

A Computational and Statistical Investigation of the Distribution of Twin Primes and Generalized Mersenne Numbers

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Abstract: We outline novel computational and statistical techniques that reveal a direct linear-logarithmic relationship between the number of twin primes and the number of primes in any interval. Using orthogonal-polynomial regression fits together with a direct linear relation of positive slope between the number of twin primes and the length of the natural-number interval, we present strong computational and statistical evidence consistent with the twin prime conjecture. We emphasize that these results constitute heuristic and empirical support rather than a formal deductive proof. We further introduce a generalized Mersenne-number framework that offers a heuristic perspective on the interconnection of primes and on why twin primes may occur infinitely often. Finally, several applications of the enumerated twin primes are outlined, including combinatorial face colorings of an N-dimensional hypercube and their potential use in cryptography.

Keywords: twin primes; computational and statistical techniques; generalized Mersenne numbers; twin primes conjecture; twin-prime distribution; heuristic and empirical evidence; twin-prime cryptography

1. Introduction

The distribution of primes, in general, and twin primes, in particular, has continued to baffle mathematicians and scientists over the last several decades [1–12]. This is especially true given that prime gaps and the intriguing pattern of twin primes have attracted the attention of mathematicians and computational scientists for centuries. This research spans disciplines ranging from number theory, statistics, computations, and combinatorics to applied areas such as cryptography. Despite extensive research on the topic, the family of twin primes remains elusive.

The twin primes infinity conjecture is one of the most popular unsolved problems in number theory. From the 19th century to the present, many authors [13–25] have pushed the boundaries of solutions toward the ultimate goal, but none has been convincing or complete enough to prove the conjecture.

This paper first investigates, through statistical and computational methods, a linear relationship between the decimal logarithms of the numbers of twin primes and primes across various intervals.

Subsequently, we examine a generalized Mersenne-number framework that offers a heuristic perspective on the interconnection of primes and on why twin primes may be infinite in number. Several extensions and applications are also discussed.



2. Twin Primes

Twin primes are a well-known and apparently simple subject in Number Theory. If $\mathbf{P} = \{3, 5, 7, \dots\}$ is the odd prime number set, and $\mathbf{N} = \{1, 2, 3, \dots\}$ the natural number set, then if two primes exist: $\exists \{P, Q\} \subset \mathbf{P}$, they can be considered twin primes if they are related by the following algorithm:

$$\{\exists P \in \mathbf{P} \subset \mathbf{N}\}: Q = P \pm 2 \rightarrow Q \in \mathbf{P} \quad (1)$$

The symbol $\pm$ stands for $+$ or $-$. In this way, one considers the occurrences simultaneously:

$$Q = P + 2 \text{ or } P = Q + 2 \quad (2)$$

2.1. The Sequence of Primes within the Natural Numbers

A characteristic of odd prime numbers is that they can only have a terminating digit from the set: $\epsilon = \{1, 3, 7, 9\}$. Therefore, one can observe the frequencies of prime numbers ending with any digit in the set ϵ as they appear at increasing indices in the sequence of the natural numbers. When the number of generated primes in large natural intervals increases sufficiently, the relative frequencies of the primes bearing the ending digits contained in ϵ , for all four classes, tend to the limit: $\frac{1}{4} = 0.25$.

If the natural numbers interval is large enough, for instance, when such an interval $[1; 2 \times 10^8]$ is studied, it holds 11,078,937 generated primes, corresponding to a prime relative frequency of 0.055394685, with respect to natural numbers. Also, under these circumstances, the arithmetic mean error of the prime relative frequency in each class ϵ lies around $\pm 2.5 \times 10^{-5}$.

Hence, one can be confident that the four prime number classes appear in the sequence of odd primes in equal proportions, as if they were randomly distributed among the natural numbers.

2.2. The Twin Prime Sequence within Natural and Prime Numbers

However, twin primes can be constructed only from three of the four classes in the set ϵ . The reason is that the class ending in 3 cannot contribute to the twin prime structure, since the result of a prime of this class plus 2 ends in 5, which is always composite, as it is divisible by 5.

Consequently, only the prime numbers ending in the digit set $\epsilon^{(a)} = \{1, 7, 9\}$ can be associated with the smaller number of a twin prime, with the larger second member of the twin ending in the digit set $\epsilon^{(b)} = \{3, 9, 1\}$, with respect to the order of the digit set $\epsilon^{(a)}$. This means that the relative frequency of generating twin primes from one of the classes of the set $\epsilon^{(a)}$ will converge asymptotically to $\frac{1}{3}$.

Such a difference between the prime and twin prime classes and their relative frequencies has to be observed with respect to both sets, not with respect to the natural numbers regarding primes, nor with respect to the primes themselves regarding twin primes.

