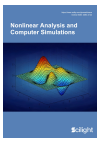
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A Degenerate q-Laplace Transform on the Jackson Grid: Fourier–Mean Recovery, Deformed Convolution, and Computational Hereditary Models

Oğuz Yağcı

Department of Mathematics, Kırıkkale University, 71450 Kırıkkale, Türkiye; oguzyagci26@gmail.com

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Abstract: We define a degenerate q-Laplace transform on the bilateral Jackson grid using Jackson's improper q-integral and the kernel $(1+\lambda t)^{-s/\lambda}$. Estimating the two grid tails separately yields explicit convergence conditions and weighted Jackson data spaces on which the transform is bounded and holomorphic. On a vertical line, the change of variable $x = \lambda^{-1}\log(1+\lambda t)$ turns the transform into an absolutely convergent pure-point Fourier–Stieltjes series; mean coefficient extraction then recovers every Jackson-grid value and identifies the exact off-grid null space. This is recovery on the sampled Jackson grid, not ordinary inversion at off-grid points. The kernel also determines the semigroup law $t \oplus_\lambda u = t + u + \lambda tu$. With this law, finite measures form a convolution Banach algebra and the transform is multiplicative. A small-total-variation hypothesis gives a Neumann resolvent, together with norm and perturbation bounds, for a linear hereditary measure equation. For the quadratic hereditary Volterra equation, local well-posedness follows from a contraction argument. Numerical experiments examine multi-parameter Neumann convergence, the $q \to 1^-$ Jackson-to-Riemann limit, the $\lambda \to 0^+$ deformed-to-additive convolution limit, symmetric and asymmetric bilateral truncations, and transform-domain Picard convergence for the nonlinear model.

Keywords: degenerate q-Laplace transform; Jackson q-integral; Fourier–mean recovery; deformed convolution; hereditary Volterra equation; numerical experiments

2020 MSC: 44A10; 42A38; 44A35; 44A55; 33D15

1. Introduction

In classical Laplace theory, the exponential kernel is compatible with ordinary addition and convolution, while inversion and uniqueness depend on the support of the underlying measure or function class [1]. A discrete integration geometry combined with a deformed kernel changes both aspects, so the corresponding properties must be established for the resulting transform.

For $\lambda > 0$, the kernel

$$K_s^{(\lambda)}(t) := (1+\lambda t)^{-s/\lambda}$$

converges to e^{-st} as $\lambda \to 0^+$, and the same algebraic kernel appears in degenerate gamma and Laplace-type integral transforms based on ordinary integration [2–5]. Related degenerate-transform developments include modified Laplace-type, Sadik, and Sumudu constructions, as well as applications to fractional kinetic equations, degenerate special functions, and viscoelastic models [6–11]. Further related developments include a modified-kernel degenerate q-Sumudu transform with a Laplace-like operational calculus [12] and a multidimensional degenerate Sumudu transform [13]. Jackson integration, by contrast, samples the geometric grid

$$\mathbb{T}_q = \{q^k : k \in \mathbb{Z}\}, \qquad 0 < q < 1,$$

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through the improper bilateral sum

$$\int_0^\infty g(t)\,d_qt = (1-q)\sum_{k\in\mathbb{Z}} q^k g(q^k),$$

whenever the series converges absolutely [14–16]. A related Jackson-integral degenerate q-Sumudu model likewise uses geometric-grid integration, but with a Sumudu-type transform rather than the present q-Laplace construction [17]. The two tails require different estimates: $k \to +\infty$ corresponds to $t \to 0^+$, whereas $k \to -\infty$ corresponds to $t \to \infty$.

Existing q-Laplace transforms use q-exponential or related kernels [18–21]; related constructions include Laplace transforms on time scales with arbitrary graininess [22] and q-Borel–Laplace methods [23]. Jackson-, quantum-, modified-, and (p,q)-integral variants appear in [24–30]. Earlier constructions also include a q-Laplace transform for products of general classes of q-polynomials and q-analogues of special functions [31]. The q-Laplace-type integral operators of Ucar and Albayrak provide another directly relevant antecedent [24]. A recent construction combines q-operational calculus with degenerate Laplace and Sumudu transforms [32]. Ernst's transform uses a one-sided Jackson integral over a finite transform-parameter-dependent interval and a q-exponential kernel [33]. In contrast, the transform considered here uses an improper bilateral Jackson integral and the degenerate algebraic kernel above. General references for q-calculus and basic hypergeometric series include [34,35].

We study

$$\mathcal{L}_q^{(\lambda)}\{f\}(s) := \int_0^\infty f(t)(1+\lambda t)^{-s/\lambda}\,d_qt = (1-q)\sum_{k\in\mathbb{Z}} q^k f(q^k)(1+\lambda q^k)^{-s/\lambda}. \tag{1}$$

The grid and the kernel play different roles. Because the grid is bilateral, the two tails must be estimated separately, and uniqueness must be formulated at the level of Jackson data. The kernel determines the nonlinear coordinate

$$\phi_\lambda(t) := \frac{1}{\lambda}\log(1+\lambda t)$$

and the semigroup law

$$t \oplus_\lambda u := t + u + \lambda tu, \qquad 1 + \lambda(t \oplus_\lambda u) = (1+\lambda t)(1+\lambda u). \tag{2}$$

Thus $K_s^{(\lambda)}(t \oplus_\lambda u) = K_s^{(\lambda)}(t)K_s^{(\lambda)}(u)$. This algebraic semigroup in the time variable is distinct from the operator and heat-kernel semigroup setting familiar from spectral geometry and heat-kernel asymptotics [36,37].

Separate tail estimates define weighted Jackson data spaces on which the transform is bounded and holomorphic. On a vertical line $s = c + i\tau$, the coordinate $x = \phi_\lambda(t)$ gives a pure-point Fourier–Stieltjes series. Mean coefficient extraction [38] recovers each sample $f(q^k)$ on the Jackson grid, not values between its nodes, identifies the null space on ordinary functions, and yields a finite-window recovery estimate. The law $\oplus_\lambda$ induces a convolution on finite measures for which the transform satisfies a product formula. Jackson data embed as atomic measures, although their convolution generally has support outside $\mathbb{T}_q$. Under a small-total-variation condition on the memory measure, the Banach-algebra equation has a unique Neumann-series solution with norm and perturbation bounds. For the quadratic convolution equation, a small-data contraction argument gives local well-posedness and uniqueness within the specified closed total-variation ball, without claiming global uniqueness. The numerical section compares the corresponding analytical estimates with computed Neumann remainders, the $q \to 1^-$ and $\lambda \to 0^+$ limits, bilateral truncation errors, and transform-domain Picard iterates.

Section 2 states the Jackson-grid analytical setting and the computational protocol. The transform theory, Fourier–mean recovery, operational identities, deformed-semigroup measure algebra, hereditary equations, and computational experiments are developed in Section 3. Section 4 records the limitations of the present analysis and related questions not treated here, followed by the conclusions in Section 5.

2. Materials and Methods

2.1. Analytical Setting and Jackson-Grid Preliminaries

Throughout, $q \in (0,1)$ and $\lambda > 0$ are fixed, and

$$\mathbb{T}_q := \{q^k : k \in \mathbb{Z}\}.$$

For functions g for which the bilateral series converges absolutely, we use the Jackson improper integral [14–16]:

$$\int_0^\infty g(t)\, d_q t = (1-q)\sum_{k\in\mathbb{Z}} q^k g(q^k). \tag{3}$$

We split bilateral sums into $k \geq 0$ (the $t \to 0^+$ tail) and $k < 0$ (the $t \to \infty$ tail). For nonnegative quantities A and B, we write $A \lesssim B$ when $A \leq CB$ for a constant independent of the relevant summation or truncation index. The notation $A \asymp B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

Since $q^k \to 0$ as $k \to +\infty$ and $q^k \to \infty$ as $k \to -\infty$,

$$(1+\lambda q^k)^{-\sigma} \to 1 \quad (k \to +\infty), \qquad (1+\lambda q^k)^{-\sigma} \sim \lambda^{-\sigma} q^{-k\sigma} \quad (k \to -\infty).$$

These asymptotics imply the following two-tail criterion.

Lemma 1. *Let $\mu, \sigma \in \mathbb{C}$. Then*

$$\sum_{k\in\mathbb{Z}} q^{k\mu}(1+\lambda q^k)^{-\sigma}$$

converges absolutely if and only if $\Re(\mu) > 0$ and $\Re(\sigma) > \Re(\mu)$. If $\Re(\sigma) \geq \Re(\mu) + \epsilon$ for some $\epsilon > 0$, then

$$\sum_{k=-\infty}^{-K-1} \left| q^{k\mu}(1+\lambda q^k)^{-\sigma} \right| \lesssim q^{(K+1)\epsilon}, \qquad K \to \infty.$$

Proof. Set $a = \Re(\sigma)$ and $b = \Re(\mu)$. For $k \geq 0$,

$$\left| q^{k\mu}(1+\lambda q^k)^{-\sigma} \right| \asymp q^{kb},$$

so the positive tail converges exactly when $b > 0$. For $k = -m$, $m \geq 1$,

$$\left| q^{-m\mu}(1+\lambda q^{-m})^{-\sigma} \right| = \lambda^{-a} q^{m(a-b)} \left(1 + \frac{q^m}{\lambda}\right)^{-a} \asymp q^{m(a-b)},$$

so the negative tail converges exactly when $a > b$. If $a - b \geq \epsilon$, summing the resulting geometric majorant gives the stated tail estimate. □

3. Results

3.1. Transform, Recovery, and Operational Calculus of the Degenerate q-Laplace Transform

We first establish absolute convergence, boundedness, holomorphy, Jackson-grid recovery, and the exact off-grid null space. The operational identities and the limits $\lambda \to 0^+$ and $q \to 1^-$ are derived afterward.

Definition 1. *Let $q \in (0,1)$ and $\lambda > 0$. For a function f such that the Jackson integral exists absolutely, define*

$$\mathcal{L}_q^{(\lambda)}\{f\}(s) := \int_0^\infty f(t)\,(1+\lambda t)^{-s/\lambda}\, d_q t, \qquad s \in \mathbb{C}. \tag{4}$$

By Equation (3), Definition 1 is equivalent to the bilateral series

$$\mathcal{L}_q^{(\lambda)}\{f\}(s) = (1-q)\sum_{k\in\mathbb{Z}} q^k\, f(q^k)\,(1+\lambda q^k)^{-s/\lambda}, \tag{5}$$

whenever the series converges absolutely.

Theorem 1. *Let $q \in (0,1)$, $\lambda > 0$, and $s \in \mathbb{C}$. Assume there exist constants $C_0, C_\infty \geq 0$ and exponents $\rho_0, \rho_\infty \in \mathbb{R}$ such that*

$$|f(t)| \leq C_0\, t^{\rho_0} \quad \textit{for } 0 < t \leq 1, \qquad |f(t)| \leq C_\infty\, t^{\rho_\infty} \quad \textit{for } t \geq 1.$$

Then $\mathcal{L}_q^{(\lambda)}\{f\}(s)$ converges absolutely provided

$$\rho_0 > -1, \qquad \Re(s) > \lambda(\rho_\infty + 1). \tag{6}$$

Proof. Split the bilateral sum (5) into $k \geq 0$ and $k \leq -1$.

