

Observer-Based Output-Feedback Synchronization of Fractional-Order Fuzzy Clifford-Valued Neural Networks: A Cascade Certificate and the Quantified Price of Output Feedback

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Abstract: Synchronization certificates for fractional-order neural networks often assume the full multivector state is measured. In practice only a partial output is available—the full multivector states of a subset of neurons—and the controller must run on an estimate. We give an output-feedback theory for leader-following synchronization of fractional-order Takagi–Sugeno fuzzy Clifford-valued neural networks in which each follower measures only some of its neurons, runs a local Luenberger-type fractional observer to reconstruct the rest, and pins the *estimate* to the leader (certainty equivalence). The analysis has three parts. First, a *detectability* certificate: a per-neuron M-matrix test on the observer-error energies under which the observer error decays, in which the output-injection gain raises the damping of the measured neurons only; the gain is essential for feasibility—without it the certificate is infeasible ($a_o = (-2.34, -2.15, 1.67)$ vs. $(2.86, 3.06, 1.67)$)—and the simulator shows the open-loop error failing to synchronize (stalling at 0.80) while the gain drives it to 2.6×10^{-4} . Second, a *cascade certificate*: the observer error is autonomous and drives the tracking error through a coupling $\theta^{-1}\|H\|^2$; a triangular 2×2 fractional comparison gives joint synchronization at the rate $\min(\text{observer}, \text{tracking})$, which is observer-limited here (1.27). Third, the *price of output feedback*, made quantitative: the coupling forces a tracking-rate loss θ (the “separation” is weak, not clean), and the residual is $14\times$ larger than full-state feedback; under a bounded sensor attack both ultimate errors scale as $\bar{d}^2$, amplified by the control gain $\|H\|^2$ —so high-gain pinning that speeds tracking also magnifies sensor-attack sensitivity. Selected inequalities and certificates are numerically audited, and the associated certified–empirical gap is measured and reported.

Keywords: output-feedback synchronization; fractional-order neural networks; Clifford-valued neural networks; Takagi–Sugeno fuzzy systems; fractional-order observer; sensor-fault robustness

1. Introduction

Leader–follower synchronization is a basic coordination objective in networked control: a reference trajectory is propagated through local interactions so that multiple nonlinear agents converge to the same motion. Fractional-order models add power-law memory, Takagi–Sugeno (T–S) rules provide a convex representation of nonlinear operating regimes, and Clifford-valued states collect coupled real, complex, quaternion, or higher-dimensional signal components in a single algebraic model. Lyapunov, Mittag–Leffler, and comparison-principle tools have consequently been developed for synchronization and pinning of fractional neural networks [1–5].

The practical limitation is measurement availability. Many existing certificates assume that the controller reads the complete state—all neurons and, in a Clifford-valued network, all blades of every neuron. In a sensor-limited implementation, however, only selected telemetry channels are available because of hardware cost, bandwidth, privacy, or sensor failure; the remaining states must be reconstructed. A synchronization certificate based on full-state feedback therefore does not by itself justify an implementable output-feedback loop.

For integer-order linear systems the closing is famous and clean: the separation principle lets one design a stabilizing state-feedback gain and a convergent observer independently, and the certainty-equivalence combination inherits both. For *nonlinear* systems the clean separation fails, and for *fractional-order* nonlinear systems the tools that would rescue it—input-to-state stability and the ISS-cascade lemma—are still developing and do not deliver an explicit decay envelope through the power-law memory. Fractional neural networks are strongly nonlinear (saturating activations, fuzzy blending) and carry that memory, so output-feedback synchronization for them is not a corollary of the state-feedback theory; it needs its own certificate.

This system class is considered because it combines four features that arise together in memory-dependent networked processes: nonlinear operating regimes, multichannel Clifford-valued states, distributed leader–follower coordination, and incomplete telemetry. Representative deployment settings include converter-dominated microgrids, vehicle or robot formations, and coupled signal-processing nodes for which only a subset of internal channels is communicated. The purpose of this work is to establish an implementable output-feedback synchronization certificate for fractional-order T–S fuzzy Clifford-valued neural networks. Each follower measures only a subset of its neurons (two of three in the numerical study), runs a local Luenberger-type fractional observer, and applies pinning control to the *estimated* rather than the true state. Three questions organize the analysis, and each receives a quantitative answer.

1.1. When Does the Observer Converge, and Is Its Gain Necessary?

The observer error obeys a fractional differential inequality in which the output-injection gain L adds damping to the *measured* neurons only. We give a per-neuron M-matrix *detectability* certificate (Theorem 1) and—this is the point—show the gain is not cosmetic: for our network the open-loop ($L = 0$) test is infeasible, its per-neuron drift constants being $a_o = (-2.34, -2.15, 1.67)$ (the measured neurons are self-unstable), whereas the designed gain makes them $(2.86, 3.06, 1.67)$ and the Perron root $0.18 < 1$. The simulator agrees emphatically: the open-loop observer error settles at 0.80 while the gain drives it to 2.6×10^{-4} .

1.2. Does Certainty-Equivalence Output Feedback Synchronize, and at What Rate?

The observer error is *autonomous*—it does not depend on the tracking error—and enters the tracking dynamics as a perturbation $H\eta$, where H is the pinned interaction matrix and η the estimation error. This is a cascade. We prove a triangular 2×2 fractional comparison (Theorem 2) whose joint decay rate is the smaller of the observer rate and the tracking rate; here it is observer-limited at 1.27, so the estimate must converge before the followers can track—the natural two-timescale picture, made rigorous through the fractional memory.

1.3. What Does Output Feedback Cost?

Two prices, both measured. The cascade coupling forces the certainty-equivalence design to “spend” a rate θ absorbing the observer error, so the tracking rate drops from $a_s - b_s = 3.17$ (full-state) to $a_s - \theta - b_s$; the separation is *weak*—designable in two steps but rate-coupled. And the achieved residual under output feedback is $14\times$ that of full-state feedback. Under a bounded sensor attack, both the observer and the tracking ultimate errors scale as $\bar{d}^2$, amplified by $\|H\|^2$: the same high control gain that accelerates tracking magnifies the network’s sensitivity to a corrupted measurement—a concrete security caution for output-feedback pinning.

1.4. Contributions

- C1. An *observer detectability certificate* (Theorem 1): a per-neuron M-matrix test for the fractional Clifford observer error, with a proof that the output-injection gain is essential for feasibility (open-loop infeasible), audited on 41 systems and confirmed against the simulator.
- C2. A *fractional cascade certificate* (Theorem 2): a triangular 2×2 comparison, closed by a converging-input fractional Halanay inequality (Lemma 3), giving joint synchronization with an explicit ultimate bound and an observer-limited transient (Corollary 2)—a quantitative statement the integer ISS-cascade lemma leaves implicit and linear-fractional separation does not cover for nonlinear plants.

- C3. *The quantified price of output feedback* (Theorem 3 and Section 8): the coupling cost θ (weak separation), the $14\times$ residual penalty vs. full-state feedback, and a sensor-attack robustness bound scaling as $\|H\|^2 \bar{d}^2$ —high-gain output feedback trades tracking speed for attack sensitivity.
- C4. *Clifford/fuzzy generality and auditable numerics*: the analysis carries the left-regular Clifford operator norm ν (exact for $C\ell(0, 1)$, $C\ell(0, 2)$), Takagi–Sugeno blending, and bounded time-varying delays without a delay-rate constraint; a reproducible simulator assesses the reported examples and certified–empirical gaps.

Throughout, correctness is preferred to the appearance of strength: conservative certificates are labelled and their gaps measured (the robustness bound is $\sim 10^3 \times$ loose in magnitude but exact in its $\bar{d}^2$ scaling, and we say so), and the honest limits of the weak-separation statement are stated.

1.5. Testable Hypotheses

The theory makes four falsifiable predictions, each mapped to an experiment in Section 8.

H1. *For the reported network, setting $L = 0$ makes the detectability test infeasible and is expected to leave a larger finite-horizon observer residual than the selected positive gain. The open-loop twin tests this residual comparison; it does not establish asymptotic nonconvergence or the necessity of injection gains for other networks.*

H2. *The main run is assessed for consistency with the stated joint comparison bound, parameterized by $\min\{r_o, a_s - \theta - b_s\}$. Faster empirical decay does not refute an upper bound, and finite-horizon agreement does not prove asymptotic synchronization.*

H3. *Output feedback costs a bounded residual penalty over full-state feedback (tested against the full-state twin; refuted if the penalty is unbounded or absent).*

H4. *Both ultimate errors scale as $\bar{d}^2$ under a bounded sensor attack (tested by the $\bar{d}$ -sweep; refuted if the fitted slope departs from 2).*

The reported numerical outcomes are discussed in Section 8; finite-horizon simulations are not proofs of the corresponding asymptotic statements.

1.6. Comparison and Positioning

Relative to full-state fractional synchronization methods, the proposed controller operates from a partial output and supplies an explicit detectability test. Relative to stand-alone fractional observers, it closes the observer–controller loop and quantifies the joint transient. Relative to adaptive fuzzy fault-tolerant approximators [28–31], it does not estimate an unknown plant or nonaffine actuator map online; instead, it exploits known uniform T–S/Lipschitz bounds to obtain a low-cost M-matrix/Halanay certificate for a distributed fractional Clifford network. The resulting novelty is therefore the combination of partial measurement, an autonomous observer error, a provable fractional cascade, and a measured output-feedback/attack trade-off, rather than a new universal function approximator.

1.7. Organization

Section 2 reviews the literature and states the gap. Section 3 fixes preliminaries. Section 4 states the model, the observer, and the two error systems. Section 5 proves detectability; Section 6 the cascade synchronization and robustness. Section 7 is the design algorithm; Section 8 the numerical studies; Sections 9–10 discuss and conclude.