Also, one can see that only a $\frac{3}{4}$ proportion of the total number of primes will potentially contribute to the construction of twin primes. It will be interesting to know the absolute value of twin primes in the above natural interval $[1; 2 \times 10^8]$ using this fraction, which should potentially be of the order: ~ 8309203 ; that is, a relative frequency with respect to the natural numbers of the interval of around: ~ 0.04154601 .

In practice, this twin number estimate may be lower, as not every prime in the $\epsilon^{(a)}$ classes will generate a twin prime. Previous estimates using the formulae by Hardy and Littlewood and, later, by Wu are consistent with the results described above; see, for example, refs. [26,27].

The benchmark empirical results, though, provide a different landscape. In a new, larger interval $[1; 2 \times 10^9]$, where 87,143,350 primes have been generated, the exact number of twin primes is 5,574,670, an order of magnitude less than the $\frac{3}{4}$ estimate in Section 2.1 above; the estimates of Hardy, Littlewood, and Wu appear to be similar.

Considering these numbers, one can observe a relative frequency of 0.002787335 with respect to the $[1; 2 \times 10^9]$ natural interval, and with respect to the number of primes obtained in the same $[1; 2 \times 10^9]$ interval, the frequency is 0.063997126. Such quantities are no longer estimates but *exact* values within the large interval used, as they are derived through our computations.

One can expect these relative frequencies to decrease further as the natural number interval increases, but even as they do, they remain above zero.

2.3. Variation of the [(Twin Prime) : (Prime)] Ratio with the Order of Magnitude of the Intervals

The estimates discussed above can be extended to all the subranges $[1; 10^N]$ that is, up to the range $[1; 10^{18}]$, because the well-known OEIS files, with no effort, provide the number of primes (A006880) and twin primes (A007508) less than a given power of 10.

In Figure 1, the ratio [(twin prime) : (prime)] is represented as a function of the order of magnitude of the interval considered, employing the information on the OEIS pages given in the previous paragraph. The results show a trend approaching an asymptotic value of ~ 0.03 . The variation of the ratio [(twin prime) : (prime)] seems to agree with an inverse exponential function. But the search for a fitting curve rules that out. In fact, an optimal fit is obtained by using an orthogonal polynomial background up to 6th order, as the following results show. Although useful to visualize trends in practical data, like the ones represented here in Figure 1, fitted functions are prone to errors when extrapolated, so they can provide [(twin prime) : (prime)] estimates with some uncertainty.

Statistical Fitting of the [(twin prime) : (prime)] ratio versus the order of magnitude N of the interval where both ratio elements are obtained to draw Figure 1

Orthogonal Polynomial Fit

$X = N$ order of magnitude of the interval $Y = [(twin\ prime) : (prime)]$ ratio

$$Y = 0.80246544 - 0.38363635 \times N + 0.091792394 \times N^2 \\ - 0.012020563 \times N^3 + 0.00086960667 \times N^4 - 3.2515874 \times 10^{-5} \times N^5 \\ + 4.8961983 \times 10^{-7} \times N^6$$

Number of data points used = 18 $\bar{X} = 9.5$ $\bar{Y} = 0.11420086$

Orthogonal Polynomial Factors: {X Shift = 9.5; X Scale = 0.23529412}

Degree: 0

Residual sum of squares (RSS) = 0.25238642

Coefficient of determination, $R^2 = 0$ Correlation coefficient, $R = 0$

$B[0] = 0.11420086$ $\alpha[0] = 0$ $\beta[0] = 0$

Degree: 1

RSS = 0.098004375 $R^2 = 0.6116892$ $R = 0.78210562$

$B[1] = -0.075864822$ $\alpha[1] = -4.9343246 \times 10^{-17}$ $\beta[1] = 1.4901961$

Degree: 2

RSS = 0.031715094 $R^2 = 0.87433914$ $R = 0.93506104$

$B[2] = 0.045742855$ $\alpha[2] = 0$ $\beta[2] = 1.1810842$

Degree: 3

RSS = 0.008776651 $R^2 = 0.96522534$ $R = 0.98245882$

$B[3] = -0.025413237$ $\alpha[3] = 0$ $\beta[3] = 1.1211073$

Degree: 4

RSS = 0.002135281 $R^2 = 0.99153964$ $R = 0.99576083$

$B[4] = 0.013141977$ $\alpha[4] = -1.0002677 \times 10^{-16}$ $\beta[4] = 1.0826605$

Degree: 5

RSS = 0.00049505918 $R^2 = 0.99803849$ $R = 0.99901876$

$B[5] = -0.0063887236$ $\alpha[5] = 0$ $\beta[5] = 1.0450526$

Degree: 6

RSS = 0.00015933652 $R^2 = 0.99936868$ $R = 0.99968429$

$B[6] = 0.0028853107$ $\alpha[6] = 4.4203414 \times 10^{-17}$ $\beta[6] = 1.0035086$

