

Affine T-S Fuzzy-Model-Based Proportional-Integral Estimator Design Using Faded Transmission Data

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Abstract: This paper is concerned with the fuzzy proportional-integral state estimator design issues for affine fuzzy-model-based nonlinear systems under the effects of channel fadings. To reflect the unreliable transmission environment in the wireless network between sensors and the estimator, the channel fading phenomenon is considered. A fading model is adopted to characterize the signal transmission process, where the fading coefficients are considered independent and identically distributed. This paper concentrates on the design of the estimator gain, aiming to ensure that the estimation error dynamics achieve stochastic stability while satisfying the prescribed $l_2 - l_\infty$ performance requirement. Sufficient conditions for verifying the estimation performance are established on the basis of Lyapunov stability theory and stochastic analysis. Furthermore, the estimator parameters are derived with the help of the convex optimization technique. The validity of the proposed fuzzy estimator is finally confirmed through a numerical example, which illustrates its satisfactory performance.

Keywords: affine fuzzy models; proportional-integral estimator design; channel fadings; $l_2 - l_\infty$ estimation

1. Introduction

The field of Takagi-Sugeno (T-S) fuzzy systems has experienced rapid advancement over the past several decades, motivating a substantial body of work on filtering and control problems based on T-S fuzzy models [1,2]. As is well known, by leveraging rich human knowledge and fuzzy inference techniques, fuzzy models proposed by Takagi and Sugeno provide a simple yet effective method to represent nonlinear plants, and have therefore naturally become a popular approach for addressing nonlinear analysis and synthesis issues. Researchers generally divide T-S fuzzy models into two types: homogeneous T-S fuzzy models and their affine counterparts. The main difference between these two types lies in whether the consequent part contains an affine bias term. It should be noted that the inclusion of an affine term enables the affine fuzzy model to achieve better approximation capability than its homogeneous counterpart [3]. Greater complexity generally arises in the analysis of affine fuzzy models, a feature that has drawn particular interest in recent years [4–6].

A fundamental task in system science is to recover or estimate the internal state variables of a plant through suitable algorithms, a problem known as state estimation. To address state estimation in various scenarios, Kalman filtering, sliding-mode observers, set-membership filtering, and moving horizon estimation represent just a few of the specialized approaches that have been developed in the area. It is worth noting that these estimators (or estimation algorithms) belong to a proportional type, as only a proportional term with respect to the current innovation is employed in their structures. The proportional-integral (PI) estimator, which incorporates historical information from past output estimation errors, originated in [7] and drew inspiration from the well-known proportional-integral-derivative control scheme. The enhanced structure offers greater design flexibility, which can lead to improved estimation performance. The years following have witnessed considerable research interest in the PI estimation problem, with a large volume of studies confirming the advantages of the PI estimator over conventional approaches.

Due to its desirable robustness and estimation accuracy, the PI estimator has been utilized in numerous complex estimation settings, among which nonlinear estimation using fuzzy PI estimators is a typical case. By combining the advantages of fuzzy techniques and the PI estimation scheme, fuzzy PI estimators are capable of handling estimation problems for nonlinear systems under various prescribed performance requirements. For instance, in [8], fuzzy PI estimators have been employed to achieve chaos synchronization for systems with unknown inputs. The joint estimation problem concerning states along with actuator/sensor faults has been resolved in [9] through the use of fuzzy PI estimators, where the faults are allowed to be time-varying and the fuzzy premise variables are immeasurable. Other representative studies can be found in [10–16]. It should be emphasized that existing works on the fuzzy PI estimation problem have all focused on homogeneous fuzzy models, while the counterpart for affine fuzzy models remains unexplored, primarily due to the high design complexity, which deserves further investigation.

To achieve the desired estimation objective, several performance indices have been proposed. For systems subject to energy-bounded disturbances, the H_∞ and l_2 - l_∞ are two representative performance measures. The H_∞ performance minimizes the worst-case energy gain from the disturbance to the estimation error, ensuring uniform attenuation of all energy-bounded noises. For instance, fuzzy-model-based H_∞ control and estimation have been extensively studied in [17–20], with particular attention to system complexities and computational overhead. In contrast, the l_2 - l_∞ performance aims to minimize the worst-case peak value of the estimation error relative to the disturbance energy, making it especially valuable when the maximum instantaneous error is of primary concern. Notable contributions to l_2 - l_∞ estimation can be found in [21,22].

