

The Global Exponential Synchronization of Delayed Fuzzy Inertial Neural Networks via the Characteristics Method

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Abstract: This study investigates the global exponential synchronization (GES) of delayed fuzzy inertial neural networks (DFINNs). Diverging from conventional approaches that rely on reduced order transformations and the construction of Lyapunov Krasovskii functionals (LKF), we employ the characteristic method to derive GES criteria for DFINNs. We introduce three sufficient conditions, each expressed as a set of linear scalar inequalities. The validity of our theoretical findings is confirmed via a numerical example utilizing three distinct controllers, supported by corresponding simulation results.

Keywords: inertial neural network; global exponential synchronization; characteristics method; parameterized method

1. Introduction

This paper focuses on the GES of the following DFINNs

$$\begin{aligned} \ddot{z}_\alpha(t) = & -m_\alpha \dot{z}_\alpha(t) - l_\alpha z_\alpha(t) + \sum_{\beta=1}^n a_{\alpha\beta}(t) h_\beta(z_\beta(t)) + \sum_{\beta=1}^n b_{\alpha\beta}(t) h_\beta(z_\beta(t - \tau_{\alpha\beta}(t))) \\ & + \sum_{\beta=1}^n c_{\alpha\beta}(t) v_\beta(t) + \bigwedge_{\beta=1}^n T_{\alpha\beta}(t) v_\beta(t) + \bigwedge_{\beta=1}^n p_{\alpha\beta}(t) h_\beta(z_\beta(t - \tau_{\alpha\beta}(t))) \\ & + \bigvee_{\beta=1}^n q_{\alpha\beta}(t) h_\beta(z_\beta(t - \tau_{\alpha\beta}(t))) + \bigvee_{\beta=1}^n S_{\alpha\beta}(t) v_\beta(t) + I_\alpha(t), \end{aligned} \quad (1)$$

$$\begin{aligned} \ddot{y}_\alpha(t) = & -m_\alpha \dot{y}_\alpha(t) - l_\alpha y_\alpha(t) + \sum_{\beta=1}^n a_{\alpha\beta}(t) h_\beta(y_\beta(t)) + \sum_{\beta=1}^n b_{\alpha\beta}(t) h_\beta(y_\beta(t - \tau_{\alpha\beta}(t))) \\ & + \sum_{\beta=1}^n c_{\alpha\beta}(t) v_\beta(t) + \bigwedge_{\beta=1}^n T_{\alpha\beta}(t) v_\beta(t) + \bigwedge_{\beta=1}^n p_{\alpha\beta}(t) h_\beta(y_\beta(t - \tau_{\alpha\beta}(t))) \\ & + \bigvee_{\beta=1}^n q_{\alpha\beta}(t) h_\beta(y_\beta(t - \tau_{\alpha\beta}(t))) + \bigvee_{\beta=1}^n S_{\alpha\beta}(t) v_\beta(t) + I_\alpha(t) + u_\alpha(t). \end{aligned} \quad (2)$$

Systems (1) and (2) constitute a drive-response configuration, where the state vectors are denoted by $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T \in \mathbb{R}^n$ and $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in \mathbb{R}^n$ with $\alpha \in \Lambda = \{1, 2, \dots, n\}$. The terms $\ddot{z}_\alpha(t)$ and $\ddot{y}_\alpha(t)$ represent the inertial (second-order) dynamics of the α -th neurons. Positive constants m_α and l_α are damping and self-feedback coefficients, respectively. The time-varying coefficients $a_{\alpha\beta}(t)$, $b_{\alpha\beta}(t)$ and $c_{\alpha\beta}(t)$ correspond to the elements of the feedback, delayed feedback, and feed-forward templates, respectively. Moreover, $p_{\alpha\beta}(t)$ and $q_{\alpha\beta}(t)$ denote the entries of the fuzzy feedback MIN and MAX templates, while $T_{\alpha\beta}(t)$ and $S_{\alpha\beta}(t)$ represent those of the fuzzy feed-forward MIN and MAX templates, respectively. The symbol $\bigwedge$ and $\bigvee$ stand for the fuzzy AND and fuzzy OR operations. The external input to the β th neuron is denoted by $v_\beta(t)$, while $I_\alpha(t)$ and $u_\alpha(t)$ represent the bias and the control input of the α th neuron, respectively. The transmission delay $\tau_{\alpha\beta}$ is non-negative and satisfies $0 \leq \tau_{\alpha\beta} \leq \tau$, where $\tau \geq 0$ is a constant. Finally, the activation function $h_\beta(t)$ is continuous and satisfies the Lipschitz condition

$$(A) \quad |h_\beta(x) - h_\beta(y)| \leq k_\beta |x - y|, \forall x, y \in \mathbb{R}, \beta \in \Lambda$$

with k_β being a Lipschitz constant.

The DFINNs models (1) and (2) draw their primary inspiration from two seminal contributions: the fuzzy cellular neural network first introduced by Chua, Yang, and Yang [1–4], and the inertial dynamical framework developed by Babcock and Westervelt [5,6]. The majority of prior studies on DFINNs have predominantly employed either reduced or non-reduced order methodologies. The former strategy [7–33] involves converting the second order inertial system into an equivalent first order differential system through variable substitution, a transformation that frequently results in increased computational complexity and a higher dimensional state space. To address these limitations, the non-reduced order approach [34–44] has been subsequently proposed.

Unfortunately, both reduced and non-reduced order approaches depend on the construction of a Lyapunov Krasovskii functional (LKF), a task widely recognized as highly intricate and technically demanding. This process typically involves handling complex linear matrix inequalities (LMIs), which are usually solved using specialized MATLAB toolboxes. To circumvent the difficulties associated with LKF construction and LMIs manipulation, the authors in [45,46] proposed a parameterized framework that characterizes neuronal states and their time derivatives via two distinct negative real parameters. Extending this idea, the studies in [47–53] further advanced the methodology by introducing the characteristic method, which generalizes the parameterization scheme to encompass not only identical negative real roots but also pairs of complex conjugate roots.

Motivated by the aforementioned studies, we investigate GES for the DFINNs described by Equations (1) and (2) using the characteristic method. Our goal is to avoid the drawbacks common to both reduced and non-reduced order methods: the requirement to construct LKFs and solve LMIs. This work presents three main contributions. First, we introduce a novel definition of GES that extends existing formulations by explicitly accounting for both the system states and their time derivatives. Second, we apply the characteristic method to analyze the error dynamics between the drive and response systems (1) and (2), thereby circumventing both reduced order transformations and the construction of LKFs. Third, we formulate the resulting stability conditions as simple linear scalar inequalities that can be directly verified without relying on MATLAB based toolboxes for solving LMIs.

The remainder of this paper unfolds as follows. In Section 2, we present essential definitions and preliminary results. Section 3 establishes three sufficient conditions that ensure GES of the inertial neural networks described by (1) and (2). In Section 4, we provide an illustrative example equipped with several distinct controllers, each accompanied by numerical simulations to verify the theoretical findings and demonstrate their practical effectiveness. Finally, in Section 5, we conclude the paper with a concise summary of our main contributions.

2. Preliminaries

For any bounded and continuous function $f(t)$, define $f^+ = \sup_{t \in \mathbb{R}} |f(t)|$. The initial conditions for systems (1) and (2) are given by

$$\begin{cases} z_\alpha(s) = \varphi_\alpha(s), -\tau \leq s \leq 0, \varphi_\alpha \in C([-\tau, 0], \mathbb{R}), \dot{z}_\alpha(0) = \bar{\varphi}_\alpha \in \mathbb{R}, \alpha \in \Lambda, \\ y_\alpha(s) = \psi_\alpha(s), -\tau \leq s \leq 0, \psi_\alpha \in C([-\tau, 0], \mathbb{R}), \dot{y}_\alpha(0) = \bar{\psi}_\alpha \in \mathbb{R}, \alpha \in \Lambda. \end{cases} \quad (3)$$

Define the synchronization error as $x_\alpha(t) = y_\alpha(t) - z_\alpha(t)$. By subtracting system (1) from system (2), the corresponding error dynamics are derived as

$$\ddot{x}_\alpha(t) + m_\alpha \dot{x}_\alpha(t) + l_\alpha x_\alpha(t) = G_\alpha(t) + u_\alpha(t), \quad (4)$$

where $G_\alpha(t) = \sum_{\beta=1}^n a_{\alpha\beta}(t)H_\beta(x_\beta(t)) + \sum_{\beta=1}^n b_{\alpha\beta}(t)H_\beta(x_\beta(t - \tau_{\alpha\beta}(t))) + \bigwedge_{\beta=1}^n p_{\alpha\beta}(t)H_\beta(x_\beta(t - \tau_{\alpha\beta}(t))) + \bigvee_{\beta=1}^n q_{\alpha\beta}(t)H_\beta(x_\beta(t - \tau_{\alpha\beta}(t)))$ and $H_\beta(x_\beta(\cdot)) = h_\beta(y_\beta(\cdot)) - h_\beta(z_\beta(\cdot))$.

We design the control inputs for the error system (4) as

$$u_\alpha(t) = \tilde{m}_\alpha \dot{x}_\alpha(t) + \tilde{l}_\alpha x_\alpha(t), \quad (5)$$

choosing the parameters such that $\tilde{m}_\alpha < m_\alpha$ and $\tilde{l}_\alpha < l_\alpha$.