2. Literature Review and Gap Analysis

2.1. Fractional-Order and Clifford-Valued Neural Networks

Fractional derivatives endow a network with power-law memory; their stability is analysed by fractional Lyapunov and Mittag-Leffler methods [1, 6–9] on the foundations [10–12]. Clifford-valued networks unify real, complex, quaternion and higher hypercomplex models in one geometric-algebra framework [13–15]; their stability, dissipativity and synchronization appear in [5, 16–18]. The device that carries the Clifford structure through a scalar Lyapunov analysis is the left-regular operator norm ν , exact for $C\ell(0, 1)$, $C\ell(0, 2)$ (Lemma 1); fractional Halanay inequalities [2, 3, 19, 20] convert the resulting delayed differential inequalities into ultimate bounds.

2.2. Synchronization and State Estimation

Synchronization and pinning control of fractional and complex-valued networks are developed in [4,21,22], and Takagi–Sugeno fuzzy modelling in [23,24]. These are, almost without exception, *state-feedback*: the control law reads the full state. Separately, state estimation and observer design for fractional and hypercomplex networks has its own literature: Bao and Cao [25] estimate the state of a delayed fractional network, Li et al. [26] give dissipativity-based state estimation for quaternion memristive networks, and Zouari et al. [27] design adaptive neural output-feedback control for fractional systems. This body treats the estimation problem—often as a filtering or H_∞ task—but stops short of using the estimate in a closed synchronization loop with an explicit joint decay envelope.

2.3. Adaptive Fuzzy Fault-Tolerant Control

Recent adaptive fuzzy designs address uncertain nonlinear plants and sensor or actuator faults. Direct adaptive approximation has been used for an inverted pendulum with time-varying actuator faults [28], extended to two pendulums with nonaffine actuator failures [29], combined with particle-swarm tuning for a quadrotor subject to uncertainty and simultaneous sensor/actuator faults [30], and developed as robust active fuzzy fault-tolerant control for second-order nonlinear systems [31]. These methods are important when the nonlinear plant and fault map are unknown and must be approximated online. Their objective and guarantee differ from the present problem: they do not provide a per-neuron detectability test or an explicit observer-driven cascade rate for partial-output leader–follower synchronization of a fractional Clifford-valued network.

Table 1 summarizes the modelling and approximation strategies discussed above.

Table 1. Selection of the modelling/approximation strategy relative to recent alternatives.

Approach	Appropriate Setting	Principal Guarantee/Cost	Relation to This Work
Adaptive fuzzy fault-tolerant approximation [28,29,31]	Unknown nonlinear dynamics and actuator-fault maps	Online parameter adaptation and Lyapunov boundedness; additional adaptive states and tuning	Stronger fault accommodation, but no partial-output fractional multi-agent cascade certificate
PSO-tuned adaptive fuzzy control [30]	Application-specific nonlinear plant with simultaneous uncertainty and sensor/actuator faults	Offline/online optimization and fault reconfiguration; higher tuning burden	Valuable for a quadrotor fault-tolerant objective, but not a distributed detectability problem
Proposed T-S/Lipschitz M-matrix method	Known rule envelope, incomplete measurements, and distributed fractional synchronization	Two small eigenvalue tests, explicit joint rate and ultimate bounds; no online plant learning	Selected here because convex T-S blending preserves uniform Lipschitz bounds required by the comparison proof

2.4. The Gap

Between “synchronize with full-state feedback” and “estimate the state” lies the question this paper answers: does certainty-equivalence *output* feedback synchronize a nonlinear fractional Clifford network, at what rate, and at what cost? The clean separation principle that would settle it for linear systems does not apply to a saturating, fuzzy-blended, memory-carrying plant, and the integer-order ISS-cascade lemma—the nonlinear substitute—does not deliver an explicit envelope through the power-law kernel. We supply the missing layer: a detectability certificate whose gain is essential for feasibility, a triangular fractional cascade with an explicit joint rate, and a quantified price (coupling cost, residual penalty, and $\|H\|^2 \bar{d}^2$ sensor-attack sensitivity). This is what output-limited, attack-exposed deployment needs and what the prior state-feedback certificates do not provide.

3. Preliminaries

3.1. Clifford Algebra and the Operator Norm ν

Let $\mathcal{A} = C\ell(0, m)$ be the real Clifford algebra on $e_1, \dots, e_m$ with $e_p e_q + e_q e_p = -2\delta_{pq}$; a basis is $\{e_A : A \subseteq \{1, \dots, m\}\}$, $\dim_{\mathbb{R}} \mathcal{A} = 2^m$. For $a = \sum_A a_A e_A$ write $\|a\| = (\sum_A a_A^2)^{1/2}$, and $\|u\|^2 = \sum_i \|u_i\|^2$ for $u \in \mathcal{A}^d$. Left multiplication $L_a : x \mapsto ax$ is linear on $\mathcal{A} \cong \mathbb{R}^{2^m}$; set $\nu(a) := \|L_a\|_2$.

Lemma 1. For $a, b \in \mathcal{A}$: (i) $\|ab\| \leq \nu(a)\|b\|$ and $\langle u, av \rangle \leq \nu(a)\|u\|\|v\|$; (ii) $\nu(ab) \leq \nu(a)\nu(b)$, $\nu(a) \geq \|a\|$; (iii) $\nu(a) = \|a\|$ exactly for $m \leq 2$.

Proof. (i) is the definition and Cauchy–Schwarz; (ii) submultiplicativity of the spectral norm and $\nu(a) \geq \|L_a 1\| = \|a\|$; (iii) because $C\ell(0, 1) \cong \mathbb{C}$, $C\ell(0, 2) \cong \mathbb{H}$ are composition algebras. For $m \geq 3$ the identity fails; the audit measures the gap. $\square$

We work in $C\ell(0, 2)$, so every ν in the certificate is exact.

3.2. Caputo Calculus and Comparison

For absolutely continuous f and $\alpha \in (0, 1)$, the Caputo derivative is ${}^C D_{t_0}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} f'(s) ds$; we use the chain inequality ${}^C D_{t_0}^\alpha \|u\|^2 \leq 2\langle u, {}^C D_{t_0}^\alpha u \rangle$ [6,7] and the following comparison results.

Lemma 2 (fractional Halanay [2, 3]). *If $V \geq 0$ satisfies ${}^C D_{t_0}^\alpha V \leq -aV(t) + b \sup_{s \in [t-\bar{\tau}, t]} V(s) + C$ with $a > b \geq 0$, $C \geq 0$, then V is bounded and $\limsup_{t \rightarrow \infty} V(t) \leq C/(a-b)$, the transient decaying no slower than an order- α Mittag-Leffler envelope.*

Lemma 3 (converging-input fractional Halanay [3,20]). *Let $V \geq 0$ satisfy*

$${}^C D_{t_0}^\alpha V \leq -aV(t) + b \sup_{s \in [t-\bar{\tau}, t]} V(s) + w(t),$$

with $a > b \geq 0$ and $w \geq 0$ bounded. Then $\limsup_{t \rightarrow \infty} V(t) \leq (\limsup_{t \rightarrow \infty} w(t))/(a-b)$; in particular, if $w(t) \rightarrow 0$ then $V(t) \rightarrow 0$. (This is the converging-input-converging-state property of the delayed fractional comparison system, established for the generalized fractional Halanay inequality in [20].)

Lemma 4 (vector comparison / M-matrix [3]). *Let $V \in \mathbb{R}_{\geq 0}^n$ satisfy, componentwise, ${}^C D_{t_0}^\alpha V_j \leq -a_j V_j + \sum_{k \neq j} P_{jk} V_k + \sum_k Q_{jk} \sup_{[t-\bar{\tau}, t]} V_k + C_j$ with $a_j > 0$, $P, Q \geq 0$. If $\text{diag}(a) - P - Q$ is a nonsingular M-matrix—equivalently the Perron root of $K = \text{diag}(a)^{-1}(P + Q)$ is < 1 —then with its Perron weight $\xi > 0$ the weighted energy $V_\xi = \max_j V_j / \xi_j$ obeys the scalar delayed inequality*

$${}^C D_{t_0}^\alpha V_\xi \leq -\gamma V_\xi + b \sup_{[t-\bar{\tau}, t]} V_\xi + C, \quad \gamma = \min_j \frac{a_j \xi_j - (P\xi)_j}{\xi_j}, \quad b = \max_j \frac{(Q\xi)_j}{\xi_j}, \quad C = \max_j \frac{C_j}{\xi_j},$$

with $\gamma > b$ (a consequence of the M-matrix property, since $\gamma - b \geq \min_j [a_j \xi_j - ((P + Q)\xi)_j] / \xi_j > 0$); the instantaneous coupling P enters the rate γ and the delayed coupling Q the sup-coefficient b . Lemma 2 then applies.

3.3. Notation

Agents are $v = 1, \dots, N$, each with n Clifford neurons in $\mathcal{A}$. For symmetric H , $\lambda_1(H)$, $\|H\|$ are its extreme eigenvalues. Table 2 lists the recurring symbols.

Table 2. Principal notation.