$p < 0.0001$

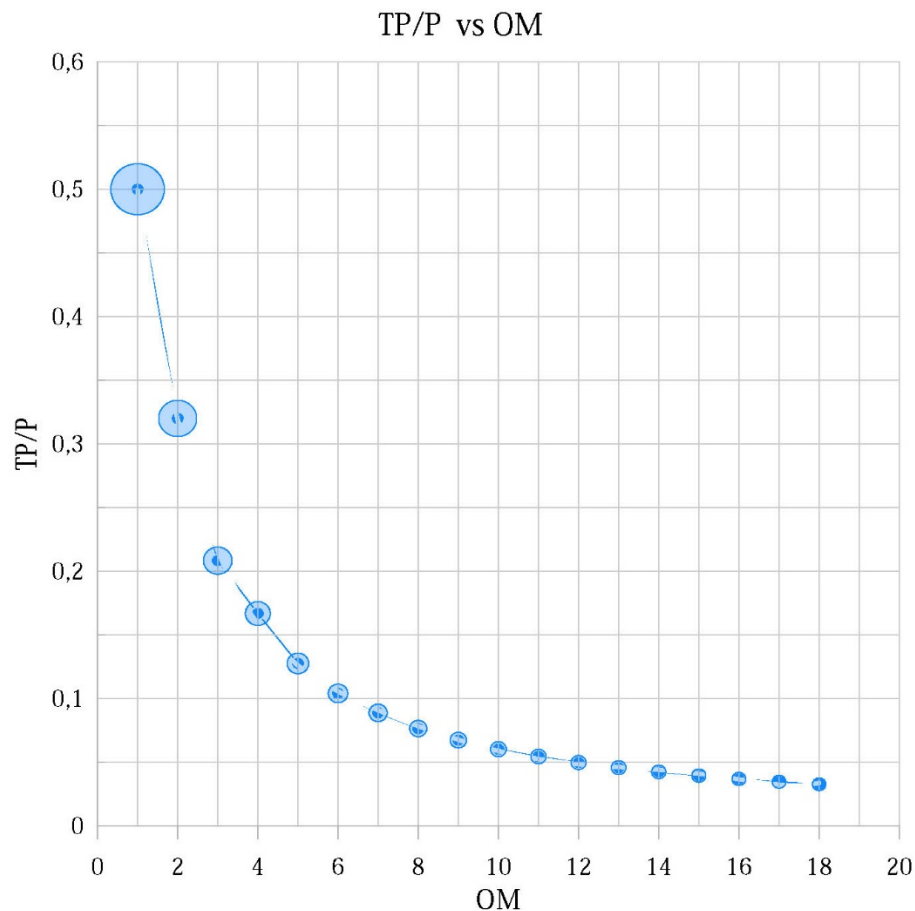


Figure 1. The ratio $Y = [(\text{twin prime}) : (\text{prime})]$ as a function of the order of magnitude $X = N$ of the interval considered, employing the information on the OEIS pages (A006880) and (A007508). The disks represent the value of the ratio.

3. The Relationship between Natural Numbers, Primes, and Twin Primes

The above considerations have prompted us to study the relationships among natural numbers, prime numbers, and twin primes. However, the most important information one could obtain if a relation between primes and twin primes exists, will correspond to the connection of such a relation with the infinite existence of the twin primes.

Hence, in our attempts to obtain evidence on a possible relation between these two natural subsets, several natural intervals have been searched for both primes and twin primes. Results are summarized in Table 1.

Table 1. Variation of prime and twin prime numbers located within increasing natural numbers intervals.

Interval: $[N]$	Number of Primes $\pi[N]$	Number of Twin Primes $\tau[N]$
2×10^2	46	15
2×10^3	303	61
2×10^4	2262	343
2×10^5	17,984	2161
2×10^6	148,933	14,872
2×10^7	1,270,607	107,408
2×10^8	11,078,937	813,372
2×10^9	87,143,350	5,574,670

Thus, as shown in Table 1, one has three variables to play with. One must also notice that the last intervals in Table 1 correspond to the intervals that have been used beforehand in Sections 2.1 and 2.2. Also, the table could be enlarged to include decimal intervals up to 10^{18} , with the corresponding values obtained from the OEIS files already mentioned. Table 1 is constructed with intervals of the type $[1; 2 \times 10^N]$, chosen in this way to provide additional information beyond the $[1; 10^N]$ OEIS intervals.

3.1. Relationship between Prime Numbers and Twin Primes

A first step toward the proposed goal of finding a relationship between prime and twin prime numbers has been to visualize and correlate the current frequencies of primes and twin primes.

