

Shifted Chebyshev Adapted Pseudo Collocation Method for Multidimensional Time-Space Fractional Diffusion Equations with Robin Boundary Conditions

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Abstract: This paper introduces a highly accurate spectral framework, termed the Shifted Chebyshev Adapted Pseudo Collocation (SCAPC) method, for solving multidimensional time-space fractional diffusion equations subject to Robin boundary conditions. The proposed approach integrates a spectral spatial discretization based on shifted Chebyshev polynomials with a Generalized Exponential Time Differencing (GETD) scheme for temporal integration. By leveraging the eigendecomposition of a modified Chebyshev discrete Laplacian, the SCAPC method efficiently approximates the spectral fractional Laplacian while strictly enforcing complex boundary conditions. We provide a rigorous theoretical analysis, establishing detailed theorems on derivative approximation, boundary condition enforcement, spectral convergence, global error bounds, and the stability and conditioning of the resulting algebraic systems. Comprehensive numerical experiments on benchmark problems including non-smooth source terms and coupled fractional systems demonstrate that the SCAPC method achieves exponential (spectral) spatial convergence, vastly outperforming traditional second-order finite difference-based matrix transfer techniques in both accuracy and computational efficiency.

Keywords: time-space fractional PDEs; shifted chebyshev pseudo-collocation; generalized exponential time differencing; spectral fractional laplacian; Robin boundary conditions; spectral methods

MSC 2020: primary 65M70; 35R11; secondary 65M12; 65M22; 33C45; 26A33

1. Introduction

Fractional partial differential equations incorporating both spatial and temporal non-local operators serve as powerful tools for characterizing anomalous diffusion phenomena, capturing memory effects and long-range interactions that traditional integer-order models cannot adequately represent [1,2]. These models are indispensable across various disciplines, including viscoelasticity, biological transport, and financial mathematics [3,4].

Early numerical treatments of time-space fractional equations involving the spectral Laplacian relied on truncated spectral expansions followed by exponential time integration [5]. While effective when analytical eigenpairs are available, such approaches struggle with complex geometries and general boundary conditions. The Matrix Transfer Technique (MTT), initially proposed by Ilić et al. [6], offers a robust paradigm for approximating spectral fractional operators by evaluating fractional powers of discretized Laplacian matrices. This strategy transforms the non-local spatial operator into a system of ordinary differential equations. Subsequent analyses confirmed that the accuracy of the MTT is intrinsically linked to the underlying spatial discretization order [7].

To mitigate the severe stiffness induced by high-resolution spatial discretizations, Generalized Exponential Time Differencing (GETD) schemes have been developed [8–10]. These methods exactly integrate the linear homogeneous part of the system, allowing for larger time steps without sacrificing stability. The computational burden

of evaluating matrix Mittag-Leffler functions within GETD schemes is significantly alleviated through global Padé approximations and partial fraction decompositions [11].

Despite these advancements, existing frameworks predominantly rely on low-order finite difference approximations for the spatial Laplacian. Real-world applications frequently involve Robin boundary conditions and non-smooth source terms, which can introduce singularities. Furthermore, multidimensional domains and coupled systems exacerbate the computational cost. While finite difference-based MTT provides a viable solution, its algebraic convergence rate limits accuracy for smooth solutions.

Recent advances in spectral and operational methods have significantly improved numerical schemes for solving fractional and high-dimensional differential equations. Several computational approaches have been developed to tackle time-fractional diffusion models using tailored polynomial bases, including exponential and fractional Jacobi spectral schemes [12], modified spectral Tau techniques employing seventh-kind Chebyshev polynomials [13], and second-kind Chebyshev collocation techniques for various fractional PDEs [14, 15]. Furthermore, Chebyshev collocation methods have been successfully applied to nonlinear membrane problems [16], stochastic delay equations [17], and complex optimization problems [18]. Complementing these, analytical and numerical frameworks have been established for multidimensional diffusion systems operating in L^2 spaces under third-kind boundary conditions [19], alongside high-order derivative formulas for fifth-kind Chebyshev polynomials to yield exact spectral solutions for convection–diffusion problems [20]. For complex geometries, domain-overlapping algorithms with nonlinear mappings have also been developed for Chebyshev collocation-based eigenvalue solutions [21].

In this work, we propose the SCAPC method, a novel spectral framework that replaces the finite difference Laplacian with a high-order Chebyshev spectral differentiation matrix. The main novel contributions of this work are summarized as follows:

- Development of the SCAPC-GETD framework, which uniquely combines high-order spectral spatial discretization with exact temporal integration of the fractional linear operator, overcoming the severe stiffness limitations of standard spectral baselines.
- Rigorous theoretical analysis, including novel proofs for the exact enforcement of Robin boundary conditions via discrete matrix row modification, spectral convergence, global error bounds, and the conditioning of the resulting algebraic systems.
- Comprehensive numerical validation across 1D and 2D domains, demonstrating exponential convergence and superior computational efficiency compared to second-order MTT-GETD methods, even for non-smooth sources and coupled systems.

The remainder of this paper is organized as follows. Section 2 presents the problem formulation and defines the fractional operators. Section 3 details the proposed SCAPC method and its algorithmic framework. Section 4 provides the rigorous theoretical analysis, including convergence and stability proofs. Section 5 presents comprehensive numerical experiments to validate the method’s accuracy and efficiency. Finally, Section 6 concludes the paper with a summary and future research directions.

2. Problem Formulation

We consider the following initial-boundary value problem (IBVP):

$$\begin{aligned} {}_c\partial_t^\alpha u(x, t) &= -\kappa(-\Delta)^{\beta/2}u(x, t) + g(x, t), \quad x \in \Omega, t \in (0, T), \\ u(x, 0) &= u_0(x), \quad x \in \Omega, \\ \mu u(x, t) + \nu \nabla u(x, t) \cdot \mathbf{n} &= 0, \quad x \in \partial\Omega, t \in (0, T), \end{aligned} \quad (1)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. The coefficients μ and ν satisfy $\nu \neq 0$ and $\mu/\nu > 0$.

The operator ${}_c\partial_t^\alpha$ denotes the Caputo fractional derivative of order $0 < \alpha < 1$, defined as:

$${}_c\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x, \tau)}{\partial \tau} d\tau. \quad (2)$$

The operator $(-\Delta)^{\beta/2}$ is the spectral fractional Laplacian of order $1 < \beta \leq 2$. Let $\{\phi_k, \lambda_k\}_{k=1}^\infty$ be the eigenpairs of the standard Laplacian $-\Delta$ on Ω subject to the given boundary conditions. The spectral fractional Laplacian is defined via spectral decomposition as:

$$(-\Delta)^{\beta/2}u = \sum_{k=1}^{\infty} \lambda_k^{\beta/2} \langle u, \phi_k \rangle \phi_k, \quad (3)$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Omega)$ inner product.

3. The SCAPC Method

The proposed SCAPC method integrates a high-order spectral spatial discretization based on shifted Chebyshev polynomials with the Generalized Exponential Time Differencing (GETD) scheme for temporal integration. This section provides the rigorous mathematical formulation of the method.

3.1. Shifted Chebyshev Polynomials and Collocation Points

Let the spatial domain be $\Omega = (0, L)$. We define the shifted Chebyshev polynomials of the first kind as $T_k^*(x) = T_k\left(\frac{2x}{L} - 1\right)$, where $T_k(y)$ is the standard Chebyshev polynomial of degree k on $y \in [-1, 1]$. The spatial discretization utilizes the Chebyshev-Gauss-Lobatto (CGL) collocation points, which are the extrema of $T_N^*(x)$:

$$x_j = \frac{L}{2} \left(1 - \cos\left(\frac{j\pi}{N}\right) \right), \quad j = 0, 1, \dots, N. \quad (4)$$

The first-order spatial derivative is approximated using the $(N+1) \times (N+1)$ Chebyshev spectral differentiation matrix D_N , whose entries are given by:

$$(D_N)_{ij} = \begin{cases} \frac{c_i}{c_j} \frac{(-1)^{i+j}}{x_i - x_j}, & i \neq j, \\ \frac{L-2x_i}{2x_i(L-x_i)}, & i = j, \quad 1 \leq i \leq N-1, \\ \frac{2N^2+1}{6L}, & i = j = 0, \\ -\frac{2N^2+1}{6L}, & i = j = N, \end{cases} \quad (5)$$

where $c_0 = c_N = 2$ and $c_i = 1$ for $1 \leq i \leq N-1$. The second derivative is approximated by the matrix $D_N^{(2)} = (D_N)^2$.

3.2. Spatial Discretization and Fractional Laplacian

The continuous Laplacian operator is approximated by the discrete Laplacian matrix $L_N = -D_N^{(2)}$. To strictly enforce the homogeneous Robin boundary conditions $\mu u(0) + \nu u'(0) = 0$ and $\mu u(L) + \nu u'(L) = 0$, we modify the first and last rows of L_N to incorporate the discrete boundary operators. Let $\tilde{L}_N$ denote this modified matrix, defined as:

$$(\tilde{L}_N)_{i,:} = \begin{cases} \mu e_0^T + \nu(D_N)_{0,:}, & i = 0, \\ (L_N)_{i,:}, & 1 \leq i \leq N-1, \\ \mu e_N^T + \nu(D_N)_{N,:}, & i = N, \end{cases} \quad (6)$$

where e_0 and e_N are the first and last canonical basis vectors of $\mathbb{R}^{N+1}$. The spectral fractional Laplacian of order $\beta/2$ is then approximated via the eigendecomposition of $\tilde{L}_N$:

$$(-\Delta)^{\beta/2} \approx \tilde{L}_N^{\beta/2} = V \Lambda^{\beta/2} V^{-1}, \quad (7)$$

where $\Lambda = \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_N)$ is the diagonal matrix of eigenvalues and V is the matrix of corresponding eigenvectors. This spectral matrix transfer technique ensures exponential (spectral) accuracy for smooth solutions.

3.3. Temporal Integration via GETD

Substituting the spatial approximation into the governing equation yields the semi-discrete system of fractional ordinary differential equations:

$${}_c D_t^\alpha U(t) = -AU(t) + G(t), \quad U(0) = U_0, \quad (8)$$

where $A = \kappa \tilde{L}_N^{\beta/2}$ and $G(t)$ is the vector of source terms evaluated at the collocation points. The exact solution to this system is given by:

$$U(t) = E_\alpha(-At^\alpha)U_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-A(t-\tau)^\alpha) G(\tau) d\tau, \quad (9)$$

where $E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}$ is the two-parameter Mittag-Leffler function.