For $k \geq 0$, the bases $1 + \lambda q^k$ lie in $[1, 1 + \lambda]$. Hence, for fixed s, the kernel is uniformly bounded by

$$C_s := \max\{1, (1 + \lambda)^{-\Re(s)/\lambda}\}.$$

Together with $|f(q^k)| \leq C_0 q^{k\rho_0}$, this bounds the positive tail by a constant multiple of $\sum_{k\geq 0} q^{k(\rho_0+1)}$, which converges if and only if $\rho_0 > -1$.

For $k = -m \leq -1$, factor the kernel as

$$(1 + \lambda q^{-m})^{-\Re(s)/\lambda} = \lambda^{-\Re(s)/\lambda} q^{m\Re(s)/\lambda} \left(1 + \frac{q^m}{\lambda}\right)^{-\Re(s)/\lambda}.$$

The last factor is uniformly bounded for $m \geq 1$ when s and λ are fixed. Combining this with $|f(q^{-m})| \leq C_\infty q^{-m\rho_\infty}$ bounds the negative tail by a constant multiple of $\sum_{m\geq 1} q^{m(\Re(s)/\lambda - (\rho_\infty+1))}$. This geometric series converges if and only if $\Re(s) > \lambda(\rho_\infty + 1)$. □

Corollary 1. *Let $\mu, a \in \mathbb{C}$ and set $f(t) = t^{\mu-1}(1 + \lambda t)^{-a/\lambda}$. Then $\mathcal{L}_q^{(\lambda)}\{f\}(s)$ converges absolutely if*

$$\Re(\mu) > 0, \qquad \Re(s) > \lambda\Re(\mu) - \Re(a),$$

and in this ROC

$$\mathcal{L}_q^{(\lambda)}\{f\}(s) = (1 - q) \sum_{k\in\mathbb{Z}} q^{k\mu} (1 + \lambda q^k)^{-(s+a)/\lambda}. \tag{7}$$

Proof. Substituting $f(t) = t^{\mu-1}(1 + \lambda t)^{-a/\lambda}$ into (5) gives (7). Applying Lemma 1 with $\sigma = (s + a)/\lambda$ gives $\Re(\mu) > 0$ and $\Re(s + a) > \lambda\Re(\mu)$, which are the stated convergence conditions. □

Definition 2. *We use the following notation for the transform of the constant function* 1*:*

$$A_{q,\lambda}(s) := \mathcal{L}_q^{(\lambda)}\{1\}(s) = (1 - q) \sum_{k\in\mathbb{Z}} q^k (1 + \lambda q^k)^{-s/\lambda}.$$

By Corollary 1 with $\mu = 1$ and $a = 0$, this bilateral series converges absolutely for $\Re(s) > \lambda$.

The Jackson-series representation (5) gives the following mapping properties of the degenerate q-Laplace transform on the q-grid.

Definition 3. *Fix $\sigma \in \mathbb{R}$. Let $\widetilde{\mathcal{X}}_{q,\lambda}(\sigma)$ denote the class of functions $f : (0, \infty) \to \mathbb{C}$ for which*

$$|f|_{\widetilde{\mathcal{X}}_{q,\lambda}(\sigma)} := (1 - q) \sum_{k\in\mathbb{Z}} q^k \, |f(q^k)| \, (1 + \lambda q^k)^{-\sigma/\lambda} < \infty. \tag{8}$$

Since the Jackson integral samples only $\mathbb{T}_q$, the quantity in (8) is generally only a seminorm on functions on $(0, \infty)$. We therefore set

$$f \sim_q g \quad \Longleftrightarrow \quad f(q^k) = g(q^k) \text{ for every } k \in \mathbb{Z},$$

and take the weighted Jackson data space to be

$$\mathcal{X}_{q,\lambda}(\sigma) := \widetilde{\mathcal{X}}_{q,\lambda}(\sigma)/\sim_q.$$

Equivalently, if

$$\mathcal{N}_q := \{h : (0, \infty) \to \mathbb{C} : \ h(q^k) = 0 \text{ for every } k \in \mathbb{Z}\},$$

then $\mathcal{X}_{q,\lambda}(\sigma)$ is the quotient by the null space $\mathcal{N}_q$. It is equipped with the norm

$$\|[f]_q\|_{\mathcal{X}_{q,\lambda}(\sigma)} := (1 - q) \sum_{k\in\mathbb{Z}} q^k \, |f(q^k)| \, (1 + \lambda q^k)^{-\sigma/\lambda}. \tag{9}$$

The norm is well defined because equivalent representatives have the same Jackson-grid samples. In what

follows, we write $f \in \mathcal{X}_{q,\lambda}(\sigma)$ for an equivalence class and use $f(q^k)$ for the corresponding grid values.

Proposition 1. *If $f \in \mathcal{X}_{q,\lambda}(\sigma)$, then $\mathcal{L}_q^{(\lambda)}\{f\}(s)$ exists absolutely for every s with $\Re(s) \geq \sigma$ and satisfies the uniform bound*

$$\left|\mathcal{L}_q^{(\lambda)}\{f\}(s)\right| \leq \|f\|_{\mathcal{X}_{q,\lambda}(\sigma)} \qquad (\Re(s) \geq \sigma). \tag{10}$$

The mapping $s \mapsto \mathcal{L}_q^{(\lambda)}\{f\}(s)$ is holomorphic on the open half-plane $\{s \in \mathbb{C} : \Re(s) > \sigma\}$.

Proof. Fix $s \in \mathbb{C}$ with $\Re(s) \geq \sigma$. For each $k \in \mathbb{Z}$ we have $(1 + \lambda q^k) > 0$ and hence

$$\left|(1+\lambda q^k)^{-s/\lambda}\right| = (1+\lambda q^k)^{-\Re(s)/\lambda} \leq (1+\lambda q^k)^{-\sigma/\lambda}.$$

Substituting this bound into (5) gives

$$\sum_{k\in\mathbb{Z}} (1-q)q^k \left|f(q^k)\right| \left|(1+\lambda q^k)^{-s/\lambda}\right| \leq (1-q)\sum_{k\in\mathbb{Z}} q^k \left|f(q^k)\right| (1+\lambda q^k)^{-\sigma/\lambda} = \|f\|_{\mathcal{X}_{q,\lambda}(\sigma)}.$$

Thus (5) converges absolutely for $\Re(s) \geq \sigma$, and the triangle inequality yields (10).

For holomorphy, let $K \Subset \{s : \Re(s) > \sigma\}$ be compact. Choose $\sigma_K > \sigma$ so that $\Re(s) \geq \sigma_K$ for all $s \in K$. For each fixed k, the function $s \mapsto (1+\lambda q^k)^{-s/\lambda}$ is entire, and for all $s \in K$ we have

$$\left|(1+\lambda q^k)^{-s/\lambda}\right| \leq (1+\lambda q^k)^{-\sigma_K/\lambda} \leq (1+\lambda q^k)^{-\sigma/\lambda}.$$

Hence the summands in (5) are uniformly dominated on K by the summable sequence

$$(1-q)q^k \left|f(q^k)\right| (1+\lambda q^k)^{-\sigma/\lambda}.$$

The Weierstrass M-test now gives uniform convergence of (5) on K. Since K was arbitrary, the series converges locally uniformly on $\{s : \Re(s) > \sigma\}$; its sum is therefore holomorphic on this half-plane. □

Classical Laplace inversion depends on the geometry of the underlying domain [1]. For discrete and time-scale transforms, the inversion procedure must therefore reflect the support [39]. In the present setting, the logarithmic kernel converts a vertical line in the s-plane into a Fourier family on the nonuniform nodes

$$x_k := \frac{1}{\lambda}\log(1+\lambda q^k), \qquad k \in \mathbb{Z}.$$

The resulting Fourier representation yields an inversion formula for the Jackson-grid data.

Theorem 2. *Let $f \in \mathcal{X}_{q,\lambda}(\sigma)$, let*

$$F(s) := \mathcal{L}_q^{(\lambda)}\{f\}(s),$$

and choose any real $c > \sigma$. Then for every $k \in \mathbb{Z}$,

$$f(q^k) = \frac{(1+\lambda q^k)^{c/\lambda}}{(1-q)q^k} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F(c+i\tau)e^{i\tau x_k}\, d\tau. \tag{11}$$

In particular, the transform determines the restriction $f|_{\mathbb{T}_q}$ uniquely on $\mathcal{X}_{q,\lambda}(\sigma)$.

Proof. For $j \in \mathbb{Z}$, set

$$a_j := (1-q)q^j f(q^j)(1+\lambda q^j)^{-c/\lambda} = (1-q)q^j f(q^j)e^{-cx_j}.$$

Since $c > \sigma$ and $1 + \lambda q^j > 1$,

$$\sum_{j\in\mathbb{Z}} |a_j| \leq (1-q)\sum_{j\in\mathbb{Z}} q^j |f(q^j)|(1+\lambda q^j)^{-\sigma/\lambda} = \|f\|_{\mathcal{X}_{q,\lambda}(\sigma)} < \infty.$$

Thus, on the vertical line $\Re(s) = c$, the Jackson series is the uniformly and absolutely convergent Fourier-type series

$$F(c+i\tau) = \sum_{j\in\mathbb{Z}} a_j e^{-i\tau x_j}, \qquad \tau \in \mathbb{R}. \tag{12}$$

Uniform absolute convergence permits termwise integration on finite intervals. For fixed k, this gives

$$\begin{aligned} M_{T,k} &:= \frac{1}{2T}\int_{-T}^{T} F(c+i\tau)e^{i\tau x_k}\,d\tau \\ &= \sum_{j\in\mathbb{Z}} a_j \left(\frac{1}{2T}\int_{-T}^{T} e^{-i\tau(x_j-x_k)}\,d\tau\right) \\ &= \sum_{j\in\mathbb{Z}} a_j\, \Phi_T(x_j-x_k), \end{aligned}$$

where

$$\Phi_T(y) := \begin{cases} 1, & y = 0, \\ \dfrac{\sin(Ty)}{Ty}, & y \neq 0. \end{cases}$$

The nodes x_j are pairwise distinct because the values q^j are pairwise distinct and $t \mapsto \lambda^{-1}\log(1+\lambda t)$ is strictly increasing. Thus, for fixed k,

$$\Phi_T(x_j - x_k) \longrightarrow \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \qquad T \to \infty.$$

Because $|\Phi_T(y)| \leq 1$ for all $y \in \mathbb{R}$ and $\sum_j |a_j| < \infty$, dominated convergence for the absolutely summable series [40] yields

$$\lim_{T\to\infty} M_{T,k} = a_k = (1-q)q^k f(q^k)(1+\lambda q^k)^{-c/\lambda}.$$

Solving for $f(q^k)$ yields (11). □

Corollary 2. *Under the hypotheses of Theorem 2, define*

$$\delta_k := \min\{x_{k-1} - x_k,\, x_k - x_{k+1}\} > 0$$

and let $M_{T,k}$ denote the finite-T mean in (11). Then

$$\left| f(q^k) - \frac{(1+\lambda q^k)^{c/\lambda}}{(1-q)q^k} M_{T,k} \right| \leq \frac{(1+\lambda q^k)^{c/\lambda}}{(1-q)q^k} \frac{\|f\|_{\mathcal{X}_{q,\lambda}(\sigma)}}{T\delta_k}. \tag{13}$$