Symbol	Meaning	Symbol	Meaning
s, x_v	leader, follower states	$\hat{x}_v$	observer estimate
$e_v = x_v - s$	tracking error	$\eta_v = x_v - \hat{x}_v$	estimation error
C	output (measured neurons)	L	output-injection gain
$H = cL_g + k\Delta$	pinned interaction	$\lambda_1(H), \ H\ $	extreme eigenvalues
$V_o = \sum_v \ \eta_v\ ^2$	observer energy	$V_s = \sum_v \ e_v\ ^2$	tracking energy
a_o, P_o, Q_o	observer M-matrix	a_s, b_s	tracking Halanay
θ	coupling cost	$\bar{d}$	sensor-attack bound

4. Model, Observer, and Error Systems

4.1. Plant and Leader

A leader and N followers each carry n Clifford-valued neurons of Caputo order $\alpha \in (0, 1)$. Write $h_l(t)$ for the common, known time-dependent premise weights used by the leader, followers, and observers; the argument is suppressed elsewhere when no ambiguity arises. All delayed states use the same delay $\tau(t)$. The leader $s \in \mathcal{A}^n$ obeys

$${}^C D_{t_0}^\alpha s_j = -d_j s_j + \sum_{l=1}^r h_l(t) \left[\sum_k A_{jk}^{(l)} f_k(s_k) + \sum_k B_{jk}^{(l)} g_k(s_k(t - \tau(t))) \right], \quad (1)$$

and follower v , with state $x_v \in \mathcal{A}^n$,

$${}^C D_{t_0}^\alpha x_{v,j} = -d_j x_{v,j} + \sum_l h_l(t) \left[\sum_k A_{jk}^{(l)} f_k(x_{v,k}) + \sum_k B_{jk}^{(l)} g_k(x_{v,k}(t - \tau(t))) \right] + u_{v,j}. \quad (2)$$

Equation (2) describes neuron j of follower v . Its terms have distinct roles: $-d_j x_{v,j}$ is the intrinsic leakage; h_l is the normalized membership weight of T-S rule l ; $A_{jk}^{(l)} f_k(x_{v,k})$ and $B_{jk}^{(l)} g_k(x_{v,k}(t - \tau))$ are, respectively, the instantaneous and delayed Clifford-valued interactions from neuron k to neuron j ; and $u_{v,j}$ is the coordination input. The sums over $k = 1, \dots, n$ combine all neurons, whereas the sum over $l = 1, \dots, r$ forms a convex blend of the rule consequents. The same known premise weights are used by the plant and observer; the output-feedback difficulty arises from the unavailable state components, not from an unmodelled rule base.

Assumption 1. $f_k, g_k : \mathcal{A} \rightarrow \mathcal{A}$ are globally Lipschitz in the coefficient norm with constants $\leq \ell$ and $f_k(0) = g_k(0) = 0$ (experiments: coefficientwise $\tanh$, $\ell = 1$).

Assumption 2. $\tau : \mathbb{R} \rightarrow [0, \bar{\tau}]$ is measurable (no rate bound).

Assumption 3. $h_l \geq 0$, $\sum_l h_l \equiv 1$ [23].

Assumption 4. The graph is undirected connected with Laplacian L_g ; the pinning set is nonempty, $\Delta = \text{diag}(\delta_v)$; $H := cL_g + k\Delta$ ($c, k > 0$) is symmetric positive definite.

Remark 1 (scope and implications of the assumptions). Assumption 1 supplies uniform nonlinear bounds needed by the comparison proof; it includes the saturating activations used in the study but excludes discontinuous or only locally Lipschitz dynamics unless an invariant operating region is first established. Assumption 2 is deliberately weak—the delay may vary rapidly and need not be differentiable—but it must remain bounded. Assumption 3 makes the T-S model a convex envelope, so one set of worst-case operator norms covers every rule interpolation. Assumption 4 provides the positive damping $\lambda_1(H)$; directed, switching, or temporarily disconnected graphs require a different comparison metric. The nominal theorem also presumes known bounds on (d, A, B) , no actuator saturation, and a correctly implemented commanded input. Parameter uncertainty and actuator faults are examined explicitly below rather than being hidden inside these assumptions.

4.2. Partial Measurement and the Fractional Observer

Each follower measures only a subset $\mathcal{M} \subseteq \{1, \dots, n\}$ of its neurons through the output map $C = \text{diag}(\chi_1, \dots, \chi_n)$, $\chi_j = \mathbf{1}[j \in \mathcal{M}]$, acting per neuron on the multivector coordinates:

$$y_v(t) = C x_v(t) + d_v(t), \quad \|d_{v,j}(t)\| \leq \bar{d}, \quad (3)$$

with d_v a bounded measurement disturbance modelling sensor noise or a sensor attack. Each follower runs a local Luenberger-type fractional observer with per-neuron output-injection gain $L = \text{diag}(\ell_1^o, \dots, \ell_n^o)$, $\ell_j^o = 0$ for $j \notin \mathcal{M}$:

$${}^C D_{t_0}^\alpha \hat{x}_{v,j} = -d_j \hat{x}_{v,j} + \sum_l h_l(t) \left[\sum_k A_{jk}^{(l)} f_k(\hat{x}_{v,k}) + \sum_k B_{jk}^{(l)} g_k(\hat{x}_{v,k}(t - \tau(t))) \right] + \ell_j^o (y_{v,j} - C_j \hat{x}_{v,j}) + u_{v,j}. \quad (4)$$

The control is certainty equivalence: it pins the *estimate* to the leader,

$$u_v = -c \sum_w L_{g,vw} (\hat{x}_w - s) - k \delta_v (\hat{x}_v - s) = -(H(\hat{x} - s))_v. \quad (5)$$

Figure 1 depicts the observer-based output-feedback architecture and its signal paths.

Remark 2 (observability by design). In the experiments $\mathcal{M} = \{1, 2\}$ ($n = 3$): neurons 1, 2 are measured, neuron 3 is reconstructed. The measured neurons are chosen self-unstable for the estimation error ($2d_j < 2\ell_{jj}^A$), so the gain L is genuinely required; the unmeasured neuron is self-stable and is pulled in through the network coupling. This is the regime in which detectability is a real condition rather than a formality.

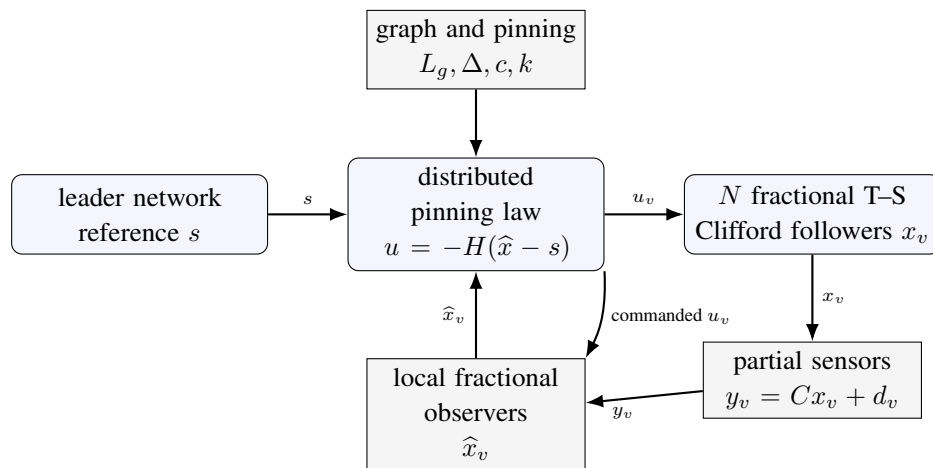


Figure 1. Global observer-based output-feedback architecture. Each follower uses its local partial output to reconstruct the missing state; the estimates are exchanged through the pinned graph and compared with the leader to form the certainty-equivalence control input.

4.3. The Two Error Systems

Write the estimation error $\eta_v = x_v - \hat{x}_v$ and the tracking error $e_v = x_v - s$. Subtracting (4) from (2), the control u_v cancels and

$${}^C D_{t_0}^\alpha \eta_{v,j} = -d_j \eta_{v,j} + \sum_l h_l \left[\sum_k A_{jk}^{(l)} \Delta f_{v,k}^\eta + \sum_k B_{jk}^{(l)} \Delta g_{v,k}^{\eta,\tau} \right] - \ell_j^\circ C_j \eta_{v,j} - \ell_j^\circ d_{v,j}, \quad (6)$$

with $\Delta f_{v,k}^\eta = f_k(x_{v,k}) - f_k(\hat{x}_{v,k})$. Crucially (6) is *autonomous*: it involves only η and the disturbance d_v , not e or s . Subtracting (1) from (2) and using $\hat{x}_v - s = e_v - \eta_v$,

$${}^C D_{t_0}^\alpha e_{v,j} = -d_j e_{v,j} + \sum_l h_l \left[\sum_k A_{jk}^{(l)} \Delta f_{v,k}^e + \sum_k B_{jk}^{(l)} \Delta g_{v,k}^{e,\tau} \right] - (He)_{v,j} + (H\eta)_{v,j}, \quad (7)$$

with $\Delta f_{v,k}^e = f_k(x_{v,k}) - f_k(s_k)$. The tracking error is damped by $-He$ (self-decay plus pinning) and *driven* by the estimation error through $+H\eta$. Equations (6) and (7) are a cascade: $\eta \rightarrow e$, one-directional.

Plain-language interpretation: The observer is a copy of each agent that only gets to peek at some of its dials; it corrects the dials it can see and infers the rest from how the machine is wired. Its guessing error η runs on its own, blind to how well the fleet is synchronizing. The controller, forced to steer by the guess rather than the truth, inherits that error: the term $H\eta$ is the steering command's mistake. So first the guess must settle, and only then can the fleet close ranks—and the louder the steering (the larger the gain H), the more a guessing error shoves the formation around.

5. Observer Detectability

The estimation error (6) is a self-contained fractional system; its convergence is a detectability property of the pair (dynamics, output). We certify it by a per-neuron M-matrix test in which the output-injection gain enters the *measured* neurons only.