Figure 2 represents such a picture, where the columns $\pi[N]$ and $\tau[N]$ are used from OEIS information, previously manipulated with a logarithmic transformation. As one can see in Figure 2, along with the attached statistical results, there is a clear quasi-linear relationship between twin primes and prime numbers, as obtained with the decimal logarithms of the variables $\pi[N]$ and $\tau[N]$. However, in Figure 2, along with the attached statistical results, there is a better second-order polynomial relationship between the distribution of twin primes and the prime numbers, the variables $\pi[N]$ and $\tau[N]$.

The statistical information on the fitted linear function in Figure 2 is as follows:

Orthogonal Polynomial Fit

$$\log(\tau[N]) = 0.0085586434 \log(\pi[N])^2 + 0.71209745 \log(\pi[N]) - 0.57780746$$

Number of data points used = 18 $\bar{X} = 9.5$ $\bar{Y} = 7.189906$

Orthogonal Polynomial Factors: X Shift = 9.5 X Scale = 0.23529412

Degree: 0

Residual sum of squares (RSS) = 371.56175

Degree: 1

RSS = 0.86086416 $R^2 = 0.99768312$ $R = 0.99884089$

Degree: 2

RSS = 0.10374825 $R^2 = 0.99972078$ $R = 0.99986038$

$p \leq 0.0001$

Note the small second-order polynomial coefficient.

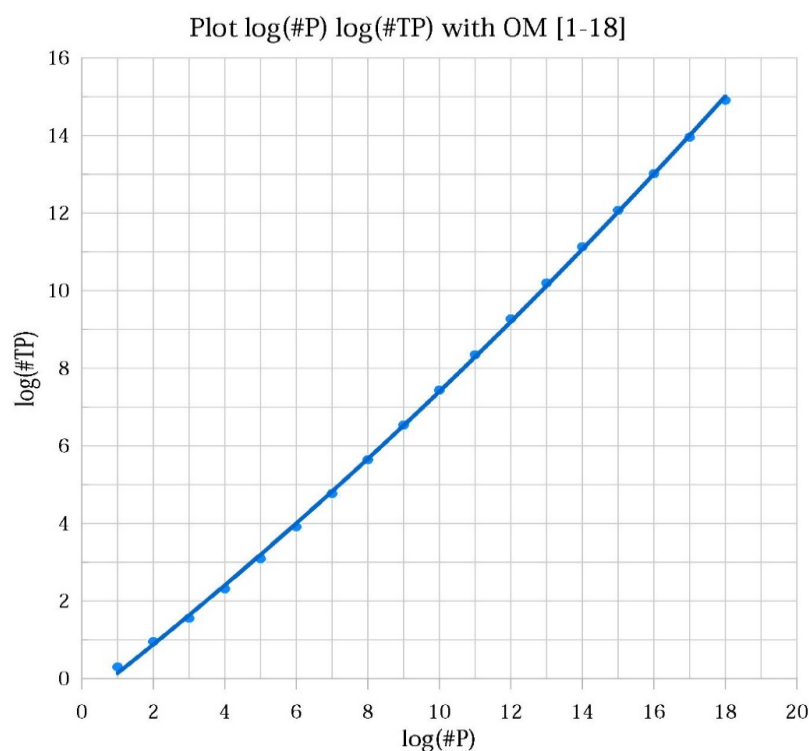


Figure 2. Plot of: $\log(\tau[N]) = A \log(\pi[N]) + B$.

3.2. The relationship between Primes, Twin Primes, and the Order of Magnitude of the Employed Intervals

Also, as the variables employed depend on the intervals used, and Figure 2 has been constructed using OEIS data, we found it interesting to construct a 3D plot that includes the two variables already shown in Figure 2, with the order of magnitude of the intervals as an additional parameter. This results in a bubble graph displayed in Figure 3.

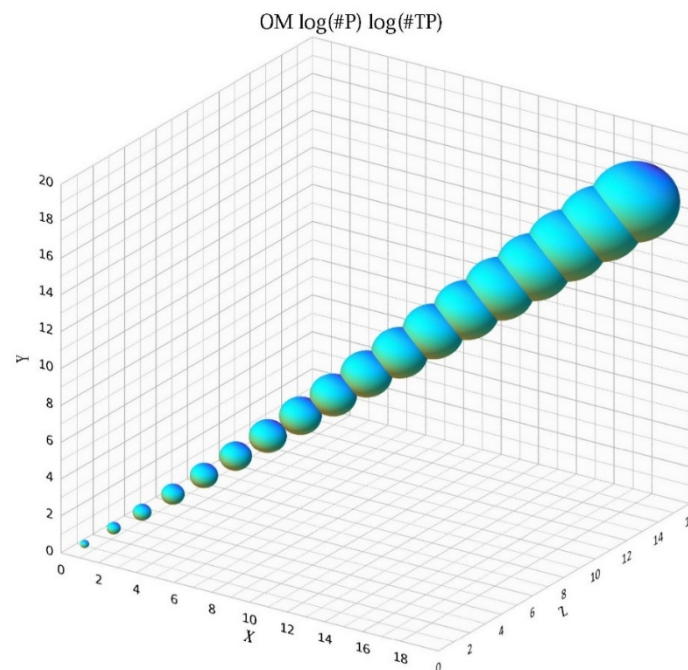


Figure 3. Plot of the logarithm transformation of the two variables $\pi[N]$, and $\tau[N]$, with the additional order of magnitude of the natural interval as a third parameter. The size of the plotted spheres corresponds to an optical effect to indicate nearness to the observer, and it is attached to the order of magnitude.