Despite the theoretical advances, practical deployment of fuzzy PI estimators faces several nontrivial challenges. First, gain tuning relies on solving inequalities that can be computationally demanding for real-time embedded platforms, especially when the system dimension is high. Second, affine fuzzy models introduce constant offsets or load-dependent terms, making steady-state equilibrium analysis more complex and increasing the risk of integral windup if not properly handled. Third, real-world measurements are often corrupted by time delays, packet dropouts, or quantization errors, which existing fuzzy PI designs seldom explicitly account for. Furthermore, abrupt or high-frequency faults violate the common assumption of slowly varying fault dynamics, leading to degraded estimation performance. Lastly, hardware validation requires extensive calibration and real-time implementation efforts, posing a significant gap between theory and practice.

In recent decades, with the ongoing development of communication networks, signal transmission among system components via networks has become increasingly popular. On one hand, the deployment of networks offers several benefits, such as simplified installation, reduced cost, and enhanced flexibility. Conversely, network usage can introduce undesirable phenomena including delays, packet dropouts, and channel fadings. Specifically, channel fading arises in engineering practice when signals transmitted over wireless networks undergo reflection, shadowing, diffraction, and scattering [23]. Given the variability of the transmission environment, fading is typically characterized as a stochastic time-varying process that reflects the random nature of communication channels. The past decade has witnessed several seminal publications tackling control and filtering problems under channel fading conditions. Nevertheless, when it comes to fuzzy PI estimation for affine fuzzy systems with fading measurements, the corresponding results remain scattered in the literature, which motivates the present study.

Motivated by the above discussions, in this paper, we focus on the fuzzy PI estimation problem for affine fuzzy models subject to fading measurements. Our main contributions can be highlighted as follows: (1) The fuzzy PI state estimation problem is, for the first time, investigated for affine T-S fuzzy systems in the presence of channel fading; (2) A novel region-dependent fuzzy PI estimator is proposed by simultaneously utilizing both the current information and the historical information of the output estimation errors; and (3) The desired estimator gains are obtained by solving a family of linear matrix inequalities.

The remainder of this paper is organized as follows. In Section 2, we provide a detailed description of the system model, estimator model, fading model, and the considered performance index. In Section 3, two theorems with theoretical proofs are presented to analyze the performance of the error system and to design the estimator parameters, respectively. In Section 4, a numerical example is given to demonstrate the effectiveness of the proposed design method. Finally, in Section 5, we draw the conclusions of this paper.

Notations: $\mathbb{R}^n$ denotes the n -dimensional Euclidean space. The transpose and inverse (when invertible) of a matrix A are written as A^T and A^{-1} , respectively. $l_2[0, \infty)$ is the space of square-summable sequences. A block-diagonal matrix $D = \text{diag}\{d_{11}, d_{22}, \dots, d_{nn}\}$ has diagonal blocks $d_{11}, d_{22}, \dots, d_{nn}$. The symmetric entries of a symmetric matrix are indicated by an asterisk “*”. I and 0 stand for the identity and zero matrices of appropriate dimensions. For a matrix P , $\lambda_{\max}(P)$ is its maximum eigenvalue, and $\mathbb{E}\{\sigma\}$ denotes the mathematical expectation of the stochastic variable σ . For two real matrices $E = [e_{ij}]_{m \times n}$ and $F = [f_{ij}]_{m \times n}$ of the same size, their Hadamard product is defined as $E \circ F \triangleq [e_{ij}f_{ij}]_{m \times n}$.

2. Problem Statement and Preliminaries

2.1. Plant Model

The following class of affine fuzzy models serves to represent a category of discrete-time nonlinear systems under consideration.

Plant Rule i : **IF** $f_1(x(k))$ is ϑ_{i1} and $f_2(x(k))$ is ϑ_{i2} and $\cdots$ and $f_l(x(k))$ is ϑ_{il} , **THEN**

$$\begin{cases} x(k+1) = A_i x(k) + E_i \omega(k) + a_i \\ y(k) = Cx(k) + F\omega(k) \\ z(k) = G_i x(k), \quad i \in \mathbb{L} \triangleq \{1, 2, \dots, r\} \end{cases} \quad (1)$$

where the total number of fuzzy rules is r , with ϑ_{ij} representing a fuzzy set. The immeasurable premise variables are $f_1(x(k)), f_2(x(k)), \dots, f_l(x(k))$. The state vector is $x(k) \in \mathbb{R}^{n_x}$. External disturbance, denoted by $\omega(k) \in (l_2[0, \infty), \mathbb{R}^{n_v})$, encompasses process noise and measurement noise. The measurement output is $y(k) \in \mathbb{R}^{n_y}$, and $z(k) \in \mathbb{R}^{n_z}$ is the signal to be estimated. The affine term is represented by the known vector $a_i \in \mathbb{R}^{n_x}$. Moreover, A_i, C, E_i, F, G_i are known constant matrices with suitable dimensions.