Next, we substitute the controller (5) into the error dynamics (4) and derive the resulting error system

$$\ddot{x}_\alpha(t) + \bar{m}_\alpha \dot{x}_\alpha(t) + \bar{l}_\alpha x_\alpha(t) = G_\alpha(t) \quad (6)$$

where $\bar{m}_\alpha = m_\alpha - \tilde{m}_\alpha$ and $\bar{l}_\alpha = l_\alpha - \tilde{l}_\alpha$.

We express the homogeneous counterpart of system (6) as $\dot{x}_\alpha(t) + \bar{m}_\alpha x_\alpha(t) + \bar{l}_\alpha x_\alpha(t) = 0$, which leads to the characteristic equation $\lambda^2 + \bar{m}_\alpha \lambda + \bar{l}_\alpha = 0$, with discriminant $\Delta_\alpha = \bar{m}_\alpha^2 - 4\bar{l}_\alpha$. Following the approach in [50,51], we explicitly derive solution formulas for the controlled error system (6) under the initial conditions (3) by analyzing three cases based on the sign of Δ_α , as detailed below.

$$x_\alpha(t) = \frac{-1}{\sqrt{\Delta_\alpha}} [(v_\alpha e^{w_\alpha t} - w_\alpha e^{v_\alpha t})x_\alpha(0) + (e^{v_\alpha t} - e^{w_\alpha t})\dot{x}_\alpha(0) + \int_0^t (e^{v_\alpha(t-s)} - e^{w_\alpha(t-s)})G_\alpha(s)ds], \quad (7)$$

where $\Delta_\alpha > 0$, $v_\alpha = \frac{-\bar{m}_\alpha - \sqrt{\Delta_\alpha}}{2}$ and $w_\alpha = \frac{-\bar{m}_\alpha + \sqrt{\Delta_\alpha}}{2}$;

$$x_\alpha(t) = [(1 + \frac{\bar{m}_\alpha}{2}t)x_\alpha(0) + t\dot{x}_\alpha(0)]e^{-\frac{\bar{m}_\alpha}{2}t} + \int_0^t (t-s)e^{-\frac{\bar{m}_\alpha}{2}(t-s)}G_\alpha(s)ds, \quad (8)$$

where $\Delta_\alpha = 0$;

$$x_\alpha(t) = [\frac{\sqrt{\bar{l}_\alpha}}{\nu_\alpha} \sin(\nu_\alpha t + \varphi_\alpha)x_\alpha(0) + \frac{\sin \nu_\alpha t}{\nu_\alpha} \dot{x}_\alpha(0)]e^{-\frac{\bar{m}_\alpha}{2}t} + \int_0^t \frac{\sin \nu_\alpha(t-s)}{\nu_\alpha} e^{-\frac{\bar{m}_\alpha}{2}(t-s)}G_\alpha(s)ds, \quad (9)$$

where $\Delta_\alpha < 0$, $\nu_\alpha = \frac{1}{2}\sqrt{-\Delta_\alpha}$, $\varphi_\alpha = \arctan \frac{2\nu_\alpha}{\bar{m}_\alpha}$.

For the sake of clarity and conciseness, we introduce the following definition and lemma.

Definition 1. We say that the drive-response systems (1) and (2) achieve GES under the designed controller (5), if for any pair of solutions $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T$ of system (1) and $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$ of system (2), there exist positive constants M , L and μ , such that for $t \geq 0$ and $\alpha \in \Omega$,

$$|x_\alpha(t)| = |y_\alpha(t) - z_\alpha(t)| < Me^{-\mu t}, \quad |\dot{x}_\alpha(t)| = |\dot{y}_\alpha(t) - \dot{z}_\alpha(t)| < Le^{-\mu t}.$$

Lemma 1 (see [3]). Let $w_\beta, r_\beta, \theta_{\alpha\beta}, \kappa_{\alpha\beta} \in \mathbb{R}$, $f_\beta : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions, and $\alpha, \beta \in \Omega$, then we have

$$\left| \bigwedge_{\beta=1}^n \theta_{\alpha\beta} f_\beta(w_\beta) - \bigwedge_{\beta=1}^n \theta_{\alpha\beta} f_\beta(r_\beta) \right| \leq \sum_{\beta=1}^n |\theta_{\alpha\beta}| |f_\beta(w_\beta) - f_\beta(r_\beta)|,$$

and

$$\left| \bigvee_{\beta=1}^n \kappa_{\alpha\beta} f_\beta(w_\beta) - \bigvee_{\beta=1}^n \kappa_{\alpha\beta} f_\beta(r_\beta) \right| \leq \sum_{\beta=1}^n |\kappa_{\alpha\beta}| |f_\beta(w_\beta) - f_\beta(r_\beta)|.$$

Remark 1. Definition 1 broadens the concept of global asymptotic synchronization (GAS) that earlier studies investigated in [10,30,43,44] for a class of fuzzy inertial neural networks with time-varying delays. Specifically, GAS means that $\lim_{t \rightarrow +\infty} |y_\alpha(t) - x_\alpha(t)| = 0$. In contrast, Definition 1 focuses on GES of systems (1) and (2). It guarantees not only GAS, but also that the synchronization error $y_\alpha(t) - x_\alpha(t)$ approaches zero at an exponential rate. In addition, Definition 1 explicitly requires the error derivative $\dot{y}_\alpha(t) - \dot{x}_\alpha(t)$ to decay exponentially to zero as well. This refinement strengthens and supplements existing results on GAS for inertial neural networks reported in [20–23,25,27–29,33,41,42].

3. Main Results

This section establishes sufficient conditions for the GES of DFNNs (1) and (2) based on the controller (5), by exploiting the explicit solution representations given in Equations (7)–(9). Because these solution forms are highly sensitive to the sign of Δ_α , three separate theorems are developed to characterize the GES of system (1) and (2) under distinct parameter regimes. For notational simplicity, we denote the state vector of error system (6) by $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ where each component satisfies the initial conditions specified in (3).

Theorem 1. Under assumption (A), then the drive system (1) and the response system (2) are globally exponentially synchronised under the controllers (5) if the following conditions hold

$$\Delta_\alpha > 0, \quad \sum_{\beta=1}^n (a_{\alpha\beta}^+ + b_{\alpha\beta}^+ + p_{\alpha\beta}^+ + q_{\alpha\beta}^+)k_\beta < \bar{l}_\alpha, \quad \alpha \in \Omega. \quad (10)$$

Proof. We define $\eta_\alpha^\mu = \sum_{\beta=1}^n (a_{\alpha\beta}^+ + b_{\alpha\beta}^+ e^{\mu\tau} + p_{\alpha\beta}^+ e^{\mu\tau} + q_{\alpha\beta}^+ e^{\mu\tau}) k_\beta$. Setting $\mu = 0$ yields $\eta_\alpha^0 = \sum_{\beta=1}^n (a_{\alpha\beta}^+ + b_{\alpha\beta}^+ + p_{\alpha\beta}^+ + q_{\alpha\beta}^+) k_\beta$. Assume that condition (7) holds, i.e., $\eta_\alpha^0 < \bar{l}_\alpha = v_\alpha w_\alpha$. Then we can choose a sufficiently large positive constant M and a sufficiently little $\mu \in (0, -\frac{1}{2} \max_{\alpha \in \Lambda} w_\alpha)$ such that the following inequality is satisfied for all $\alpha \in \Lambda$

$$\frac{|v_\alpha| + 1}{M\sqrt{\Delta_\alpha}} \chi_{\varphi, \psi} + \frac{\eta_\alpha^\mu}{(v_\alpha + \mu)(w_\alpha + \mu)} < 1, \quad (11)$$

where $\chi_{\varphi, \psi} = \max_{\alpha \in \Lambda} \{|\varphi_\alpha(0) - \psi_\alpha(0)|, \max_{\alpha \in \Lambda} |\bar{\varphi}_\alpha - \bar{\psi}_\alpha|\}$.

Step 1: Given the initial conditions (3), we have $|x_\alpha(t)| < Me^{-\mu t}$ for $t \in [-\tau, 0]$ and $\alpha \in \Lambda$. To establish that this bound remains valid for all $t \geq 0$, we proceed by contradiction, assume there exist an index $\alpha_1 \in \Lambda$ and a time $\sigma_1 > 0$ such that

$$\begin{cases} |x_{\alpha_1}(t)| < Me^{-\mu t}, t \in [0, \sigma_1], \alpha \in \Lambda, \\ |x_{\alpha_1}(\sigma_1)| = Me^{-\mu\sigma_1}. \end{cases} \quad (12)$$