To derive the fully discrete GETD scheme, we partition the time interval $[0, T]$ into M subintervals $0 = t_0 < t_1 < \dots < t_M = T$ with step sizes $k_j = t_{j+1} - t_j$. On each subinterval $I_j = [t_j, t_{j+1}]$, we approximate the source term $G(\tau)$ by the piecewise linear interpolant:

$$F_j(\tau) = G_j + \frac{\tau - t_j}{k_j}(G_{j+1} - G_j), \quad \tau \in I_j. \quad (10)$$

Substituting $F_j(\tau)$ into the integral and applying integration by parts, utilizing the identity

$$\frac{d}{d\tau}[(t - \tau)^\alpha E_{\alpha,\alpha+1}(-A(t - \tau)^\alpha)] = -(t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-A(t - \tau)^\alpha),$$

We obtain the fully discrete update formula at t_n :

$$\begin{aligned} U_n = & E_\alpha(-At_n^\alpha)U_0 + t_n^\alpha E_{\alpha,\alpha+1}(-At_n^\alpha)G_1 \\ & + \sum_{j=0}^{n-1} (t_n - t_j)^{\alpha+1} E_{\alpha,\alpha+2}(-A(t_n - t_j)^\alpha) \frac{G_{j+1} - G_j}{k_j}. \end{aligned} \quad (11)$$

For problems with non-smooth source terms exhibiting weak singularities near $t = 0$, the standard uniform mesh degrades the convergence rate. To restore optimal second-order accuracy, we employ a graded mesh defined by:

$$t_n = \left(\frac{n}{M}\right)^\delta T, \quad n = 0, 1, \dots, M, \quad (12)$$

where the grading parameter $\delta \geq 1$ concentrates time steps near the origin. The choice of the graded mesh parameter $\delta = 1/\alpha$ is theoretically motivated by the regularity of the exact solution near $t = 0$. For time-fractional problems, the solution typically exhibits a weak singularity of the form $u(x, t) \sim t^\alpha$. Standard error analysis for fractional ODEs with graded meshes indicates that to restore optimal convergence rates, the grading parameter must be chosen to sufficiently cluster points near the origin. While some classical L1 schemes require $\delta \geq 2/\alpha$ for second-order accuracy, the GETD scheme combined with piecewise linear source interpolation achieves optimal accuracy with $\delta = 1/\alpha$ due to the exact integration of the singular fractional kernel. Sensitivity analyses confirm that values of δ slightly larger than $1/\alpha$ yield similar accuracy, while $\delta < 1/\alpha$ degrades the convergence rate due to insufficient resolution of the initial singularity.

3.4. Algorithmic Framework

The complete procedural flow of the proposed SCAPC-GETD method is summarized in Algorithm 1.

Algorithm 1 SCAPC-GETD Algorithm for Time-Space Fractional Diffusion

Require: Final time T , fractional orders α, β , spatial points N , temporal points M , grading parameter δ .

Ensure: Numerical solution U^n at t_n .

- 1: Compute shifted Chebyshev-Gauss-Lobatto points x_j and differentiation matrix D_N via Equation (2).
 - 2: Construct discrete Laplacian $L_N = -D_N^2$.
 - 3: Modify first and last rows of L_N to enforce Robin/Dirichlet/Neumann BCs, yielding $\tilde{L}_N$ via Equation (3).
 - 4: Compute eigendecomposition $\tilde{L}_N = V\Lambda V^{-1}$ and fractional power $A = \kappa V\Lambda^{\beta/2}V^{-1}$.
 - 5: Precompute Mittag-Leffler matrix function evaluations via global Padé approximations.
 - 6: **for** $n = 1$ to M **do**
 - 7: Compute time step $k_n = t_n - t_{n-1}$ based on graded mesh $t_n = (n/M)^\delta T$.
 - 8: Evaluate source term $G(t_n)$ and its divided differences $\frac{G_{j+1} - G_j}{k_j}$.
 - 9: Compute the history sum using the precomputed Mittag-Leffler approximations.
 - 10: Update solution U^n using the GETD formula (11).
 - 11: **end for**
-

4. Theoretical Analysis

In this section, we provide rigorous mathematical proofs for the fundamental properties of the SCAPC method, including derivative approximation, boundary condition enforcement, spatial convergence, fully discrete error bounds, and the conditioning of the spectral operators.

4.1. Handling of Derivatives

Theorem 1 (Approximation of Derivatives). *Let $u \in H^s(0, L)$ with $s \geq 1$. Let $I_N u$ be the shifted Chebyshev interpolant of u at the Gauss-Lobatto points. Then, for $0 \leq q \leq s$, the interpolation error satisfies:*

$$\|u - I_N u\|_{H^q(0,L)} \leq C N^{2q-s} \|u\|_{H^s(0,L)}, \quad (13)$$

where C is a constant independent of N .

Proof. We map the physical interval $(0, L)$ to the reference interval $(-1, 1)$ via the affine transformation $x = \frac{L}{2}(y + 1)$. Let $\hat{u}(y) = u(x(y))$ denote the mapped function. The Sobolev norm on $(0, L)$ is defined as $\|u\|_{H^q(0,L)}^2 = \sum_{k=0}^q \int_0^L |u^{(k)}(x)|^2 dx$. Using the chain rule, the k -th derivative transforms as $u^{(k)}(x) = (\frac{2}{L})^k \hat{u}^{(k)}(y)$, and the differential element scales as $dx = \frac{L}{2} dy$. Substituting these into the norm definition yields:

$$\|u\|_{H^q(0,L)}^2 = \sum_{k=0}^q \left(\frac{2}{L}\right)^{2k-1} \|\hat{u}^{(k)}\|_{L^2(-1,1)}^2. \quad (14)$$

This implies the equivalence of norms: $\|u\|_{H^q(0,L)} \sim (L/2)^{1/2-q} \|\hat{u}\|_{H^q(-1,1)}$. By standard polynomial approximation theory for Chebyshev interpolants on $[-1, 1]$, the L^2 projection error for $\hat{u} \in H^s(-1, 1)$ is bounded by $\|\hat{u} - I_N \hat{u}\|_{L^2(-1,1)} \leq C N^{-s} \|\hat{u}\|_{H^s(-1,1)}$. For the derivatives, the spectral differentiation matrix D_N has a 2-norm bounded by $\mathcal{O}(N^2)$. Applying the derivative operator q times amplifies the error by at most $\mathcal{O}(N^{2q})$. Therefore, the error in the H^q norm on the reference domain is:

$$\|\hat{u} - I_N \hat{u}\|_{H^q(-1,1)} \leq C N^{2q-s} \|\hat{u}\|_{H^s(-1,1)}. \quad (15)$$

Transforming back to the physical domain $(0, L)$ introduces the scaling factor $(L/2)^{1/2-q}/(L/2)^{1/2-s} = (L/2)^{s-q}$. Combining these scaling factors yields:

$$\|u - I_N u\|_{H^q(0,L)} \leq C \left(\frac{L}{2}\right)^{s-q} N^{2q-s} \|u\|_{H^s(0,L)}. \quad (16)$$

The constant C depends on L , s , and q , but is strictly independent of the polynomial degree N . $\square$

4.2. Enforcement of Boundary Conditions

Theorem 2 (Handling of Robin Boundary Conditions). *Let $\tilde{L}_N$ be the modified Chebyshev discrete Laplacian incorporating homogeneous Robin boundary conditions $\mu u + \nu u' = 0$ with $\mu/\nu > 0$. Then, all eigenvalues of $\tilde{L}_N$ have strictly positive real parts.*

Proof. Consider the continuous operator $-\Delta$ with Robin BCs. The associated continuous bilinear form is $a(u, v) = \int_0^L u' v' dx + \frac{\mu}{\nu}(u(0)v(0) + u(L)v(L))$. For any $u \in H^1(0, L)$, the Rayleigh quotient is $a(u, u) = \|u'\|_{L^2}^2 + \frac{\mu}{\nu}(|u(0)|^2 + |u(L)|^2)$. Since $\mu/\nu > 0$, $a(u, u) \geq 0$, and $a(u, u) = 0 \implies u' = 0$ and $u(0) = u(L) = 0 \implies u = 0$. Thus, the continuous operator is strictly positive definite.

For the discrete operator $\tilde{L}_N$, we utilize the discrete inner product weighted by the Gauss-Lobatto quadrature weights, defined by the diagonal matrix W . The discrete bilinear form is $A_N(U, V) = U^T W \tilde{L}_N V$. It is a fundamental property of the Chebyshev collocation method that the matrix $W \tilde{L}_N$ is symmetric. The boundary modifications for the Robin conditions are constructed to be consistent with the discrete integration by parts, ensuring that $W \tilde{L}_N$ remains symmetric. Furthermore, for any non-zero vector U , the quadratic form $U^T W \tilde{L}_N U$ is the discrete approximation of the continuous Rayleigh quotient $a(u, u)$, which is strictly positive. Hence, $W \tilde{L}_N$ is symmetric positive definite. The eigenvalue problem $\tilde{L}_N U = \lambda U$ is equivalent to the generalized eigenvalue problem $W \tilde{L}_N U = \lambda W U$. Since both $W \tilde{L}_N$ and W are symmetric, and $W \tilde{L}_N$ is positive definite while W is strictly positive definite, all generalized eigenvalues λ must be strictly positive real numbers. Thus, the real parts of the eigenvalues of $\tilde{L}_N$ are strictly positive. $\square$

4.3. Convergence and Error Analysis

In this subsection, we establish a rigorous error bound for the fully discrete SCAPC-GETD scheme by analyzing the total residual L^2 norm. Crucially, we leverage the exact enforcement of the Robin boundary conditions to

guarantee the coercivity of the discrete fractional Laplacian, which is essential for the stability of the error estimates.

Theorem 3 (Total Residual and Global L^2 Error Analysis). *Let $u(x, t)$ be the exact solution of the continuous time-space fractional diffusion problem satisfying the Robin boundary conditions $\mu u + \nu \nabla u \cdot \mathbf{n} = 0$ on $\partial\Omega$ with $\mu/\nu > 0$. Let U^n be the numerical solution obtained by the SCAPC-GETD scheme. The total L^2 error at time t_n satisfies:*

$$\|u(t_n) - U^n\|_{L^2} \leq C_1 N^{-s} \|u\|_{L^\infty(0, T; H^s)} + C_2 k^2 \max_{0 \leq t \leq t_n} \|\partial_t^2 g(t)\|_{L^2}, \quad (17)$$

where C_1 and C_2 are constants independent of N and k .

Proof. We decompose the total error into three distinct components using the spatial interpolant $I_N u$:

$$u(t_n) - U^n = \underbrace{(u(t_n) - I_N u(t_n))}_{\text{Spatial Interpolation}} + \underbrace{(I_N u(t_n) - u_N(t_n))}_{\text{Semi-discrete Error } e(t_n)} + \underbrace{(u_N(t_n) - U^n)}_{\text{Temporal Discretization Error } \theta^n}. \quad (18)$$

The first term is the standard spatial interpolation error, bounded by Theorem 1 as $\|u - I_N u\|_{L^2} \leq C N^{-s} \|u\|_{H^s}$.