Proof. With the notation of the proof of Theorem 2,

$$M_{T,k} - a_k = \sum_{j\neq k} a_j \frac{\sin(T(x_j - x_k))}{T(x_j - x_k)}.$$

The monotonicity of x_j implies $|x_j - x_k| \geq \delta_k$ for $j \neq k$, while $|\sin y| \leq 1$. Hence

$$|M_{T,k} - a_k| \leq \frac{1}{T\delta_k} \sum_{j\neq k} |a_j| \leq \frac{\|f\|_{\mathcal{X}_{q,\lambda}(\sigma)}}{T\delta_k},$$

and multiplication by the recovery prefactor gives (13). □

Remark 1. *Theorem 2 uses Fourier coefficient extraction on the discrete support rather than Bromwich inversion. On $\Re(s) = c$, the transform is the Fourier–Stieltjes transform of the finite pure-point measure*

$$\omega_c := \sum_{j\in\mathbb{Z}} a_j \delta_{x_j}, \qquad \sum_j |a_j| < \infty,$$

and the standard orthogonality average for distinct frequencies [38] shows that the mean in (11) extracts the atom $\omega_c(\{x_k\})$. *Thus the frequencies* $x_k = \lambda^{-1}\log(1+\lambda q^k)$ *and the masses* $a_k = (1-q)q^k f(q^k)e^{-cx_k}$ *determine the recovered Jackson data.*

Corollary 3. *Let* $f, g \in \mathcal{X}_{q,\lambda}(\sigma)$. *If for some* $c > \sigma$,

$$\mathcal{L}_q^{(\lambda)}\{f\}(c+i\tau) = \mathcal{L}_q^{(\lambda)}\{g\}(c+i\tau) \qquad (\tau \in \mathbb{R}),$$

then $f(q^k) = g(q^k)$ *for every* $k \in \mathbb{Z}$, *hence* $f = g$ *in the quotient space* $\mathcal{X}_{q,\lambda}(\sigma)$. *In particular, equality of the transforms on any right half-plane contained in their common region of convergence implies equality of the corresponding Jackson-grid data.*

Proof. Apply Theorem 2 to $f - g$. Since the transform of $f - g$ vanishes on $\Re(s) = c$, formula (11) gives $(f-g)(q^k) = 0$ for every $k \in \mathbb{Z}$. Definition 3 then gives equality in $\mathcal{X}_{q,\lambda}(\sigma)$. □

Proposition 2. *Fix* $\sigma \in \mathbb{R}$ *and* $c > \sigma$. *On the raw function class* $\widetilde{\mathcal{X}}_{q,\lambda}(\sigma)$, *define*

$$\mathcal{T}_c f(\tau) := \mathcal{L}_q^{(\lambda)}\{f\}(c+i\tau), \qquad \tau \in \mathbb{R}.$$

Then

$$\ker(\mathcal{T}_c) = \mathcal{N}_q = \{h : \ h(q^k) = 0 \text{ for every } k \in \mathbb{Z}\}. \tag{14}$$

Hence $\mathcal{L}_q^{(\lambda)}$ *is not injective on ordinary functions on* $(0,\infty)$, *even if one restricts to smooth compactly supported functions, whereas the induced transform on the quotient space* $\mathcal{X}_{q,\lambda}(\sigma)$ *is injective.*

Proof. Let $h \in \mathcal{N}_q$. By definition, every summand in the bilateral representation (5) vanishes. Hence

$$\mathcal{T}_c h(\tau) = 0 \qquad (\tau \in \mathbb{R}).$$

Thus $\mathcal{N}_q \subseteq \ker(\mathcal{T}_c)$.

Conversely, let $h \in \widetilde{\mathcal{X}}_{q,\lambda}(\sigma)$ satisfy $\mathcal{T}_c h(\tau) = 0$ for all $\tau \in \mathbb{R}$. The equivalence class $[h]_q$ lies in $\mathcal{X}_{q,\lambda}(\sigma)$, so applying Theorem 2 to $[h]_q$ gives

$$h(q^k) = \frac{(1+\lambda q^k)^{c/\lambda}}{(1-q)q^k} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathcal{T}_c h(\tau) e^{i\tau x_k}\, d\tau = 0$$

for every $k \in \mathbb{Z}$. Hence $h \in \mathcal{N}_q$, proving (14).

Strong classical regularity does not remove this null space. Fix $m \in \mathbb{Z}$ and consider the standard bump function

$$h_m(t) := \begin{cases} \exp\left[-\dfrac{1}{(t-q^{m+1})(q^m - t)}\right], & q^{m+1} < t < q^m, \\ 0, & \text{otherwise.} \end{cases}$$

Because q^{m+1} and q^m are consecutive points of $\mathbb{T}_q$, the interval (q^{m+1}, q^m) contains no Jackson-grid node. Consequently, h_m is a nonzero element of $C_c^\infty(0,\infty)$ satisfying $h_m(q^k) = 0$ for every k. It follows that $h_m \in \mathcal{N}_q \setminus \{0\}$ and that its transform vanishes identically. Thus off-grid injectivity fails; Corollary 3 gives injectivity only after quotienting by this null space. □

Corollary 4. *Let* $f, g \in \widetilde{\mathcal{X}}_{q,\lambda}(\sigma)$, *and suppose that both admit holomorphic extensions to a connected open set* $U \subset \mathbb{C}$ *containing* $[0,\infty)$. *If for some* $c > \sigma$,

$$\mathcal{L}_q^{(\lambda)}\{f\}(c+i\tau) = \mathcal{L}_q^{(\lambda)}\{g\}(c+i\tau) \qquad (\tau \in \mathbb{R}),$$

then $f(t) = g(t)$ *for every* $t > 0$.

Proof. By Proposition 2, $f(q^k) = g(q^k)$ for every $k \in \mathbb{Z}$. Let H denote the difference of the holomorphic extensions

of f and g. Then $H(q^k) = 0$ for every k, and $q^k \to 0$ as $k \to +\infty$. Since $0 \in U$, the zeros of H accumulate inside the domain of holomorphy. The identity theorem [40] gives $H \equiv 0$ on the connected set U, and restriction to $(0, \infty)$ yields $f = g$. □

Remark 2. *The preceding result distinguishes grid uniqueness from off-grid uniqueness. Classical Laplace uniqueness is formulated on the underlying continuous function class, up to the usual almost-everywhere qualification [1]. On noncontinuous time scales, by contrast, the support geometry enters the uniqueness statement [39]. Since Proposition 2 identifies the null space as $\mathcal{N}_q$, no additional Jackson-grid information is lost.*

A transform-domain equation determines the unknown in $\mathcal{X}_{q,\lambda}(\sigma)$, and therefore fixes every value $f(q^k)$, but it does not specify an extension between consecutive nodes. Values at arbitrary $t > 0$, whether imposed as point conditions or interpreted as a continuous trajectory, require an additional interpolation or regularity model. Smoothness alone does not suffice: Proposition 2 provides compactly supported functions h_m in the null space. Even real-analyticity on the open half-line is insufficient. For example, the nonzero function

$$h(t) = \sin\left(\pi \frac{\log t}{\log q}\right), \qquad t > 0,$$

satisfies $h(q^k) = 0$ for every $k \in \mathbb{Z}$. Corollary 4 establishes full uniqueness under the sufficient condition of holomorphic continuation through $t = 0$, which couples the grid to an interior accumulation point.

The deformed-semigroup convolution is measure-valued because combining q-grid atoms through $t \oplus_\lambda u = t + u + \lambda t u$ generally produces support outside the Jackson grid. Thus the transform cannot select a unique ordinary off-grid representative. The obstruction concerns reconstruction on $(0, \infty)$ rather than injectivity on the Jackson-data quotient: formula (11) recovers exactly the data sampled by the Jackson integral.

Linearity follows from absolute convergence of the Jackson series. For the following shift identities, each transform on the right-hand side is understood as an absolutely convergent Jackson series at its displayed argument; holomorphy claims apply only on domains where the corresponding weighted convergence hypotheses hold.

Theorem 3. *Let $a \in \mathbb{C}$. If $\mathcal{L}_q^{(\lambda)}\{f\}(s + a)$ exists, then*

$$\mathcal{L}_q^{(\lambda)}\{(1 + \lambda t)^{-a/\lambda} f(t)\}(s) = \mathcal{L}_q^{(\lambda)}\{f\}(s + a). \tag{15}$$

Proof. We use the kernel identity

$$(1 + \lambda t)^{-a/\lambda}(1 + \lambda t)^{-s/\lambda} = (1 + \lambda t)^{-(s+a)/\lambda}.$$

Multiplication by $f(t)$ followed by Jackson integration gives (15). □

Theorem 4. *Let $n \in \mathbb{N}_0$. Assume that $\mathcal{L}_q^{(\lambda)}\{f\}(s - j\lambda)$ exists for all $j = 0, 1, \ldots, n$. Then*

$$\mathcal{L}_q^{(\lambda)}\{t^n f(t)\}(s) = \lambda^{-n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \mathcal{L}_q^{(\lambda)}\{f\}(s - j\lambda). \tag{16}$$

Proof. Use the binomial identity

$$t^n = \lambda^{-n}\big((1 + \lambda t) - 1\big)^n = \lambda^{-n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (1 + \lambda t)^j.$$

Multiplying by $(1 + \lambda t)^{-s/\lambda}$ yields

$$\begin{aligned} t^n (1 + \lambda t)^{-s/\lambda} &= \lambda^{-n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (1 + \lambda t)^{j - s/\lambda} \\ &= \lambda^{-n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (1 + \lambda t)^{-(s - j\lambda)/\lambda}. \end{aligned}$$

By hypothesis, each term is Jackson-integrable against $f(t)$. Integrating the displayed identity term by term and using linearity gives (16). □

Corollary 5. *If $\mathcal{L}_q^{(\lambda)}\{f\}(s)$ and $\mathcal{L}_q^{(\lambda)}\{f\}(s-\lambda)$ exist, then*

$$\mathcal{L}_q^{(\lambda)}\{tf(t)\}(s) = \frac{1}{\lambda}\Big(\mathcal{L}_q^{(\lambda)}\{f\}(s-\lambda) - \mathcal{L}_q^{(\lambda)}\{f\}(s)\Big). \tag{17}$$

Proof. Apply Theorem 4 with $n = 1$. □

Definition 4. *Let $\sigma \in \mathbb{R}$ and $n \in \mathbb{N}_0$. For $f \in \mathcal{X}_{q,\lambda}(\sigma)$, we write $f \in \mathcal{X}_{q,\lambda}^{\log,n}(\sigma)$ if*

$$(1-q)\sum_{k\in\mathbb{Z}} q^k\,|f(q^k)|\,(1+\lambda q^k)^{-\sigma/\lambda}\,\big(1+|\log(1+\lambda q^k)|\big)^n < \infty. \tag{18}$$

Theorem 5. *Let $n \in \mathbb{N}_0$ and assume $f \in \mathcal{X}_{q,\lambda}^{\log,n}(\sigma)$ for some $\sigma \in \mathbb{R}$. Then $\mathcal{L}_q^{(\lambda)}\{f\}$ is n times complex differentiable on $\{s : \Re(s) > \sigma\}$ and*

$$\frac{d^n}{ds^n}\mathcal{L}_q^{(\lambda)}\{f\}(s) = (-1)^n\lambda^{-n}\,\mathcal{L}_q^{(\lambda)}\{\big(\log(1+\lambda t)\big)^n f(t)\}(s), \qquad \Re(s) > \sigma. \tag{19}$$

Proof. Write (5) as

$$\mathcal{L}_q^{(\lambda)}\{f\}(s) = (1-q)\sum_{k\in\mathbb{Z}} a_k\,\exp\Big(-\frac{s}{\lambda}\log(1+\lambda q^k)\Big), \qquad a_k := q^k f(q^k).$$

For each fixed k, the summand is entire in s and satisfies

$$\frac{d^n}{ds^n}(1+\lambda q^k)^{-s/\lambda} = (-1)^n\lambda^{-n}\big(\log(1+\lambda q^k)\big)^n(1+\lambda q^k)^{-s/\lambda}.$$

Fix a compact set $K \Subset \{s : \Re(s) > \sigma\}$ and choose $\sigma_K > \sigma$ with $\Re(s) \geq \sigma_K$ on K.
For $s \in K$,

$$\big|(1+\lambda q^k)^{-s/\lambda}\big| \leq (1+\lambda q^k)^{-\sigma_K/\lambda} \leq (1+\lambda q^k)^{-\sigma/\lambda}.$$

For $j = 0, 1, \ldots, n$, the absolute value of the jth derivative summand is bounded by

$$(1-q)\,|a_k|\,\max\{1,\lambda^{-n}\}\big(1+|\log(1+\lambda q^k)|\big)^n(1+\lambda q^k)^{-\sigma/\lambda},$$

and the sum of these bounds is finite by (18). The Weierstrass M-test gives uniform convergence of the derivative series through order n on K. Hence termwise differentiation is valid through order n and yields (19). □

Table 1 summarizes linearity together with Theorems 3–5 and Corollary 5.