In both Figures 2 and 3, one can observe a close relationship between the number of primes and the number of twin primes as the natural number interval increases. Both figures suggest that the relative abundance of twin primes and primes increases as the natural number interval increases. This is yet another confirmation of the direct proportionality relation between the abundance of primes versus twin primes.

Although decimal logarithms have been used in Figures 2–4, Napierian logarithms have also been tested and produce similar pictures. The most notable difference is that the order-of-magnitude N will be a rational number rather than an integer.

3.3. Relationship Heuristics and Causality

Such a relationship heuristically suggests, rather than proves, the infinitude of twin primes, reflecting the empirical observation that the number of twin primes grows together with the number of primes.

In any case, the association implied by fitting the two variables, as shown in Figure 2, can at most be regarded as indirect, heuristic support for the twin prime conjecture, not as a proof. We note that, because (a) both variables are connected, (b) they depend on the interval over which they are obtained, and (c) they are derived from the same natural-number source, the almost linear relationship between their logarithms is suggestive of a genuine structural connection rather than of random variation or chance. We stress, however, that such a statistical association does not by itself constitute a formal proof of a statement about infinitely many objects. See [17,18] for more information about causality in relationships among observed variables.

3.4. Extrapolation Abilities of the Relationship between Twin Primes and the Order of Magnitude of the Intervals Used

Beyond this, one can use the relationships obtained, as discussed previously, to apply a fitted equation to extrapolate values that do not appear in the plot or in the equation. It is well known that fitted equations like the ones obtained may not be very accurate for extrapolation, but even so, they offer another way to approach the variation of twin primes as the interval's order of magnitude increases, as valid as those mentioned earlier of Hardy,

Littlewood, and Wu. Or at least the linear plot of the next Figure 4 suggests that the twin prime sequence continues to increase without a stopping point.

Figure 4 shows the variation in the number of twin primes $\tau[N]$ with the order of magnitude of the considered interval N .

Linear Fitting Properties of the drawn Figure 4

$$\tau[N] = 0.93743793 * N - 0.62193151$$

Number of data points used = 18 $\bar{N} = 9.5$; $\bar{\tau}[N] = 8.2837288$

Residual sum of squares (SS) = 0.21857682 Regression SS = 425.77369 $R^2 = 0.9994869$ $R = 0.99974342$

Residual mean square: $\bar{\sigma}^2 = 0.013661051$

Standard error of intercept (A) = ± 0.057477412

Standard error of slope (B) = ± 0.0053100079

$p \leq 0.0001$

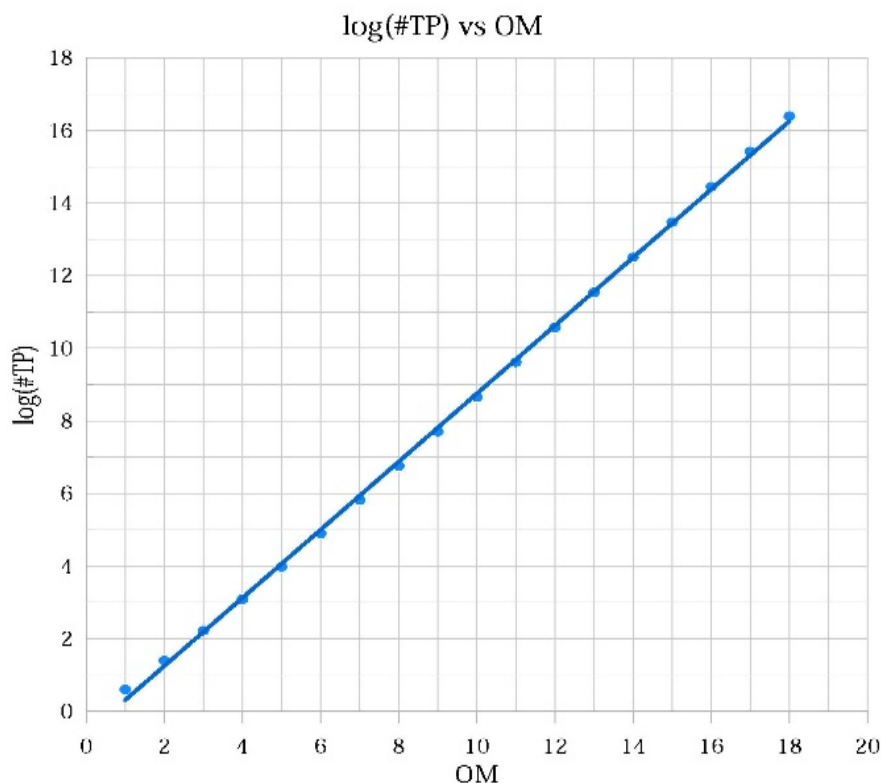


Figure 4. Variation of the twin prime number $\tau[N]$ as a function of the order of magnitude of the considered natural interval N .