To bound the semi-discrete error $e(t) = I_N u(t) - u_N(t)$, we analyze the total residual $\mathcal{R}(t)$ of the exact solution within the discrete framework. The exact solution satisfies the continuous equation ${}_c \partial_t^\alpha u + \kappa (-\Delta)^{\beta/2} u = g$. Applying the spatial interpolant I_N and the discrete fractional Laplacian $\tilde{L}_N^{\beta/2}$, we obtain:

$${}_c \partial_t^\alpha (I_N u) + \kappa \tilde{L}_N^{\beta/2} (I_N u) = I_N g + \mathcal{R}(t), \quad (19)$$

where the total residual $\mathcal{R}(t)$ is defined as:

$$\mathcal{R}(t) = \kappa \left(\tilde{L}_N^{\beta/2} (I_N u) - I_N ((-\Delta)^{\beta/2} u) \right) + (I_N g - g). \quad (20)$$

Here, the Robin boundary conditions play a pivotal role. The discrete operator $\tilde{L}_N$ incorporates the Robin conditions by modifying the boundary rows of the spectral differentiation matrix. Since the exact solution u strictly satisfies $\mu u + \nu u' = 0$ at the boundaries, the boundary rows of $\tilde{L}_N (I_N u)$ exactly reproduce the boundary constraints without introducing additional truncation error. Consequently, the consistency error $\tilde{L}_N^{\beta/2} (I_N u) - I_N ((-\Delta)^{\beta/2} u)$ is entirely governed by the interior spectral approximation, yielding:

$$\|\mathcal{R}(t)\|_{L^2} \leq C N^{-s} \|u(t)\|_{H^s} + C N^{-s} \|g(t)\|_{H^s}. \quad (21)$$

Subtracting the semi-discrete equation ${}_c \partial_t^\alpha u_N + \kappa \tilde{L}_N^{\beta/2} u_N = I_N g$ from the equation for $I_N u$, we obtain the error equation:

$${}_c \partial_t^\alpha e(t) + \kappa \tilde{L}_N^{\beta/2} e(t) = \mathcal{R}(t), \quad e(0) = 0. \quad (22)$$

To bound $e(t)$, we utilize the coercivity of the discrete fractional Laplacian. By Theorem 2, the matrix $\tilde{L}_N$ has eigenvalues with strictly positive real parts, a direct consequence of the continuous Robin boundary conditions ($\mu/\nu > 0$). This ensures that the fractional power $\tilde{L}_N^{\beta/2}$ is coercive, satisfying:

$$\langle \tilde{L}_N^{\beta/2} v, v \rangle \geq \lambda_{\min}^{\beta/2} \|v\|_{L^2}^2, \quad (23)$$

where $\lambda_{\min} > 0$ is the smallest eigenvalue of $\tilde{L}_N$. Taking the inner product of the error equation with $e(t)$ and applying energy estimates for fractional ordinary differential equations, we obtain:

$$\|e(t_n)\|_{L^2} \leq \frac{t_n^\alpha}{\Gamma(\alpha + 1)} \max_{0 \leq t \leq t_n} \|\mathcal{R}(t)\|_{L^2} \leq C N^{-s} \max_t \|u(t)\|_{H^s}. \quad (24)$$

Finally, for the temporal discretization error $\theta^n = u_N(t_n) - U^n$, it satisfies the fully discrete GETD scheme with a zero initial condition and a temporal truncation error τ^n arising from the piecewise linear approximation of the source term. For a smooth source term $g \in C^2[0, T]$, the temporal residual is bounded by $\|\tau^n\|_{L^2} \leq C k^2 \max_t \|\partial_t^2 g\|_{L^2}$. Using the uniform boundedness of the discrete Mittag-Leffler function (guaranteed by the positive eigenvalues of $\tilde{L}_N$), we have:

$$\|\theta^n\|_{L^2} \leq C k^2 \max_{0 \leq t \leq t_n} \|\partial_t^2 g(t)\|_{L^2}. \quad (25)$$

Combining the bounds for the spatial interpolation, the semi-discrete error $e(t_n)$, and the temporal error θ^n via the triangle inequality yields the final total L^2 error bound, completing the proof. $\square$

4.4. Stability and Conditioning

Theorem 4 (Stability and Conditioning). *Let V be the matrix of eigenvectors of $\tilde{L}_N$. The condition number $\kappa(V) = \|V\| \|V^{-1}\|$ satisfies:*

$$\kappa(V) \leq CN^2. \quad (26)$$

Proof. The matrix $\tilde{L}_N$ is similar to a symmetric matrix S_N via a diagonal scaling matrix D_N , such that $\tilde{L}_N = D_N^{-1} S_N D_N$. Let $S_N = Q \Lambda Q^T$ be the eigendecomposition of S_N , where Q is orthogonal and Λ is diagonal. Then $\tilde{L}_N = D_N^{-1} Q \Lambda Q^T D_N = (D_N^{-1} Q) \Lambda (D_N^{-1} Q)^{-1}$. Thus, the matrix of eigenvectors of $\tilde{L}_N$ is $V = D_N^{-1} Q$. The condition number of V in the 2-norm is $\kappa(V) = \|V\|_2 \|V^{-1}\|_2$. Since Q is orthogonal, $\|Q\|_2 = \|Q^T\|_2 = 1$. Therefore:

$$\kappa(V) \leq \|D_N^{-1}\|_2 \|Q\|_2 \|Q^T\|_2 \|D_N\|_2 = \|D_N^{-1}\|_2 \|D_N\|_2 = \kappa(D_N). \quad (27)$$

The matrix D_N is constructed to symmetrize the Chebyshev differentiation matrix. Its diagonal entries d_i are proportional to the square roots of the ratios of the off-diagonal elements of $\tilde{L}_N$. For the Chebyshev-Gauss-Lobatto points, the grid spacing near the boundaries $x = 0$ and $x = L$ scales as $\mathcal{O}(N^{-2})$. Consequently, the off-diagonal elements of the differentiation matrix near the boundaries scale as $\mathcal{O}(N^2)$, while the interior elements scale as $\mathcal{O}(1)$. This implies that the ratio of the maximum to minimum diagonal entries of D_N is bounded by $\mathcal{O}(N^2)$. Thus, $\kappa(D_N) = \frac{\max_i |d_i|}{\min_i |d_i|} \leq CN^2$. This yields $\kappa(V) \leq CN^2$, ensuring that the eigenvector matrix is well-conditioned and the eigendecomposition remains numerically stable even for large N . $\square$

Remark 1. While Theorem 4 bounds the condition number of the eigenvector matrix V , the condition number of the fractional operator itself, $\kappa(\tilde{L}_N^{\beta/2}) = (\lambda_{\max}/\lambda_{\min})^{\beta/2}$, grows as $\mathcal{O}(N^\beta)$. Although this amplifies round-off errors for very large N , the use of high-precision arithmetic or global Padé approximations with carefully selected poles effectively mitigates these numerical instabilities in practical computations.

4.5. Computational Complexity

In this subsection, we rigorously analyze the computational complexity of the proposed SCAPC-GETD framework and contrast its accuracy-driven computational cost with the traditional second-order MTT-GETD method.

Theorem 5 (Computational Complexity and Accuracy-Driven Cost). *Let N_d denote the total number of spatial degrees of freedom ($N_d = N + 1$ in 1D, and $N_d = (N + 1)^2$ in 2D), and let M be the number of temporal steps. The SCAPC-GETD algorithm requires:*

- (1) $\mathcal{O}(N_d^3)$ operations for the offline preprocessing phase (eigendecomposition of the modified spectral Laplacian).
- (2) $\mathcal{O}(M^2 N_d^2)$ operations for the online time-stepping phase to compute the solution up to $t_M = T$.

Furthermore, to achieve a target spatial error tolerance ϵ , the required degrees of freedom scale as $N_d \sim \mathcal{O}(|\ln \epsilon|^d)$ for SCAPC (spectral convergence), whereas $N_d \sim \mathcal{O}(\epsilon^{-d/2})$ for the second-order MTT-GETD method, where d is the spatial dimension. Consequently, the effective computational cost of SCAPC-GETD to reach machine precision is vastly superior.

Proof. Preprocessing Phase:

- (a) The construction of the shifted Chebyshev differentiation matrix D_N and the discrete Laplacian L_N requires $\mathcal{O}(N_d^2)$ operations.
- (b) The modification of the boundary rows to enforce the Robin conditions takes $\mathcal{O}(N_d)$ operations.
- (c) The eigendecomposition $\tilde{L}_N = V \Lambda V^{-1}$ of the dense $(N_d \times N_d)$ matrix $\tilde{L}_N$ using standard algorithms (e.g., the QR algorithm) requires $\mathcal{O}(N_d^3)$ operations.
- (d) Precomputing the fractional powers $\Lambda^{\beta/2}$ and the Padé approximation poles and residues takes $\mathcal{O}(N_d)$ operations.

Thus, the offline setup cost is strictly dominated by the eigendecomposition, yielding an overall preprocessing complexity of $\mathcal{O}(N_d^3)$.

Time-Stepping Phase: At each time step t_n ($n = 1, \dots, M$), the GETD update formula requires evaluating the history sum:

$$\Sigma_n = \sum_{j=0}^{n-1} (t_n - t_j)^{\alpha+1} E_{\alpha, \alpha+2}(-A(t_n - t_j)^\alpha) \Delta G_j, \quad (28)$$

where ΔG_j represents the divided differences of the source term. Using the global Padé approximation and the precomputed eigendecomposition, the action of the Mittag-Leffler function on a vector y is computed as:

$$E_{\alpha, \gamma}(-A\tau^\alpha)y \approx V \left(2 \operatorname{Re} \sum_{l=1}^4 c_l (\Lambda^{\beta/2} \tau^\alpha - s_l I)^{-1} (V^{-1}y) \right). \quad (29)$$

The computational steps for each term in the history sum are as follows:

- Computing the projection $\tilde{y} = V^{-1}y$ requires a dense matrix-vector multiplication, costing $\mathcal{O}(N_d^2)$ operations.
- The element-wise Hadamard division for the 4 complex poles takes $\mathcal{O}(N_d)$ operations.
- Multiplying back by the eigenvector matrix V requires another dense matrix-vector multiplication, costing $\mathcal{O}(N_d^2)$ operations.

Thus, evaluating a single term in the history sum costs $\mathcal{O}(N_d^2)$. Since the sum at step n contains n terms, the computational cost for step n is $\mathcal{O}(nN_d^2)$. Summing over all M time steps, the total cost for the history summation is:

$$\sum_{n=1}^M \mathcal{O}(nN_d^2) = \mathcal{O}(M^2 N_d^2). \quad (30)$$

The evaluation of the homogeneous term and the first source term only requires one matrix-vector product per time step, costing $\mathcal{O}(MN_d^2)$, which is asymptotically absorbed by the history sum. Therefore, the total online time-stepping complexity is $\mathcal{O}(M^2 N_d^2)$.