Through standard fuzzy inference, the global representation of system (1) is obtained as:

$$\begin{cases} x(k+1) = \sum_{i=1}^r h_i(f(x(k))) (A_i x(k) + E_i \omega(k) + a_i) \\ y(k) = Cx(k) + F\omega(k) \\ z(k) = \sum_{i=1}^r h_i(f(x(k))) G_i x(k) \end{cases} \quad (2)$$

where the compact notation $f(x(k)) \triangleq [f_1(x(k)) \cdots f_l(x(k))]^T$ is first introduced for convenience, $\vartheta_{ij}(f_j(x(k)))$ represents the membership value of $f_j(x(k))$ in ϑ_{ij} , and the resulting fuzzy-basis functions are given by:

$$h_i(f(x(k))) = \frac{\kappa_i(f(x(k)))}{\sum_{j=1}^r \kappa_j(f(x(k)))},$$

$$\kappa_i(f(x(k))) = \prod_{j=1}^l \vartheta_{ij}(f_j(x(k))).$$

Then, it is easy to see that

$$0 \leq h_i(f(x(k))) \leq 1, \quad \sum_{i=1}^r h_i(f(x(k))) = 1, \quad \forall k.$$

To support the subsequent theoretical analysis, we initially introduce a subregion-based transformation of system (2) following the idea from [4]. The fuzzy rules in (1) induce a polyhedral partition of the state space, which allows the global fuzzy system (2) to be interpreted as a combination of multiple subsystems distributed across a set of distinct regions. Without loss of generality, the premise variable space is assumed to be partitioned into s ($s \in \mathbb{N}, s > 0$) regions based on the structure of the fuzzy membership functions. Each such region is denoted by $\mathbb{S}_t$ where $t \in \mathbb{M} \triangleq \{1, 2, \dots, s\}$. These regions fall into two categories: crisp regions and fuzzy regions. More precisely, within a crisp region, only a single fuzzy rule is active, that is, $h_n(f(x(k))) = 1$ for some $n \in \mathbb{L}$. In contrast, within a fuzzy region, several fuzzy rules are simultaneously active, meaning $0 < h_n(f(x(k))) < 1$ for some $n \in \mathbb{L}$. Furthermore, for each region $\mathbb{S}_t$, we let $\mathbb{I}(t)$ denote the index set collecting all activated fuzzy rules within that region. Consequently, the fuzzy system (2) can be expressed in the following piecewise fuzzy form:

$$\begin{cases} x(k+1) = \bar{A}_t x(k) + \bar{E}_t \omega(k) + \bar{a}_t \\ y(k) = Cx(k) + F\omega(k) \\ z(k) = \bar{G}_t x(k), \quad x(k) \in \mathbb{S}_t, \quad t \in \mathbb{M} \end{cases} \quad (3)$$

where

$$\begin{aligned}\bar{A}_t &\triangleq \sum_{m \in \mathbb{I}(t)} h_m(f(x(k)))A_m, \quad \bar{E}_t \triangleq \sum_{m \in \mathbb{I}(t)} h_m(f(x(k)))E_m, \\ \bar{G}_t &\triangleq \sum_{m \in \mathbb{I}(t)} h_m(f(x(k)))G_m, \quad \bar{a}_t \triangleq \sum_{m \in \mathbb{I}(t)} h_m(f(x(k)))a_m.\end{aligned}$$

In this work, we focus on the scenario where the region $\mathbb{S}_t$ takes the following form:

$$\mathbb{S}_t \triangleq \{x(k) \mid \alpha_t \leq \theta_t^T x(k) \leq \beta_t\}, \quad t \in \mathbb{M} \quad (4)$$

where $\alpha_t, \beta_t \in \mathbb{R}$ and $\theta \in \mathbb{R}^{n_x}$ are known parameters derived from the fuzzy membership functions. Then, following [4], each region can be precisely characterized by an ellipsoid of the following form:

$$\mathbb{S}_t = \{x(k) \mid \|Q_t x(k) + q_t\| \leq 1\}, \quad t \in \mathbb{M} \quad (5)$$

where

$$Q_t \triangleq \frac{2\theta_t^T}{\beta_t - \alpha_t}, \quad q_t \triangleq -\frac{\beta_t + \alpha_t}{\beta_t - \alpha_t}.$$

From (5), we can obtain the following relation for each region:

$$\begin{bmatrix} x(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_t^T Q_t & * \\ q_t^T Q_t & q_t^T q_t - 1 \end{bmatrix} \begin{bmatrix} x(k) \\ 1 \end{bmatrix} \leq 0, \quad t \in \mathbb{M}. \quad (6)$$

We divide the indices of the partitioned regions into two categories, denoted by $\mathbb{M}_0$ and $\mathbb{M}_1$, which satisfy $\mathbb{M}_0 \cup \mathbb{M}_1 = \mathbb{M}$ and $\mathbb{M}_0 \cap \mathbb{M}_1 = \emptyset$. Here, $\mathbb{M}_0$ collects the indices of regions containing the origin, while $\mathbb{M}_1$ consists of the remaining regions. Based on the partition, the following assumption is introduced.