Using the fitted linear equation, keeping in mind the present uncertainty of extrapolating fitted functions, as previously noted in Figure 1, the number $\tau[N]$ can be approximated with some error, as in the Hardy-Littlewood estimates.

For example: $\log(\tau[100]) \approx 93 \rightarrow \tau[100] \approx 10^{93}$. That corresponds to a large number (for comparison, the estimated number of stars in the observable universe is actually $\sim 10^{24}$), signaling, no doubt, the heuristic practical infinity of twin primes. Even if the above estimated value is subject to a 50% error.

As shown in Figure 4, the number of twin primes increases linearly with the length of the natural number interval, with a slope of 0.93743793. Consequently, in the limit as N goes to infinity, the number of twin primes goes to infinity as per the linear fitting of Figure 4. In particular, the small constant term—0.62193151 becomes insignificant; thus, in the limit as N goes to ∞ , we have

$$\lim_{N \rightarrow \infty} \tau[N] = 0.93743793 \times N = \infty$$

The above relation provides strong statistical and computational support consistent with the twin prime conjecture, while not, on its own, tantamount to a deductive proof.

It must be stressed that the linear and polynomial fits presented above, and the behavior of the fitted models as N grows without bound, are derived from a necessarily finite body of computational data. They therefore provide

strong heuristic and empirical evidence for the persistence and continued growth of the twin-prime count, but they do not, and cannot, constitute a deductive proof of the twin prime conjecture in the context of analytic number theory. We present them in this spirit, as a computational and statistical investigation of the distribution of twin primes.

4. Generalized Mersenne Numbers

On the other hand, one can define the so-called generalized Mersenne numbers, recently proposed [28], which can be constructed by the row vector (used for typographic ease) algorithm:

$$\{\forall P \in \mathbf{P} \wedge K \in \mathbf{N}\}: \langle \mathbf{M}_{(P;K)} | = \langle P^{\mathbf{N}} | \pm 2K \langle \mathbf{1} | = (P, P^2, P^3, \dots, P^N, \dots) \pm 2K(1, 1, 1, \dots, 1, \dots) \quad (3)$$

Considering that, when using the minus sign, the origin shift parameter K is restricted so as not to produce negative results in the elements of the generalized Mersenne number vector $\langle \mathbf{M}_{(P;K)} |$, so its elements remain natural numbers.

5. Twin Primes as a Simple Case of Generalized Mersenne Numbers

Accordingly, twin primes are but a particular case of generalized Mersenne prime numbers, belonging to some of the elements of a vector that can be written as:

$$\langle \mathbf{M}_{(P,1)} | = \langle \mathbf{P} | \pm \langle \mathbf{2} | \rightarrow \exists M_{(P,1);Q} = P \pm 2 = Q \in \mathbf{P} \quad (4)$$

The vector $\langle \mathbf{M}_{(P,1)} |$ is infinite-dimensional, like the odd-prime vector $\langle \mathbf{P} | = (3, 5, 7, \dots, P, \dots)$, and the vector: $\langle \mathbf{2} | = (2, 2, 2, \dots, 2, \dots) = 2 \langle \mathbf{1} |$.

6. Iterative Algorithm to Generate Twin Primes

That is, after finding a prime P associated with a twin prime Q , one can continue searching the elements of this particular generalized Mersenne vector $\langle \mathbf{M}_{(P,1)} |$ until finding a new prime R , that might be associated with a twin prime S .

So, one can iterate in this manner until a new prime appears with the finding of a possible twin again.

6.1. The Nature of the Iterative Procedure for Generating Twin Primes

One can study such a possibility by rewriting the twin prime vector iteration, but from now on use only the sum $+$, instead of the expression $\pm$, to avoid the need for repeated explanations.

Renaming the vector $\langle \mathbf{M}_{(P,1)} |$ as $\langle \mathbf{T}_1 |$, then one can write a sequence of iterations like:

$$\langle \mathbf{T}_1 | = \langle \mathbf{P} | + \langle \mathbf{2} | \rightarrow \langle \mathbf{T}_2 | = \langle \mathbf{T}_1 | + \langle \mathbf{2} | \rightarrow \langle \mathbf{T}_3 | = \langle \mathbf{T}_2 | + \langle \mathbf{2} | \rightarrow \dots \rightarrow \langle \mathbf{T}_N | = \langle \mathbf{T}_{N-1} | + \langle \mathbf{2} | \rightarrow \dots \quad (5)$$

Such that in the elements of the vectors: $\{\langle \mathbf{T}_1 |, \langle \mathbf{T}_2 |, \dots, \langle \mathbf{T}_N |, \dots\}$, twin primes will appear at the same positions in consecutive vectors.