Based on assumption (A), (7), (12) and Lemma 1, we obtain

$$\begin{aligned} |x_{\alpha_1}(\sigma_1)| &= \left| \frac{-1}{\sqrt{\Delta_{\alpha_1}}} [(v_{\alpha_1} e^{w_{\alpha_1}\sigma_1} - w_{\alpha_1} e^{v_{\alpha_1}\sigma_1}) x_{\alpha_1}(0) + (e^{v_{\alpha_1}\sigma_1} - e^{w_{\alpha_1}\sigma_1}) \dot{x}_{\alpha_1}(0) \right. \\ &\quad \left. + \int_0^{\sigma_1} (e^{v_{\alpha_1}(\sigma_1-s)} - e^{w_{\alpha_1}(\sigma_1-s)}) G_{\alpha_1}(s) ds] \right| \\ &\leq \frac{1}{\sqrt{\Delta_{\alpha_1}}} [(|v_{\alpha_1}| + 1) \chi_{\varphi, \psi} e^{w_{\alpha_1}\sigma_1} + \int_0^{\sigma_1} |e^{v_{\alpha_1}(\sigma_1-s)} - e^{w_{\alpha_1}(\sigma_1-s)}| \\ &\quad \times (| \sum_{\beta=1}^n a_{\alpha_1\beta}(s) H_\beta(x_\beta(s)) | + | \sum_{\beta=1}^n b_{\alpha_1\beta}(s) H_\beta(x_\beta(s - \tau_{\alpha_1\beta}(s))) | \\ &\quad + | \sum_{\beta=1}^n p_{\alpha_1\beta}(s) H_\beta(x_\beta(s - \tau_{\alpha_1\beta}(s))) | + | \sum_{\beta=1}^n q_{\alpha_1\beta}(s) H_\beta(x_\beta(s - \tau_{\alpha_1\beta}(s))) |) ds] \\ &\leq \frac{1}{\sqrt{\Delta_{\alpha_1}}} [(|v_{\alpha_1}| + 1) \chi_{\varphi, \psi} e^{-\mu\sigma_1} + \int_0^{\sigma_1} (e^{w_{\alpha_1}(\sigma_1-s)} - e^{v_{\alpha_1}(\sigma_1-s)}) \\ &\quad \times (\sum_{\beta=1}^n |a_{\alpha_1\beta}(s)| |H_\beta(x_\beta(s))| + \sum_{\beta=1}^n |b_{\alpha_1\beta}(s)| |H_\beta(x_\beta(s - \tau_{\alpha_1\beta}(s)))| \\ &\quad + \sum_{\beta=1}^n |p_{\alpha_1\beta}(s)| |H_\beta(x_\beta(s - \tau_{\alpha_1\beta}(s)))| + \sum_{\beta=1}^n |q_{\alpha_1\beta}(s)| |H_\beta(x_\beta(s - \tau_{\alpha_1\beta}(s)))|) ds] \\ &\leq \frac{1}{\sqrt{\Delta_{\alpha_1}}} [(|v_{\alpha_1}| + 1) \chi_{\varphi, \psi} e^{-\mu\sigma_1} + \int_0^{\sigma_1} (e^{w_{\alpha_1}(\sigma_1-s)} - e^{v_{\alpha_1}(\sigma_1-s)}) \\ &\quad \times (\sum_{\beta=1}^n a_{\alpha_1\beta}^+ k_\beta |x_\beta(s)| + \sum_{\beta=1}^n b_{\alpha_1\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha_1\beta}(s))| \\ &\quad + \sum_{\beta=1}^n p_{\alpha_1\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha_1\beta}(s))| + \sum_{\beta=1}^n q_{\alpha_1\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha_1\beta}(s))|) ds] \\ &\leq \frac{1}{\sqrt{\Delta_{\alpha_1}}} [(|v_{\alpha_1}| + 1) \chi_{\varphi, \psi} e^{-\mu\sigma_1} + \int_0^{\sigma_1} (e^{w_{\alpha_1}(\sigma_1-s)} - e^{v_{\alpha_1}(\sigma_1-s)}) \\ &\quad \times (\sum_{\beta=1}^n a_{\alpha_1\beta}^+ k_\beta Me^{-\mu s} + \sum_{\beta=1}^n b_{\alpha_1\beta}^+ k_\beta Me^{-\mu(s - \tau_{\alpha_1\beta}(s))} \\ &\quad + \sum_{\beta=1}^n p_{\alpha_1\beta}^+ k_\beta Me^{-\mu(s - \tau_{\alpha_1\beta}(s))} + \sum_{\beta=1}^n q_{\alpha_1\beta}^+ k_\beta Me^{-\mu(s - \tau_{\alpha_1\beta}(s))}) ds] \\ &\leq \frac{M}{\sqrt{\Delta_{\alpha_1}}} e^{-\mu\sigma_1} \left[\frac{|v_{\alpha_1}| + 1}{M} \chi_{\varphi, \psi} + \int_0^{\sigma_1} (e^{(w_{\alpha_1} + \mu)(\sigma_1-s)} - e^{(v_{\alpha_1} + \mu)(\sigma_1-s)}) \left(\sum_{\beta=1}^n a_{\alpha_1\beta}^+ k_\beta \right. \right. \\ &\quad \left. \left. + \sum_{\beta=1}^n b_{\alpha_1\beta}^+ k_\beta e^{\mu\tau_{\alpha_1\beta}(s)} + \sum_{\beta=1}^n p_{\alpha_1\beta}^+ k_\beta e^{\mu\tau_{\alpha_1\beta}(s)} + \sum_{\beta=1}^n q_{\alpha_1\beta}^+ k_\beta e^{\mu\tau_{\alpha_1\beta}(s)} \right) ds \right] \\ &\leq Me^{-\mu\sigma_1} \left[\frac{|v_{\alpha_1}| + 1}{M\sqrt{\Delta_{\alpha_1}}} \chi_{\varphi, \psi} + \frac{\eta_{\alpha_1}^\mu}{\sqrt{\Delta_{\alpha_1}}} \int_0^{\sigma_1} (e^{(w_{\alpha_1} + \mu)(\sigma_1-s)} - e^{(v_{\alpha_1} + \mu)(\sigma_1-s)}) ds \right] \\ &= Me^{-\mu\sigma_1} \left[\frac{|v_{\alpha_1}| + 1}{M\sqrt{\Delta_{\alpha_1}}} \chi_{\varphi, \psi} + \frac{\eta_{\alpha_1}^\mu}{(v_{\alpha_1} + \mu)(w_{\alpha_1} + \mu)} (1 - g(\sigma_1)) \right] \end{aligned} \quad (13)$$

where $g(t) = \frac{1}{\sqrt{\Delta_{\alpha_1}}} [(w_{\alpha_1} + \mu)e^{(v_{\alpha_1} + \mu)t} - (v_{\alpha_1} + \mu)e^{(w_{\alpha_1} + \mu)t}]$, $t \in [0, +\infty)$.

Because $g'(t) \leq 0$ for all $t \in [0, +\infty)$, the function $g(t)$ is non-increasing on this interval. Moreover, since $g(t)$ decreases strictly on $[0, \sigma_1]$, and given the boundary conditions $\lim_{t \rightarrow +\infty} g(t) = 0$ and $g(0) = 1$, we conclude that $g(\sigma_1) \in [0, 1)$. Combining this result with inequalities (11) and (13), we obtain

$$|x_{\alpha_1}(\sigma_1)| < Me^{-\mu\sigma_1} \left[\frac{|v_{\alpha_1}| + 1}{M\sqrt{\Delta_{\alpha_1}}} \chi_{\varphi, \psi} + \frac{\eta_{\alpha_1}^\mu}{(v_{\alpha_1} + \mu)(w_{\alpha_1} + \mu)} \right] < Me^{-\mu\sigma_1}.$$

This clearly leads to a contradiction. Therefore, we conclude that the inequality

$$|x_\alpha(t)| < Me^{-\mu t}, \alpha \in \Lambda \quad (14)$$

holds for all $t \in [0, +\infty)$.

Step 2: We now prove that there exists a positive constant L such that

$$|\dot{x}_\alpha(t)| < Le^{-\mu t}, \alpha \in \Lambda \quad (15)$$

holds for all $t \in [0, +\infty)$.

Because $\Delta_\alpha > 0$, we differentiate both sides of Equation (7) with respect to t and obtain

$$\begin{aligned} \dot{x}_\alpha(t) &= \frac{-1}{\sqrt{\Delta_\alpha}} [\bar{l}_\alpha(e^{w_\alpha t} - e^{v_\alpha t})x_\alpha(0) + (v_\alpha e^{v_\alpha t} - w_\alpha e^{w_\alpha t})\dot{x}_\alpha(0) \\ &\quad + \int_0^t (v_\alpha e^{v_\alpha(t-s)} - w_\alpha e^{w_\alpha(t-s)})G_\alpha(s)ds]. \end{aligned} \quad (16)$$