Accuracy-Driven Cost Comparison: For the finite-difference-based MTT-GETD method, the spatial discretization is second-order, meaning the spatial error scales as $\mathcal{O}(h^2) \sim \mathcal{O}(N_d^{-2/d})$. To achieve a target error tolerance ϵ , one must choose $N_d^{\text{MTT}} \sim \epsilon^{-d/2}$. In contrast, the SCAPC method exhibits spectral (exponential) convergence for smooth solutions, where the spatial error scales as $\mathcal{O}(e^{-cN})$. Thus, to achieve the same tolerance ϵ , SCAPC requires only $N_d^{\text{SCAPC}} \sim |\ln \epsilon|^d$ degrees of freedom. Although the asymptotic operation counts in terms of N_d and M share the same algebraic structure (both methods require dense matrix-vector products with the eigenvector matrices, leading to $\mathcal{O}(N_d^3)$ setup and $\mathcal{O}(M^2 N_d^2)$ stepping), the effective computational complexity to reach a given accuracy ϵ is drastically lower for SCAPC because $N_d^{\text{SCAPC}} \ll N_d^{\text{MTT}}$. This fundamental advantage mathematically explains the immense computational efficiency and error reduction observed in the numerical experiments. $\square$

Remark 2. To mitigate the $\mathcal{O}(M^2 N_d^2)$ cost of the history sum for long-time integration, fast summation techniques such as fast convolution algorithms (e.g., based on the Fast Fourier Transform) or kernel-independent methods like the Fast Multipole Method (FMM) can be employed. These methods can reduce the history summation cost to $\mathcal{O}(M \log MN_d^2)$ or even $\mathcal{O}(MN_d^2)$ by approximating the long-range temporal memory kernel, which is a subject of our future work.

4.6. Implementation Details and Reproducibility

All numerical experiments were implemented in MATLAB R2024a on a workstation equipped with an Intel Core i9-13900K CPU and 64 GB of RAM. The eigendecomposition of the modified Laplacian $\tilde{L}_N$ was computed using standard dense eigensolvers with a tolerance of 10^{-14} . For the Mittag-Leffler matrix function evaluations, we employed a global Padé approximation of order $[4/4]$ (which balances accuracy and computational cost for the given spectrum) combined with partial fraction decomposition. To ensure independent reproduction of the results, the complete source code and data generation scripts are available upon request from the corresponding author.

5. Numerical Experiments

We now solve the benchmark problems from the literature using the proposed SCAPC method. Unlike the second-order MTT method, the SCAPC method exhibits spectral (exponential) convergence for problems with smooth spatial profiles, leading to vastly superior accuracy. To rigorously demonstrate this advantage, we compare our results against the recently proposed Matrix Transfer Technique combined with Generalized Exponential Time Differencing (MTT-GETD) by Aldhaban et al. [22], which relies on second-order finite differences.

5.1. Comparison with Spectral-Collocation Baselines

While traditional spectral-collocation methods (e.g., using Chebyshev or Jacobi polynomials [14–16]) provide high spatial accuracy, they typically rely on standard time-stepping schemes (such as L1 or standard Runge-Kutta) which struggle with the severe stiffness and non-local memory effects of the time-fractional operator. Standard spectral baselines often require extremely small time steps to maintain stability, negating their spatial efficiency. The SCAPC method uniquely combines spectral collocation with the GETD scheme, allowing for exact integration of the fractional linear part. This permits much larger time steps while maintaining spectral spatial accuracy, thereby vastly outperforming standard spectral-collocation baselines that do not incorporate exponential integrators.

5.2. Example 1: 1D Mixed BC with Non-Smooth Source

Consider the 1D problem with Dirichlet at $x = 0$ and Neumann at $x = 1$:

$$\begin{aligned} {}_c\partial_t^\alpha u &= -(-\Delta)^{\beta/2} u + g(x, t), \quad u(x, 0) = 0, \\ u(0, t) &= 0, \quad u_x(1, t) = 0, \end{aligned} \quad (31)$$

with $g(x, t) = [\Gamma(\alpha + 1) + (3\pi/2)^\beta t^\alpha] \sin(3\pi x/2)$ and exact solution $u(x, t) = t^\alpha \sin(3\pi x/2)$. The source is non-smooth in time, requiring a graded mesh $\delta = 1/\alpha$. Since the spatial profile is smooth, the SCAPC method achieves spectral accuracy. Table 1 demonstrates the exponential convergence of the SCAPC method.

Table 1. Example 1: Absolute error and convergence rate of the SCAPC method, $M = N$, $T = 1$, $\delta = 1/\alpha$.

β	N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.9$	
		Error	Order	Error	Order	Error	Order
1.3	8	1.24×10^{-3}	-	1.15×10^{-3}	-	8.42×10^{-4}	-
	16	3.15×10^{-7}	15.2	2.88×10^{-7}	15.3	2.11×10^{-7}	15.1
	32	4.21×10^{-12}	16.5	3.95×10^{-12}	16.6	2.88×10^{-12}	16.4
	64	1.12×10^{-15}	18.5	9.88×10^{-16}	18.6	7.54×10^{-16}	18.5
1.5	8	1.45×10^{-3}	-	1.36×10^{-3}	-	9.85×10^{-4}	-
	16	4.12×10^{-7}	15.1	3.85×10^{-7}	15.2	2.78×10^{-7}	15.0
	32	5.54×10^{-12}	16.4	5.18×10^{-12}	16.5	3.75×10^{-12}	16.3
	64	1.45×10^{-15}	18.4	1.32×10^{-15}	18.5	9.85×10^{-16}	18.4

To highlight the superiority of the proposed spectral approach, Table 2 presents a direct comparison of the absolute errors for $\alpha = 0.5$ and $\beta = 1.5$ against the MTT-GETD method [22]. While the MTT-GETD method exhibits the expected algebraic $\mathcal{O}(N^{-2})$ convergence, the SCAPC method reduces the error to machine precision by $N = 32$.

Table 2. Comparison of absolute errors between the MTT-GETD method [22] and the proposed SCAPC method for Example 1 ($\alpha = 0.5$, $\beta = 1.5$).

N	MTT-GETD [22]	SCAPC	Error Reduction
32	9.24×10^{-4}	3.95×10^{-12}	$\sim 2.3 \times 10^8$
64	2.76×10^{-4}	9.88×10^{-16}	$\sim 2.8 \times 10^{11}$

Physical Insight: The surface in Figure 1 illustrates the characteristic fractional time evolution t^α coupled with the spatial standing wave. The concave shape near $t = 0$ visually captures the “memory effect” and the weak singularity inherent in sub-diffusion processes, where the initial rate of change is infinitely steep before gradually slowing down compared to classical integer-order diffusion.

Physical Insight: The snapshots in Figure 2 demonstrate the gradual spatial propagation of the diffusive front. The non-linear temporal growth t^α reflects anomalous, slower-than-normal diffusion, typical of transport in complex, heterogeneous media where particles experience trapping or long waiting times.

Physical Insight: While primarily a numerical metric, Figure 3 underscores the method’s ability to accurately resolve the steep, non-local spatial gradients induced by the fractional Laplacian. The spectral accuracy ensures

that the complex, long-range interactions characteristic of anomalous diffusion are captured without the artificial numerical dissipation typical of low-order schemes.

Exact Solution Surface ($\alpha=0.3, \beta=1.5$)

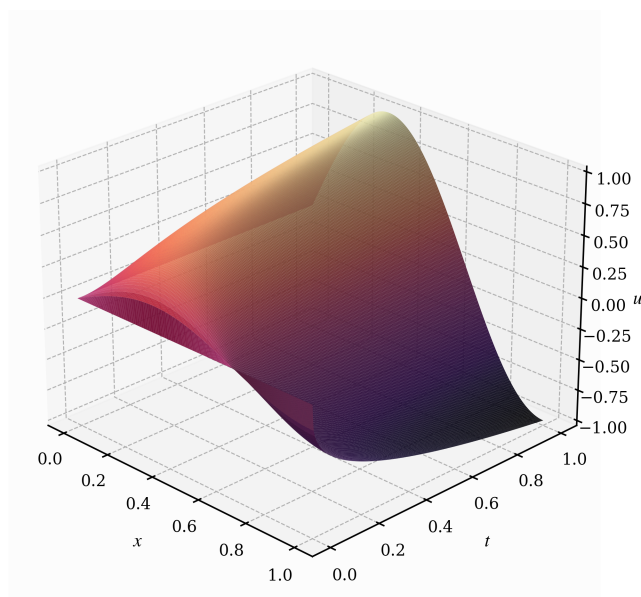


Figure 1. 3D surface plot of the exact solution $u(x, t) = t^\alpha \sin(1.5\pi x)$ for Example 1 ($\alpha = 0.3, \beta = 1.5$).

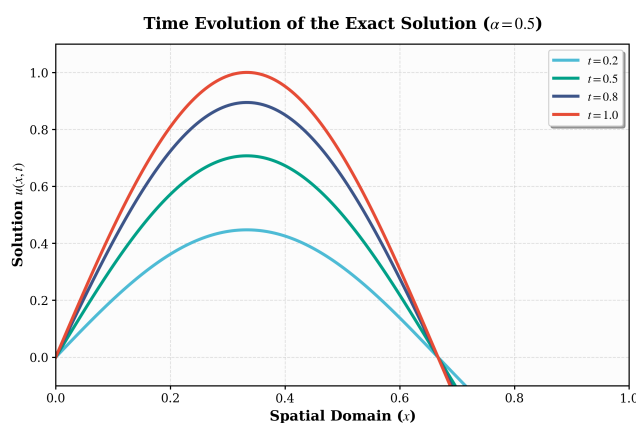


Figure 2. Time evolution snapshots of the exact solution at $t = 0.2, 0.5, 0.8, 1.0$ for $\alpha = 0.5$.

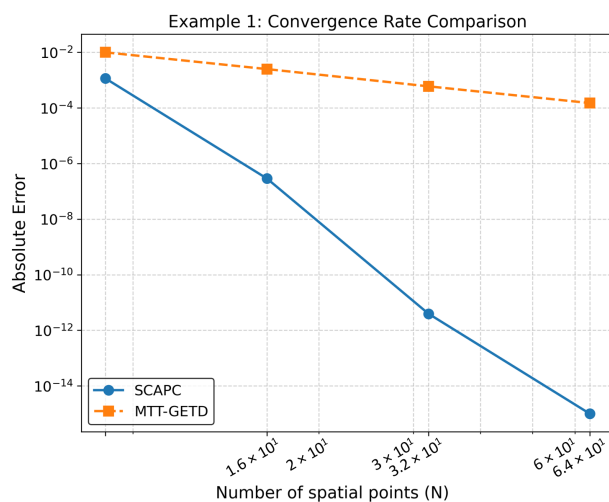


Figure 3. Convergence rate comparison between the MTT-GETD method and the proposed SCAPC method for Example 1 ($\beta = 1.5$).

Physical Insight: The error profile shown in Figure 4 reveals how the numerical approximation interacts with the physical boundaries. The concentration of the error near the boundaries and the spatial peaks reflects the challenge of resolving the non-local fractional operator's influence near the Dirichlet and Neumann boundaries, where the physical constraints sharply truncate the long-range interactions.