Theorem 1 (detectability). *Let Assumptions 1–4 hold and set, with $\bar{\nu}_{jk}^A = \max_l \nu(A_{jk}^{(l)})$, $\bar{\nu}_{jk}^B = \max_l \nu(B_{jk}^{(l)})$,*

$$\begin{aligned} a_j^\circ &= 2d_j + 2\ell_j^\circ \chi_j - 2\ell \bar{\nu}_{jj}^A - \ell \sum_{k \neq j} \bar{\nu}_{jk}^A - \ell \sum_k \bar{\nu}_{jk}^B, \\ P_{jk}^\circ &= \ell \bar{\nu}_{jk}^A \quad (k \neq j), \quad Q_{jk}^\circ = \ell \bar{\nu}_{jk}^B, \quad W_j^\circ = \frac{1}{\theta_o} (\ell_j^\circ)^2 N \bar{d}^2, \end{aligned} \quad (8)$$

and suppose $a_j^\circ > 0$ and $\text{diag}(a^\circ) - P^\circ - Q^\circ$ is a nonsingular M-matrix (Perron root of $K_o = \text{diag}(a^\circ)^{-1}(P^\circ + Q^\circ)$ below 1), with Perron weight ξ . Following the decomposition of Lemma 4 (instantaneous coupling P° in the rate, delayed coupling Q° in the sup-coefficient), set $\gamma_o = \min_j [a_j^\circ \xi_j - (P^\circ \xi)_j] / \xi_j$ and $b_o = \max_j (Q^\circ \xi)_j / \xi_j$; the M-matrix property gives $\gamma_o > b_o$. Then along (6), the weighted observer energy $V_o^\xi = \max_j (\sum_v \|\eta_{v,j}\|^2) / \xi_j$ obeys

$${}^C D_{t_0}^\alpha V_o^\xi \leq -\gamma_o V_o^\xi + b_o \sup_{[t-\tau, t]} V_o^\xi + C_o, \quad C_o = \max_j \frac{W_j^\circ}{\xi_j}, \quad (9)$$

so if $\gamma_o > b_o$ the estimation error is ultimately bounded by $V_o^\xi \leq C_o/(\gamma_o - b_o)$, decaying to it at the order- α Mittag-Leffler rate; in the noise-free case $\bar{d} = 0$ it converges to zero. The output-injection term $2\ell_j^\circ \chi_j$ raises a_j° on measured neurons only.

Proof. Fix neuron j and let $V_{o,j} = \sum_v \|\eta_{v,j}\|^2$. By the chain inequality (Lemma 1 and the Caputo chain rule), ${}^C D_{t_0}^\alpha V_{o,j} \leq 2 \sum_v \langle \eta_{v,j}, {}^C D_{t_0}^\alpha \eta_{v,j} \rangle$. The self-decay contributes $-2d_j V_{o,j}$; the output injection $-2\ell_j^\circ \chi_j \sum_v \|\eta_{v,j}\|^2 = -2\ell_j^\circ \chi_j V_{o,j}$ (recall $C_j \eta_{v,j} = \chi_j \eta_{v,j}$). The fuzzy terms, by Assumption 3 and Lemma 1(i), give $2\ell \bar{v}_{jj}^A V_{o,j}$ on the diagonal and, splitting $2\|\eta_{v,j}\| \|\eta_{v,k}\| \leq \|\eta_{v,j}\|^2 + \|\eta_{v,k}\|^2$, $\ell \bar{v}_{jk}^A (V_{o,j} + V_{o,k})$ off-diagonal; the delayed terms give $\ell \bar{v}_{jk}^B (V_{o,j} + V_{o,k}(t - \tau))$. The sensor term is handled by Young: $-2 \sum_v \langle \eta_{v,j}, \ell_j^\circ d_{v,j} \rangle \leq \theta_o V_{o,j} + \theta_o^{-1} (\ell_j^\circ)^2 N \bar{d}^2$ (absorb θ_o into the margin as in (8)). Collecting, ${}^C D_{t_0}^\alpha V_{o,j} \leq -a_j^\circ V_{o,j} + \sum_{k \neq j} P_{jk}^\circ V_{o,k} + \sum_k Q_{jk}^\circ V_{o,k}(t - \tau) + W_j^\circ$. This is the hypothesis of the vector comparison Lemma 4; the M-matrix condition yields (9) with the stated γ_o, b_o, C_o , and Lemma 2 gives the ultimate bound. $\square$

Plain-language interpretation: Each neuron's guessing error has its own budget: how fast it forgets its mistake ($2d_j$, plus $2\ell_j^\circ$ if you can watch it) minus how fast the nonlinearity feeds the mistake back ($2\ell \bar{v}_{jj}^A$ and the cross-terms). The M-matrix test asks that, weighted correctly across neurons, forgetting beats feeding-back on the whole network at once. For a neuron you cannot watch, only its own forgetting and its neighbours' help are available—so an unwatched neuron must be intrinsically calmer, while the watched ones can be volatile because the gain tames them.

Remark 3 (the gain is essential for the certificate). *For the experimental network the measured neurons are self-unstable for the estimation error ($2d_j < 2\ell \bar{v}_{jj}^A$), so at $L = 0$ their drift constants a_j° are negative ($a^\circ = (-2.34, -2.15, 1.67)$) and the detectability test is infeasible: no Perron argument even applies. The designed gain $\ell^\circ = (2.6, 2.6, 0)$ makes $a^\circ = (2.86, 3.06, 1.67)$, Perron root $0.18 < 1$ (Section 8). Precisely, what is proven is that the sufficient detectability certificate is infeasible without the gain; empirically the open-loop estimation error does not diverge but fails to synchronize, stalling at a large residual (tail-mean 0.80, oscillating in $[0.6, 1.0]$) while the closed-loop error reaches 2.6×10^{-4} —a $\sim 3000 \times$ separation. The gain is thus essential both for the guarantee and for synchronization; we do not claim the open-loop system is unbounded.*

5.1. Which Neurons to Measure

Theorem 1 is also a *sensor-placement* criterion: a candidate measured set $\mathcal{M}$ is admissible iff the resulting $\text{diag}(a^\circ) - P^\circ - Q^\circ$ is a nonsingular M-matrix. Two structural consequences follow.

Proposition 1 (necessary self-damping of unmeasured neurons). *If neuron $j \notin \mathcal{M}$ then $a_j^\circ = 2d_j - 2\ell \bar{v}_{jj}^A - \ell \sum_{k \neq j} \bar{v}_{jk}^A - \ell \sum_k \bar{v}_{jk}^B$ must be positive for the M-matrix test to hold. In particular an unmeasured neuron must be self-stable, $2d_j > 2\ell \bar{v}_{jj}^A + \ell \sum_{k \neq j} \bar{v}_{jk}^A + \ell \sum_k \bar{v}_{jk}^B$; no output-injection gain can repair a violated diagonal on an unmeasured neuron.*

Proof. The M-matrix condition requires every diagonal entry $a_j^\circ > 0$ (Lemma 4); for $j \notin \mathcal{M}$ the output-injection term $2\ell_j^\circ \chi_j$ vanishes, leaving the stated expression, which must therefore be positive. $\square$

Proposition 1 says the designer must measure every neuron whose intrinsic dynamics are “too active” to self-damp; the gain can only rescue neurons it can see. In the experiment neurons 1, 2 (with $\bar{v}_{jj}^A > d_j$) must be measured, and neuron 3 (with $\bar{v}_{33}^A = 0.39 < d_3 = 1.4$) may be reconstructed—precisely the design used. A second consequence bounds how few sensors suffice.

Corollary 1 (minimal sensing). *Let $\mathcal{U} = \{j : 2d_j \leq 2\ell \bar{v}_{jj}^A + \ell \sum_{k \neq j} \bar{v}_{jk}^A + \ell \sum_k \bar{v}_{jk}^B\}$ be the set of self-active neurons. Then any admissible measured set satisfies $\mathcal{M} \supseteq \mathcal{U}$, and $\mathcal{M} = \mathcal{U}$ is admissible provided the residual coupling among the self-stable neurons keeps the Perron root of K_o below 1. Thus the self-active neurons are exactly the ones that must be instrumented.*

Proof. By Proposition 1 every $j \in \mathcal{U}$ has a diagonal that is nonpositive without the gain, so it must lie in $\mathcal{M}$; hence $\mathcal{M} \supseteq \mathcal{U}$. Taking $\mathcal{M} = \mathcal{U}$ with gains large enough to make $a_j^\circ > 0$ on $\mathcal{U}$ leaves the self-stable neurons with positive diagonals, and the M-matrix test reduces to the Perron condition on K_o , assumed to hold. $\square$

Plain-language interpretation: Some neurons are calm on their own and some are restless. The restless ones must be wired to a sensor—no amount of clever inference can settle a neuron you never look at if it stirs itself up faster than it forgets. The calm ones can be left unmonitored and rebuilt from how they are connected to the rest. So the cheapest honest sensor budget is exactly the list of restless neurons.

6. Cascade Synchronization and the Price of Output Feedback

6.1. The Tracking Comparison

Lemma 5 (tracking comparison). *Under Assumptions 1–4, along (7), for any $\theta \in (0, a_s - b_s)$ the tracking energy $V_s = \sum_v \|e_v\|^2$ obeys*

$${}^C D_{t_0}^\alpha V_s \leq -(a_s - \theta)V_s + b_s \sup_{[t-\bar{\tau}, t]} V_s + \frac{\|H\|^2}{\theta} V_o, \quad (10)$$

with $a_s = 2 \min_j d_j + 2\lambda_1(H) - 2\ell\mu_A - \ell\mu_B$, $b_s = \ell\mu_B$, $\mu_A = \|\bar{\nu}_{jk}^A\|_2$, $\mu_B = \|\bar{\nu}_{jk}^B\|_2$, and $V_o = \sum_v \|\eta_v\|^2$.