Table 1. Operational identities for $\mathcal{L}_q^{(\lambda)}$ under the stated absolute-convergence assumptions.

Time-Domain Operation	Transform-Domain Identity
$af + bg$	$\mathcal{L}_q^{(\lambda)}\{af+bg\}(s) = a\mathcal{L}_q^{(\lambda)}\{f\}(s) + b\mathcal{L}_q^{(\lambda)}\{g\}(s)$
$(1+\lambda t)^{-a/\lambda}f(t)$	$\mathcal{L}_q^{(\lambda)}\{(1+\lambda t)^{-a/\lambda}f\}(s) = \mathcal{L}_q^{(\lambda)}\{f\}(s+a)$
$t^n f(t)$	$\mathcal{L}_q^{(\lambda)}\{t^n f\}(s) = \lambda^{-n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\mathcal{L}_q^{(\lambda)}\{f\}(s-j\lambda)$
$(\log(1+\lambda t))^n f(t)$	$\mathcal{L}_q^{(\lambda)}\{(\log(1+\lambda t))^n f\}(s) = (-\lambda)^n\frac{d^n}{ds^n}\mathcal{L}_q^{(\lambda)}\{f\}(s)$

Each row is understood on the common domain where the corresponding Jackson series are absolutely convergent.

The kernel $(1+\lambda t)^{-s/\lambda}$ tends to the exponential kernel as $\lambda \to 0^+$ [2,3]. Under suitable hypotheses, Jackson integration on the q-grid $\mathbb{T}_q$ converges to ordinary integration as $q \to 1^-$ [14–16]. We now state the assumptions used for these two limits.

Lemma 2. *The function $x \mapsto \log(1+x)/x$ is strictly decreasing on $(0,\infty)$. Hence, for every $t \geq 0$ and every s with $\Re(s) > 0$, the map $\lambda \mapsto (1+\lambda t)^{-\Re(s)/\lambda}$ is increasing on $(0,\infty)$ and satisfies*

$$e^{-\Re(s)t} \leq (1+\lambda t)^{-\Re(s)/\lambda} \leq 1 \qquad (\lambda > 0).$$

Proof. For $x > 0$,

$$\frac{d}{dx}\left(\frac{\log(1+x)}{x}\right) = \frac{x/(1+x) - \log(1+x)}{x^2} < 0,$$

because $\log(1+x) > x/(1+x)$. For $t > 0$,

$$(1+\lambda t)^{-\Re(s)/\lambda} = \exp\left[-\Re(s)t\,\frac{\log(1+\lambda t)}{\lambda t}\right],$$

and the first assertion gives the required monotonicity. The bound $0 < \log(1+x)/x \leq 1$ gives the displayed two-sided estimate. The case $t = 0$ is immediate. □

Proposition 3. *Fix $q \in (0,1)$ and define the exponential-kernel Jackson transform*

$$\mathcal{L}_q^{(0)}\{f\}(s) := \int_0^\infty f(t)e^{-st}\,d_qt = (1-q)\sum_{k\in\mathbb{Z}} q^k f(q^k)e^{-sq^k}.$$

This ordinary-exponential kernel differs from the q-exponential and related kernels used in established q-Laplace theories [18–20, 23, 33]. Assume that for some $\lambda_0 > 0$ and some s with $\Re(s) > 0$ the series defining $\mathcal{L}_q^{(\lambda_0)}\{f\}(s)$ converges absolutely. Then $\mathcal{L}_q^{(\lambda)}\{f\}(s)$ converges absolutely for every $\lambda \in (0,\lambda_0]$ and

$$\lim_{\lambda\to 0^+} \mathcal{L}_q^{(\lambda)}\{f\}(s) = \mathcal{L}_q^{(0)}\{f\}(s).$$

Proof. By Lemma 2, for each $k \in \mathbb{Z}$ and each $\lambda \in (0,\lambda_0]$ one has

$$\left|(1+\lambda q^k)^{-s/\lambda}\right| \leq (1+\lambda_0 q^k)^{-\Re(s)/\lambda_0}.$$

This estimate dominates the summand of $\mathcal{L}_q^{(\lambda)}\{f\}(s)$ by the corresponding absolutely summable term at λ_0, so dominated convergence applies to the bilateral series. The required pointwise limit

$$\lim_{\lambda\to 0^+}(1+\lambda q^k)^{-s/\lambda} = e^{-sq^k}$$

follows from $\log(1+\lambda q^k) \sim \lambda q^k$ as $\lambda \downarrow 0$. □

Remark 3. *Proposition 3 can be combined with the finite-difference identity in Theorem 4 to obtain the s-differentiation limit under an additional derivative-convergence hypothesis. Set $F_\lambda(s) = \mathcal{L}_q^{(\lambda)}\{f\}(s)$. Assume, in addition to the hypotheses ensuring the transform limit, that F_λ' converges locally uniformly to F_0' in a neighborhood of the point s. Then*

$$\frac{F_\lambda(s-\lambda) - F_\lambda(s)}{\lambda} = -\int_0^1 F_\lambda'(s-\theta\lambda)\,d\theta \longrightarrow -F_0'(s),$$

so the identity $\mathcal{L}_q^{(0)}\{tf(t)\}(s) = -\frac{d}{ds}\mathcal{L}_q^{(0)}\{f\}(s)$ is recovered under this derivative-convergence hypothesis.

Lemma 3. *Let $0 < a < b < \infty$ and let $h \in C([a,b])$. For each $q \in (0,1)$ sufficiently close to 1, choose integers $m_q \leq n_q$ so that*

$$q^{m_q} \leq b < q^{m_q-1}, \qquad q^{n_q+1} < a \leq q^{n_q}.$$

Then

$$S_q[h;a,b] := (1-q)\sum_{k=m_q}^{n_q} q^k h(q^k)$$

is well defined, and

$$\lim_{q\to 1^-} S_q[h;a,b] = \int_a^b h(t)\,dt.$$

The final Jackson cell extends below the left endpoint, but its weighted contribution vanishes as the mesh tends to zero. No values of the function outside the stated interval are required.

Proof. To avoid evaluating the function below its domain, separate the last cell from the cells entirely contained in the interval. Write

$$S_q[h;a,b] = \sum_{k=m_q}^{n_q-1} (q^k - q^{k+1})h(q^k) + (1-q)q^{n_q}h(q^{n_q}).$$

The first sum is a right-endpoint Riemann sum on the interior interval $[q^{n_q}, q^{m_q}] \subseteq [a,b]$. Its mesh satisfies

$$\max_{m_q \le k < n_q} (q^k - q^{k+1}) \le (1-q)b \longrightarrow 0.$$

Since h is uniformly continuous on $[a,b]$, this sum differs from $\int_{q^{n_q}}^{q^{m_q}} h(t)\,dt$ by a quantity tending to zero.

The contributions from both mismatched endpoints can be estimated explicitly using only values inside the interval. Set $M = \|h\|_{C([a,b])}$. The grid inequalities imply

$$0 \le q^{n_q} - a < (1-q)q^{n_q}, \qquad 0 \le b - q^{m_q} < (q^{-1}-1)q^{m_q},$$

and hence

$$\begin{aligned}\left|(1-q)q^{n_q}h(q^{n_q}) - \int_a^{q^{n_q}} h(t)\,dt - \int_{q^{m_q}}^b h(t)\,dt\right| &\le M\big((1-q)q^{n_q} + q^{n_q} - a + b - q^{m_q}\big)\\ &\le M\big(2(1-q)b + (q^{-1}-1)b\big) \longrightarrow 0.\end{aligned}$$

Combining the interior Riemann-sum limit with these endpoint bounds gives the assertion. In particular, the proof does not invoke continuity or an extension of the function below the left endpoint. □

Proposition 4. *Fix $\lambda > 0$ and $s \in \mathbb{C}$ with $\Re(s) > 0$, and set*

$$g(t) := f(t)(1+\lambda t)^{-s/\lambda}.$$

Assume that $g \in C([0,\infty))$ and that there exist $A > 0$ and a nonincreasing function $H : [A,\infty) \to [0,\infty)$ such that $|g(t)| \le H(t)$ for all $t \ge A$ and

$$\int_A^\infty H(t)\,dt < \infty.$$

Then the Jackson integrals $\int_0^\infty g(t)\,d_qt$ exist absolutely for all q sufficiently close to 1, and

$$\lim_{q\to 1^-} \mathcal{L}_q^{(\lambda)}\{f\}(s) = \lim_{q\to 1^-} \int_0^\infty g(t)\,d_qt = \int_0^\infty g(t)\,dt = \int_0^\infty f(t)(1+\lambda t)^{-s/\lambda}\,dt.$$

Proof. Split both the improper integral and the Jackson sum into three regions: a neighborhood of 0, a compact middle interval, and the tail at ∞.

Fix $\varepsilon > 0$. Continuity of g at 0 gives $\delta \in (0,1)$ and $M_0 > 0$ such that $|g(t)| \le M_0$ on $[0,\delta]$. After decreasing δ if necessary, assume

$$\int_0^\delta |g(t)|\,dt < \varepsilon, \qquad M_0\delta < \varepsilon.$$

For every $q \in (0,1)$,

$$(1-q)\sum_{q^k<\delta} q^k|g(q^k)| \le M_0(1-q)\sum_{q^k\le\delta} q^k \le M_0\delta < \varepsilon.$$

Hence the ordinary and Jackson contributions from the near-zero region are both uniformly small.

Next choose $B \geq \max\{A, 2\}$ so that

$$\int_{B/2}^{\infty} H(t)\,dt < \varepsilon.$$

If $q \in (1/2, 1)$ and k satisfies $q^k > B$, monotonicity of H gives

$$(q^k - q^{k+1})H(q^k) \leq \int_{q^{k+1}}^{q^k} H(t)\,dt.$$

Hence

$$\begin{aligned}(1-q)\sum_{q^k>B} q^k|g(q^k)| &\leq (1-q)\sum_{q^k>B} q^k H(q^k)\\ &= \sum_{q^k>B}(q^k - q^{k+1})H(q^k)\\ &\leq \int_{Bq}^{\infty} H(t)\,dt\\ &\leq \int_{B/2}^{\infty} H(t)\,dt < \varepsilon.\end{aligned}$$

The same majorant gives

$$\int_{B}^{\infty} |g(t)|\,dt \leq \int_{B}^{\infty} H(t)\,dt < \varepsilon.$$

Hence both tail contributions beyond B are uniformly small for q near 1.