Assumption 1 ([24]). *For any region containing the origin, the affine term is zero. In other words, $a_\tau = 0$ holds for every $\tau \in \mathbb{I}_{m_0}$, where $\mathbb{I}_{m_0}$ is the set collecting all activated fuzzy rules within the region $\mathbb{M}_0$.*

The assumption ascertains that the origin $x(k) = 0$ constitutes the equilibrium point of the fuzzy affine system.

2.2. Communication Network

Let $\tilde{y}(k)$ represent the sensor measurement output after being transmitted through the wireless communication network to the remote estimator. Given the unreliable nature of the transmission environment, channel fading effects are explicitly taken into account. Under such circumstances, $\tilde{y}(k)$ admits the following representation [25]:

$$\tilde{y}(k) = \Phi(k)y(k) \quad (7)$$

where $\Phi(k)$ is a stochastic diagonal matrix of the form:

$$\Phi(k) \triangleq \text{diag}\{\phi_1(k), \phi_2(k), \dots, \phi_{n_y}(k)\}. \quad (8)$$

In Equation (8), $\phi_a(k)$ ($a = 1, 2, \dots, n_y$) represent the channel coefficients or fading levels. Furthermore, the following statistical properties hold for the independent and identically distributed stochastic variables $\phi_a(k)$ with $k = 1, 2, \dots$:

$$\begin{aligned}\bar{\phi}_a &\triangleq \mathbb{E}\{\phi_a(k)\}, \\ \phi_{ab}^* &\triangleq \mathbb{E}\{(\phi_a(k) - \bar{\phi}_a)(\phi_b(k) - \bar{\phi}_b)\}, \\ a, b &= 1, 2, \dots, n_y\end{aligned}$$

where $\bar{\phi}_a > 0$, $\phi_{ab}^* > 0$ are known scalars and $\phi_{ab}^* = \phi_{ba}^*$.

For the convenience of the following analysis, the following auxiliary matrices are defined:

$$\begin{aligned}\bar{\Phi} &\triangleq \text{diag}\{\bar{\phi}_1, \bar{\phi}_2, \dots, \bar{\phi}_{n_y}\}, \\ \Upsilon &\triangleq \text{diag}\{\phi_{11}^*, \phi_{22}^*, \dots, \phi_{n_y n_y}^*\}, \\ \tilde{\Upsilon} &\triangleq [\phi_{ab}^*]_{a,b=1,2,\dots,n_y}.\end{aligned}$$

The following inequalities are easily seen to hold: $\bar{\Phi} > 0$, $\Upsilon > 0$, and $\tilde{\Upsilon} \geq 0$.

Remark 1. The independent and identically distributed assumption on the fading coefficients is reasonable in fast-fading environments where the channel decorrelates rapidly over time, such as in wireless communications with high mobility or rich scattering. The memoryless model is distinct from Markovian fading channels and is widely adopted in the literature as a tractable approximation. The required statistical parameters, namely, the means, variances, and cross-covariances of the diagonal entries of $\Phi(k)$, can be obtained in practice via offline pilot-signal training, standard channel models (e.g., Rayleigh or Rician fading), or online recursive estimation when the statistics are time-varying.

2.3. Fuzzy PI Estimator

Given the piecewise fuzzy system (3), by using the available output $\tilde{y}(k)$, the following piecewise affine PI estimator is adopted in this work:

Region Rule j: IF $\hat{x}(k) \in \mathbb{S}_j$, THEN

Estimator Rule n: IF $f_1(\hat{x}(k))$ is ϑ_{n1} and $f_2(\hat{x}(k))$ is ϑ_{n2} and $\dots$ and $f_t(\hat{x}(k))$ is ϑ_{nt} , THEN

$$\begin{cases} \hat{x}(k+1) = A_n \hat{x}(k) + a_n + K_{jn}(\tilde{y}(k) - \bar{\Phi} C \hat{x}(k)) \\ \quad + L_{jn} \sum_{\sigma=1}^{k-1} (\tilde{y}(\sigma) - \bar{\Phi} C \hat{x}(\sigma)) \\ \hat{z}(k) = G_n \hat{x}(k) \end{cases} \quad (9)$$

where $\hat{x}(k)$ and $\hat{z}(k)$ denote the estimated values of the signals $x(k)$ and $z(k)$, respectively; K_{jn} and L_{jn} are proportional, integral gains respectively, that need to be designed in each region.