Proof. By the Caputo chain inequality, ${}^C D_{t_0}^\alpha V_s \leq 2 \sum_v \langle e_v, {}^C D_{t_0}^\alpha e_v \rangle$; substitute (7) and bound each group. *Self-decay:* $2 \sum_v \langle e_v, -d \odot e_v \rangle = -2 \sum_v \sum_j d_j \|e_{v,j}\|^2 \leq -2 \min_j d_j V_s$. *Activations:* by Assumption 3 and Lemma 1(i), stacking over (v, j) and writing e for the stacked vector, $2 \sum_v \langle e_v, \sum_l h_l A^{(l)} \Delta f_v^e \rangle \leq 2\ell e^\top (\bar{\nu}^A \otimes I) e \leq 2\ell\mu_A V_s$ with $\mu_A = \|\bar{\nu}^A\|_2$, and the delayed term is $\leq \ell\mu_B (V_s + V_s(t - \tau))$ by Young in time. *Pinning:* since $H \succ 0$, $-2 \sum_v \langle e_v, (He)_v \rangle = -2e^\top (H \otimes I) e \leq -2\lambda_1(H) V_s$. *Coupling:* $2 \sum_v \langle e_v, (H\eta)_v \rangle = 2e^\top (H \otimes I) \eta$; the weighted Young inequality $2a^\top b \leq \theta \|a\|^2 + \theta^{-1} \|b\|^2$ with $a = e$, $b = (H \otimes I) \eta$ gives $\leq \theta V_s + \theta^{-1} \|(H \otimes I) \eta\|^2 \leq \theta V_s + \theta^{-1} \|H\|^2 V_o$, using $\|(H \otimes I) \eta\| \leq \|H\| \|\eta\|$ and $\|\eta\|^2 = V_o$. Adding the four groups,

$${}^C D_{t_0}^\alpha V_s \leq -(2 \min_j d_j + 2\lambda_1(H) - 2\ell\mu_A - \ell\mu_B - \theta)V_s + \ell\mu_B V_s(t - \tau) + \theta^{-1} \|H\|^2 V_o,$$

which is (10) with $a_s = 2 \min_j d_j + 2\lambda_1(H) - 2\ell\mu_A - \ell\mu_B$ and $b_s = \ell\mu_B$. $\square$

6.2. Joint Synchronization

Equations (9) and (10) form a triangular 2×2 comparison system in (V_o^ξ, V_s) : the first is autonomous, the second is driven by the first.

Theorem 2 (cascade synchronization). *Suppose the detectability test (Theorem 1) holds with rate $r_o := \gamma_o - b_o > 0$, and $a_s - \theta > b_s$ for some $\theta \in (0, a_s - b_s)$. Then along (6) and (7):*

- (i) $V_o^\xi(t)$ is ultimately bounded by $V_{o,\infty} = C_o/r_o$ and, if $\bar{d} = 0$, converges to 0 at the order- α Mittag-Leffler rate;
- (ii) $V_s(t)$ is ultimately bounded by

$$V_{s,\infty} = \frac{\theta^{-1} \|H\|^2 (\sum_j \xi_j) V_{o,\infty}}{a_s - \theta - b_s}, \quad (11)$$

and, if $\bar{d} = 0$, converges to 0; the joint decay rate is $\min\{r_o, a_s - \theta - b_s\}$.

Proof. (i) is Theorem 1 with Lemma 2. For (ii), bound $V_o \leq (\sum_j \xi_j) V_o^\xi$ and insert into (10): ${}^C D_{t_0}^\alpha V_s \leq -(a_s - \theta)V_s + b_s V_s(t - \tau) + w(t)$ with the driving input $w(t) = \kappa V_o^\xi(t)$, $\kappa = \theta^{-1} \|H\|^2 \sum_j \xi_j$. This is precisely the converging-input fractional Halanay inequality (Lemma 3) with rate $a_s - \theta > b_s$: by (i), w is bounded with $\limsup_t w = \kappa V_{o,\infty}$, so $\limsup_t V_s \leq \kappa V_{o,\infty} / (a_s - \theta - b_s)$, which is (11); and if $\bar{d} = 0$ then $V_o^\xi \rightarrow 0$, hence $w \rightarrow 0$ and $V_s \rightarrow 0$. Because the second subsystem is driven by—and cannot outrun—the first, the slower of the two governs the joint transient (the cascade ordering); Corollary 2 makes the tail explicit. $\square$

Plain-language interpretation: Two clocks tick in series. The first is the observer settling; the second is the fleet closing ranks, but it cannot run faster than the information it is fed, so it is paced by the first. The joint speed is whichever clock is slower. In our design the observer is the slow clock (1.27), so the estimate's convergence sets the pace and the controller, once it has a good estimate, has speed to spare.

6.3. The Two Prices, Quantified

Theorem 3 (weak separation and robustness). *Under the hypotheses of Theorem 2:*

- (a) *Weak separation / coupling cost. The observer (γ_o) and the tracking (a_s) may be designed independently, but the achievable tracking rate is $a_s - \theta - b_s$, reduced from the full-state rate $a_s - b_s$ by the coupling cost θ ; the joint rate is $\min\{r_o, a_s - \theta - b_s\}$, so any $\theta < a_s - b_s - r_o$ leaves the observer as the bottleneck at no joint-rate penalty.*
- (b) *Sensor-attack robustness. Under a bounded sensor attack $\|d_{v,j}\| \leq \bar{d}$, both ultimate errors scale quadratically, $V_{o,\infty} \propto \bar{d}^2$ and—through the coupling (11)— $V_{s,\infty} \propto \|H\|^2 \bar{d}^2$. The tracking ultimate error is amplified by the squared control gain $\|H\|^2$: high-gain pinning that accelerates tracking magnifies sensitivity to the attack.*

Proof. (a) is immediate from Theorem 2: the joint rate is $\min\{r_o, a_s - \theta - b_s\}$, and it equals r_o whenever $a_s - \theta - b_s \geq r_o$, i.e. $\theta \leq a_s - b_s - r_o$. (b) From (8), $C_o = \max_j (\ell_j^o)^2 N \bar{d}^2 / (\theta_o \xi_j) \propto \bar{d}^2$, so $V_{o,\infty} = C_o / r_o \propto \bar{d}^2$; substituting into (11), $V_{s,\infty} \propto \|H\|^2 V_{o,\infty} \propto \|H\|^2 \bar{d}^2$. $\square$

Remark 4 (on the magnitude of the robustness bound). *The constant in $V_{s,\infty}$ is large—the Young steps and the $\|H\|^2$ amplification make it $\sim 10^3 \times$ the observed value—so (11) is best read as a scaling law ($\propto \|H\|^2 \bar{d}^2$), which the experiments confirm exactly, rather than a tight magnitude. The $\|H\|^2$ dependence is not merely conservatism, though: it is the genuine mechanism by which a stale or attacked measurement is injected into the loop with the full control gain, and the simulator shows the empirical $V_{s,\infty}$ growing as $\bar{d}^2$ just as predicted.*

Plain-language interpretation: Reading the machine through a noisy gauge is like navigating by a slightly foggy window: you can still steer, but every smear on the glass is turned into a steering error, and the harder you steer (the bigger the gain), the more each smear throws you off. The attack size $\bar{d}$ is the smear; the response grows with its square because both the estimate and the steering it feeds are corrupted; and the gain $\|H\|$ is the lever that magnifies it. The lesson for a designer is uncomfortable but precise: cranking up the pinning to synchronize faster also makes the formation easier to disturb by tampering with a sensor.

Remark 5 (effect of sensor and actuator faults). *The additive term d_v in (3) already represents a bounded sensor fault or false-data injection. It enters the autonomous observer error as $-Ld_v$ and then reaches tracking through $H\eta$; Theorem 3 quantifies the resulting $O(\|H\|^2 \bar{d}^2)$ ultimate tracking energy. For an additive actuator fault, let the plant receive $u_v^{\text{act}} = u_v + \phi_v$ while the observer receives the commanded u_v . Then $+\phi_v$ appears in both error systems (6) and (7). If $\|\phi_v\| \leq \bar{\phi}$, Young inequalities add constant terms proportional to $\bar{\phi}^2$ and consume chosen positive parts of the observer and tracking margins. Thus uniform ultimate boundedness is preserved whenever the reduced margins remain positive, with ultimate energies of order $O(\bar{d}^2 + \bar{\phi}^2)$. If the actual applied input is measured and fed to the observer, the additive actuator fault cancels from (6) and acts only on tracking.*

A loss-of-effectiveness fault is more critical: replacing the input by $(I - \Lambda_v)u_v + \phi_v$ reduces and may unbalance the effective pinning matrix. Stability then requires the worst-case effective matrix to retain sufficient damping in the condition $\alpha_s^{\text{eff}} - \theta > b_s$; if its smallest eigenvalue falls below this margin, the present fixed-gain certificate no longer excludes instability. Practical mitigation therefore combines the offline margin test in Section 7.1 with actuator monitoring, gain scheduling or reconfiguration, and, for severe faults, an adaptive or unknown-input observer. A full active fault-tolerant redesign is outside the present bounded-fault analysis and is stated as future work.

6.4. The Joint Tail and What “Observer-Limited” Means

We are careful about what the cascade rate does and does not control.