On the middle interval, the boundary cells are controlled by the endpoint estimates in the preceding compact-interval lemma; the function is evaluated only at nodes in the interval. It remains to consider $[\delta, B]$. Applying Lemma 3 to the continuous function g gives

$$\lim_{q\to 1^-}(1-q)\sum_{\delta\leq q^k\leq B} q^k g(q^k) = \int_{\delta}^{B} g(t)\,dt.$$

Combining the estimates from the three regions gives, for q sufficiently close to 1,

$$\left|\int_0^{\infty} g(t)\,d_q t - \int_0^{\infty} g(t)\,dt\right| \leq \left|(1-q)\sum_{\delta\leq q^k\leq B} q^k g(q^k) - \int_{\delta}^{B} g(t)\,dt\right| + 4\varepsilon.$$

Letting $q \to 1^-$, the limit superior is at most 4ε. Since $\varepsilon > 0$ was arbitrary, the stated limit follows.

The same estimates give absolute convergence of the Jackson integral for all q sufficiently close to 1: the near-zero and large-t contributions are finite, and the middle sum has finitely many terms. □

Corollary 6. *Fix $s \in \mathbb{C}$ with $\Re(s) > 0$ and $\lambda_0 > 0$. For each $\lambda \in (0, \lambda_0]$, set*

$$g_\lambda(t) := f(t)(1+\lambda t)^{-s/\lambda}.$$

Assume that each g_λ satisfies the hypotheses of Proposition 4 and that

$$\int_0^{\infty} |f(t)|(1+\lambda_0 t)^{-\Re(s)/\lambda_0}\,dt < \infty.$$

Then the successive limits give the classical Laplace transform:

$$\lim_{\lambda\to 0^+}\lim_{q\to 1^-} \mathcal{L}_q^{(\lambda)}\{f\}(s) = \int_0^{\infty} f(t)e^{-st}\,dt.$$

Proof. Fix $\lambda \in (0, \lambda_0]$. Proposition 4 gives

$$\lim_{q\to 1^-} \mathcal{L}_q^{(\lambda)}\{f\}(s) = \int_0^\infty f(t)(1+\lambda t)^{-s/\lambda}\,dt.$$

By Lemma 2, for $0 < \lambda \le \lambda_0$,

$$\left|f(t)(1+\lambda t)^{-s/\lambda}\right| \le |f(t)|(1+\lambda_0 t)^{-\Re(s)/\lambda_0},$$

and the majorant on the right is integrable by hypothesis. Since $(1+\lambda t)^{-s/\lambda} \to e^{-st}$ pointwise as $\lambda \to 0^+$, dominated convergence yields the outer limit. □

3.2. Deformed-Semigroup Measure Algebra and Product Theorem

In the original t-variable, define the composition law

$$t \oplus_\lambda u := t + u + \lambda t u, \qquad t, u \ge 0. \tag{20}$$

Then

$$1 + \lambda(t \oplus_\lambda u) = (1+\lambda t)(1+\lambda u), \tag{21}$$

and hence

$$K_s^{(\lambda)}(t \oplus_\lambda u) = K_s^{(\lambda)}(t) K_s^{(\lambda)}(u), \qquad K_s^{(\lambda)}(t) = (1+\lambda t)^{-s/\lambda}. \tag{22}$$

The coordinate map turns deformed addition into ordinary addition, so pushforward by this map converts the associated measure convolution into classical additive convolution. Accordingly, the product rule is formulated on the commutative semigroup $([0,\infty), \oplus_\lambda)$, without requiring the Jackson grid to be closed under this operation.

Lemma 4. *The map*

$$\phi_\lambda(t) := \frac{1}{\lambda}\log(1+\lambda t), \qquad t \ge 0,$$

is a continuous strictly increasing semigroup isomorphism from $([0,\infty), \oplus_\lambda)$ *onto* $([0,\infty), +)$. *Its inverse is*

$$\psi_\lambda(x) = \frac{e^{\lambda x} - 1}{\lambda}.$$

Hence $\oplus_\lambda$ *is associative and commutative with identity* 0, *and* $t \oplus_\lambda u \to t + u$ *for each* $t, u \ge 0$ *as* $\lambda \to 0^+$.

Proof. Taking logarithms in (21) gives

$$\phi_\lambda(t \oplus_\lambda u) = \lambda^{-1}\log\big((1+\lambda t)(1+\lambda u)\big) = \phi_\lambda(t) + \phi_\lambda(u).$$

The explicit inverse proves bijectivity and continuity; associativity, commutativity, the identity, and the pointwise limit follow from $t \oplus_\lambda u = t + u + \lambda t u$. □

Let $\mathcal{M}([0,\infty))$ be the Banach space of finite complex Borel measures with total-variation norm [40]. For Jackson data define the atomic measure in the original variable by

$$\eta_f := (1-q)\sum_{k\in\mathbb{Z}} q^k f(q^k)\,\delta_{q^k}. \tag{23}$$

The condition $\eta_f \in \mathcal{M}([0,\infty))$ is equivalent to absolute summability of the unweighted Jackson masses.

Proposition 5. *Let* $f \in \mathcal{X}_{q,\lambda}(\sigma)$. *If* $\sigma \le 0$, *then* (23) *defines a finite measure and*

$$\|\eta_f\|_{\mathrm{TV}} = (1-q)\sum_{k\in\mathbb{Z}} q^k |f(q^k)| \le \|f\|_{\mathcal{X}_{q,\lambda}(\sigma)}.$$

At $\sigma = 0$ *the two norms are equal. If* $\sigma > 0$, *membership in* $\mathcal{X}_{q,\lambda}(\sigma)$ *alone does not imply* $\eta_f \in \mathcal{M}([0,\infty))$.

Proof. If $\sigma \le 0$, the inequality $(1+\lambda q^k)^{-\sigma/\lambda} \ge 1$ gives the claimed norm bound immediately. For $\sigma > 0$, set $a = \sigma/(2\lambda) > 0$ and take $f(t) = t^{a-1}$. Applying Lemma 1 gives

$$(1-q)\sum_{k\in\mathbb{Z}} q^{ka}(1+\lambda q^k)^{-\sigma/\lambda} < \infty,$$

whereas $(1-q)\sum_{k\in\mathbb{Z}} q^{ka} = \infty$ because the negative tail diverges. □

For $\eta, \zeta \in \mathcal{M}([0,\infty))$, define their λ-deformed convolution by

$$(\eta \circledast_\lambda \zeta)(B) := \int_{[0,\infty)}\int_{[0,\infty)} \mathbf{1}_B(t \oplus_\lambda u)\, d\eta(t)\, d\zeta(u) \tag{24}$$

for Borel sets $B \subset [0,\infty)$. Equivalently, for every bounded Borel function φ,

$$\int \varphi(v)\, d(\eta \circledast_\lambda \zeta)(v) = \iint \varphi(t \oplus_\lambda u)\, d\eta(t)\, d\zeta(u). \tag{25}$$

Proposition 6. *The space $\mathcal{M}([0,\infty))$, equipped with $\circledast_\lambda$, is a commutative unital Banach algebra with identity δ_0 and*

$$\|\eta \circledast_\lambda \zeta\|_{\mathrm{TV}} \le \|\eta\|_{\mathrm{TV}}\|\zeta\|_{\mathrm{TV}}. \tag{26}$$

If $\phi_{\lambda\#}\eta$ denotes the pushforward of η by ϕ_λ, then

$$\phi_{\lambda\#}(\eta \circledast_\lambda \zeta) = (\phi_{\lambda\#}\eta) * (\phi_{\lambda\#}\zeta), \tag{27}$$

where $*$ is ordinary additive convolution.

Proof. Associativity, commutativity, and the identity follow from Lemma 4. Total variation together with Fubini–Tonelli gives (26). To verify the pushforward identity, let g be a bounded Borel test function. Then

$$\int g(x)\, d\phi_{\lambda\#}(\eta \circledast_\lambda \zeta)(x) = \iint g(\phi_\lambda(t \oplus_\lambda u))\, d\eta(t) d\zeta(u),$$

and Lemma 4 changes the integrand to $g(\phi_\lambda(t) + \phi_\lambda(u))$, which is precisely the test-function representation of the right-hand side of (27). □

For a finite measure η, define its degenerate Laplace character by

$$\mathfrak{L}_\lambda[\eta](s) := \int_{[0,\infty)} (1+\lambda t)^{-s/\lambda}\, d\eta(t), \qquad \Re(s) \ge 0. \tag{28}$$

Lemma 5. *If η_f in (23) is finite, then*

$$\mathfrak{L}_\lambda[\eta_f](s) = \mathcal{L}_q^{(\lambda)}\{f\}(s), \qquad \Re(s) \ge 0. \tag{29}$$

For any finite measure,

$$\mathfrak{L}_\lambda[\eta](s) = \int_{[0,\infty)} e^{-sx}\, d(\phi_{\lambda\#}\eta)(x).$$

Proof. Substituting (23) into (28) gives the bilateral Jackson series. For the second identity, use $(1+\lambda t)^{-s/\lambda} = e^{-s\phi_\lambda(t)}$ and the definition of pushforward. □

Theorem 6. *For $\eta, \zeta \in \mathcal{M}([0,\infty))$ and $\Re(s) \ge 0$,*

$$\mathfrak{L}_\lambda[\eta \circledast_\lambda \zeta](s) = \mathfrak{L}_\lambda[\eta](s)\, \mathfrak{L}_\lambda[\zeta](s). \tag{30}$$

In particular, if η_f and η_g are finite Jackson measures, then

$$\mathfrak{L}_\lambda[\eta_f \circledast_\lambda \eta_g](s) = \mathcal{L}_q^{(\lambda)}\{f\}(s)\mathcal{L}_q^{(\lambda)}\{g\}(s). \tag{31}$$

Proof. Using (25), (22), and Fubini's theorem,

$$\begin{aligned}\mathfrak{L}_\lambda[\eta \circledast_\lambda \zeta](s) &= \iint K_s^{(\lambda)}(t \oplus_\lambda u)\, d\eta(t) d\zeta(u) \\ &= \left(\int K_s^{(\lambda)}(t)\, d\eta(t)\right)\left(\int K_s^{(\lambda)}(u)\, d\zeta(u)\right).\end{aligned}$$

Since the integrands have modulus at most one for $\Re(s) \geq 0$, finite total variation justifies the use of Fubini's theorem. □

Remark 4. *The semigroup change of coordinates already implies the abstract convolution product identity for finite measures. The additional content here is its precise interpretation for Jackson-grid data: the weighted samples define an atomic measure, the hypotheses identify when that measure has finite total variation, and the product identity applies even though convolution usually leaves the original sampling grid. In particular, the grid-specific statement (31) is obtained from (27), (29) and (30); it is not a claim that convolution closes on $\mathbb{T}_q$. Thus the product formula supplies a rigorous operational bridge from bilateral Jackson transforms to an ambient measure algebra, rather than a new convolution law independent of ordinary additive convolution.*

Remark 5. *For Jackson-supported measures,*

$$\operatorname{supp}(\eta_f \circledast_\lambda \eta_g) \subseteq \{q^i \oplus_\lambda q^j : i, j \in \mathbb{Z}\} = \{q^i + q^j + \lambda q^{i+j} : i, j \in \mathbb{Z}\},$$

which is generally not contained in $\mathbb{T}_q$. We therefore formulate the convolution on finite measures rather than require closure on the Jackson grid. By Proposition 6 and Lemma 5, the map ϕ_λ transports the product formula to the classical additive convolution formula for finite measures. In the original variable, $\oplus_\lambda$ is the semigroup law induced by the kernel, and convolution of Jackson-supported measures may produce off-grid support.