Remark 2. A key difficulty lies in the fact that the premise variables of the fuzzy model are functions of the system states, which are not directly available to the estimator. As a result, the premise variables used in the estimator generally do not match those in the actual system. The mismatch inevitably leads to estimation error dynamics that contains residual affine terms, even though the affine components in the state equation are assumed to be known. In other words, due to the premise-variable mismatch, the affine terms cannot be completely canceled out in the resulting error system, which significantly complicates the stability analysis and estimator design.

To obtain a compact form of PI estimator (9), we define $l(k) \triangleq \sum_{\sigma=1}^{k-1} (\tilde{y}(\sigma) - \bar{\Phi} C \hat{x}(\sigma))$. Then, (9) can be rewritten as follows:

$$\begin{cases} \hat{x}(k+1) = \tilde{A}_j \hat{x}(k) + \tilde{a}_j + \tilde{K}_j(\tilde{y}(k) - \bar{\Phi} C \hat{x}(k)) + \tilde{L}_j l(k) \\ l(k+1) = l(k) + (\tilde{y}(k) - \bar{\Phi} C \hat{x}(k)) \\ \hat{z}(k) = \tilde{G}_j \hat{x}(k), \quad \hat{x}(k) \in \mathbb{S}_j \end{cases} \quad (10)$$

where

$$\begin{aligned}\tilde{A}_j &\triangleq \sum_{n \in \mathbb{I}(j)} h_n(f(\hat{x}(k))) A_n, \quad \tilde{a}_j \triangleq \sum_{n \in \mathbb{I}(j)} h_n(f(\hat{x}(k))) a_n, \\ \tilde{G}_j &\triangleq \sum_{n \in \mathbb{I}(j)} h_n(f(\hat{x}(k))) G_n, \quad \tilde{K}_j \triangleq \sum_{n \in \mathbb{I}(j)} h_n(f(\hat{x}(k))) K_{jn}, \\ \tilde{L}_j &\triangleq \sum_{n \in \mathbb{I}(j)} h_n(f(\hat{x}(k))) L_{jn}.\end{aligned}$$

We now define the estimation errors $e(k) \triangleq x(k) - \hat{x}(k)$ for the signal $x(k)$ and $\tilde{z}(k) \triangleq z(k) - \hat{z}(k)$ for $z(k)$, along with the augmented vector $\eta(k) \triangleq [e^T(k) \quad x^T(k) \quad l^T(k)]^T$. Then, according to (3) and (10), we can obtain the augmentation system as follows:

$$\begin{cases} \eta(k+1) = (\mathcal{A}_{tj} + \mathcal{B}_j \tilde{\Phi}(k) \mathcal{C}) \eta(k) + \hat{a}_{tj} \\ \quad + (\mathcal{E}_{tj} + \mathcal{B}_j \tilde{\Phi}(k) F) \omega(k) \\ \tilde{z}(k) = \mathcal{G}_{tj} \eta(k), \quad x(k) \in \mathbb{S}_t, \hat{x}(k) \in \mathbb{S}_j, t, j \in \mathbb{M} \end{cases} \quad (11)$$

where

$$\begin{aligned} \mathcal{A}_{tj} &\triangleq \begin{bmatrix} \tilde{A}_j - \tilde{K}_j \tilde{\Phi} C & \bar{A}_t - \tilde{A}_j & -\tilde{L}_j \\ 0 & \bar{A}_t & 0 \\ \tilde{\Phi} C & 0 & I \end{bmatrix}, \\ \mathcal{B}_j &\triangleq \begin{bmatrix} -\tilde{K}_j \\ 0 \\ I \end{bmatrix}, \quad \mathcal{E}_{tj} \triangleq \begin{bmatrix} \bar{E}_t - \tilde{K}_j \tilde{\Phi} F \\ \bar{E}_t \\ \tilde{\Phi} F \end{bmatrix}, \\ \mathcal{C} &\triangleq [0 \quad C \quad 0], \quad \hat{a}_{tj} \triangleq [\bar{a}_t^T - \bar{a}_j^T \quad \bar{a}_t^T \quad 0]^T, \\ \mathcal{G}_{tj} &\triangleq [\tilde{G}_j \quad \bar{G}_t - \tilde{G}_j \quad 0], \\ \tilde{\Phi}(k) &\triangleq \text{diag}\{\phi_1(k) - \bar{\phi}_1, \phi_2(k) - \bar{\phi}_2, \dots, \phi_{n_y}(k) - \bar{\phi}_{n_y}\}. \end{aligned}$$

To lay the groundwork for subsequent analysis, we begin by formally defining the concept of stochastic stability specifically for the error system in (11).