Based on assumption (A), (14), (16) and Lemma 1, we deduce that

$$\begin{aligned} |\dot{x}_\alpha(t)| &= \frac{1}{\sqrt{\Delta_\alpha}} |\bar{l}_\alpha(e^{w_\alpha t} - e^{v_\alpha t})x_\alpha(0) + (v_\alpha e^{v_\alpha t} - w_\alpha e^{w_\alpha t})\dot{x}_\alpha(0) \\ &\quad + \int_0^t (v_\alpha e^{v_\alpha(t-s)} - w_\alpha e^{w_\alpha(t-s)})G_\alpha(s)ds| \\ &\leq \frac{1}{\sqrt{\Delta_\alpha}} [(\bar{l}_\alpha + |v_\alpha|)\chi_{\varphi,\psi}e^{w_\alpha t} + \int_0^t |v_\alpha e^{v_\alpha(t-s)} - w_\alpha e^{w_\alpha(t-s)}| \\ &\quad \times (|\sum_{\beta=1}^n a_{\alpha\beta}(s)H_\beta(x_\beta(s))| + |\sum_{\beta=1}^n b_{\alpha\beta}(s)H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))| \\ &\quad + |\sum_{\beta=1}^n p_{\alpha\beta}(s)H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))| + |\sum_{\beta=1}^n q_{\alpha\beta}(s)H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))|)ds] \\ &\leq \frac{1}{\sqrt{\Delta_\alpha}} [(\bar{l}_\alpha + |v_\alpha|)\chi_{\varphi,\psi}e^{-\mu t} + \int_0^t |v_\alpha e^{v_\alpha(t-s)} - w_\alpha e^{w_\alpha(t-s)}| \\ &\quad \times (\sum_{\beta=1}^n |a_{\alpha\beta}(s)| |H_\beta(x_\beta(s))| + \sum_{\beta=1}^n |b_{\alpha\beta}(s)| |H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))| \\ &\quad + \sum_{\beta=1}^n |p_{\alpha\beta}(s)| |H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))| + \sum_{\beta=1}^n |q_{\alpha\beta}(s)| |H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))|)ds] \\ &\leq \frac{1}{\sqrt{\Delta_\alpha}} [(\bar{l}_\alpha + |v_\alpha|)\chi_{\varphi,\psi}e^{-\mu t} + \int_0^t |v_\alpha e^{v_\alpha(t-s)} - w_\alpha e^{w_\alpha(t-s)}| \\ &\quad \times (\sum_{\beta=1}^n a_{\alpha\beta}^+ k_\beta |x_\beta(s)| + \sum_{\beta=1}^n b_{\alpha\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha\beta}(s))| \\ &\quad + \sum_{\beta=1}^n p_{\alpha\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha\beta}(s))| + \sum_{\beta=1}^n q_{\alpha\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha\beta}(s))|)ds] \\ &\leq e^{-\mu t} [\frac{\bar{l}_\alpha + |v_\alpha|}{\sqrt{\Delta_\alpha}} \chi_{\varphi,\psi} + \frac{1}{\sqrt{\Delta_\alpha}} \int_0^t (|v_\alpha|e^{(v_\alpha + \mu)(t-s)} + |w_\alpha|e^{(w_\alpha + \mu)(t-s)}) \\ &\quad \times (\sum_{\beta=1}^n a_{\alpha\beta}^+ k_\beta Me^{-\mu s} + \sum_{\beta=1}^n b_{\alpha\beta}^+ k_\beta Me^{-\mu(s - \tau_{\alpha\beta}(s))} \\ &\quad + \sum_{\beta=1}^n p_{\alpha\beta}^+ k_\beta Me^{-\mu(s - \tau_{\alpha\beta}(s))} + \sum_{\beta=1}^n q_{\alpha\beta}^+ k_\beta Me^{-\mu(s - \tau_{\alpha\beta}(s))})ds] \\ &\leq e^{-\mu t} [\frac{\bar{l}_\alpha + |v_\alpha|}{\sqrt{\Delta_\alpha}} \chi_{\varphi,\psi} + \frac{M\eta_\alpha^\mu}{\sqrt{\Delta_\alpha}} \int_0^t (|v_\alpha|e^{(v_\alpha + \mu)(t-s)} + |w_\alpha|e^{(w_\alpha + \mu)(t-s)})ds] \\ &= e^{-\mu t} [\frac{\bar{l}_\alpha + |v_\alpha|}{\sqrt{\Delta_\alpha}} \chi_{\varphi,\psi} + \frac{M\eta_\alpha^\mu}{\sqrt{\Delta_\alpha}} (\frac{|v_\alpha|(1 - e^{(v_\alpha + \mu)t})}{|v_\alpha + \mu|} + \frac{|w_\alpha|(1 - e^{(w_\alpha + \mu)t})}{|w_\alpha + \mu|})] \\ &< Le^{-\lambda t}, \end{aligned}$$

where $L = \max_{\alpha \in \Lambda} \{ \frac{\bar{l}_\alpha + |v_\alpha|}{\sqrt{\Delta_\alpha}} \chi_{\varphi,\psi} + \frac{M\eta_\alpha^\mu}{\sqrt{\Delta_\alpha}} (\frac{|v_\alpha|}{|v_\alpha + \mu|} + \frac{|w_\alpha|}{|w_\alpha + \mu|}) \}$. We have thus proved Theorem 1. $\square$

Theorem 2. Under assumption (A), then the drive system (1) and the response system (2) are globally exponentially synchronised under the controllers (5) if the following conditions hold

$$\Delta_\alpha = 0, \sum_{\beta=1}^n (a_{\alpha\beta}^+ + b_{\alpha\beta}^+ + p_{\alpha\beta}^+ + q_{\alpha\beta}^+) k_\beta < \bar{l}_\alpha, \alpha \in \Lambda. \quad (17)$$

Proof. Assume condition (17) holds, i.e., $\eta_\alpha^0 < \bar{l}_\alpha = \frac{\bar{m}_\alpha^2}{4}$. Then we can choose a sufficiently large positive constant M and a sufficient little $\mu \in (0, \frac{1}{4} \min_{\alpha \in \Lambda} \bar{m}_\alpha)$ such that for every $\alpha \in \Lambda$, the following inequality is satisfied

$$(1 + \frac{2 + \bar{m}_\alpha}{e(\bar{m}_\alpha - 2\mu)}) \frac{\chi_{\Phi,\Psi}}{M} + \frac{4\eta_\alpha^\mu}{(\bar{m}_\alpha - 2\mu)^2} < 1. \quad (18)$$

Step 1: The initial condition (3) ensures that $|x_\alpha(t)| < Me^{-\mu t}$ holds for all $t \in [-\tau, 0]$ and every $\alpha \in \Lambda$. To prove that this bound continues to hold for all $t \geq 0$, we argue by contradiction, suppose there exist an index $\alpha_2 \in \Lambda$ and a time $\sigma_2 > 0$ such that

$$\begin{cases} |x_{\alpha_2}(t)| < Me^{-\mu t}, t \in [0, \sigma_2), \alpha \in \Lambda, \\ |x_{\alpha_2}(\sigma_2)| = Me^{-\mu\sigma_2}. \end{cases} \quad (19)$$

By applying assumption (A), (8), (18), (19) and Lemma 1, as well as $\max_{x \geq 0} xe^{-x} = \frac{1}{e}$, we have

$$\begin{aligned} |x_{\alpha_2}(\sigma_2)| &= |[(1 + \frac{\bar{m}_{\alpha_2}}{2}\sigma_2)x_{\alpha_2}(0) + \sigma_2\dot{x}_{\alpha_2}(0)]e^{-\frac{\bar{m}_{\alpha_2}}{2}\sigma_2} + \int_0^{\sigma_2}(\sigma_2 - s)e^{-\frac{\bar{m}_{\alpha_2}}{2}(\sigma_2 - s)}G_{\alpha_2}(s)ds| \\ &\leq [e^{-(\frac{\bar{m}_{\alpha_2}}{2} - \mu)\sigma_2} + (1 + \frac{\bar{m}_{\alpha_2}}{2})\sigma_2e^{-(\frac{\bar{m}_{\alpha_2}}{2} - \mu)\sigma_2}]\chi_{\Phi, \Psi}e^{-\mu\sigma_2} + \int_0^{\sigma_2}(\sigma_2 - s)e^{-\frac{\bar{m}_{\alpha_2}}{2}(\sigma_2 - s)} \\ &\quad \times (|\sum_{\beta=1}^n a_{\alpha_2\beta}(s)H_\beta(x_\beta(s))| + |\sum_{\beta=1}^n b_{\alpha_2\beta}(s)H_\beta(x_\beta(s - \tau_{\alpha_2\beta}(s)))| \\ &\quad + |\sum_{\beta=1}^n p_{\alpha_2\beta}(s)H_\beta(x_\beta(s - \tau_{\alpha_2\beta}(s)))| + |\sum_{\beta=1}^n q_{\alpha_2\beta}(s)H_\beta(x_\beta(s - \tau_{\alpha_2\beta}(s)))|)ds \\ &\leq (1 + \frac{2 + \bar{m}_{\alpha_2}}{e(\bar{m}_{\alpha_2} - 2\mu)})\chi_{\Phi, \Psi}e^{-\mu\sigma_2} + \int_0^{\sigma_2}(\sigma_2 - s)e^{-\frac{\bar{m}_{\alpha_2}}{2}(\sigma_2 - s)} \\ &\quad \times (\sum_{\beta=1}^n |a_{\alpha_2\beta}(s)||H_\beta(x_\beta(s))| + \sum_{\beta=1}^n |b_{\alpha_2\beta}(s)||H_\beta(x_\beta(s - \tau_{\alpha_2\beta}(s)))| \\ &\quad + \sum_{\beta=1}^n |p_{\alpha_2\beta}(s)||H_\beta(x_\beta(s - \tau_{\alpha_2\beta}(s)))| + \sum_{\beta=1}^n |q_{\alpha_2\beta}(s)||H_\beta(x_\beta(s - \tau_{\alpha_2\beta}(s)))|)ds \\ &\leq (1 + \frac{2 + \bar{m}_{\alpha_2}}{e(\bar{m}_{\alpha_2} - 2\mu)})\chi_{\Phi, \Psi}e^{-\mu\sigma_2} + \int_0^{\sigma_2}(\sigma_2 - s)e^{-\frac{\bar{m}_{\alpha_2}}{2}(\sigma_2 - s)} \\ &\quad \times (\sum_{\beta=1}^n a_{\alpha_2\beta}^+ k_\beta |x_\beta(s)| + \sum_{\beta=1}^n b_{\alpha_2\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha_2\beta}(s))| \\ &\quad + \sum_{\beta=1}^n p_{\alpha_2\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha_2\beta}(s))| + \sum_{\beta=1}^n q_{\alpha_2\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha_2\beta}(s))|)ds \\ &\leq (1 + \frac{2 + \bar{m}_{\alpha_2}}{e(\bar{m}_{\alpha_2} - 2\mu)})\chi_{\Phi, \Psi}e^{-\mu\sigma_2} + \int_0^{\sigma_2}(\sigma_2 - s)e^{-\frac{\bar{m}_{\alpha_2}}{2}(\sigma_2 - s)} \\ &\quad \times (\sum_{\beta=1}^n a_{\alpha_2\beta}^+ k_\beta Me^{-\mu s} + \sum_{\beta=1}^n b_{\alpha_2\beta}^+ k_\beta Me^{-\mu(s - \tau_{\alpha_2\beta}(s))} \\ &\quad + \sum_{\beta=1}^n p_{\alpha_2\beta}^+ k_\beta Me^{-\mu(s - \tau_{\alpha_2\beta}(s))} + \sum_{\beta=1}^n q_{\alpha_2\beta}^+ k_\beta Me^{-\mu(s - \tau_{\alpha_2\beta}(s))})ds \\ &\leq Me^{-\mu\sigma_2}[(1 + \frac{2 + \bar{m}_{\alpha_2}}{e(\bar{m}_{\alpha_2} - 2\mu)})\frac{\chi_{\Phi, \Psi}}{M} + \eta_{\alpha_2}^\mu \int_0^{\sigma_2}(\sigma_2 - s)e^{-(\frac{\bar{m}_{\alpha_2}}{2} - \mu)(\sigma_2 - s)}ds] \\ &= Me^{-\mu\sigma_2}[(1 + \frac{2 + \bar{m}_{\alpha_2}}{e(\bar{m}_{\alpha_2} - 2\mu)})\frac{\chi_{\Phi, \Psi}}{M} + \eta_{\alpha_2}^\mu \frac{4 - 2(2 + (\bar{m}_{\alpha_2} - 2\mu)\sigma_2)e^{-(\frac{\bar{m}_{\alpha_2}}{2} - \mu)\sigma_2}}{(\bar{m}_{\alpha_2} - 2\mu)^2}] \\ &< Me^{-\mu\sigma_2}[(1 + \frac{2 + \bar{m}_{\alpha_2}}{e(\bar{m}_{\alpha_2} - 2\mu)})\frac{\chi_{\Phi, \Psi}}{M} + \frac{4\eta_{\alpha_2}^\mu}{(\bar{m}_{\alpha_2} - 2\mu)^2}] \\ &< Me^{-\mu\sigma_2}. \end{aligned}$$