3.3. A Hereditary Measure Equation on Deformed Time

We now apply the measure algebra of Section 3.2 to a causal memory equation. Since $x = \phi_\lambda(t)$ converts $\oplus_\lambda$ into ordinary addition, consider

$$\eta_u + \eta_h \circledast_\lambda \eta_u = \eta_f, \tag{32}$$

where η_h is a finite memory measure, η_f is a forcing measure, and η_u is the response. Passing to the additive coordinate x, Proposition 6 converts (32) into an ordinary convolution Volterra equation for finite measures [41].

Theorem 7. *Let $\eta_h, \eta_f \in \mathcal{M}([0,\infty))$ and assume*

$$\|\eta_h\|_{\mathrm{TV}} \leq r < 1. \tag{33}$$

Then (32) has a unique solution $\eta_u \in \mathcal{M}([0,\infty))$, given by the norm-convergent resolvent series

$$\eta_u = \sum_{n=0}^{\infty} (-1)^n \eta_h^{\circledast_\lambda n} \circledast_\lambda \eta_f, \qquad \eta_h^{\circledast_\lambda 0} := \delta_0. \tag{34}$$

It satisfies

$$\|\eta_u\|_{\mathrm{TV}} \leq \frac{\|\eta_f\|_{\mathrm{TV}}}{1-r}. \tag{35}$$

If $(\eta_{h,i}, \eta_{f,i})$, $i = 1, 2$, satisfy $\|\eta_{h,i}\|_{\mathrm{TV}} \leq r < 1$, and $\eta_{u,i}$ are the corresponding solutions, then

$$\|\eta_{u,1} - \eta_{u,2}\|_{\mathrm{TV}} \leq \frac{\|\eta_{f,1} - \eta_{f,2}\|_{\mathrm{TV}}}{1-r} + \frac{\|\eta_{f,2}\|_{\mathrm{TV}}}{(1-r)^2} \|\eta_{h,1} - \eta_{h,2}\|_{\mathrm{TV}}. \tag{36}$$

For $\Re(s) \geq 0$, if

$$U(s) = \mathfrak{L}_\lambda[\eta_u](s), \quad H(s) = \mathfrak{L}_\lambda[\eta_h](s), \quad F(s) = \mathfrak{L}_\lambda[\eta_f](s),$$

then

$$U(s) = \frac{F(s)}{1 + H(s)}, \qquad |1 + H(s)| \geq 1 - r. \tag{37}$$

Proof. By Proposition 6, the convolution operator

$$T_h\eta := \eta_h \circledast_\lambda \eta$$

is bounded on $\mathcal{M}([0,\infty))$ with $\|T_h\| \le \|\eta_h\|_{\mathrm{TV}} \le r < 1$. The strict norm bound above allows the Neumann series to invert $I + T_h$, giving (34). Applying (26) term by term yields

$$\|\eta_u\|_{\mathrm{TV}} \le \sum_{n=0}^{\infty} r^n \|\eta_f\|_{\mathrm{TV}} = \frac{\|\eta_f\|_{\mathrm{TV}}}{1-r}.$$

For the perturbation estimate, subtract the two equations and write

$$(I + T_{h,1})(\eta_{u,1} - \eta_{u,2}) = (\eta_{f,1} - \eta_{f,2}) - (\eta_{h,1} - \eta_{h,2}) \circledast_\lambda \eta_{u,2}.$$

The bound $\|(I + T_{h,1})^{-1}\| \le (1-r)^{-1}$, together with Proposition 6 and (35) for $\eta_{u,2}$, gives (36). Applying Theorem 6 to (32) gives $(1 + H(s))U(s) = F(s)$. Since $|H(s)| \le \|\eta_h\|_{\mathrm{TV}} \le r$ for $\Re(s) \ge 0$, the denominator is uniformly separated from zero, and (37) follows. □

Remark 6. *Since $t \oplus_\lambda u \to t + u$ as $\lambda \to 0^+$, the deformed hereditary equation formally approaches the usual additive measure-convolution equation, while the transform character tends to e^{-st}. When the source measures arise from Jackson data, the parameter q enters through their atomic sampling. The response is allowed to have off-grid support because $\circledast_\lambda$ is defined on finite measures rather than on Jackson-supported measures alone.*

Example 1. *Fix $m, n \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C}$, and set*

$$\eta_h = \alpha\delta_{q^m}, \qquad \eta_f = \beta\delta_{q^n}.$$

If $|\alpha| < 1$, Theorem 7 gives

$$\eta_u = \beta \sum_{r=0}^{\infty} (-\alpha)^r \delta_{t_{n,r}}, \qquad t_{n,r} := \frac{(1+\lambda q^n)(1+\lambda q^m)^r - 1}{\lambda}. \tag{38}$$

By associativity, $t_{n,r} = q^n \oplus_\lambda q^m \oplus_\lambda \cdots \oplus_\lambda q^m$, with r copies of q^m. The transform is

$$U(s) = \frac{\beta(1+\lambda q^n)^{-s/\lambda}}{1 + \alpha(1+\lambda q^m)^{-s/\lambda}}, \tag{39}$$

and the truncation after order N obeys the exact total-variation tail formula

$$\left\| \eta_u - \beta \sum_{r=0}^{N} (-\alpha)^r \delta_{t_{n,r}} \right\|_{\mathrm{TV}} = \frac{|\beta|\,|\alpha|^{N+1}}{1 - |\alpha|}. \tag{40}$$

For $q = 1/2$, $\lambda = 1$, $m = 0$, $n = 2$, $\alpha = 0.6$, and $\beta = 1$, the first support points are $0.25, 1.5, 4, 9, 19, 39, \ldots$, which are not Jackson nodes after the first term. Figure 1 compares the uniform total-variation tail bound with the actual transform remainder at $s = 0.5$.

3.3.1. A Quadratic Nonlinear Hereditary Volterra Equation

The same convolution algebra gives the nonlinear memory equation

$$\eta_u + \eta_h \circledast_\lambda \eta_u + \gamma\, \eta_u \circledast_\lambda \eta_u = \eta_f, \qquad \gamma \in \mathbb{C}. \tag{41}$$

The quadratic term is nonlinear in the unknown measure. Causality is nevertheless preserved because $\circledast_\lambda$ is transported by ϕ_λ to ordinary Volterra convolution on $[0,\infty)$.

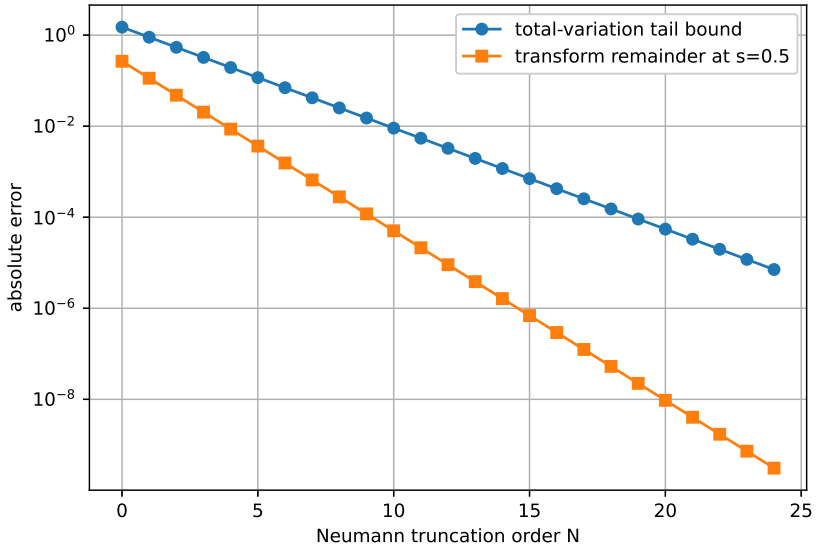


Figure 1. Convergence of the Neumann approximation in Example 1. The total-variation tail is given exactly by (40) and is therefore a uniform bound; the transform-domain remainder at $s = 0.5$ decays faster because the kernel supplies an additional damping factor for each omitted atom.

Theorem 8. *Let $\eta_h, \eta_f \in \mathcal{M}([0,\infty))$, assume $\|\eta_h\|_{\mathrm{TV}} \le r < 1$, and let $\gamma \neq 0$. If*

$$\frac{4|\gamma|\,\|\eta_f\|_{\mathrm{TV}}}{(1-r)^2} < 1, \tag{42}$$

then (41) has a unique solution in the closed ball $\{\eta : \|\eta\|_{\mathrm{TV}} \le R\}$, where

$$R = \frac{1-\sqrt{1-4ac}}{2c}, \qquad a = \frac{\|\eta_f\|_{\mathrm{TV}}}{1-r}, \qquad c = \frac{|\gamma|}{1-r}. \tag{43}$$

Starting from any point in this ball, the Picard iteration

$$\eta_u^{(j+1)} = (I + T_h)^{-1}\left(\eta_f - \gamma\,\eta_u^{(j)} \circledast_\lambda \eta_u^{(j)}\right) \tag{44}$$

converges in total variation to the solution. Uniqueness here is restricted to the indicated closed ball; no uniqueness statement outside that ball is asserted. A valid contraction factor is

$$\rho = 2cR = 1 - \sqrt{1-4ac} < 1. \tag{45}$$

For $\Re(s) \ge 0$, the transformed solution satisfies

$$\gamma U(s)^2 + (1 + H(s))U(s) - F(s) = 0. \tag{46}$$

Proof. By Theorem 7, $\mathcal{R}_h := (I + T_h)^{-1}$ exists and $\|\mathcal{R}_h\| \le (1-r)^{-1}$. Define

$$\Phi(\eta) = \mathcal{R}_h\eta_f - \gamma\mathcal{R}_h(\eta \circledast_\lambda \eta).$$

If $\|\eta\|_{\mathrm{TV}} \le R$, Proposition 6 gives

$$\|\Phi(\eta)\|_{\mathrm{TV}} \le a + cR^2 = R,$$

where the last equality follows from the definition of R. For two measures in the same ball,

$$\begin{aligned}\|\Phi(\eta) - \Phi(\zeta)\|_{\mathrm{TV}} &\le c\,\|\eta \circledast_\lambda \eta - \zeta \circledast_\lambda \zeta\|_{\mathrm{TV}} \\ &\le c(\|\eta\|_{\mathrm{TV}} + \|\zeta\|_{\mathrm{TV}})\|\eta - \zeta\|_{\mathrm{TV}} \\ &\le 2cR\,\|\eta - \zeta\|_{\mathrm{TV}}.\end{aligned}$$

Condition (42) gives $2cR = \rho < 1$. The Banach fixed-point theorem then yields existence, uniqueness in the ball, and convergence of (44). Applying Theorem 6 to (41) gives (46). □

Proposition 7. *Continuous dependence on the forcing and memory measures within a common contraction regime. Fix $\gamma \neq 0$, $0 \leq r < 1$, and $F \geq 0$ such that*

$$4|\gamma|F < (1-r)^2.$$

Set

$$R = \frac{1 - \sqrt{1 - 4|\gamma|F/(1-r)^2}}{2|\gamma|/(1-r)}, \qquad \rho = \frac{2|\gamma|R}{1-r} < 1.$$

For $i = 1, 2$, let $\eta_{h_i}, \eta_{f_i} \in \mathcal{M}([0,\infty))$ satisfy $\|\eta_{h_i}\|_{\mathrm{TV}} \leq r$ and $\|\eta_{f_i}\|_{\mathrm{TV}} \leq F$. Denote by η_{u_i} the unique solution of (41) in the common closed ball of radius R, with the corresponding forcing and memory measures. Then

$$\|\eta_{u_1} - \eta_{u_2}\|_{\mathrm{TV}} \leq \frac{\|\eta_{f_1} - \eta_{f_2}\|_{\mathrm{TV}} + R\|\eta_{h_1} - \eta_{h_2}\|_{\mathrm{TV}}}{1 - r - 2|\gamma|R}. \tag{47}$$

Consequently, the solution depends Lipschitz continuously on both data measures while the common smallness margin remains positive. This statement concerns the solutions selected within the prescribed ball and does not assert global uniqueness.