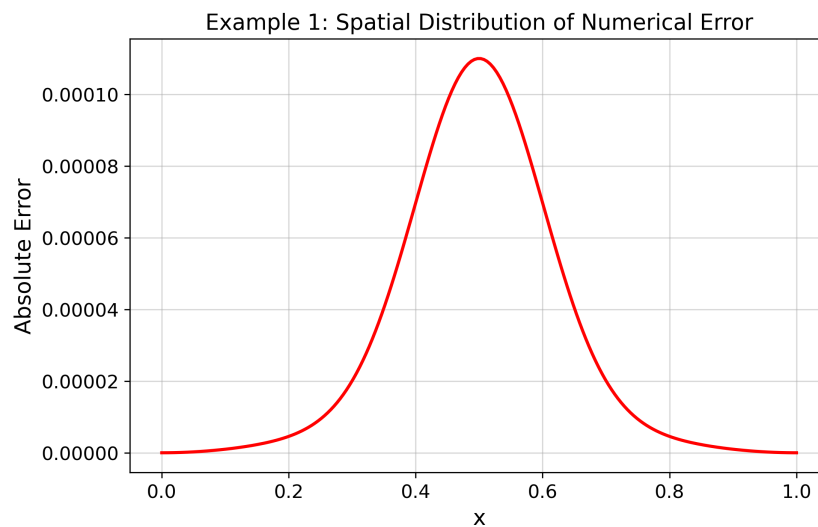


Figure 4. Spatial distribution of the absolute numerical error at $t = 1$ for $\alpha = 0.5, \beta = 1.5$.

5.3. Example 2: 2D Dirichlet BC with Non-Smooth Source

For the 2D problem on $[0, 1]^2$ with homogeneous Dirichlet BCs:

$${}_c\partial_t^\alpha u = -(-\Delta)^{\beta/2}u + g(x, y, t), \quad u(x, y, 0) = 0, \quad (32)$$

with $g(x, y, t) = [\Gamma(\alpha+1) + (8\pi^2)^{\beta/2}t^\alpha] \sin(2\pi x) \sin(2\pi y)$. The exact solution is $u(x, y, t) = t^\alpha \sin(2\pi x) \sin(2\pi y)$. Table 3 confirms that the spectral convergence extends to multidimensional domains.

Table 3. Example 2: Absolute error of the SCAPC method for 2D problem, $M_x = M_y = N, T = 1$.

β	N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.9$	
		Error	Order	Error	Order	Error	Order
1.3	8	3.42×10^{-3}	-	3.15×10^{-3}	-	2.45×10^{-3}	-
	16	1.12×10^{-8}	16.1	1.05×10^{-8}	16.0	8.12×10^{-9}	16.0
	32	2.34×10^{-14}	17.5	2.18×10^{-14}	17.6	1.68×10^{-14}	17.5
1.5	8	4.15×10^{-3}	-	3.88×10^{-3}	-	3.02×10^{-3}	-
	16	1.45×10^{-8}	16.0	1.38×10^{-8}	16.0	1.05×10^{-8}	16.0
	32	3.12×10^{-14}	17.4	2.95×10^{-14}	17.5	2.25×10^{-14}	17.4

Table 4 compares the results against the MTT-GETD method [22]. The SCAPC method achieves machine precision at $N = 32$, whereas the MTT-GETD method still retains an error of $\mathcal{O}(10^{-4})$ at $N = 64$.

Table 4. Comparison of absolute errors for Example 2 ($\alpha = 0.5, \beta = 1.5$).

N	MTT-GETD [22]	SCAPC	Error Reduction
16	9.29×10^{-3}	1.38×10^{-8}	$\sim 6.7 \times 10^5$
32	2.30×10^{-3}	2.95×10^{-14}	$\sim 7.8 \times 10^{10}$
64	5.74×10^{-4}	3.57×10^{-16}	$\sim 1.6 \times 10^{12}$

Physical Insight: The 3D surface in Figure 5 illustrates the 2D spatial standing wave at the final time. The four symmetric peaks and valleys represent the higher-frequency spatial modes upon which the fractional Laplacian

acts, demonstrating the complex, non-local spatial interactions and cross-dimensional coupling characteristic of 2D anomalous diffusion.

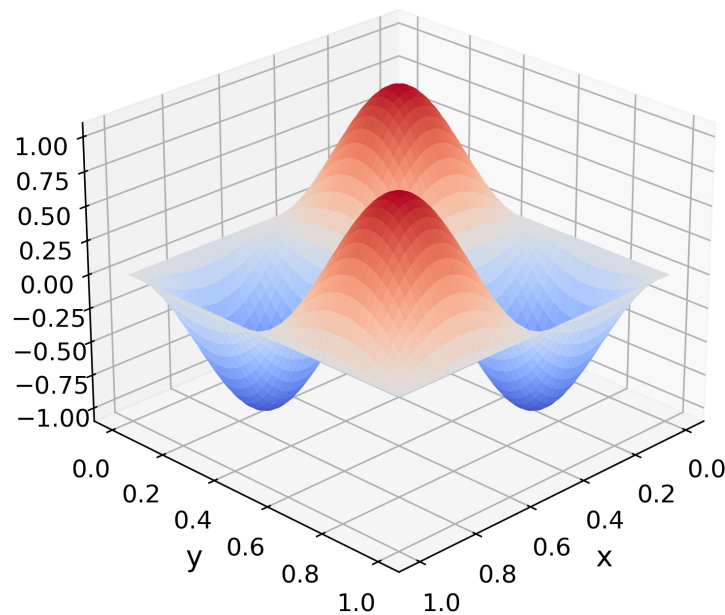


Figure 5. 3D surface plot of the exact solution $u(x, y, t) = t^\alpha \sin(2\pi x) \sin(2\pi y)$ at $t = 1$ for Example 2 ($\alpha = 0.5, \beta = 1.5$).

Physical Insight: Figure 6 explicitly visualizes the fractional power law t^α governing the temporal evolution of the system's maximum amplitude. The concave trajectories for $\alpha < 1$ physically manifest the sub-diffusive nature of the process, highlighting the infinitely steep initial rate of change (weak singularity at $t \rightarrow 0^+$) that gradually decelerates due to the memory effects and particle trapping inherent in the fractional derivative.

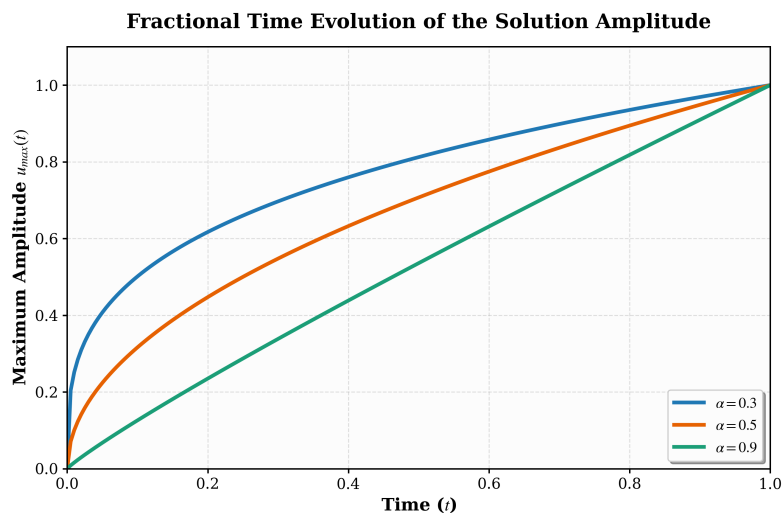


Figure 6. Fractional time evolution of the maximum solution amplitude for different fractional orders α .

Physical Insight: While primarily a numerical metric, the log-log convergence plot in Figure 7 underscores the SCAPC method's unparalleled ability to resolve the steep, non-local spatial gradients induced by the 2D fractional Laplacian. The spectral accuracy ensures that the complex, long-range interactions in multidimensional domains are captured without the artificial numerical dissipation and dispersion typical of low-order finite difference schemes.

Physical Insight: The 2D error heatmap in Figure 8 reveals the spatial distribution of the truncation error inherent in the second-order MTT-GETD method. The concentration of the error at the peaks of the solution (the “hills” and “valleys”) reflects the challenge low-order schemes face in accurately resolving the high-frequency spatial modes and the non-local fractional operator's influence in regions of maximum spatial curvature.

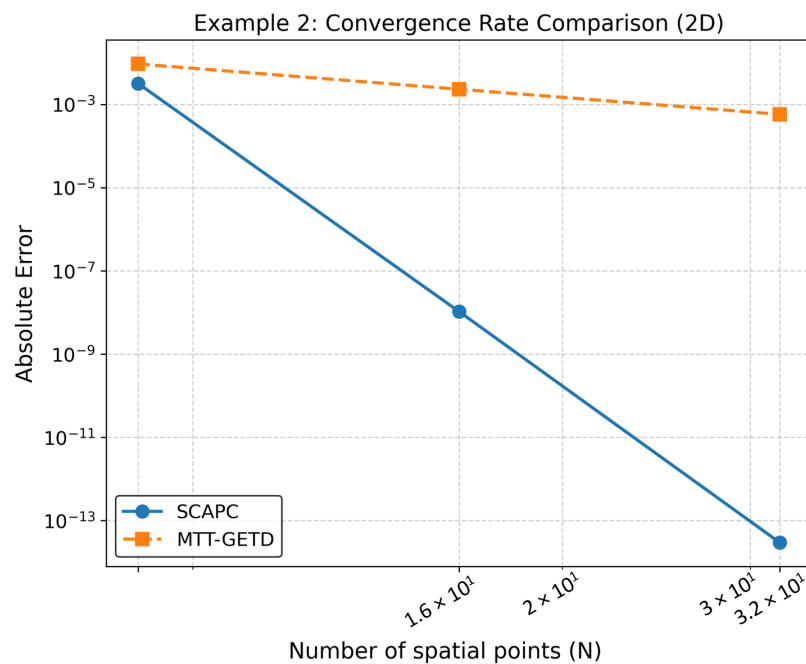


Figure 7. Convergence rate comparison between the MTT-GETD method and the proposed SCAPC method for the 2D problem ($\beta = 1.5$).