The above leads to a contradiction. Therefore, we conclude that inequality (14) is valid for all $t \in [0, +\infty)$ and every $\alpha \in \Lambda$.

Step 2: We now establish the existence of a positive constant L for which inequality (15) is satisfied. Assuming $\Delta_\alpha = 0$, we differentiate both sides of Equation (8) with respect to t and obtain

$$\dot{x}_\alpha(t) = [-\bar{l}_\alpha t x_\alpha(0) + (1 - \frac{\bar{m}_\alpha}{2}t)\dot{x}_\alpha(0)]e^{-\frac{\bar{m}_\alpha}{2}t} + \int_0^t (1 - \frac{\bar{m}_\alpha}{2}(t - s))e^{-\frac{\bar{m}_\alpha}{2}(t - s)}G_\alpha(s)ds. \quad (20)$$

It follows from assumption (A), (14), (20), Lemma 1 and $\max_{x \geq 0} xe^{-x} = \frac{1}{e}$ that

$$\begin{aligned} |\dot{x}_\alpha(t)| &= |[-\bar{l}_\alpha t x_\alpha(0) + (1 - \frac{\bar{m}_\alpha}{2}t)\dot{x}_\alpha(0)]e^{-\frac{\bar{m}_\alpha}{2}t} + \int_0^t (1 - \frac{\bar{m}_\alpha}{2}(t - s))e^{-\frac{\bar{m}_\alpha}{2}(t - s)}G_\alpha(s)ds| \\ &\leq [e^{-(\frac{\bar{m}_\alpha}{2} - \mu)t} + (\bar{l}_\alpha + \frac{\bar{m}_\alpha}{2})te^{-(\frac{\bar{m}_\alpha}{2} - \mu)t}]\chi_{\Phi, \Psi}e^{-\mu t} + \int_0^t |1 - \frac{\bar{m}_\alpha}{2}(t - s)|e^{-\frac{\bar{m}_\alpha}{2}(t - s)} \\ &\quad \times (|\sum_{\beta=1}^n a_{\alpha\beta}(s)H_\beta(x_\beta(s))| + |\sum_{\beta=1}^n b_{\alpha\beta}(s)H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))| \\ &\quad + |\sum_{\beta=1}^n p_{\alpha\beta}(s)H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))| + |\sum_{\beta=1}^n q_{\alpha\beta}(s)H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))|)ds \\ &\leq (1 + \frac{2\bar{l}_\alpha + \bar{m}_\alpha}{e(\bar{m}_\alpha - 2\mu)})\chi_{\Phi, \Psi}e^{-\mu t} + \int_0^t (1 + \frac{\bar{m}_\alpha}{2}(t - s))e^{-\frac{\bar{m}_\alpha}{2}(t - s)} \\ &\quad \times (\sum_{\beta=1}^n |a_{\alpha\beta}(s)||H_\beta(x_\beta(s))| + \sum_{\beta=1}^n |b_{\alpha\beta}(s)||H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))| \\ &\quad + \sum_{\beta=1}^n |p_{\alpha\beta}(s)||H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))| + \sum_{\beta=1}^n |q_{\alpha\beta}(s)||H_\beta(x_\beta(s - \tau_{\alpha\beta}(s)))|)ds \end{aligned}$$

$$\begin{aligned}
&\leq (1 + \frac{2\bar{l}_\alpha + \bar{m}_\alpha}{e(\bar{m}_\alpha - 2\mu)})\chi_{\Phi, \Psi} e^{-\mu t} + \int_0^t (1 + \frac{\bar{m}_\alpha}{2}(t-s))e^{-\frac{\bar{m}_\alpha}{2}(t-s)} \\
&\quad \times (\sum_{\beta=1}^n a_{\alpha\beta}^+ k_\beta |x_\beta(s)| + \sum_{\beta=1}^n b_{\alpha\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha\beta}(s))| \\
&\quad + \sum_{\beta=1}^n p_{\alpha\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha\beta}(s))| + \sum_{\beta=1}^n q_{\alpha\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha\beta}(s))|) ds \\
&\leq (1 + \frac{2\bar{l}_\alpha + \bar{m}_\alpha}{e(\bar{m}_\alpha - 2\mu)})\chi_{\Phi, \Psi} e^{-\mu t} + \int_0^t (1 + \frac{\bar{m}_\alpha}{2}(t-s))e^{-\frac{\bar{m}_\alpha}{2}(t-s)} \\
&\quad \times (\sum_{\beta=1}^n a_{\alpha\beta}^+ k_\beta M e^{-\mu s} + \sum_{\beta=1}^n b_{\alpha\beta}^+ k_\beta M e^{-\mu(s - \tau_{\alpha\beta}(s))} \\
&\quad + \sum_{\beta=1}^n p_{\alpha\beta}^+ k_\beta M e^{-\mu(s - \tau_{\alpha\beta}(s))} + \sum_{\beta=1}^n q_{\alpha\beta}^+ k_\beta M e^{-\mu(s - \tau_{\alpha\beta}(s))}) ds \\
&\leq e^{-\mu t} [(1 + \frac{2\bar{l}_\alpha + \bar{m}_\alpha}{e(\bar{m}_\alpha - 2\mu)})\chi_{\Phi, \Psi} + M\eta_\alpha^\mu \int_0^t (1 + \frac{\bar{m}_\alpha}{2}(t-s))e^{-(\frac{\bar{m}_\alpha}{2} - \mu)(t-s)} ds] \\
&= e^{-\mu t} [(1 + \frac{2\bar{l}_\alpha + \bar{m}_\alpha}{e(\bar{m}_\alpha - 2\mu)})\chi_{\Phi, \Psi} + M\eta_\alpha^\mu (-\frac{(\bar{m}_\alpha - 2\mu)\bar{m}_\alpha t + 4\bar{m}_\alpha - 4\mu}{(\bar{m}_\alpha - 2\mu)^2} e^{-(\frac{\bar{m}_\alpha}{2} - \mu)t} \\
&\quad + \frac{4(\bar{m}_\alpha - \mu)}{(\bar{m}_\alpha - 2\mu)^2})] \\
&< L e^{-\mu t},
\end{aligned}$$

where $L = \max_{\alpha \in \Lambda} \{(1 + \frac{2\bar{l}_\alpha + \bar{m}_\alpha}{e(\bar{m}_\alpha - 2\mu)})\chi_{\Phi, \Psi} + \frac{4M\eta_\alpha^\mu(\bar{m}_\alpha - \mu)}{(\bar{m}_\alpha - 2\mu)^2}\}$. The proof of Theorem 2 is therefore complete. $\square$

Theorem 3. Under assumption (A), then the drive system (1) and the response system (2) are globally exponentially synchronised under the controllers (5) if the following conditions hold

$$\Delta_\alpha < 0, \quad \sum_{\beta=1}^n (a_{\alpha\beta}^+ + b_{\alpha\beta}^+ + p_{\alpha\beta}^+ + q_{\alpha\beta}^+) k_\beta < \frac{\bar{m}_\alpha \sqrt{-\Delta_\alpha}}{4}, \quad \alpha \in \Lambda. \quad (21)$$

Proof. Assume condition (21) holds, i.e., $4\eta_\alpha^0 < \bar{m}_\alpha \sqrt{-\Delta_\alpha}$. Then we can choose a sufficiently large positive constant M and a sufficient little $\mu \in (0, \frac{1}{4} \min_{\alpha \in \Lambda} \bar{m}_\alpha)$ such that for every $\alpha \in \Lambda$, the following inequality is satisfied

$$\frac{\sqrt{\bar{l}_\alpha} + 1}{M\nu_\alpha} \chi_{\Phi, \Psi} + \frac{4\eta_\alpha^\mu}{(\bar{m}_\alpha - 2\mu)\sqrt{-\Delta_\alpha}} < 1. \quad (22)$$