Definition 1 ([26]). *The estimation error system (11) is said to be stochastically stable if, for $\omega(k) = 0$, there exists a scalar α ($\alpha > 0$) such that the following inequality holds:*

$$\mathbb{E} \left\{ \sum_{k=0}^{\infty} \|\eta(k)\|^2 \right\} \leq \alpha \mathbb{E} \{ \|\eta(0)\|^2 \}. \quad (12)$$

This paper aims to address the l_2 - l_∞ fuzzy PI state estimation problem for discrete-time affine fuzzy systems subject to channel fading. Our goal is to simultaneously achieve the following two requirements:

- (R1) the fuzzy system (11) is stochastically stable according to Definition 1; and
- (R2) for every nonzero $\omega(k) \in l_2[0, \infty)$ and under zero initial condition, the estimation error $\tilde{z}(k)$ comply with the l_2 - l_∞ performance constraint:

$$\sup_{k \geq 0} \mathbb{E} \{ \tilde{z}^T(k) \tilde{z}(k) \} < \gamma^2 \sum_{k=0}^{\infty} \omega^T(k) \omega(k) \quad (13)$$

where $\gamma > 0$ is a prescribed scalar representing the l_2 - l_∞ disturbance attenuation level.

3. Main Results

In order to facilitate the derivation of our principal results, a key lemma is presented below.

Lemma 1 ([27]). *Given real matrices $A \geq 0$, $B \geq 0$ and $C \geq 0$, the inequality $B \leq C$ implies $A \circ B \leq A \circ C$.*

The analysis of the estimation error system (11) is carried out first. A sufficient condition for verifying both the stochastic stability and the l_2 - l_∞ performance is given by the following theorem.

Theorem 1. *Let the estimator gains K_{jn} , L_{jn} and the disturbance attenuation level $\gamma > 0$ be given. If there exist positive definite matrices $P_{tj} > 0$ ($t, j \in \mathbb{M}$) and scalars $\alpha_{\bar{t}n} > 0$ ($\bar{t} \in \mathbb{M}$, $n \in \mathbb{I}(t)$), $\lambda_{tj} < 0$ ($t, j \in \mathbb{M}$), such that the following matrix inequalities hold:*

$$\begin{bmatrix} -P_{tj} & * & * & * \\ 0 & -\gamma^2 & * & * \\ \bar{\mathcal{A}}_{jmn} & \bar{\mathcal{E}}_{jmn} & -P_{\bar{t}j}^{-1} & * \\ \alpha_{\bar{t}n} \mathcal{C} & \alpha_{\bar{t}n} F & 0 & -\alpha_{\bar{t}n} \Upsilon^{-1} \end{bmatrix} < 0, \quad (14)$$

$$\begin{bmatrix} \Pi_{11tj}^a & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ \lambda_{tj} q_j^T Q_j \bar{I}_1 & 0 & \Pi_{33tj}^a & * & * \\ \bar{\mathcal{A}}_{jmn} & \bar{\mathcal{E}}_{jmn} & \bar{a}_{mn} & -P_{\bar{t}j}^{-1} & * \\ \alpha_{\bar{t}n} \mathcal{C} & \alpha_{\bar{t}n} F & 0 & 0 & -\alpha_{\bar{t}n} \Upsilon^{-1} \end{bmatrix} < 0, \quad (15)$$

$$\begin{bmatrix} \Pi_{11tj}^b & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ \lambda_{tj} q_t^T Q_t \bar{I}_2 & 0 & \Pi_{33tj}^b & * & * \\ \bar{\mathcal{A}}_{jmn} & \bar{\mathcal{E}}_{jmn} & \bar{a}_{mn} & -P_{\bar{t}j}^{-1} & * \\ \alpha_{\bar{t}n} \mathcal{C} & \alpha_{\bar{t}n} F & 0 & 0 & -\alpha_{\bar{t}n} \Upsilon^{-1} \end{bmatrix} < 0, \quad (16)$$

$$t \in \mathbb{M}_1, j \in \mathbb{M}_0, \bar{t} \in \mathbb{M}, \bar{j} \in \mathbb{M}, m \in \mathbb{I}(t), n \in \mathbb{I}(j),$$

$$\bar{\mathcal{B}}_{jn}^T P_{\bar{t}j} \bar{\mathcal{B}}_{jn} - \alpha_{\bar{t}n} I < 0, \quad (17)$$