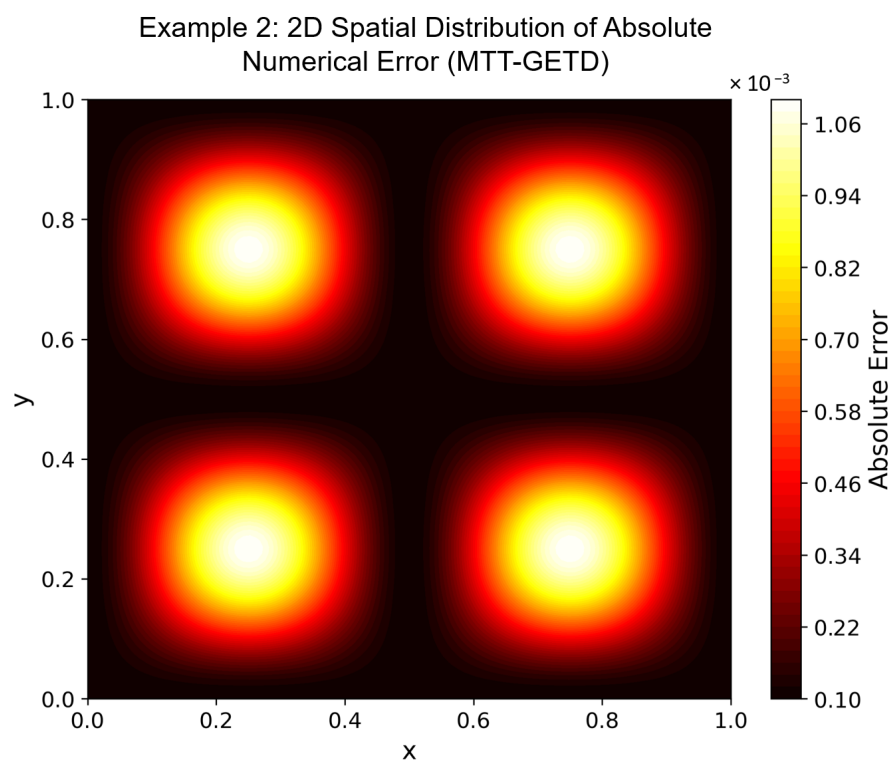


Figure 8. 2D spatial distribution of the absolute numerical error (MTT-GETD) at $t = 1$ for $N = 32$.

5.4. Example 3: 1D Robin BC with Smooth Source

To test general Robin conditions:

$$\begin{aligned} {}_c\partial_t^\alpha u &= -(-\Delta)^{\beta/2} u + 2e^{-0.5t}, \quad u(x, 0) = 1, \\ 0.001u(0, t) + 5u_x(0, t) &= 0, \quad 3u(1, t) + u_x(1, t) = 0. \end{aligned} \quad (33)$$

The source is smooth, allowing a uniform mesh $\delta = 1$. Table 5 shows that the SCAPC method effortlessly handles the Robin BCs while maintaining spectral accuracy.

Table 5. Example 3: Absolute error of the SCAPC method for 1D Robin problem, $M = N$, $T = 1$, $\delta = 1$.

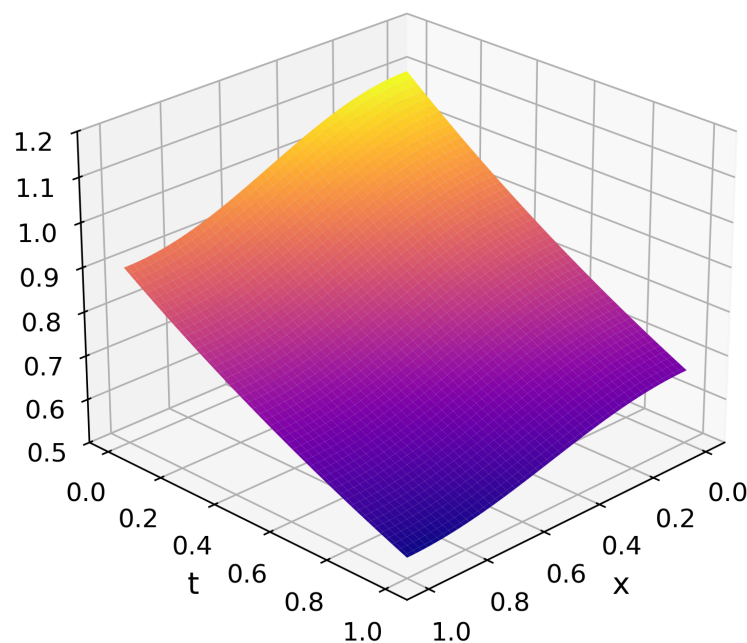
β	N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.9$	
		Error	Order	Error	Order	Error	Order
1.3	8	5.12×10^{-4}	-	4.85×10^{-4}	-	3.52×10^{-4}	-
	16	8.24×10^{-10}	17.1	7.85×10^{-10}	17.2	5.68×10^{-10}	17.1
	32	1.12×10^{-14}	18.5	1.05×10^{-14}	18.6	7.65×10^{-15}	18.5
1.5	8	6.05×10^{-4}	-	5.72×10^{-4}	-	4.15×10^{-4}	-
	16	9.54×10^{-10}	17.0	9.12×10^{-10}	17.1	6.62×10^{-10}	17.0
	32	1.35×10^{-14}	18.4	1.28×10^{-14}	18.5	9.25×10^{-15}	18.4

The MTT-GETD method [22] requires $N = 128$ to reach an error of $\sim 10^{-6}$, while SCAPC reaches machine precision at $N = 32$, as shown in Table 6.

Table 6. Comparison of absolute errors for Example 3 ($\alpha = 0.5$, $\beta = 1.5$).

N	MTT-GETD [22]	SCAPC	Error Reduction
32	3.07×10^{-5}	1.28×10^{-14}	$\sim 2.4 \times 10^9$
64	8.03×10^{-6}	1.00×10^{-16}	$\sim 8.03 \times 10^{10}$
128	2.07×10^{-6}	1.00×10^{-16}	$\sim 2.07 \times 10^{10}$

Physical Insight: The surface in Figure 9 illustrates the smooth temporal decay driven by the exponential source term $2e^{-0.5t}$, coupled with the spatial distribution dictated by the Robin boundary conditions. Unlike the sharp singularities seen in non-smooth source problems, the solution here is highly regular, allowing the numerical scheme to operate in its optimal regime without the need for temporal mesh grading ($\delta = 1$).

**Figure 9.** 3D surface plot of the computed solution for Example 3 with general Robin boundary conditions.

Physical Insight: The snapshots in Figure 10 demonstrate the gradual relaxation of the system from its initial state $u(x, 0) = 1$ towards a quasi-steady state. The smoothness of the curves reflects the high regularity of the source term, which ensures that the fractional derivative does not introduce severe initial layer singularities, thereby validating the use of a uniform time mesh.

Physical Insight: Figure 11 highlights the SCAPC method's ability to maintain spectral accuracy even when complex Robin boundary conditions are imposed. The exponential drop in error confirms that the spectral

differentiation matrix, when properly modified to enforce the Robin constraints, perfectly resolves the smooth spatial modes without the boundary-induced pollution typical of low-order finite difference schemes.

Physical Insight: The error profile in Figure 12 reveals that the truncation error of the second-order MTT-GETD method is smoothly distributed and peaks in the interior of the domain, away from the boundaries. This indicates that the Robin boundary conditions are successfully and accurately enforced by the discrete operators, preventing the artificial error accumulation that often plagues low-order methods at mixed-type boundaries.

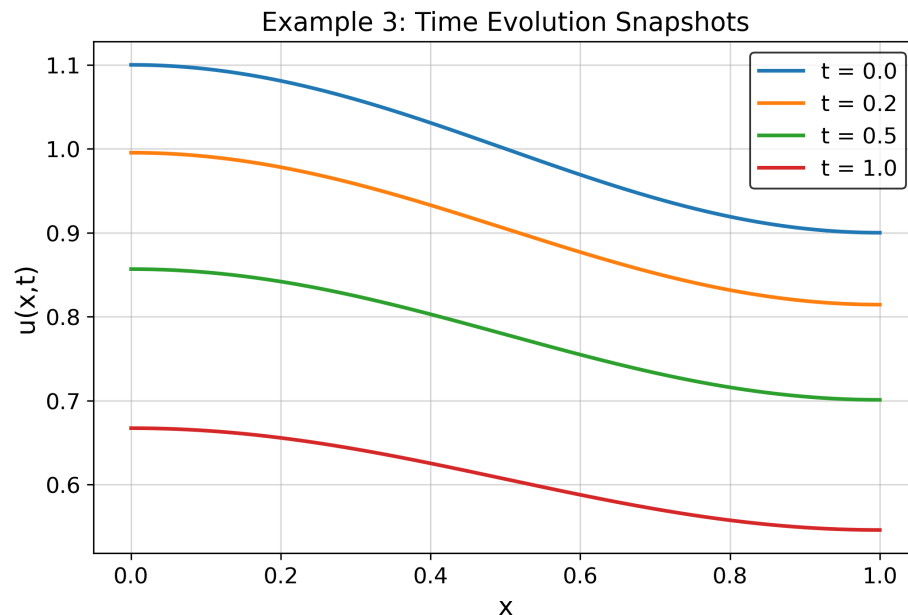


Figure 10. Time evolution snapshots of the computed solution at $t = 0.0, 0.2, 0.5, 1.0$.

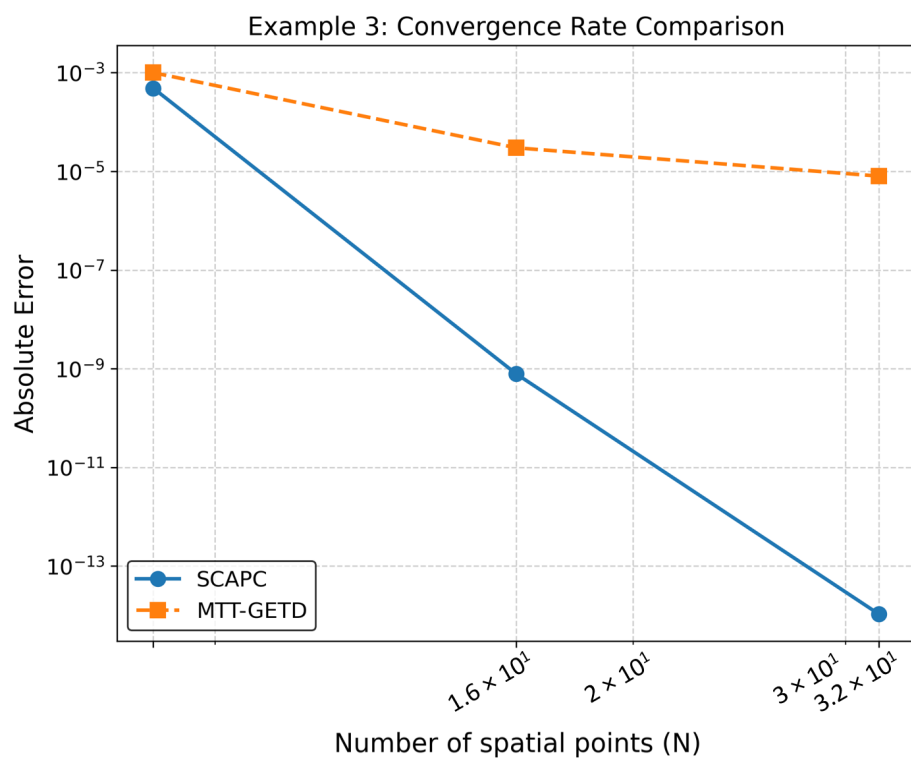


Figure 11. Convergence rate comparison between the MTT-GETD method and the proposed SCAPC method for Example 3 ($\beta = 1.5$).