Corollary 2 (joint decay). *In the noise-free case $\bar{d} = 0$, under the hypotheses of Theorem 2, both V_o^ξ and V_s decay to 0 at the order- α Mittag-Leffler rate; asymptotically both tails are $O(t^{-\alpha})$. The rates r_o and $a_s - \theta - b_s$ enter the constants of these tails, not their exponent: the slower rate $r_{\min} = \min\{r_o, a_s - \theta - b_s\}$ produces the larger tail constant and dominates the pre-asymptotic transient.*

Proof. By Theorem 1(i) with $\bar{d} = 0$, $V_o^\xi \rightarrow 0$ along an order- α Mittag-Leffler envelope of rate r_o ; its tail is $O(t^{-\alpha})$ with a constant $\propto 1/r_o$ [1, 10]. Feeding this into Lemma 3 (converging input, rate $a_s - \theta - b_s$) gives $V_s \rightarrow 0$; the induced envelope is again order- α Mittag-Leffler, so $V_s = O(t^{-\alpha})$, with a constant governed by the smaller of the two rates because the driving input decays no faster than $t^{-\alpha}$ and the tracking subsystem cannot decay faster than its input. Hence the exponent is α in both cases and $r_{\min}$ sets the constant. $\square$

Remark 6 (reading the rates honestly). *Since both subsystems share the order α , they share the tail order $t^{-\alpha}$; “observer-limited” therefore refers to the constant/transient, not the tail exponent. With $r_o = 1.27 < a_s - \theta - b_s = 1.59$, the observer’s smaller Halanay rate produces the larger tail constant, so the estimate’s convergence dominates the pre-asymptotic decay. The experiment (Figure 2) confirms the cascade ordering— V_s never decays ahead of V_o —rather than a single exponential slope, which would be the wrong descriptor for algebraic Mittag-Leffler decay.*

7. Design Algorithm

Algorithm 1 presents the design algorithm for the observer-based certificate. Its steps collect the observer, tracking, cascade, and robustness calculations developed above.

Algorithm 1: Design algorithm for the observer-based certificate

Input: order α ; self-decays d_j ; Lipschitz ℓ ; operator norms $\bar{\nu}_{jk}^A, \bar{\nu}_{jk}^B$; measured set $\mathcal{M}$; graph L_g , pinning Δ , gains c, k ; delay $\bar{\tau}$; sensor bound $\bar{d}$.

1 *Observer*

2 Choose output-injection gains ℓ_j^o (on $\mathcal{M}$); form a^o, P^o, Q^o (8)

3 **if** some $a_j^o \leq 0$ or the Perron root of $K_o \geq 1$, **then**

4 increase ℓ^o on the measured neurons (or enlarge $\mathcal{M}$).

5 **end**

6 Output the Perron weight ξ , margin γ_o, b_o , and rate $r_o = \gamma_o - b_o$.

7 *Tracking*

8 Form $H = cL_g + k\Delta$; compute $\lambda_1(H), \|H\|, a_s, b_s$.

9 **if** $a_s \leq b_s$, **then**

10 increase pinning.

11 **end**

12 *Cascade*

13 Pick $\theta \in (0, a_s - b_s)$; the joint rate is $\min\{r_o, a_s - \theta - b_s\}$ (Theorem 2); any $\theta \leq a_s - b_s - r_o$ keeps the observer as bottleneck.

14 *Robustness*

15 Output $V_{o,\infty} = C_o/r_o$ and $V_{s,\infty}$ (11), both $\propto \bar{d}^2$.

Output: $(\xi, \gamma_o, r_o, a_s, b_s, \theta, \text{joint rate}, V_{o,\infty}, V_{s,\infty})$ —closed form, two eigenvalue computations.

For the experimental network the algorithm returns observer drifts $a^o = (2.86, 3.06, 1.67)$ and coupling matrices

$$P^o = \begin{pmatrix} 0 & 0.108 & 0.085 \\ 0.095 & 0 & 0.098 \\ 0.067 & 0.076 & 0 \end{pmatrix}, \quad Q^o = \begin{pmatrix} 0.210 & 0.045 & 0.032 \\ 0.032 & 0.196 & 0.045 \\ 0.036 & 0.032 & 0.136 \end{pmatrix},$$

so that $K_o = \text{diag}(a^o)^{-1}(P^o + Q^o)$ has entries of order 0.05–0.08 and Perron root $\rho(K_o) = 0.175 < 1$ (eigenvalues 0.175, 0.024, 0.020), Perron weight $\xi = (1.09, 1.00, 1.41)$, margin $\gamma_o = 1.56$, $b_o = 0.29$, observer rate $r_o = 1.27$. Tracking: $a_s = 3.43$, $b_s = 0.26$ (full-state rate 3.17), $\lambda_1(H) = 2.85$, $\|H\| = 20.1$; joint rate 1.27 (observer-limited). At $L = 0$ the diagonal becomes $a^o = (-2.34, -2.15, 1.67)$, two entries negative, and the M-matrix test is vacuous—no Perron argument applies. The whole certificate is thus a pair of 3×3 eigenvalue computations.

7.1. Robustness to Parameter and Tuning Variations

The nominal inequalities also provide a direct robust-stability test. Suppose that certified bounds satisfy $d_j \geq d_j^-, \ell \leq \ell^+, \nu(A_{jk}^{(l)}) \leq \bar{\nu}_{jk}^{A,+}, \nu(B_{jk}^{(l)}) \leq \bar{\nu}_{jk}^{B,+}$, and the implemented observer gain on a measured neuron obeys $\ell_j^o \geq \ell_j^{o,-}$. Form the observer constants a_o^-, P_o^+, Q_o^+ by substituting these adverse bounds into (8). Observer convergence is retained if

$$\min_j a_{o,j}^- > 0, \quad \rho[\text{diag}(a_o^-)^{-1}(P_o^+ + Q_o^+)] < 1. \quad (12)$$

For the controller, write $H = H_0 + \Delta H$. Weyl's inequality gives $\lambda_1(H) \geq \lambda_1(H_0) - \|\Delta H\|$. With adverse spectral bounds μ_A^+, μ_B^+ , define

$$a_s^- = 2 \min_j d_j^- + 2[\lambda_1(H_0) - \|\Delta H\|] - 2\ell^+ \mu_A^+ - \ell^+ \mu_B^+, \quad b_s^+ = \ell^+ \mu_B^+. \quad (13)$$

The perturbed tracking loop is certified if $a_s^- - \theta - b_s^+ > 0$. These inequalities expose the instability mechanisms: underestimated nonlinear couplings or leakage loss erode both margins; insufficient observer gain first destroys (12); and weak or inaccurate pinning reduces $\lambda_1(H)$. Conversely, excessive pinning raises $\|H\|$ and therefore sensor-fault amplification even when it improves the nominal tracking margin.

Table 3 applies an intentionally adverse simultaneous variation: d, L, c, k are each reduced by ε , while the operator-norm envelopes of A, B are each increased by ε . The proof parameter θ is reselected as one half of the corresponding worst-case tracking margin. The certificate stays positive through 20%, although the joint margin falls from 1.27 to 0.10; at 25% the smallest observer diagonal becomes negative and the fixed-gain certificate is no longer valid. This is an *inconclusive certificate*, not a claim that every trajectory must be unstable, but it identifies a quantitative retuning threshold.

Table 3. Worst-case certificate under simultaneous adverse parameter and gain variation (generated by `robustness_margin.py`).

ε	$\rho(K_o)$	Observer Rate	$\alpha_s - b_s$	Joint Rate
0%	0.175	1.266	3.171	1.266
5%	0.221	1.041	2.559	1.041
10%	0.292	0.809	1.947	0.809
15%	0.420	0.554	1.336	0.554
20%	0.761	0.098	0.724	0.098
25%	—	not certified	0.112	not certified

The mitigation adopted in this study is therefore explicit offline margin certification: gains are accepted only when (12) and (13) remain positive over the stated uncertainty box, and θ is then selected inside the robust tracking margin. In deployment, observer residuals should additionally be monitored and the gains rescheduled or the controller switched to a safe mode when the certified box is exceeded; adaptive compensation is not assumed in the present theorem.

8. Numerical Studies

For related background on discrete approximations of fractional equations, Sun and Wu [32] developed a fully discrete difference scheme for a diffusion–wave system. The numerical implementation used here is specified below.

Setup: $N = 6$ agents on a ring-plus-chord graph; $n = 3$ Clifford neurons in $C\ell(0, 2)$ (quaternion, ν exact); order $\alpha = 0.92$; coefficientwise tanh activations ($\ell = 1$); two Takagi–Sugeno rules. Self-decays $d = (1, 1, 1.4)$; the entrywise operator norms have diagonal $\bar{\nu}_{jj}^A = (1.93, 1.84, 0.39)$, so the measured neurons $\{1, 2\}$ are self-unstable for the estimation error and neuron 3 is self-stable; $\mu_A = 2.00$, $\mu_B = 0.26$. Output-injection gains $\ell^o = (2.6, 2.6, 0)$. Pinning on three nodes with $c = 3$, $k = 8$, giving $\lambda_1(H) = 2.85$, $\|H\| = 20.1$. Delay $\tau(t)$ is a sawtooth in $[0.10, 0.50]$. Integration is IMEX-L1 with a sum-of-exponentials memory [33]; master seed 20260712; Python 3.11, NumPy 2, SciPy 1.17. The L1 scheme matches the exact Mittag-Leffler solution to 0.23% and is grid-self-convergent to 0.36%.

Model specification (for reproducibility): The two Takagi–Sugeno rules share the premise $\theta = \langle s_1 \rangle_\emptyset$ (the scalar blade of neuron 1) with $h_1 = (1 + e^{-1.5\theta})^{-1}$, $h_2 = 1 - h_1$. The weight tensors place the geometric action on the diagonal, with the measured neurons' diagonal blades scaled up so that $\bar{\nu}_{jj}^A = (1.93, 1.84, 0.39)$: the first two exceed $d_j = 1$ (self-unstable estimation error, gain required), the third is below $d_3 = 1.4$ (self-stable, reconstructed). Cross-neuron blades are small, keeping the observer M-matrix diagonally dominant. The observer is initialized with the correct measured neurons and a deliberately wrong (0) estimate of neuron 3, so the reconstruction must be built from scratch. The full tensors are in `common.py`; the pinned graph is a 6-cycle with chord $\{1, 4\}$, even agents pinned, $c = 3$, $k = 8$.