For example:

$$\begin{aligned} \langle \mathbf{T}_1 | &= \langle \mathbf{P} | + \langle \mathbf{2} | = (3, 5, 7, 9, 11, \dots) + (2, 2, 2, 2, 2, \dots) = (5, 7, 9, 11, 13, \dots) \\ \langle \mathbf{T}_2 | &= \langle \mathbf{T}_1 | + \langle \mathbf{2} | = (5, 7, 9, 11, 13, \dots) + (2, 2, 2, 2, 2, \dots) = (7, 9, 11, 13, 15, \dots) \\ \langle \mathbf{T}_3 | &= \langle \mathbf{T}_2 | + \langle \mathbf{2} | = (7, 9, 11, 13, 15, \dots) + (2, 2, 2, 2, 2, \dots) = (9, 11, 13, 15, 17, \dots) \end{aligned}$$

In fact, the iteration steps can be written in a compact manner, because it can be shown without difficulty that one can also write the equalities:

$$\exists N \in \mathbf{N}: \langle \mathbf{T}_N | = \langle \mathbf{P} | + N \langle \mathbf{2} | = \langle \mathbf{P} | + 2N \langle \mathbf{1} | \quad (6)$$

From this result, it can be understood that each vector, $\langle \mathbf{T}_N |$ for instance, in the N -th iteration step, included in the infinite steps within the iterative scheme (5), corresponds to an origin shift of the odd prime vector $\langle \mathbf{P} |$ by an even homothety $2N$ of the unity vector $\langle \mathbf{1} |$.

6.2. Iteration Disc Definition

Accepting such a homothetic parameter as the diameter of a disk, $D(N)$ say, which one can call the *iteration disk*, then the radius N of this disk increases by 1 at each iteration.

7. Heuristic Reasoning Leading to Infinitely Many Twin Primes and Prime Interconnectedness

It is evident that the infinitely many elements of each of the infinitely many new iterated vectors $\langle \mathbf{T}_I |$ in the infinite iteration scheme (5) produce primes, that is:

$$\{\forall I \rightarrow \exists S\} : \Rightarrow T_{I,S} \in \mathbf{P} \wedge T_{I-1,S} \in \mathbf{P} \quad (7)$$

Hence, for all iterations in the sequence (5), there exists a vector element in some S position such that it is in the same position as the vector in the previous iteration, thus coinciding in being both primes, and therefore they are twin primes. Thus, Equation (7) appears to be a possible heuristic explanation for why infinitely many twin primes exist.

Furthermore, although the twin primes are repeated in subsequent vectors of the iteration (5) path, one can suppose that the vector $\langle \mathbf{T}_1 |$, being infinite-dimensional, like the associated odd-prime vector, already contains, in the first iteration, all possible twin primes that might be generated from the odd prime vector $\langle \mathbf{P} |$.

8. Further Conjectures on Prime Numbers Interconnectedness

Because of Equation (6) associated with the iteration sequence (5), one can easily see that it can also be written as:

$$\{\forall N \in \mathbf{N} \wedge \exists R \in \mathbf{N}\} : P_R \in \mathbf{P} \rightarrow T_{N,R} = P_R + 2N \quad (8)$$

an expression that can also transform into the following conjecture:

$$\{\exists P \in \mathbf{P} \wedge \exists N \in \mathbf{N}\} : Q = P + 2N \rightarrow Q \in \mathbf{P} \quad (9)$$

meaning that, given a prime number P , there exists an even number $2N$ such that their sum is a prime number.

A similar conjectural relationship was described earlier in a paper by the authors on prime numbers, which showed their interconnectedness or *pairing*, as they named this property, via powers of 2 [29], that is:

$$\{\exists P \in \mathbf{P} \wedge \exists N \in \mathbf{N}\} : Q = P + 2^N \rightarrow Q \in \mathbf{P} \quad (10)$$

Such a relationship might be considered a particular case of the conjecture implicit in the expression (9).

From there, one can also construct an infinite non-symmetric matrix or tensor from two infinite-dimensional vectors, not in the well-known way of a tensor product, but by an operation that can be called a *tensor sum*. It can be easily defined by:

$$\mathbf{C}_{\oplus} = \langle \mathbf{A} | \oplus \langle \mathbf{B} | \rightarrow \{\forall R, S \in \mathbf{N}\} : C_{RS} = A_R + B_S \quad (11)$$

then, using this definition on the vectors: $\{\langle \mathbf{P} |; \langle \mathbf{E} | = 2\langle \mathbf{N} | \}$, the following tensor sum is simply obtained:

$$\mathbf{S}_{\oplus} = \langle \mathbf{P} | \oplus \langle \mathbf{E} | \rightarrow \{\forall R, S \in \mathbf{N}\} : S_{\oplus,RS} = P_R + E_S = P_R + 2N_S \quad (12)$$

That is, the matrix $\mathbf{S}_{\oplus}$ elements contain all the possible results of the Equations (8) and (9).