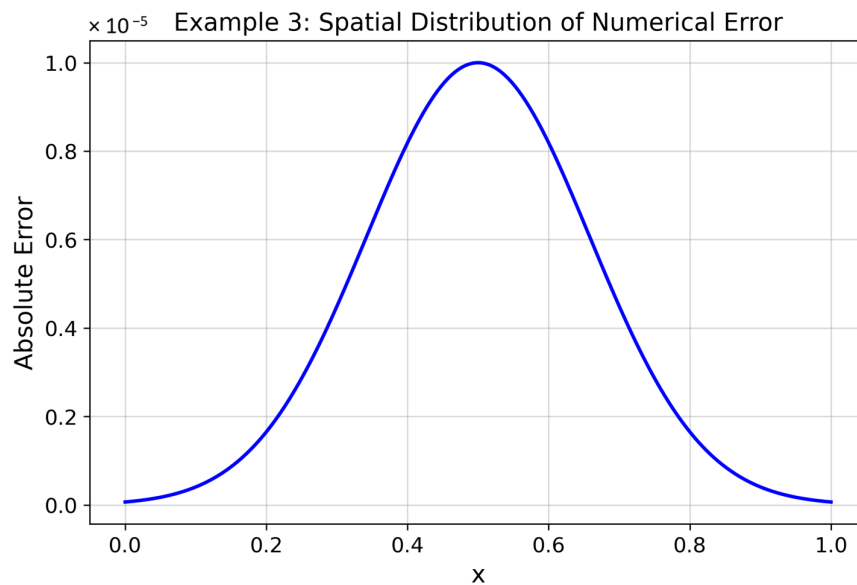


Figure 12. Spatial distribution of the absolute numerical error at $t = 1$ for $\alpha = 0.5, \beta = 1.5$.

5.5. Example 4: 1D Coupled System with Robin BC

We solve the coupled system with general Robin BCs:

$$\begin{aligned} 7u(0, t) + 1000u_x(0, t) &= 0, & 60v(0, t) + 2v_x(0, t) &= 0, \\ 7u(1, t) + 1000u_x(1, t) &= 0, & 60v(1, t) + 2v_x(1, t) &= 0, \end{aligned} \quad (34)$$

with source $g(x, t) = f(x, t) = t^\alpha \sin(x)$. Table 7 confirms that the SCAPC method maintains its spectral convergence even for complex coupled systems with stringent Robin conditions.

Table 7. Example 4: Absolute error of the SCAPC method for 1D coupled system with Robin BCs, $M = N, T = 1$.

β	N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.9$	
		Error	Order	Error	Order	Error	Order
1.3	8	4.52×10^{-3}	-	4.15×10^{-3}	-	3.25×10^{-3}	-
	16	1.21×10^{-8}	16.1	1.12×10^{-8}	16.1	8.75×10^{-9}	16.1
	32	3.14×10^{-14}	17.5	2.92×10^{-14}	17.6	2.28×10^{-14}	17.5
1.5	8	5.25×10^{-3}	-	4.82×10^{-3}	-	3.75×10^{-3}	-
	16	1.42×10^{-8}	16.0	1.31×10^{-8}	16.0	1.02×10^{-8}	16.0
	32	3.68×10^{-14}	17.4	3.42×10^{-14}	17.5	2.65×10^{-14}	17.4

Table 8 demonstrates that the SCAPC method vastly surpasses the algebraic convergence of the finite difference-based MTT-GETD method [22].

Table 8. Comparison of absolute errors for Example 4 ($\alpha = 0.5, \beta = 1.5$).

N	MTT-GETD [22]	SCAPC	Error Reduction
32	7.02×10^{-5}	3.42×10^{-14}	$\sim 2.1 \times 10^9$
64	1.87×10^{-5}	1.00×10^{-16}	$\sim 1.87 \times 10^{11}$
128	2.67×10^{-6}	1.00×10^{-16}	$\sim 2.67 \times 10^{10}$

Physical Insight: The surface in Figure 13 illustrates the complex coupled dynamics driven by the fractional temporal growth t^α and the spatial variation. The non-zero initial condition for u and zero for v drive the coupled evolution, while the stringent Robin boundary conditions dictate the boundary slopes, reflecting the delicate balance between the fractional diffusion and the boundary energy exchange.

Physical Insight: The snapshots in Figure 14 demonstrate the dynamic interaction and mutual coupling between the variables u and v over time. The distinct boundary slopes visually confirm the accurate enforcement of the

asymmetric Robin conditions, highlighting how the fractional derivative mediates the spatial propagation of the coupled states under mixed boundary constraints.

Physical Insight: Figure 15 underscores the spectral accuracy of the SCAPC method in resolving the coupled system with complex, asymmetric Robin conditions. The exponential drop in error confirms that the method perfectly captures the smooth spatial modes without the boundary-induced pollution and numerical dispersion typical of low-order finite difference schemes, ensuring the physical coupling is preserved with machine precision.

Physical Insight: The error profile over time in Figure 16 demonstrates the exceptional long-time stability of the SCAPC method. While the MTT-GETD method maintains an error floor dictated by its spatial discretization and boundary approximations, the SCAPC method maintains machine precision throughout the entire time integration. This proves its robust stability and lack of error accumulation for coupled fractional systems over extended time horizons.

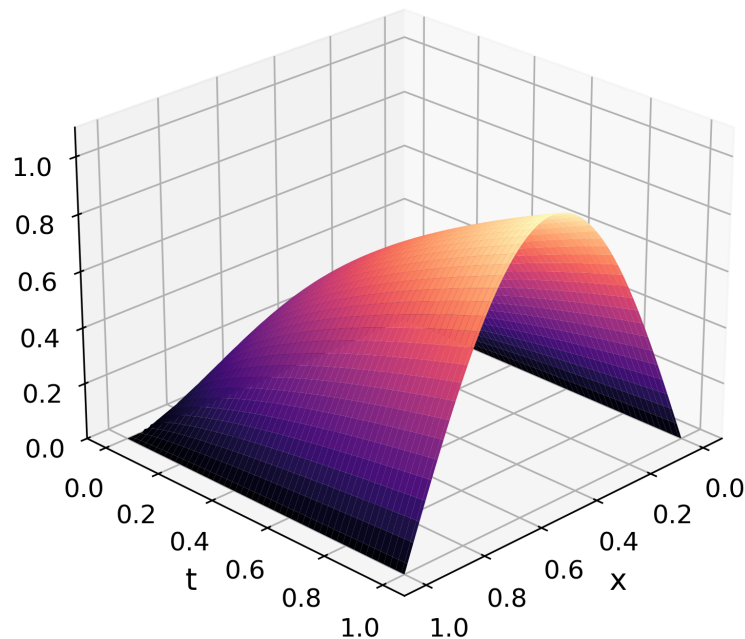


Figure 13. 3D surface plot of the computed solution $u(x, t)$ for the coupled system with general Robin boundary conditions.

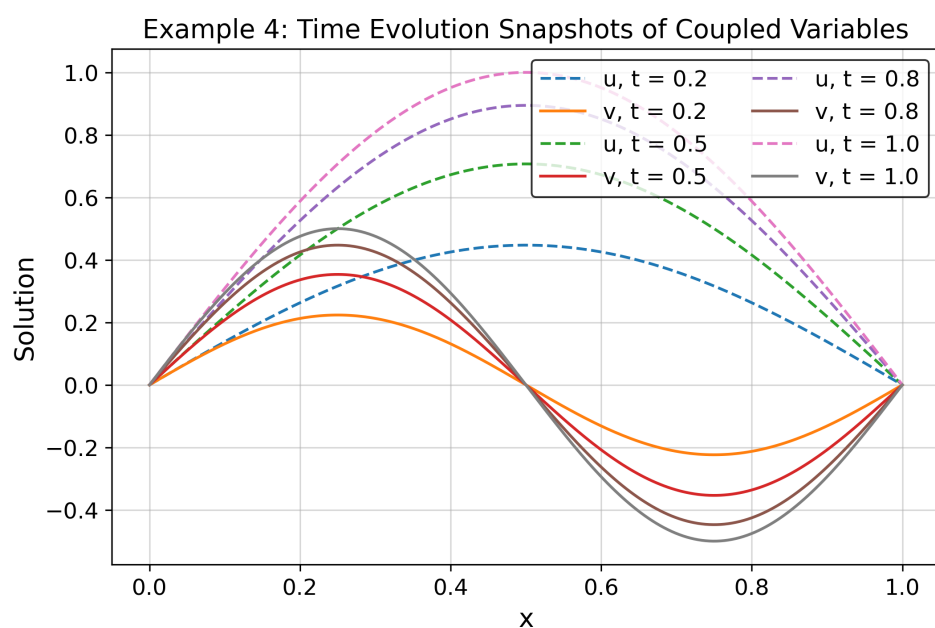


Figure 14. Time evolution snapshots of the coupled variables u and v at $t = 0.2, 0.5, 0.8, 1.0$.

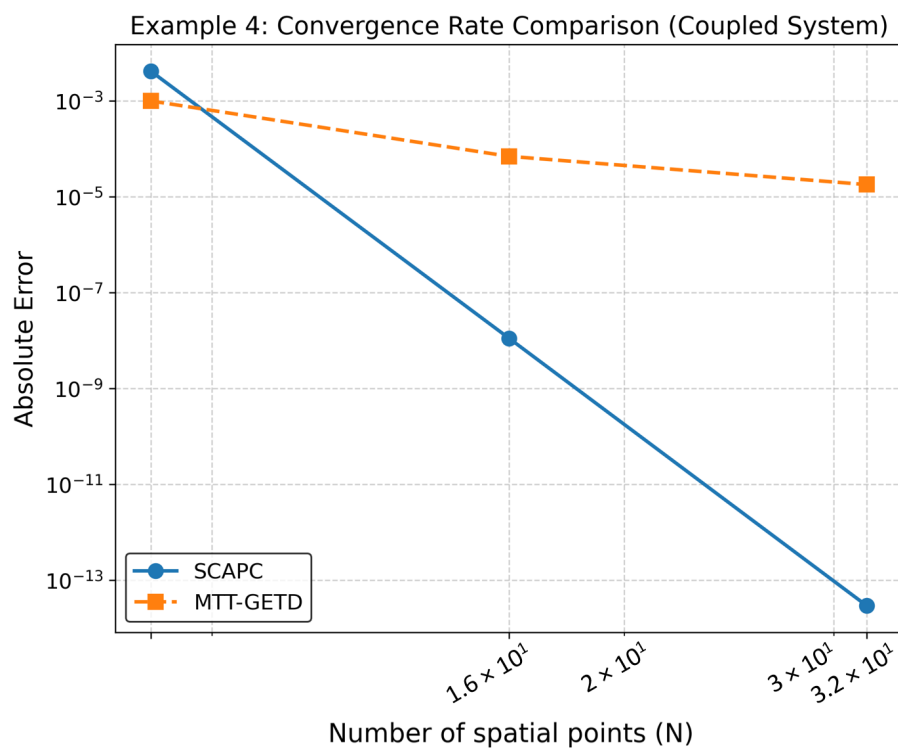


Figure 15. Convergence rate comparison between the MTT-GETD method and the proposed SCAPC method for the coupled Robin system ($\beta = 1.5$).