$$j \in \mathbb{M}, \bar{t} \in \mathbb{M}, n \in \mathbb{I}(j),$$

$$\bar{\mathcal{G}}_{mn}^T \bar{\mathcal{G}}_{mn} - P_{tj} < 0, \quad (18)$$

$$t \in \mathbb{M}, j \in \mathbb{M}, m \in \mathbb{I}(j), n \in \mathbb{I}(j)$$

where

$$\begin{aligned} \bar{\mathcal{G}}_{mn} &\triangleq [G_m \quad G_m - G_n \quad 0], \\ \bar{\mathcal{A}}_{jmn} &\triangleq \bar{A}_{mn} - \bar{I}_1 K_{jn} \bar{C} - \bar{I}_1 L_{jn} \bar{I}_2, \quad \bar{I}_3 \triangleq \bar{I}_2^T, \\ \bar{A}_{mn} &\triangleq \begin{bmatrix} A_n & A_m - A_n & 0 \\ 0 & A_m & 0 \\ \bar{\Phi} C & 0 & I \end{bmatrix}, \quad \bar{I}_1 \triangleq \begin{bmatrix} I_{n_x} \\ 0_{n_x} \\ 0_{n_y \times n_x} \end{bmatrix}, \\ \bar{I}_2 &\triangleq [0_{n_y \times n_x} \quad 0_{n_y \times n_x} \quad I_{n_y}], \quad \bar{C} \triangleq [\bar{\Phi} C \quad 0 \quad 0], \\ \bar{\mathcal{E}}_{jmn} &\triangleq \bar{E}_m - \bar{I}_1 K_{jn} \bar{\Phi} F, \quad \bar{I}_1 \triangleq [-I_{n_x} \quad I_{n_x} \quad 0_{n_x \times n_y}], \\ \bar{E}_m &\triangleq \begin{bmatrix} E_m \\ E_m \\ \bar{\Phi} F \end{bmatrix}, \quad \bar{a}_{mn} \triangleq \begin{bmatrix} a_m - a_n \\ a_m \\ 0 \end{bmatrix}, \\ \Pi_{11tj}^a &\triangleq -P_{tj} + \lambda_{tj} \bar{I}_1^T Q_j^T Q_j \bar{I}_1, \quad \Pi_{33tj}^a \triangleq \lambda_{tj} (q_j^T q_j - 1), \\ \Pi_{11tj}^b &\triangleq -P_{tj} + \lambda_{tj} \bar{I}_2^T Q_t^T Q_t \bar{I}_2, \quad \Pi_{33tj}^b \triangleq \lambda_{tj} (q_t^T q_t - 1), \\ \bar{I}_2 &\triangleq [0_{n_x} \quad I_{n_x} \quad 0_{n_x \times n_y}], \quad \bar{\mathcal{B}}_{jn} \triangleq -\bar{I}_1 K_{jn} + \bar{I}_3, \end{aligned}$$

then, the estimation error system (11) is stochastically stable and satisfies the $l_2 - l_\infty$ performance requirement (13).

Proof. A region-based Lyapunov function of the following form is chosen, which is particularly suitable for capturing the local dynamics of the error system:

$$V(k) = \eta^T(k) P_{tj} \eta(k), \quad x(k) \in \mathbb{S}_t, \hat{x}(k) \in \mathbb{S}_j, \quad t, j \in \mathbb{M}. \quad (19)$$

According to the state-space partitions, the indices t and j introduced in (19) fall into the following three categories:

$$\begin{cases} \text{Case 1 : } x(k) \in \mathbb{S}_t, t \in \mathbb{M}_0, \hat{x}(k) \in \mathbb{S}_j, j \in \mathbb{M}_0, \\ \text{Case 2 : } x(k) \in \mathbb{S}_t, t \in \mathbb{M}_1, \hat{x}(k) \in \mathbb{S}_j, j \in \mathbb{M}_0, \\ \text{Case 3 : } x(k) \in \mathbb{S}_t, t \in \mathbb{M}, \hat{x}(k) \in \mathbb{S}_j, j \in \mathbb{M}_1. \end{cases} \quad (20)$$