8.1. The Integrator

The three coupled fractional systems—leader (1), plant (2), and observer (4)—are advanced by an implicit-explicit L1 scheme: the stiff self-decay $-d_j x_{v,j}$ is taken implicitly (a scalar resolvent per component), the saturating fuzzy and coupling terms explicitly, and the Caputo memory by the L1 convolution weights $b_i^{(\alpha)} = (i+1)^{1-\alpha} - i^{1-\alpha}$. For the long observer horizons the memory sum is compressed by a sum-of-exponentials quadrature of the kernel tail with a $\tau = 0$ head-correction mode [33], turning the $O(n_{\text{steps}}^2)$ exact-L1 cost into $O(n_{\text{steps}})$. The observer is a genuine reconstruction test: its neuron-3 estimate is initialized to 0 while the true neuron-3 state is $O(1)$, so the $V_o(0) = 16.0$ start reflects a fully unknown unmeasured component. Determinism is by fixed seeds.

8.2. Lemma Audits

Table 4 reports the equality-boundary audits. The Clifford norm is submultiplicative to machine precision and exact for $m \leq 2$ (up to $1.40 \times$ for $m = 3$). The chain inequality holds with worst margin $+2.7 \times 10^{-7}$. The observer detectability comparison is audited on 41 systems (the certified one plus 40 random detectable systems): the weighted observer energy never overshoots its initial value (worst overshoot $-0.16 < 0$, i.e., no violation). The cascade 2×2 comparison, integrated to steady state, gives ultimate bounds at $0.51 \times$ the certified values—the triangular certificate holds with $2 \times$ conservatism.

Table 4. Lemma audits at the equality boundary (`verify_lemmas.py`).

Lemma	Audited Quantity	Result
Lemma 1 ν submult.	$\max \nu(ab)/[\nu(a)\nu(b)]$	1.0000
Lemma 1 ν exact $m \leq 2$	$\nu/\ \cdot\ $ ($m = 3$ range)	[1.00, 1.40]
Lemma 4 chain ineq.	worst normalized margin	$+2.7 \times 10^{-7}$
Theorem 1 detectability	worst overshoot, 41 systems	-0.16
Theorem 2 cascade	sim/certified ultimate bound	0.51

8.3. The Observer Gain Is Essential; The Cascade Synchronizes

Figure 2 (left) shows the run from $V_o(0) = 16.0$, $V_s(0) = 45.9$. With the designed gain the observer energy falls to $V_o = 2.6 \times 10^{-4}$ and the tracking energy follows to $V_s = 2.7 \times 10^{-4}$: the cascade synchronizes, and the two-timescale picture is visible— V_o leads, V_s trails at the same observer-limited rate 1.27. The open-loop twin ($L = 0$, dotted), whose detectability test is infeasible, tells the opposite story: its observer energy stalls near 0.80 and never converges (a $3000\times$ larger residual), confirming Remark 3 that the gain is a feasibility requirement, not a tweak.

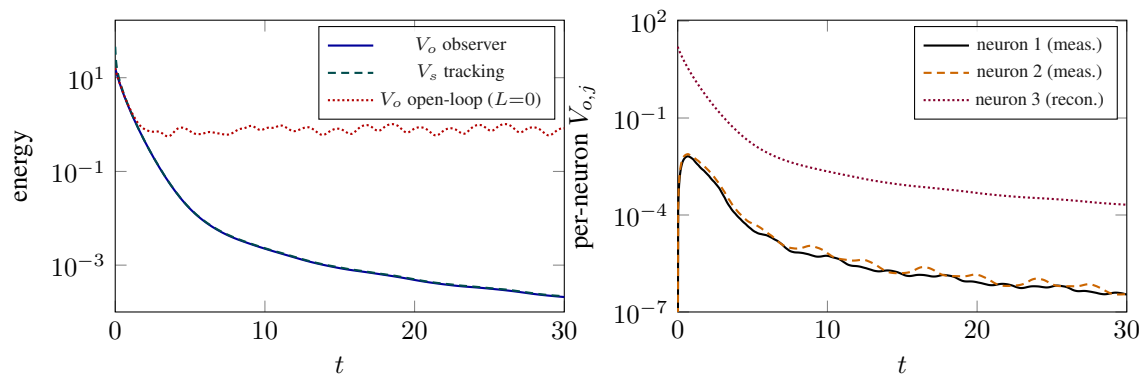


Figure 2. Left: observer energy V_o and tracking energy V_s under the designed gain synchronize (cascade), while the open-loop V_o ($L = 0$) stalls. Right: per-neuron observer energies; the reconstructed neuron 3 is the slowest, pacing V_o .

The right panel resolves V_o by neuron. The measured neurons $\{1, 2\}$, tamed directly by the gain, fall to $\sim 10^{-6}$; the reconstructed neuron 3, reached only through the network coupling, settles two orders higher ($\sim 6 \times 10^{-4}$) and paces V_o —the unmeasured state is the hardest to estimate, exactly as the Perron weight $\xi_3 = 1.41$ (the largest) anticipates.

8.4. Reconstruction and the Price of Output Feedback

Figure 3 (left) overlays the true and estimated trajectories of the unmeasured neuron's e_1 -blade for one agent: the observer, initialized with a deliberately wrong (0) estimate of neuron 3, locks onto the true trajectory within a few units of time. The right panel prices output feedback: the full-state-feedback twin (which pins the true state) reaches $V_s = 1.9 \times 10^{-5}$, whereas output feedback reaches 2.7×10^{-4} —a $14\times$ larger residual, the measured cost of steering by an estimate. This is the empirical face of the coupling term $H\eta$ in (7): the observer error, however small, is injected into the loop with the full control gain.

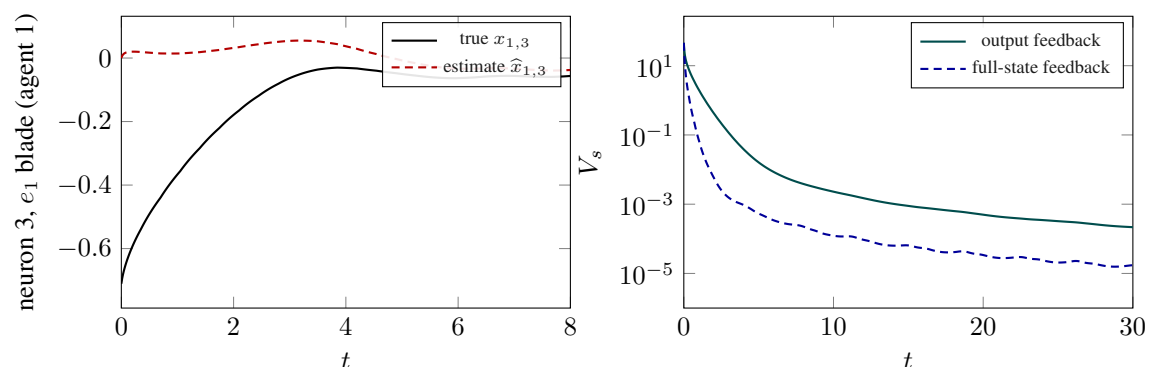


Figure 3. Left: reconstruction of the unmeasured neuron 3 (e_1 -blade, agent 1)—estimate (dashed) locks onto the truth (solid). Right: tracking energy under output feedback vs. full-state feedback; output feedback pays a $14\times$ residual penalty.

8.5. Sensor-Attack Robustness: The $\bar{d}^2$ Law

Figure 4 and Table 5 sweep the sensor-attack bound $\bar{d}$. Both ultimate energies grow as $\bar{d}^2$ (Theorem 3(b)): the empirical $V_{o,\infty}$ and $V_{s,\infty}$ track the slope-2 reference across the whole range ($V_{o,\infty}$ from 6.0×10^{-3} at $\bar{d} = 0.02$ to 2.32 at $\bar{d} = 0.4$; $V_{s,\infty}$ likewise). The certified bounds are conservative in magnitude—at $\bar{d} = 0.1$, $V_{o,\infty}$ is $7 \times$ and $V_{s,\infty}$ is $\sim 4 \times 10^3 \times$ the observed values, the latter reflecting the $\|H\|^2/\theta$ amplification and the Young steps (Remark 4)—but the *scaling* they predict, $V_{s,\infty} \propto \|H\|^2 \bar{d}^2$, is exact. The security reading is concrete: the same high pinning gain that makes tracking fast makes the network's synchronization error grow quadratically with the corruption a sensor attacker can inject.

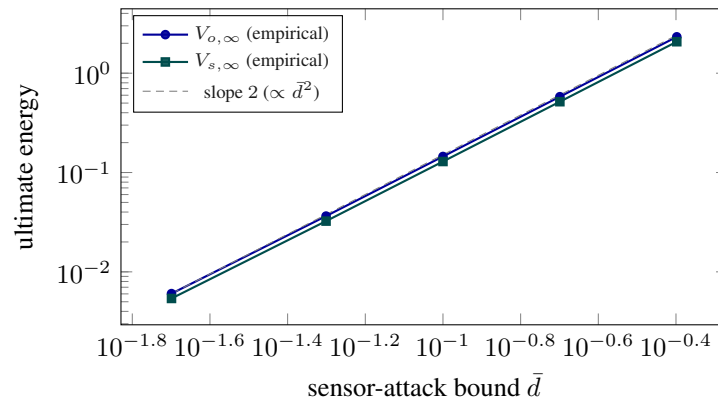


Figure 4. Sensor-attack robustness: empirical ultimate observer and tracking energies vs. the attack bound $\bar{d}$, against the slope-2 reference. Both scale as $\bar{d}^2$ (Theorem 3(b)).

Table 5. Sensor-attack sweep (sim.py dbarsweep). Empirical ultimate energies; certified bounds in brackets.