Proof. Subtract the two nonlinear measure equations, writing $\Delta u = \eta_{u_1} - \eta_{u_2}$, $\Delta h = \eta_{h_1} - \eta_{h_2}$, and $\Delta f = \eta_{f_1} - \eta_{f_2}$. Bilinearity and commutativity give

$$\Delta u + \eta_{h_1} \circledast_\lambda \Delta u + \Delta h \circledast_\lambda \eta_{u_2} + \gamma\, \Delta u \circledast_\lambda (\eta_{u_1} + \eta_{u_2}) = \Delta f.$$

By the total-variation convolution inequality and $\|\eta_{u_i}\|_{\mathrm{TV}} \leq R$,

$$(1 - r - 2|\gamma|R)\|\Delta u\|_{\mathrm{TV}} \leq \|\Delta f\|_{\mathrm{TV}} + R\|\Delta h\|_{\mathrm{TV}}.$$

The coefficient on the left is strictly positive by the assumed smallness condition, yielding (47). □

Remark 7. *Equation (41) is a nonlinear Volterra-type equation in the deformed time variable. For $\gamma = 0$ it reduces to the linear hereditary equation, and as $\lambda \to 0^+$ its convolution terms approach ordinary additive measure convolutions. Section 3.4 considers transform-domain Picard iteration under (42).*

3.4. Computational Experiments, Limit Validation, and Bilateral Truncation

The computations use the formulas established above. Reference values are obtained either from exact transform formulas or from independently converged high-precision sums and integrals; no convergence rate is fitted. We cross-checked the numerical references at 70-digit precision. In the Jackson-to-Riemann test, the production sum used grid points with $10^{-14} \leq q^k \leq 60$, whereas the verification run used $10^{-25} \leq q^k \leq 100$; both runs produced the displayed digits. The nonlinear experiment is evaluated on a uniform 401-point transform grid over $0 \leq s \leq 4$.

3.4.1. Neumann Resolvent across q and λ

For the atomic model of Example 1, the exact transform remainder after order N is

$$R_N(s) = \frac{|\beta| A_s\, |\alpha B_s|^{N+1}}{|1 + \alpha B_s|}, \qquad A_s = (1 + \lambda q^n)^{-\Re(s)/\lambda}, \quad B_s = (1 + \lambda q^m)^{-\Re(s)/\lambda}, \tag{48}$$

for positive real parameters. Table 2 lists the corresponding geometric factors and threshold orders, while Figure 2 displays the remainder curves at $s = 0.5$, $m = 1$, $n = 2$, $\alpha = 0.6$, and $\beta = 1$ for four parameter pairs. The total-variation bound from (40) is shown simultaneously and remains valid uniformly in q, λ, and s.

Table 2. Parameter sensitivity of the atomic Neumann approximation at $s = 0.5$. The quantity $|\alpha B_s|$ is the exact geometric factor in (48); $N_{10^{-8}}$ is the first order for which $R_N(0.5) < 10^{-8}$.

q	λ	$\lvert\alpha B_{0.5}\rvert$	$N_{10^{-8}}$
0.35	0.50	0.510638	26
0.50	1.00	0.489898	25
0.70	0.50	0.444444	21
0.70	1.50	0.472317	23

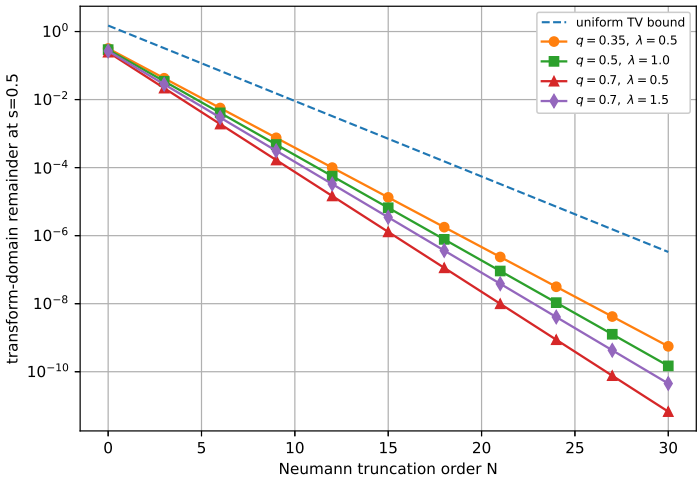


Figure 2. Transform-domain Neumann remainder for four (q, λ) configurations in the atomic hereditary model with $m = 1$, $n = 2$, $\alpha = 0.6$, $\beta = 1$, and $s = 0.5$. The dashed curve is the uniform total-variation bound (40). Parameter dependence enters the actual geometric factor through the deformed kernel.

3.4.2. Numerical Jackson-to-Riemann Limit as $q \to 1^-$

Proposition 4 is tested with $f(t) = e^{-t}$, $\lambda = 1$, and $s = 2$. The continuous limiting integral is

$$I = \int_0^\infty e^{-t}(1+t)^{-2}\,dt = 0.4036526376768059257\ldots. \tag{49}$$

Direct bilateral Jackson summation gives the values in Table 3. Over the sampled values, the error decreases monotonically as q approaches one, as shown in Figure 3.

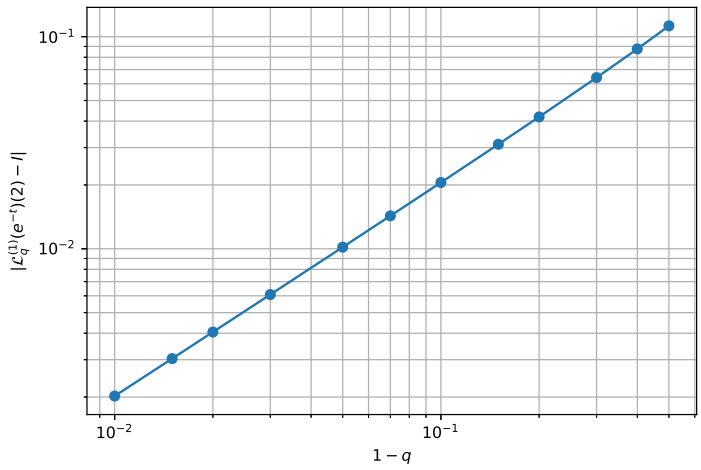


Figure 3. Absolute error in the Jackson-to-Riemann experiment of Table 3, plotted against $1 - q$ on logarithmic axes. For the displayed parameter values, the computed error decreases toward zero; Proposition 4 gives the corresponding limiting result.

Table 3. Numerical verification of the $q \to 1^-$ limit for $f(t) = e^{-t}$, $\lambda = 1$, and $s = 2$.

q	$\mathcal{L}_q^{(1)}\{e^{-t}\}(2)$	Absolute Error to I
0.80	0.361787410211476	4.187×10^{-2}
0.90	0.383115662595766	2.054×10^{-2}
0.95	0.393475056044332	1.018×10^{-2}
0.98	0.399602519932321	4.050×10^{-3}
0.99	0.401630993790290	2.022×10^{-3}

3.4.3. Deformed-to-Additive Convolution Limit as $\lambda \to 0^+$

For the parameters of Example 1, the transform tends to the additive-convolution value

$$U_0(0.5) = \frac{e^{-0.5q^n}}{1 + \alpha e^{-0.5q^m}} = 0.6470305740316126504\ldots. \tag{50}$$

The deformed support points $t_{n,r}$ in (38) also tend to $q^n + rq^m$. Table 4 records both limits; Figure 4 shows the transform error.

Table 4. The $\lambda \to 0^+$ limit for $q = 1/2$, $m = 0$, $n = 2$, $\alpha = 0.6$, $\beta = 1$, and $s = 0.5$. The last three columns are the first nontrivial support points of the Neumann response.

λ	$\lvert U_\lambda(0.5) - U_0(0.5)\rvert$	$t_{n,1}$	$t_{n,2}$	$t_{n,3}$
1	1.904×10^{-2}	1.500000	4.000000	9.000000
0.25	6.992×10^{-3}	1.312500	2.640625	4.300781
0.0625	1.976×10^{-3}	1.265625	2.344727	3.491272
0	0	1.250000	2.250000	3.250000

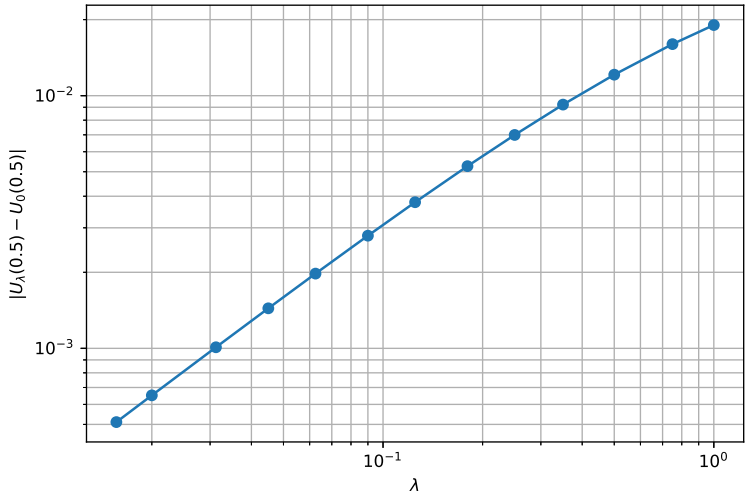


Figure 4. Numerical convergence of the atomic hereditary transform to its additive-convolution limit as $\lambda \to 0^+$. The plot quantifies the limit in Remark 6 at $s = 0.5$.

3.4.4. Asymmetric Bilateral Truncation and Computational Cost

The small-t and large-t tails of the bilateral Jackson representation generally decay at different rates, so we truncate them independently.