Additionally, among all the possible prime-interconnection occurrences, twin primes are but a particular, the simplest case.

9. A General Conjecture about Prime Interconnectivity

Yet, perhaps a further generalization of the analysis of the twin primes problem allows one to apply a similar technique to prime sets, starting from the Equations (9) and (10).

Therefore, one can also formulate the extended conjecture that the prime set $\mathbf{P}$ is interconnected via the addition of specific even numbers.

A very simple example can demonstrate this intricate prime interrelationship through even numbers:

$$\begin{aligned} 3 + 2 = 5 & \therefore 3 + 4 = 7 \therefore 3 + 8 = 11 \therefore 3 + 10 = 13 \therefore \dots \\ 5 + 2 = 7 & \therefore 5 + 6 = 11 \therefore 5 + 8 = 13 \therefore 5 + 12 = 17 \therefore \dots \\ 7 + 4 = 11 & \therefore 7 + 6 = 13 \therefore 7 + 10 = 17 \therefore 7 + 12 = 19 \therefore \dots \end{aligned}$$

In this manner, one can conjecture that the set of prime numbers can be considered a completely connected set of numbers through the elements of the set of even numbers.

The matrix $\mathbf{S}_{\oplus}$, as defined by the Equation (12), contains, via the even numbers, all of these prime interconnections.

10. An Application and Outlook: Twin Primes as Labels for N-Dimensional Hypercubes

Twin primes form very interesting and non-uniform sequences. That is, their occurrence does not follow any prescribed pattern, nor do they conform to a simple closed-form rule. As strongly suggested by the present computational and statistical analysis, and by extensive prior numerical evidence, twin primes are believed to occur infinitely often. All of these features make twin primes attractive for many applications, including cryptography. As an illustration, and independently of the distributional analysis above, we expand here on one such combinatorial application.

Consider the combinatorial enumeration of face colorings of an N -dimensional hypercube [30–32]. As the dimension of the hypercube increases, the combinatorial complexity of coloring the faces also dramatically increases, as the number of N_q -hyperplanes for any integer q in the interval $\{1, N\}$ is given by:

$$N_q = \binom{N}{q} 2^q \quad (13)$$

For example, there are 672 faces for a 7-dimensional hypercube. The coloring of the faces of such an N -dimensional cube is then equivalent to the problem of unique ways to label the hypercube with labels chosen from a set under the action of the automorphism of the N -hypercube, which is the wreath product group $S_n[S_2]$. Consequently, the resulting face labeling provides a combinatorially rich set of possibilities ideally suited for cryptography.

Here, we chose the set of colors as the set of twin primes so that the faces of the N -hypercube are labeled with unique twin primes. The string of faces labeled by the twin primes would then offer a cryptic means of communication, especially within the automorphism group of the N -hypercube, allowing permutations of the face labeling with the twin primes. Such applications can be significantly expanded to encompass various cryptographic applications and could become the subject of future studies. Such applications go beyond cryptography to the symmetry properties of nonrigid water clusters [32]. It is evident from the present study that twin primes offer a unique platform for future explorations of applications of number theory, combinatorics, applied mathematics, and statistics.

11. Conclusions

The present computational and statistical study revealed fascinating trends and facts regarding the abundance of twin primes, primes, and generalized Mersenne numbers. Our computations revealed a direct relationship between the decimal logarithm of the abundance of prime numbers and the decimal logarithm of the abundance of twin primes. Furthermore, our statistically derived regression relations indicated a positive slope between the number of twin primes and the interval length, suggesting that the number of twin primes continues to grow without an apparent upper bound as the interval length increases. We emphasize that these computational trends, together with the heuristic interconnectedness of twin primes, primes, and generalized Mersenne numbers, constitute strong empirical and heuristic evidence consistent with the twin prime conjecture, rather than a formal deductive proof. We also showed that the combinatorially enumerated twin-prime sets can serve as face-colorings of N -dimensional hypercubes, and consequently, the developed techniques provide powerful new combinatorial tools for twin primes in cryptography. Twin primes offer yet another dimension for expanding the emerging field of artificial intelligence through novel twin-prime-based applications, such as the ones demonstrated in this work.

Author Contributions

R.C.-D.: Conceptualization, methodology, data curation, writing—original draft preparation; K.B.: Data curation, visualization, investigation, supervision, validation, writing—reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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