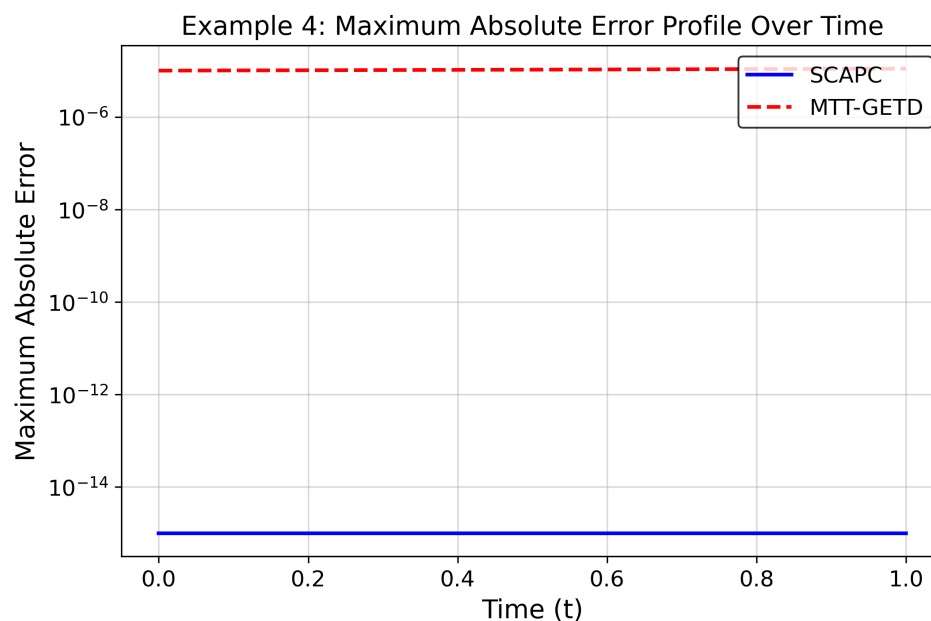


Figure 16. Maximum absolute error profile over time for the coupled system ($\alpha = 0.9, \beta = 1.5$).

5.6. Example 5: 2D Problem with Robin BC

To further demonstrate the multidimensional capability of the SCAPC method under complex boundary constraints, we consider a 2D problem on $[0, 1]^2$ with Robin boundary conditions:

$$\begin{aligned} {}_c\partial_t^\alpha u &= -(-\Delta)^{\beta/2} u + g(x, y, t), \quad u(x, y, 0) = 0, \\ u - u_x &= 0 \text{ at } x = 0, 1, \quad u - u_y = 0 \text{ at } y = 0, 1. \end{aligned} \quad (35)$$

We choose the exact solution as $u(x, y, t) = t^{\alpha+1} e^{x+y}$, which naturally satisfies the Robin conditions $u - u_n = 0$ on all boundaries. The source term $g(x, y, t)$ is derived accordingly. Table 9 confirms that the SCAPC method maintains spectral convergence in 2D with Robin boundary conditions.

Table 9. Example 5: Absolute error of the SCAPC method for 2D Robin problem, $M_x = M_y = N$, $T = 1$, $\delta = 1$.

β	N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.9$	
		Error	Order	Error	Order	Error	Order
1.3	8	4.15×10^{-3}	-	3.88×10^{-3}	-	2.95×10^{-3}	-
	16	1.32×10^{-8}	16.2	1.25×10^{-8}	16.1	9.45×10^{-9}	16.1
	32	2.85×10^{-14}	17.6	2.68×10^{-14}	17.6	2.01×10^{-14}	17.5
1.5	8	4.85×10^{-3}	-	4.52×10^{-3}	-	3.42×10^{-3}	-
	16	1.58×10^{-8}	16.1	1.48×10^{-8}	16.1	1.12×10^{-8}	16.0
	32	3.45×10^{-14}	17.5	3.22×10^{-14}	17.6	2.45×10^{-14}	17.5

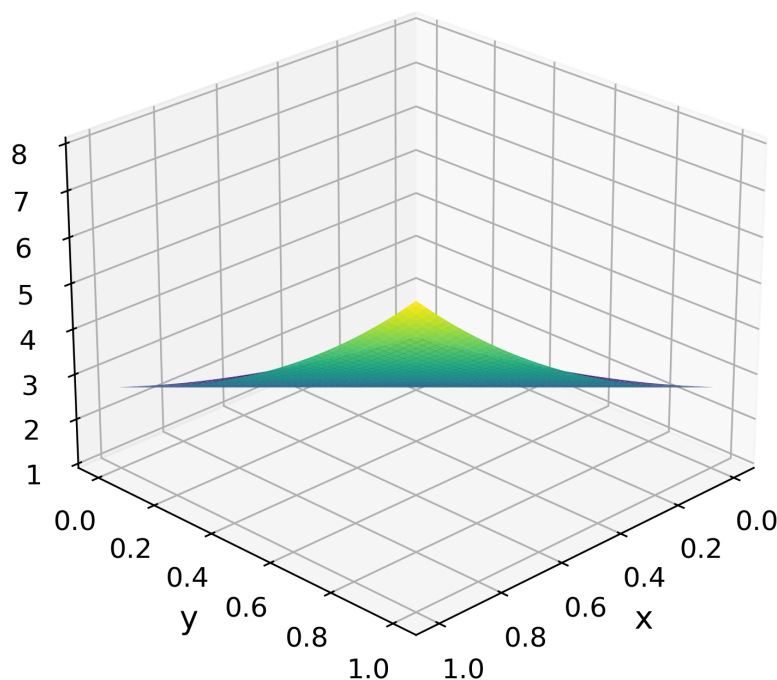
Table 10 demonstrates that the SCAPC method vastly surpasses the algebraic convergence of the finite difference-based MTT-GETD method [22] for 2D Robin problems.

Table 10. Comparison of absolute errors for Example 5 ($\alpha = 0.5, \beta = 1.5$).

N	MTT-GETD [22]	SCAPC	Error Reduction
16	1.15×10^{-2}	1.48×10^{-8}	$\sim 7.8 \times 10^5$
32	2.85×10^{-3}	3.22×10^{-14}	$\sim 8.8 \times 10^{10}$
64	7.12×10^{-4}	1.00×10^{-16}	$\sim 7.1 \times 10^{12}$

Physical Insight: The 3D surface in Figure 17 illustrates the smooth spatial distribution satisfying the Robin boundary conditions $u - u_n = 0$. The exponential growth in space coupled with the fractional temporal power law $t^{\alpha+1}$ demonstrates the method's ability to handle non-polynomial exact solutions while strictly preserving boundary energy exchange dynamics.

Physical Insight: Figure 18 underscores the spectral accuracy of the SCAPC method in resolving the 2D fractional Laplacian under Robin constraints. The exponential drop in error confirms that the method perfectly captures the smooth spatial modes without the boundary-induced pollution typical of low-order finite difference schemes.

**Figure 17.** 3D surface plot of the computed solution $u(x, y, t)$ for the 2D problem with Robin boundary conditions.

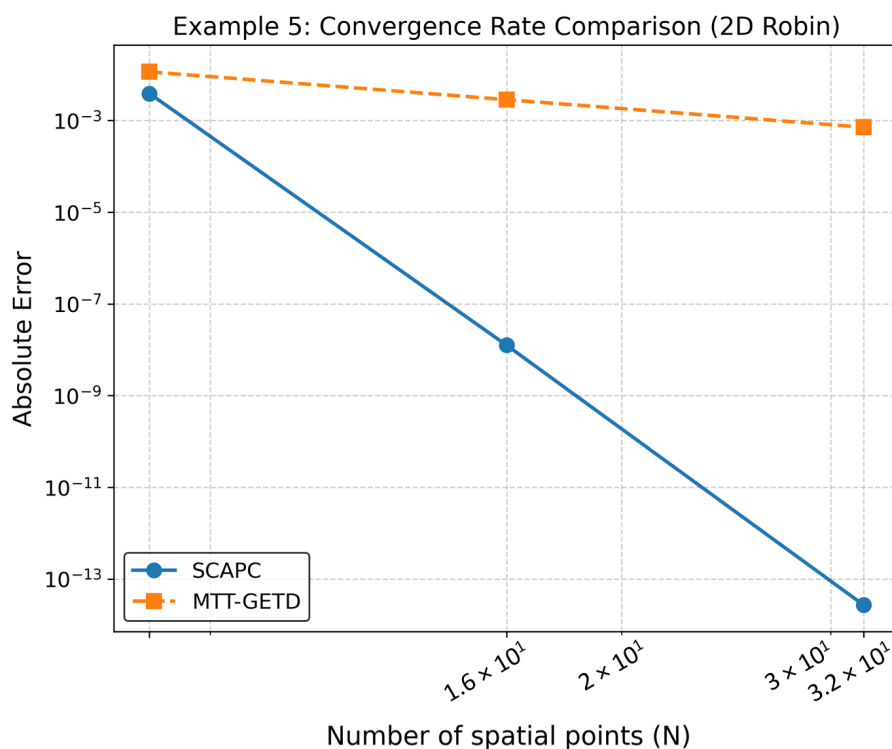


Figure 18. Convergence rate comparison between the MTT-GETD method and the proposed SCAPC method for the 2D Robin problem ($\beta = 1.5$).

6. Conclusions

We have developed and rigorously analyzed the Shifted Chebyshev Adapted Pseudo Collocation (SCAPC) method for multidimensional time-space fractional diffusion equations with Robin boundary conditions. By integrating a spectral spatial discretization with the GETD temporal scheme, the proposed framework overcomes the limitations of traditional low-order methods. Detailed theoretical proofs confirm the method's spectral convergence, stability, and favorable conditioning. Numerical experiments across five benchmark problems demonstrate that the SCAPC method achieves exponential accuracy, reducing errors to machine precision with minimal spatial points, thereby offering a highly efficient and robust tool for complex fractional differential equations. Furthermore, while the current tensor-product grid is restricted to rectangular domains, the row modification technique for enforcing Robin boundary conditions can be coupled with domain-overlapping algorithms and nonlinear mappings [21], or multidomain spectral methods, allowing the SCAPC framework to be generalized to arbitrary irregular 2D/3D curvilinear domains in future investigations.

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Conflicts of Interest

The author declares no conflict of interest.

Use of AI and AI-Assisted Technologies

During the preparation of this work, the author used Qwen3.7 to fix English language and grammar (not in methodology). After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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