Step 1: The initial condition guarantees that $|x_\alpha(t)| < M e^{-\mu t}$ holds for all $t \in [-\tau, 0]$ and every $\alpha \in \Lambda$. To show that this estimate remains valid for all $t \geq 0$, we argue by contradiction, suppose there exist an index $\alpha_3 \in \Lambda$ and a time $\sigma_3 > 0$ such that

$$\begin{cases} |x_{\alpha_3}(t)| < M e^{-\mu t}, t \in [0, \sigma_3], \alpha \in \Lambda, \\ |x_{\alpha_3}(\sigma_3)| = M e^{-\mu \sigma_3}. \end{cases} \quad (23)$$

By combining assumption (A), (9), (22) and (23), along with Lemma 1, we deduce that

$$\begin{aligned}
|x_{\alpha_3}(\sigma_3)| &= |[\frac{\sqrt{\bar{l}_{\alpha_3}}}{\nu_{\alpha_3}} \sin(\nu_{\alpha_3} \sigma_3 + \varphi_{\alpha_3}) x_{\alpha_3}(0) + \frac{\sin \nu_{\alpha_3} \sigma_3}{\nu_{\alpha_3}} \dot{x}_{\alpha_3}(0)] e^{-\frac{\bar{m}_{\alpha_3}}{2} \sigma_3} \\
&\quad + \int_0^{\sigma_3} \frac{\sin \nu_{\alpha_3}(\sigma_3 - s)}{\nu_{\alpha_3}} e^{-\frac{\bar{m}_{\alpha_3}}{2}(\sigma_3 - s)} G_{\alpha_3}(s) ds| \\
&\leq \frac{\sqrt{\bar{l}_{\alpha_3}} + 1}{\nu_{\alpha_3}} \chi_{\Phi, \Psi} e^{-\frac{\bar{m}_{\alpha_3}}{2} \sigma_3} + \int_0^{\sigma_3} \frac{\sin \nu_{\alpha_3}(\sigma_3 - s)}{\nu_{\alpha_3}} |e^{-\frac{\bar{m}_{\alpha_3}}{2}(\sigma_3 - s)} \\
&\quad \times (|\sum_{\beta=1}^n a_{\alpha_3\beta}(s) H_\beta(x_\beta(s))| + |\sum_{\beta=1}^n b_{\alpha_3\beta}(s) H_\beta(x_\beta(s - \tau_{\alpha_3\beta}(s)))| \\
&\quad + |\sum_{\beta=1}^n p_{\alpha_3\beta}(s) H_\beta(x_\beta(s - \tau_{\alpha_3\beta}(s)))| + |\sum_{\beta=1}^n q_{\alpha_3\beta}(s) H_\beta(x_\beta(s - \tau_{\alpha_3\beta}(s)))|) ds \\
&\leq \frac{\sqrt{\bar{l}_{\alpha_3}} + 1}{\nu_{\alpha_3}} \chi_{\Phi, \Psi} e^{-\mu \sigma_3} + \frac{1}{\nu_{\alpha_3}} \int_0^{\sigma_3} e^{-\frac{\bar{m}_{\alpha_3}}{2}(\sigma_3 - s)} \\
&\quad \times (\sum_{\beta=1}^n a_{\alpha_3\beta}(s) |H_\beta(x_\beta(s))| + \sum_{\beta=1}^n |b_{\alpha_3\beta}(s)| |H_\beta(x_\beta(s - \tau_{\alpha_3\beta}(s)))| \\
&\quad + \sum_{\beta=1}^n |p_{\alpha_3\beta}(s)| |H_\beta(x_\beta(s - \tau_{\alpha_3\beta}(s)))| + \sum_{\beta=1}^n |q_{\alpha_3\beta}(s)| |H_\beta(x_\beta(s - \tau_{\alpha_3\beta}(s)))|) ds \\
&\leq \frac{\sqrt{\bar{l}_{\alpha_3}} + 1}{\nu_{\alpha_3}} \chi_{\Phi, \Psi} e^{-\mu \sigma_3} + \frac{1}{\nu_{\alpha_3}} \int_0^{\sigma_3} e^{-\frac{\bar{m}_{\alpha_3}}{2}(\sigma_3 - s)} \\
&\quad \times (\sum_{\beta=1}^n a_{\alpha_3\beta}^+ k_\beta |x_\beta(s)| + \sum_{\beta=1}^n b_{\alpha_3\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha_3\beta}(s))| \\
&\quad + \sum_{\beta=1}^n p_{\alpha_3\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha_3\beta}(s))| + \sum_{\beta=1}^n q_{\alpha_3\beta}^+ k_\beta |x_\beta(s - \tau_{\alpha_3\beta}(s))|) ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\sqrt{l_{\alpha_3}+1}}{\nu_{\alpha_3}} \chi_{\Phi, \Psi} e^{-\mu \sigma_3} + \frac{1}{\nu_{\alpha_3}} \int_0^{\sigma_3} e^{-\frac{\bar{m}_{\alpha_3}}{2}(\sigma_3-s)} \\
&\quad \times \left(\sum_{\beta=1}^n a_{\alpha_3\beta}^+ k_{\beta} M e^{-\mu s} + \sum_{\beta=1}^n b_{\alpha_3\beta}^+ k_{\beta} M e^{-\mu(s-\tau_{\alpha_3\beta}(s))} \right. \\
&\quad \left. + \sum_{\beta=1}^n p_{\alpha_3\beta}^+ k_{\beta} M e^{-\mu(s-\tau_{\alpha_3\beta}(s))} + \sum_{\beta=1}^n q_{\alpha_3\beta}^+ k_{\beta} M e^{-\mu(s-\tau_{\alpha_3\beta}(s))} \right) ds \\
&\leq M e^{-\mu \sigma_3} \left[\frac{\sqrt{l_{\alpha_3}+1}}{M \nu_{\alpha_3}} \chi_{\Phi, \Psi} + \frac{\eta_{\alpha_3}^{\mu}}{\nu_{\alpha_3}} \int_0^{\sigma_3} e^{-\left(\frac{\bar{m}_{\alpha_3}}{2}-\mu\right)(\sigma_3-s)} ds \right. \\
&= M e^{-\mu \sigma_3} \left[\frac{\sqrt{l_{\alpha_3}+1}}{M \nu_{\alpha_3}} \chi_{\Phi, \Psi} + \frac{4\eta_{\alpha_3}^{\mu}(1-e^{-\left(\frac{\bar{m}_{\alpha_3}}{2}-\mu\right)\sigma_3})}{(\bar{m}_{\alpha_3}-2\mu)\sqrt{-\Delta_{\alpha_3}}} \right] \\
&< M e^{-\mu \sigma_3} \left[\frac{\sqrt{l_{\alpha_3}+1}}{M \nu_{\alpha_3}} \chi_{\Phi, \Psi} + \frac{4\eta_{\alpha_3}^{\mu}}{(\bar{m}_{\alpha_3}-2\mu)\sqrt{-\Delta_{\alpha_3}}} \right] \\
&< M e^{-\mu \sigma_3}.
\end{aligned}$$

This leads to a contradiction. Hence, we conclude that inequality (14) is satisfied for all $t \in [0, +\infty)$ and every $\alpha \in \Lambda$.

Step 2: We now show that inequality (15) holds for some positive constant L . Since $\Delta_{\alpha} < 0$, differentiating Equation (9) with respect to t yields

$$\begin{aligned}
\dot{x}_{\alpha}(t) &= -\left[\frac{\bar{l}_{\alpha}}{\nu_{\alpha}} \sin \nu_{\alpha} t x_{\alpha}(0) + \frac{\sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \sin(\nu_{\alpha} t - \varphi_{\alpha}) \dot{x}_{\alpha}(0)\right] e^{-\frac{\bar{m}_{\alpha}}{2} t} \\
&\quad - \frac{\sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \int_0^t \sin(\nu_{\alpha}(t-s) - \varphi_{\alpha}) e^{-\frac{\bar{m}_{\alpha}}{2}(t-s)} G_{\alpha}(s) ds.
\end{aligned} \tag{24}$$