Proposition 8. *Let $a > 0$, let $s \in \mathbb{C}$ satisfy*

$$b := \frac{\Re(s)}{\lambda} - a > 0,$$

and suppose

$$|f(t)| \le Mt^{a-1} \qquad (t > 0) \tag{51}$$

for some $M > 0$. For integers $K_+, K_- \ge 0$, define

$$S_{K_-,K_+}(s) := (1-q) \sum_{k=-K_-}^{K_+} q^k f(q^k)(1+\lambda q^k)^{-s/\lambda}. \tag{52}$$

Then

$$\mathrm{Err}_+(K_+) \le C_+ q^{(K_++1)a}, \qquad C_+ := \frac{(1-q)M}{1-q^a}, \tag{53}$$

$$\mathrm{Err}_-(K_-) \le C_- q^{(K_-+1)b}, \qquad C_- := \frac{(1-q)M\lambda^{-\Re(s)/\lambda}}{1-q^b}. \tag{54}$$

Therefore,

$$\left| \mathcal{L}_q^{(\lambda)}\{f\}(s) - S_{K_-,K_+}(s) \right| \le C_+ q^{(K_++1)a} + C_- q^{(K_-+1)b}. \tag{55}$$

Given $\varepsilon > 0$, it is sufficient to choose

$$K_+ \ge \max\left\{0, \frac{\log_{1/q}(2C_+/\varepsilon)}{a} - 1\right\}, \tag{56}$$

$$K_- \ge \max\left\{0, \frac{\log_{1/q}(2C_-/\varepsilon)}{b} - 1\right\}, \tag{57}$$

and take ceilings.

Proof. For $k \ge K_+ + 1$, combine $(1+\lambda q^k)^{-\Re(s)/\lambda} \le 1$ with (51) to obtain

$$(1-q)q^k |f(q^k)|(1+\lambda q^k)^{-\Re(s)/\lambda} \le (1-q)Mq^{ka}.$$

Summing this geometric majorant gives (53). For the negative tail, write $k = -m$ with $m \ge K_- + 1$. The inequality $1 + \lambda q^{-m} \ge \lambda q^{-m}$ gives

$$(1+\lambda q^{-m})^{-\Re(s)/\lambda} \le \lambda^{-\Re(s)/\lambda} q^{m\Re(s)/\lambda}.$$

Combining this estimate with $|f(q^{-m})| \le Mq^{-m(a-1)}$, we obtain

$$(1-q)M\lambda^{-\Re(s)/\lambda} q^{mb}.$$

Summing the resulting geometric series yields (54). The triangle inequality gives the total bound, and the stated choices of cutoffs make each tail at most $\varepsilon/2$. □

The symmetric truncation $S_K = S_{K,K}$ corresponds to $K_+ = K_- = K$. Proposition 8 permits the two cutoffs to be chosen independently. In the example below, where a and b differ, this choice reduces the number of retained terms.

For comparison, take $q = 1/2$, $\lambda = 1$, $a = 1/2$, $s = 2$, and $f(t) = t^{-1/2}$. Here $M = 1$ in (51) and $b = 3/2$. An independently converged high-precision bilateral summation gives

$$\mathcal{L}_{1/2}^{(1)}\{t^{-1/2}\}(2) = 1.13309003545874217553561724698\ldots. \tag{58}$$

At the symmetric cutoff $K_+ = K_- = 80$, Proposition 8 gives an error bound of 1.098×10^{-12}, and direct comparison with (58) gives the same value to the displayed precision. For these parameters the positive-grid tail dominates, so the a priori estimate is nearly attained.

The cutoff formulas (56) and (57) also permit a direct cost comparison. For a prescribed ε, the asymmetric method selects (K_+, K_-) independently, while the symmetric method must take $K = \max\{K_+, K_-\}$. Since each retained node contributes one kernel evaluation and one summand accumulation, we use the number of retained terms as an operation-count proxy for the arithmetic cost. The resulting counts appear in Table 5 and Figure 5.

Table 5. Symmetric versus asymmetric bilateral truncation for the test problem (58). "Actual error" refers to the asymmetric sum.

ε	K_+	K_-	Asymmetric Terms	Symmetric Terms	Actual Error
10^{-6}	43	13	57	87	7.758×10^{-7}
10^{-9}	63	20	84	127	6.521×10^{-10}
10^{-12}	83	26	110	167	8.856×10^{-13}

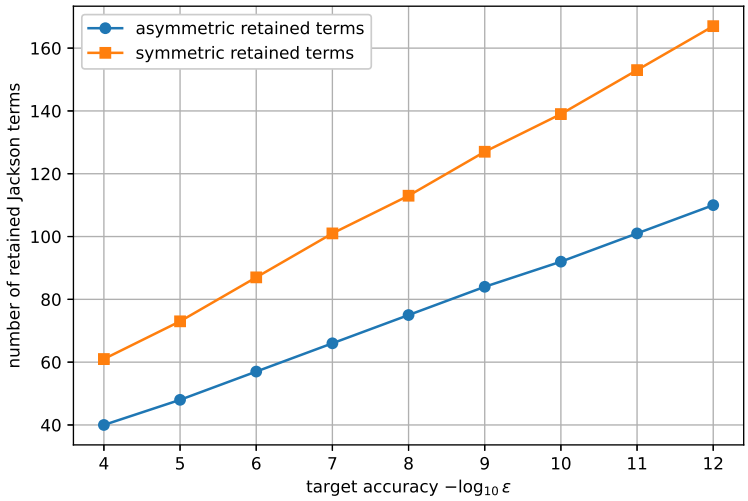


Figure 5. Number of retained Jackson terms required by the sufficient error bounds for symmetric and asymmetric truncation. For the unequal tail exponents $a = 1/2$ and $b = 3/2$, independent cutoffs reduce the count from 167 to 110 terms at a target of 10^{-12}.

3.4.5. Nonlinear Hereditary Picard Experiment

For the nonlinear equation (41), choose

$$q = \frac{1}{2}, \quad \lambda = 1, \quad m = 0, \quad n = 2, \quad \eta_h = 0.25\,\delta_{q^m}, \quad \eta_f = 0.2\,\delta_{q^n}, \quad \gamma = 0.5.$$

Here $r = 0.25$ and

$$\frac{4|\gamma|\|\eta_f\|_{\mathrm{TV}}}{(1-r)^2} = 0.711111\ldots < 1,$$

so Theorem 8 applies. Its ball radius and guaranteed contraction factor are

$$R = 0.3468871125850725\ldots, \qquad \rho = 0.4625161501134300\ldots.$$

For real $s \in [0, 4]$, all parameters are positive and the solution branch of (46) that is continuous at $\gamma = 0$ is

$$U_*(s) = \frac{\sqrt{(1+H(s))^2 + 4\gamma F(s)} - (1+H(s))}{2\gamma}. \tag{59}$$

Starting from the zero iterate, we use the transformed Picard map

$$U_{j+1}(s) = \frac{F(s) - \gamma U_j(s)^2}{1+H(s)}, \qquad U_0(s) = 0, \tag{60}$$

and compare it with (59) on a uniform 401-point grid over $[0, 4]$. Figure 6 reports both the maximum solution error and the maximum quadratic-equation residual. The latter decreases from 2.0×10^{-1} at $j = 0$ to 4.97×10^{-9} at $j = 8$ and to 7.35×10^{-11} at $j = 10$.

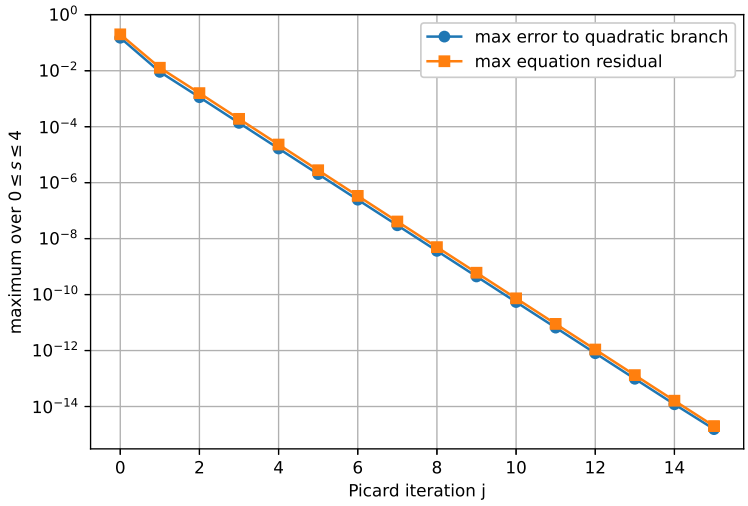


Figure 6. Transform-domain Picard convergence for the quadratic nonlinear hereditary equation. Both the maximum error to the explicit positive quadratic branch and the equation residual are measured over $0 \le s \le 4$. Both quantities decrease geometrically over the displayed iterations.

4. Discussion

For the kernel $(1+\lambda t)^{-s/\lambda}$, the two tails of the bilateral Jackson series lead to distinct convergence conditions. On the resulting weighted data spaces, the transform is bounded and holomorphic. Fourier–mean extraction recovers every grid value; on ordinary functions, its null space consists exactly of functions vanishing on $\mathbb{T}_q$. For a fixed grid node, Equation (13) gives a finite-window error estimate, with a constant that grows as neighboring logarithmic frequencies approach one another.

The kernel induces the semigroup law $t \oplus_\lambda u = t + u + \lambda tu$, and the associated convolution turns the finite measures into a commutative unital Banach algebra on which the transform is multiplicative. Jackson data embed as atomic measures, although convolution generally produces off-grid support. Under the condition $\|\eta_h\|_{\mathrm{TV}} < 1$, Section 3.3 obtains a Neumann-series solution and perturbation bounds for the linear hereditary equation. For the quadratic extension (41), assumption (42) yields a unique solution in the stated ball and convergence of the Picard iteration. Proposition 7 also quantifies continuous dependence on the forcing and memory measures under a shared strict smallness margin.

For the parameter sets tested, the computed quantities satisfy the analytical bounds and exhibit the limiting behavior proved above. In particular, the Neumann-remainder curves are bounded by the geometric total-variation tail estimate, while the observed Picard decay is consistent with the contraction factor from the closed-ball fixed-point theorem; the plots illustrate these bounds rather than establish new uniform rates. The Neumann remainders remain below the uniform total-variation bound. The Jackson sums approach the continuous integral as $q \to 1^-$, and the atomic deformed resolvent approaches its additive-convolution limit as $\lambda \to 0^+$. In the truncation example with unequal tail exponents, separate cutoffs retain fewer grid points than the symmetric cutoff. In the nonlinear example, the chosen parameters satisfy the contraction criterion, and both the solution error and the equation residual decrease geometrically over the computed iterates.

Propositions 3 and 4 establish the separate one-parameter limits under their stated hypotheses. With the additional domination assumption in Corollary 6, these limits recover the classical Laplace transform successively. The present analysis does not provide a range characterization for $\mathcal{L}_q^{(\lambda)}$, grid-uniform or noise-stable recovery estimates, off-grid reconstruction under weaker assumptions, nonlinear well-posedness beyond the small-data ball, or resolvent criteria without the small-total-variation condition.

5. Conclusions

For the bilateral Jackson-grid transform, we establish explicit convergence conditions, recovery of all Jackson-grid data, and a deformed convolution compatible with the transform. The Fourier-mean formula reconstructs the sampled grid values only; it is not an inversion formula for arbitrary off-grid function values, and uniqueness on the Jackson-data quotient does not imply injectivity on ordinary functions. On finite measures, the induced convolution yields a Neumann-series solution for the linear hereditary equation under a small-total-variation condition and a contraction result for the quadratic equation under the stated small-data hypothesis with uniqueness confined

to the prescribed closed ball. A Lipschitz estimate further establishes continuous dependence on both forcing and memory measures for data in a common contraction regime. In the computed examples, the observed errors satisfy the derived convergence bounds, the $q \to 1^-$ and $\lambda \to 0^+$ limits approach the corresponding analytically established values, asymmetric cutoffs retain fewer terms in the unequal-tail test, and the nonlinear Picard iterates converge geometrically.

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Conflicts of Interest

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Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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