesi, B.; Iandiorio, C.; Belardi, V.G.; et al. Spinodal decomposition-inspired metamaterial: Tailored homogenized elastic properties via the dimensionless Cahn-Hilliard equation. *Eur. J. Mech. A Solids* **2025**, *112*, 105615. https://doi.org/10.1016/j.euromechsol.2025.105615.
10. Bell-Navas, E.; Portillo, D.; Romero, I. A fully variational numerical method for structural topology optimization based on a Cahn-Hilliard model. *Comput. Methods Appl. Mech. Eng.* **2025**, *445*, 118233. https://doi.org/10.1016/j.cma.2025.118233.
11. Madan, R.; Bhowmick, S.; Hadji, L.; et al. Limit angular speed analysis of porous functionally graded rotating disk under thermo-mechanical loading. *Multidiscip. Model. Mater. Struct.* **2023**, *19*, 311–323. https://doi.org/10.1108/mmms-09-2022-0197.
12. Islam, M.; Kumar Sinha, R. A unified framework for the direct discontinuous Galerkin method and its variants for the Cahn-Hilliard equations. *Commun. Nonlinear Sci. Numer. Simul.* **2026**, *160*, 109974. https://doi.org/10.1016/j.cnsns.2026.109974.
13. Conti, M.; Gatti, S.; Giorgini, A.; et al. On the Cahn-Hilliard equation with nonlinear diffusion: The non-convex case. *J. Funct. Anal.* **2026**, *291*, 111599. https://doi.org/10.1016/j.jfa.2026.111599.
14. Mehrjoo, H.; Monfared, A.E.F.; Jafari, S.; et al. Simulation of Newtonian and non-Newtonian reactive fluids during the dissolution process in porous media using Lattice Boltzmann method. *Int. Commun. Heat Mass Transf.* **2026**, *170*, 110061. https://doi.org/10.1016/j.icheatmasstransfer.2025.110061.
15. Zahid, F.; Cunningham, J.A. Review of the Color Gradient Lattice Boltzmann Method for Simulating Multi-Phase Flow in Porous Media: Viscosity, Gradient Calculation, and Fluid Acceleration. *Fluids* **2025**, *10*, 128. https://doi.org/10.3390/fluids10050128.
16. Molla, M.M. Development and numerical validations of a two-dimensional cumulant-driven thermal lattice Boltzmann method. *Results Eng.* **2025**, *27*, 106225. https://doi.org/10.1016/j.rineng.2025.106225.
17. Alamian, R.; Nayak, A.K.; Safdari Shadloo, M. CooLBM: A GPU-accelerated collaborative open-source reactive multi-phase/component simulation code via lattice Boltzmann method. *Comput. Phys. Commun.* **2025**, *315*, 109711. https://doi.org/10.1016/j.cpc.2025.109711.
18. Asaei, A.; Siavashi, M.; Hosseini, M.; et al. Pore-scale simulation of hydrogen adsorption in a metal-organic framework (MOF-5) using lattice Boltzmann method. *Int. J. Hydrogen Energy* **2025**, *168*, 151147. https://doi.org/10.1016/j.ijhydene.2025.151147.
19. Ibrahim, R.W.; Baleanu, D.; Salahshour, S. Integrating Experimental Imaging and (Quantum-Deformation)-Curvature Dynamics in Bleb Morphogenesis. *Eng. Rep.* **2026**, *8*, e70726. https://doi.org/10.1002/eng2.70726.
20. Ibrahim, R.W.; Baleanu, D. (Quantum, Deformed)-Fractional Complex Steps: Theory, Entropy Analysis, and Imaging Applications. *J. Appl. Anal. Comput.* **2026**, *16*, 2639–2667. https://doi.org/10.11948/20250379.
21. Ibrahim, R.W.; Baleanu, D.; Salahshour, S. Weighted Deformed Gram Matrices for Quantum-Inspired Learning with Applications to Biomedical Signal Analysis. *Eng. Rep.* **2026**, *8*, e70855. https://doi.org/10.1002/eng2.70855.