It follows from assumption (A), (14), (24) and Lemma 1 that

$$\begin{aligned}
|\dot{x}_{\alpha}(t)| &= \left| -\left[\frac{\bar{l}_{\alpha}}{\nu_{\alpha}} \sin \nu_{\alpha} t x_{\alpha}(0) + \frac{\sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \sin(\nu_{\alpha} t - \varphi_{\alpha}) \dot{x}_{\alpha}(0)\right] e^{-\frac{\bar{m}_{\alpha}}{2} t} \right. \\
&\quad \left. - \frac{\sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \int_0^t \sin(\nu_{\alpha}(t-s) - \varphi_{\alpha}) e^{-\frac{\bar{m}_{\alpha}}{2}(t-s)} G_{\alpha}(s) ds \right| \\
&\leq \frac{\bar{l}_{\alpha} + \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \chi_{\Phi, \Psi} e^{-\frac{\bar{m}_{\alpha}}{2} t} + \frac{\sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \int_0^t e^{-\frac{\bar{m}_{\alpha}}{2}(t-s)} \\
&\quad \times \left(\left| \sum_{\beta=1}^n a_{\alpha\beta}(s) H_{\beta}(x_{\beta}(s)) \right| + \left| \sum_{\beta=1}^n b_{\alpha\beta}(s) H_{\beta}(x_{\beta}(s - \tau_{\alpha\beta}(s))) \right| \right. \\
&\quad \left. + \left| \sum_{\beta=1}^n p_{\alpha\beta}(s) H_{\beta}(x_{\beta}(s - \tau_{\alpha\beta}(s))) \right| + \left| \sum_{\beta=1}^n q_{\alpha\beta}(s) H_{\beta}(x_{\beta}(s - \tau_{\alpha\beta}(s))) \right| \right) ds \\
&\leq \frac{\bar{l}_{\alpha} + \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \chi_{\Phi, \Psi} e^{-\mu t} + \frac{\sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \int_0^t e^{-\frac{\bar{m}_{\alpha}}{2}(t-s)} \\
&\quad \times \left(\sum_{\beta=1}^n |a_{\alpha\beta}(s)| |H_{\beta}(x_{\beta}(s))| + \sum_{\beta=1}^n |b_{\alpha\beta}(s)| |H_{\beta}(x_{\beta}(s - \tau_{\alpha\beta}(s)))| \right. \\
&\quad \left. + \sum_{\beta=1}^n |p_{\alpha\beta}(s)| |H_{\beta}(x_{\beta}(s - \tau_{\alpha\beta}(s)))| + \sum_{\beta=1}^n |q_{\alpha\beta}(s)| |H_{\beta}(x_{\beta}(s - \tau_{\alpha\beta}(s)))| \right) ds \\
&\leq \frac{\bar{l}_{\alpha} + \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \chi_{\Phi, \Psi} e^{-\mu t} + \frac{\sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \int_0^t e^{-\frac{\bar{m}_{\alpha}}{2}(t-s)} \\
&\quad \times \left(\sum_{\beta=1}^n a_{\alpha\beta}^+ k_{\beta} |x_{\beta}(s)| + \sum_{\beta=1}^n b_{\alpha\beta}^+ k_{\beta} |x_{\beta}(s - \tau_{\alpha\beta}(s))| \right. \\
&\quad \left. + \sum_{\beta=1}^n p_{\alpha\beta}^+ k_{\beta} |x_{\beta}(s - \tau_{\alpha\beta}(s))| + \sum_{\beta=1}^n q_{\alpha\beta}^+ k_{\beta} |x_{\beta}(s - \tau_{\alpha\beta}(s))| \right) ds \\
&\leq \frac{\bar{l}_{\alpha} + \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \chi_{\Phi, \Psi} e^{-\mu t} + \frac{\sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \int_0^t e^{-\frac{\bar{m}_{\alpha}}{2}(t-s)} \\
&\quad \times \left(\sum_{\beta=1}^n a_{\alpha\beta}^+ k_{\beta} M e^{-\mu s} + \sum_{\beta=1}^n b_{\alpha\beta}^+ k_{\beta} M e^{-\mu(s-\tau_{\alpha\beta}(s))} \right. \\
&\quad \left. + \sum_{\beta=1}^n p_{\alpha\beta}^+ k_{\beta} M e^{-\mu(s-\tau_{\alpha\beta}(s))} + \sum_{\beta=1}^n q_{\alpha\beta}^+ k_{\beta} M e^{-\mu(s-\tau_{\alpha\beta}(s))} \right) ds \\
&\leq e^{-\mu t} \left[\frac{\bar{l}_{\alpha} + \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \chi_{\Phi, \Psi} + \frac{M \eta_{\alpha}^{\mu} \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \int_0^t e^{-\left(\frac{\bar{m}_{\alpha}}{2}-\mu\right)(t-s)} ds \right] \\
&= e^{-\mu t} \left[\frac{\bar{l}_{\alpha} + \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \chi_{\Phi, \Psi} + \frac{2M \eta_{\alpha}^{\mu} \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}(\bar{m}_{\alpha}-2\mu)} (1 - e^{-\left(\frac{\bar{m}_{\alpha}}{2}-\mu\right)t}) \right] \\
&< L e^{-\mu t},
\end{aligned}$$

where $L = \max_{\alpha \in \Lambda} \left\{ \frac{\bar{l}_{\alpha} + \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}} \chi_{\Phi, \Psi} + \frac{2M \eta_{\alpha}^{\mu} \sqrt{\bar{l}_{\alpha}}}{\nu_{\alpha}(\bar{m}_{\alpha}-2\mu)} \right\}$. Hence the proof of Theorem 3 is complete. $\square$

Remark 2. Researchers across multiple scientific fields have increasingly highlighted the synchronization between the driven and response DFNNs (1) and (2) as a key problem. Numerous studies in [7–44] have addressed it using reduced order and non-reduced order frameworks, both of which typically require constructing LKFs and solving

LMIs. By contrast, we avoid both reduced order schemes and standard LKFs based designs. Instead, we employ a characteristic based approach and derive three delay independent sufficient conditions that ensure GES of systems (1) and (2). This strategy offers a clear benefit: it bypasses the intricacies of reduced order transformations and conventional LKFs formulations, thereby eliminating lengthy derivations and reducing computational burden. We compare our work with four references [22, 30, 37, 44] in terms of time delay types, research methods, application tools and obtained results. See the Table 1 below for details.

Table 1. Comparison of the methods and results between this paper and related references.

Comparison	[22]	[30]	[37]	[44]	This Paper
Type of delay	Constant delay	Constant delay	Bounded time-varying delay	Constant delay	Bounded time-varying delay
Whether to construct LKFs	Yes	Yes	Yes	Yes	No
Whether to solve LMIs	Yes	No	Yes	Yes	No
Type of method	Reduced order (RO)	RO	Non-Reduced order (N-RO)	N-RO	Characteristic method
Delay-dependent (DD) or delay-independent (DI) criteria	DD	DI	DD	DI	DI
Type of synchronization	Global exponential (GE)	Global asymptotic (GA)	GE	GA	GE
Improvement of this paper	Avoidance of the reduced order method; No requirement for LKFs construction; Independence from LMIs solutions; Delay-independent criteria verifiable via linear scalar inequalities.				

4. A simulation Example with Three Different Controllers

To further validate our theoretical findings, we carry out a numerical experiment in which we design three distinct controllers to achieve GES. Each controller is tailored to one of the three possible cases for the discriminant Δ_α : positive, zero, and negative.

Example 1. For $n = 2$, consider the drive-response DFNNs (1) and (2) with the parameters: $m_1 = 3, m_2 = 3, l_1 = 2, l_2 = 3, a_{11}(t) = -0.3 \sin 2t, a_{12}(t) = 0.2 \cos(t - 1), a_{21}(t) = 0.2 \cos 2t, a_{22}(t) = -0.4 \sin(t + 1), b_{11}(t) = -0.4 \sin t, b_{12}(t) = 0.5 \cos(t - 2), b_{21}(t) = 0.5 \cos 2t, b_{22}(t) = -0.6 \sin 3t, c_{11}(t) = 0.2 \sin \sqrt{2}t, c_{12}(t) = -0.2 \cos \sqrt{3}t, c_{21}(t) = 0.3 \sin \sqrt{3}t, c_{22}(t) = -0.3 \cos \sqrt{2}t, p_{11}(t) = -0.3 \sin t, p_{12}(t) = 0.3 \cos t, p_{21}(t) = 0.3 \sin \sqrt{2}t, p_{22}(t) = -0.3 \cos \sqrt{3}t, q_{11}(t) = 0.3 \cos t, q_{12}(t) = -0.3 \sin t, q_{21}(t) = -0.3 \cos \sqrt{2}t, q_{22}(t) = 0.3 \sin \sqrt{3}t, T_{11}(t) = 0.2, T_{12}(t) = -0.2, T_{21}(t) = -0.1, T_{22}(t) = 0.1, S_{11}(t) = -0.1, S_{12}(t) = 0.1, S_{21}(t) = 0.2, S_{22}(t) = -0.2, v_1(t) = -1, v_2(t) = 1, I_1 = -2, I_2 = 2, \tau_{11}(t) = |\sin t|, \tau_{12}(t) = |\sin 2t|, \tau_{21}(t) = |\cos t|, \tau_{22}(t) = |\cos 2t|, h_1(x) = 0.4 \sin x + 0.6x, h_2(x) = 0.6 \sin x + 0.4x.$

Moreover, we design three controllers as follows

$$u_1(t) = -\dot{x}_1(t) - x_1(t), u_2(t) = -2\dot{x}_2(t) - x_2(t), \quad (25)$$

$$u_1(t) = (3 - \sqrt{12})\dot{x}_1(t) - x_1(t), u_2(t) = -\dot{x}_2(t) - x_2(t) \quad (26)$$

$$u_1(t) = -3\dot{x}_1(t) - 8x_1(t), u_2(t) = -3\dot{x}_2(t) - 7x_2(t) \quad (27)$$

We directly verify that the activation functions h_1 and h_2 fulfill the global Lipschitz condition (A) with Lipschitz constants $k_1 = k_2 = 1$. Furthermore, under each of the above controllers, the corresponding conditions derived in Theorems 1–3 hold, as shown below.