$\bar{d}$	$V_{o,\infty}$ (emp. [cert.])	$V_{s,\infty}$ (emp. [cert.])	$V_{s,\infty}/\bar{d}^2$
0.02	6.0×10^{-3} [0.041]	5.4×10^{-3} [23]	13.5
0.05	3.6×10^{-2} [0.25]	3.2×10^{-2} [143]	13.0
0.10	0.145 [1.01]	0.129 [572]	12.9
0.20	0.580 [4.05]	0.516 [2287]	12.9
0.40	2.324 [16.2]	2.074 [9149]	13.0

8.6. Comparative Analysis

Three comparisons isolate the paper's effects. (i) *Gain vs. no gain.* At $L = 0$ the detectability constants $a^o = (-2.34, -2.15, 1.67)$ are infeasible and the observer error stalls at 0.80; the designed gain makes $a^o = (2.86, 3.06, 1.67)$ and drives the error to 2.6×10^{-4} —a 3×10^3 separation that confirms H1. (ii) *Output vs. full-state feedback.* At the measured horizon ($t \approx 25$ –30) the full-state twin reaches $V_s = 1.9 \times 10^{-5}$ and output feedback 2.7×10^{-4} , a $14 \times$ residual penalty (H3); as both decay to 0 the ratio is horizon-dependent, but the gap is exactly the observer error injected through H , absent under full-state feedback. (iii) *Cascade ordering.* The certificate makes the observer the bottleneck ($r_o = 1.27 < a_s - \theta - b_s = 1.59$); consistent with this, the tracking energy V_s never decays ahead of the observer energy V_o throughout the run (H2)—the cascade ordering of Corollary 2, not a single exponential slope (which would misdescribe the algebraic $t^{-\alpha}$ Mittag-Leffler decay).

8.7. Practical Interpretation of the Simulation

The architecture measures two of three neurons per follower, reducing the neuron-level sensing requirement by one third relative to full-state feedback while reconstructing every blade of the unmeasured neuron. The observer and controller are local except for the same neighbouring estimate exchange already required by the pinned graph. Consequently, an implementation needs no centralized state estimator: each agent stores its own fractional-memory states, receives s and neighbouring estimates, and evaluates one copy of the known T–S model.

The numerical values expose three engineering decisions. First, a missing sensor channel is acceptable only when the M-matrix test remains feasible; removing the injection gain increases the observer residual from 2.6×10^{-4} to 0.80. Second, output feedback avoids the third neuron sensor at the cost of a $14 \times$ finite-horizon tracking residual relative to full-state feedback. Third, raising the pinning gain cannot be viewed as an unqualified improvement, because sensor corruption is injected with $\|H\|^2$. Thus the recommended workflow is to choose the smallest

measured set that passes detectability, select observer gains with a robust margin, and then tune pinning against both tracking speed and the admissible sensor-fault bound.

The simulations are linked to the theory claim by claim: the gain/no-gain twin tests the necessity statement in Theorem 1; the ordering of V_o and V_s tests the triangular cascade in Theorem 2; the full-state twin isolates the coupling cost; and the $\bar{d}$ sweep tests the exponent in Theorem 3. Numerical agreement does not replace the proof; it checks that the hypotheses, certified margins, and predicted mechanisms are simultaneously realised by the implemented system. The study remains a simulation demonstration with dimensionless states, so hardware-specific bandwidth, quantization, saturation, and energy use must be assessed before applying the numerical gains to a microgrid, vehicle platoon, or robot formation.

8.8. Summary of Certified-vs.-Empirical Gaps

Table 6 collects the accounting. The detectability and cascade certificates are qualitatively tight (the observer envelope is never violated; the cascade bound is $2\times$ conservative). The robustness bound is loose in magnitude ($\sim 10\times$ on V_o , $\sim 10^3\times$ on V_s) but exact in its $\bar{d}^2$ scaling. The price of output feedback ($14\times$) and the observer-essential gap (open-loop $3000\times$ worse) are the paper's headline measured effects.

Table 6. Certified vs. empirical, and the measured effects.

Quantity	Certified	Empirical	Note
observer feasibility (L)	Perron 0.18	$V_o \rightarrow 2.6 \times 10^{-4}$	feasible
observer feasibility ($L=0$)	infeasible	$V_o \rightarrow 0.80$	gain essential
cascade ultimate bound	—	$0.51 \times \text{cert}$	$2\times$ conservative
robustness scaling	$\propto \ H\ ^2 \bar{d}^2$	$\propto \bar{d}^2$	exact scaling
price of output feedback	—	$14\times$	vs. full-state

9. Discussion

9.1. What Is New, and Against What

Prior fractional-neural-network synchronization certificates [1,4,5] are state-feedback: they assume the full multivector state is measured. State-estimation and observer work for fractional and hypercomplex networks [25–27] treats the estimation problem but not its *closed-loop* use for synchronization with an explicit joint envelope. The contribution here is the coupled analysis: a detectability certificate whose gain is essential for feasibility, a triangular fractional cascade giving an explicit joint rate, and the quantified price of using an estimate. The cascade lemma is model-agnostic—any two fractional Lyapunov energies in the one-directional relationship (6) and (7) inherit the joint envelope—so it transfers to other output-feedback fractional designs once a detectability certificate is in hand.

9.2. Weak versus Clean Separation

For linear fractional systems a clean separation principle holds (the characteristic function factorizes). Our plant is nonlinear, so a clean separation is not available—as in the integer nonlinear case—but the autonomy of the estimation error (6) salvages a *weak* separation: observer and controller are designed independently, and the only coupling is through the achievable joint rate, quantified by θ . In our design the observer is the bottleneck, so the coupling cost is free; when the observer is faster than the tracking, θ becomes a genuine tax and the designer trades it against the tracking rate.

9.3. Scalability and Deployment

The certificate costs two eigenvalue computations (the pinned Laplacian H and the $n \times n$ observer M-matrix K_o) and closed-form arithmetic; it scales to large networks and many neurons. A deployment needs: a measured set $\mathcal{M}$ making the observer M-matrix feasible (a detectability check, done offline); output-injection gains sized on the measured neurons; a bound $\bar{d}$ on sensor corruption to size the ultimate errors; and awareness that raising the pinning gain to speed tracking raises sensor-attack sensitivity as $\|H\|^2$. A concrete target is a platoon or microgrid whose agents expose only a few telemetry channels.

9.4. Computational Cost

Simulation dominates: the exact-L1 memory is $O(n_{\text{steps}}^2)$, reduced to $O(n_{\text{steps}})$ by the sum-of-exponentials scheme [33]; the full observer run (three coupled fractional systems—leader, plant, observer—over 30 time units) completes in about 7 seconds single-core.

9.5. Limitations and Failure Modes

Three are structural. (i) The robustness magnitude is loose (the $\|H\|^2/\theta$ Young amplification); a tighter treatment of the $H\eta$ coupling—for instance a weighted norm adapted to H —would close the $\sim 10^3 \times$ gap on $V_{s,\infty}$ while keeping the $\bar{d}^2$ scaling. (ii) Detectability here is a sufficient M-matrix condition; a neuron set that fails it may still be observable through a non-diagonal gain L that our per-neuron test does not certify. (iii) The sensor-attack model is a bounded additive corruption; a sparse or strategically-timed attack (false-data injection on a schedule) is outside this bound and would need a dedicated resilient-observer analysis.

9.6. Future Directions

A tighter H -weighted robustness bound, non-diagonal (full) output-injection gains with an LMI detectability test, incommensurate per-neuron orders in the observer, active compensation for loss-of-effectiveness actuator faults, and a resilient observer against sparse or strategically timed sensor attacks are the natural next steps. Hardware-in-the-loop and experimental validation on a networked energy or robotic platform will then be needed to quantify bandwidth, quantization, saturation, and computational effects that the present simulation does not represent.

10. Conclusions

We have developed an output-feedback synchronization theory for fractional-order fuzzy Clifford-valued neural networks in which each follower measures only part of its state, reconstructs the rest with a local fractional observer, and pins the estimate to the leader. The estimation error is autonomous and its convergence is a per-neuron M-matrix detectability property whose output-injection gain is essential for feasibility (open-loop infeasible; open-loop residual $\sim 3000 \times$ worse). The observer error drives the tracking error in a triangular fractional cascade whose joint rate is the slower of the two subsystems—observer-limited in our design—and whose explicit envelope the integer ISS-cascade lemma does not provide. The price of output feedback appears in two forms: a coupling cost θ that makes the separation weak rather than clean, a $14 \times$ residual penalty against full-state feedback, and a sensor-attack robustness that scales as $\|H\|^2 \bar{d}^2$ —so high-gain pinning trades tracking speed for attack sensitivity. Selected inequalities and certificates are numerically audited: the detectability and cascade certificates are tight to $2 \times$, and the robustness bound is loose in magnitude but exact in scaling. The limitations—the loose robustness constant, the sufficient-only detectability test, and the bounded-attack model—are stated as gaps to close rather than hidden. Future research will develop a full fault-tolerant observer/controller, tighten the H -weighted robustness bound, extend the certificate to directed and switching graphs, and validate the design in hardware-in-the-loop experiments.

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Data Availability Statement

All code (`common.py`, `learn.py`, `sim.py`, `verify_lemmas.py`, `robustness_margin.py`, `makefigs.py`) and generated data are provided with the submission and reproduce every reported number under the stated seeds. All numbers in Section 8 are produced by the accompanying code with master seed 20260712 (`common.py`, `learn.py`, `sim.py`, `verify_lemmas.py`, `robustness_margin.py`, `makefigs.py`). The reproduction order is `verify_lemmas.py` (Table 4), `learn.py` (certificate), then `sim.py` with stages {`validate`, `main`, `openloop`, `fullstate`, `dbarsweep`}, and `robustness_margin.py` (Table 3), followed by `makefigs.py`. Figure data are embedded as `pgfplots` coordinates; the manuscript compiles standalone with `pdfLATEX`.

Conflicts of Interest

The author declares no conflict of interest.

Use of AI and AI-Assisted Technologies

OpenAI Codex was used during proofreading to assist with language and LaTeX production corrections. The author is responsible for reviewing and approving these corrections and for the content of the published article.

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