For controller (25) ($\Delta_1 > 0, \Delta_2 > 0$):

$$\begin{cases} \Delta_1 = 4 > 0, 2.6 = \sum_{\beta=1}^2 (a_{1\beta}^+ + b_{1\beta}^+ + p_{1\beta}^+ + q_{1\beta}^+)k_\beta < \bar{l}_1 = 3, \\ \Delta_2 = 9 > 0, 2.9 = \sum_{\beta=1}^2 (a_{2\beta}^+ + b_{2\beta}^+ + p_{2\beta}^+ + q_{2\beta}^+)k_\beta < \bar{l}_1 = 4. \end{cases}$$

For controller (26) ($\Delta_1 = 0, \Delta_2 = 0$):

$$\begin{cases} \Delta_1 = 0, 2.6 = \sum_{\beta=1}^2 (a_{1\beta}^+ + b_{1\beta}^+ + p_{1\beta}^+ + q_{1\beta}^+) k_{\beta} < \bar{l}_1 = 3, \\ \Delta_2 = 0, 2.9 = \sum_{\beta=1}^2 (a_{2\beta}^+ + b_{2\beta}^+ + p_{2\beta}^+ + q_{2\beta}^+) k_{\beta} < \bar{l}_2 = 4. \end{cases}$$

For controller (27) ($\Delta_1 < 0, \Delta_2 < 0$):

$$\begin{cases} \Delta_1 = -4 < 0, 2.6 = \sum_{\beta=1}^2 (a_{1\beta}^+ + b_{1\beta}^+ + p_{1\beta}^+ + q_{1\beta}^+) k_{\beta} < \frac{\bar{m}_1 \sqrt{-\Delta_1}}{4} = 3, \\ \Delta_2 = -4 < 0, 2.9 = \sum_{\beta=1}^2 (a_{2\beta}^+ + b_{2\beta}^+ + p_{2\beta}^+ + q_{2\beta}^+) k_{\beta} < \frac{\bar{m}_2 \sqrt{-\Delta_2}}{4} = 3. \end{cases}$$

Consequently, by Theorems 1–3, Example 1 achieves GES under each of the controllers (25)–(27). The numerical simulations presented in Figures 1–3 further confirm this theoretical conclusion.

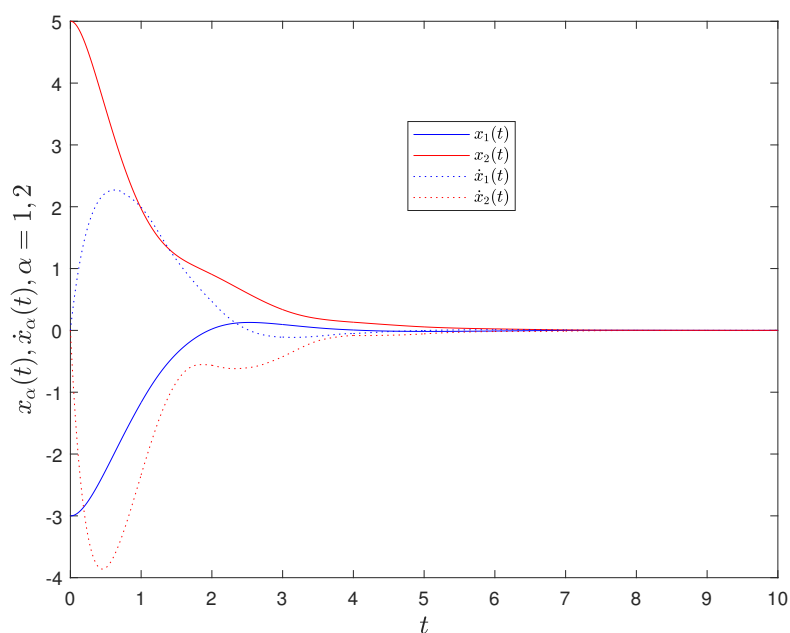


Figure 1. Synchronization errors of drive-response Example 1 with controller (25).

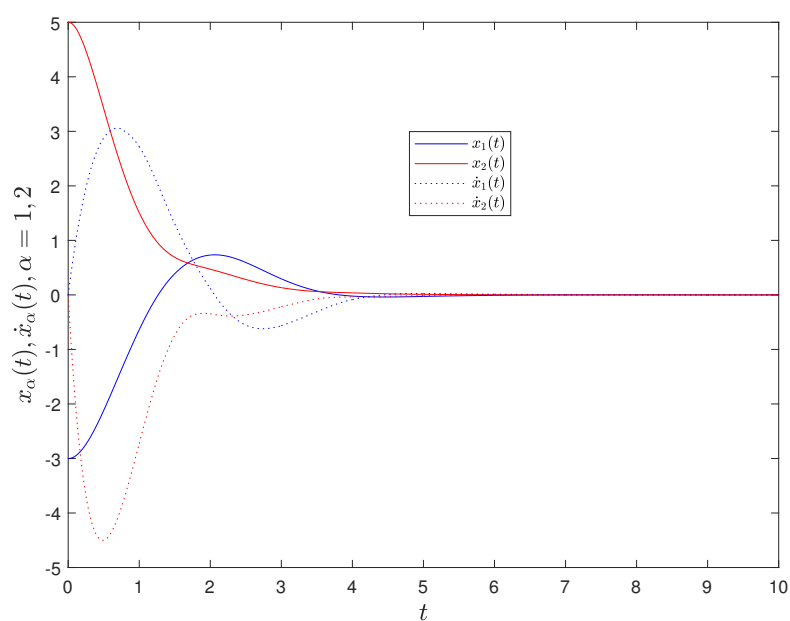


Figure 2. Synchronization errors of drive-response Example 1 with controller (26).

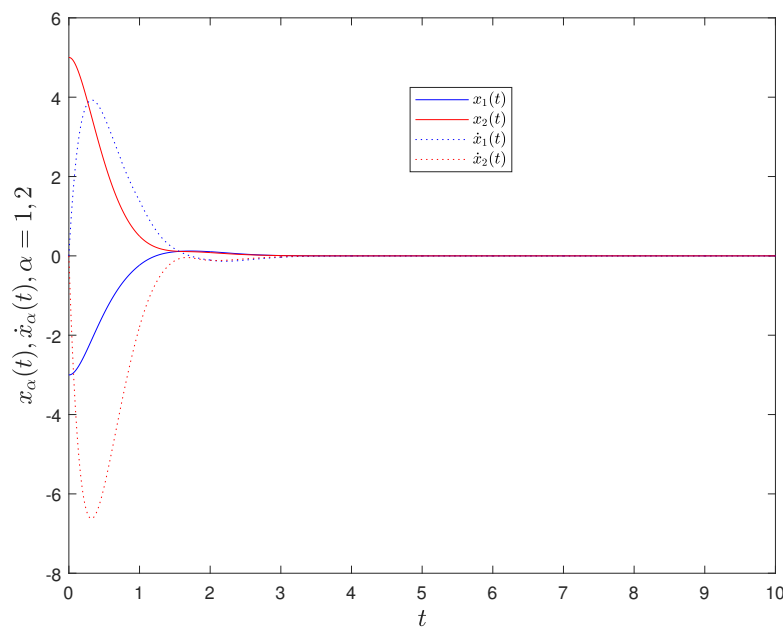


Figure 3. Synchronization errors of drive-response Example 1 with controller (27).

Remark 3. To the best of our knowledge, researchers have not yet investigated GES for the drive-response DFINNs given by Example 1 via the characteristics method. Our approach actively avoids both the reduced order transformation and the construction of LKFs. As a result, the methods developed in [7–44] cannot directly verify GES in Example 1 under the three distinct controllers (25)–(27). Moreover, when we remove these controllers, the synchronization errors between the drive and response systems in Example 1 no longer converge to the origin, which we demonstrate this behavior in Figure 4.

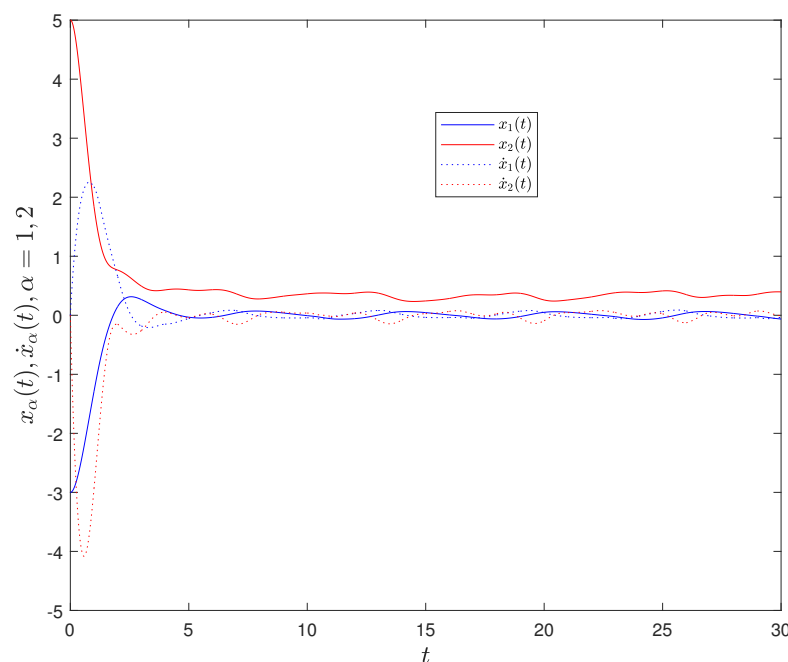


Figure 4. Synchronization errors of drive-response Example 1 without controller.

5. Conclusions

Using the characteristics method, we investigate GES for a class of DFINNs. By conducting a detailed analysis based on the discriminants Δ_α , we derive explicit solution representations for the error system (6). Moreover, we establish several sufficient conditions that guarantee GES in drive-response DFINNs configurations. To validate our theoretical results, we provide a numerical example equipped with three distinct controllers. Drawing upon the approaches presented in [54–58], future research will explore GES for DFINNs with less stringent constraints on activation functions and fractional inertial neural network models.

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W.W. was responsible for drafting the main manuscript text. W.C. conducted the numerical simulations for all three examples and participated in the review and editing process. All authors have read and agreed to the published version of the manuscript.

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No AI tools were utilized for this paper.

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