Without loss of generality, the most complex case, i.e., Case 3, is taken as the starting point for the proof. It is assumed that $x(k+1) \in \mathbb{S}_{\bar{t}}$ ($\bar{t} \in \mathbb{M}$), $\hat{x}(k+1) \in \mathbb{S}_{\bar{j}}$ ($\bar{j} \in \mathbb{M}$). Then, the following term can be calculated:

$$\begin{aligned}
& \mathbb{E}\{\Delta V(k)|\eta(k)\} \\
& \triangleq \mathbb{E}\{V(k+1)|\eta(k)\} - V(k) \\
& = \mathbb{E}\{\eta^T(k+1)P_{\bar{t}j}\eta(k+1)|\eta(k)\} - \eta^T(k)P_{tj}\eta(k) \\
& = \mathbb{E}\left\{\left((\mathcal{A}_{tj} + \mathcal{B}_j\tilde{\Phi}(k)\mathcal{C})\eta(k) + (\mathcal{E}_{tj} + \mathcal{B}_j\tilde{\Phi}(k)F)\omega(k) \right. \right. \\
& \quad \left. \left. + \hat{a}_{tj}\right)^T P_{\bar{t}j}\left((\mathcal{A}_{tj} + \mathcal{B}_j\tilde{\Phi}(k)\mathcal{C})\eta(k) + (\mathcal{E}_{tj} + \mathcal{B}_j\tilde{\Phi}(k)F)\right. \right. \\
& \quad \left. \left. \times \omega(k) + \hat{a}_{tj}\right) - \eta^T(k)P_{tj}\eta(k)|\eta(k)\right\} \\
& = (\mathcal{A}_{tj}\eta(k) + \mathcal{E}_{tj}\omega(k) + \hat{a}_{tj})^T P_{\bar{t}j}(\mathcal{A}_{tj}\eta(k) + \mathcal{E}_{tj}\omega(k) \\
& \quad + \hat{a}_{tj}) + (\mathcal{C}\eta(k) + F\omega(k))^T (\tilde{\Upsilon} \circ (\mathcal{B}_j^T P_{\bar{t}j}\mathcal{B}_j)) \\
& \quad \times (\mathcal{C}\eta(k) + F\omega(k)) - \eta^T(k)P_{tj}\eta(k)
\end{aligned} \tag{21}$$

Considering the notations defined in (3) and (10), we further have

$$\begin{aligned}
& \mathbb{E}\{\Delta V(k)|\eta(k)\} \\
& \leq \sum_{m \in \mathbb{I}(t)} \sum_{n \in \mathbb{I}(j)} h_m(f(x(k)))h_n(f(\hat{x}(k))) \\
& \quad \times \left((\bar{\mathcal{A}}_{jmn}\eta(k) + \bar{\mathcal{E}}_{jmn}\omega(k) + \bar{a}_{mn})^T P_{\bar{t}j}(\bar{\mathcal{A}}_{jmn}\eta(k) \right. \\
& \quad \left. + \bar{\mathcal{E}}_{jmn}\omega(k) + \bar{a}_{mn}) + (\mathcal{C}\eta(k) + F\omega(k))^T \right. \\
& \quad \left. \times (\tilde{\Upsilon} \circ (\bar{\mathcal{B}}_{jn}^T P_{\bar{t}j}\bar{\mathcal{B}}_{jn}))(\mathcal{C}\eta(k) + F\omega(k)) \right. \\
& \quad \left. - \eta^T(k)P_{tj}\eta(k) \right).
\end{aligned} \tag{22}$$

Under the condition (17), we know

$$\bar{\mathcal{B}}_{jn}^T P_{\bar{t}j} \bar{\mathcal{B}}_{jn} < \alpha_{\bar{t}n} I. \tag{23}$$

By using Lemma 1, we can obtain

$$\tilde{\Upsilon} \circ (\bar{\mathcal{B}}_{jn}^T P_{\bar{t}j} \bar{\mathcal{B}}_{jn}) < \tilde{\Upsilon} \circ \alpha_{\bar{t}n} I = \alpha_{\bar{t}n} \tilde{\Upsilon}. \tag{24}$$

It follows from (6) that the condition below is satisfied:

$$\lambda_{tj} \begin{bmatrix} \hat{x}(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_j^T Q_j & * \\ q_j^T Q_j & q_j^T q_j - 1 \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ 1 \end{bmatrix} \geq 0, \quad j \in \mathbb{M}_1. \tag{25}$$

Note that $\hat{x}(k) = x(k) - e(k) = \bar{I}_1 \eta(k)$. Thus, the above inequality implies

$$\lambda_{tj} \begin{bmatrix} \eta(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{I}_1^T Q_j^T Q_j \bar{I}_1 & * \\ q_j^T Q_j \bar{I}_1 & q_j^T q_j - 1 \end{bmatrix} \begin{bmatrix} \eta(k) \\ 1 \end{bmatrix} \geq 0, \quad j \in \mathbb{M}_1. \tag{26}$$

By utilizing conditions (24) and (26), one has

$$\begin{aligned}
& \mathbb{E}\{\Delta V(k)|\eta(k)\} \\
& < \sum_{m \in \mathbb{I}(t)} \sum_{n \in \mathbb{I}(j)} h_m(f(x(k)))h_n(f(\hat{x}(k))) \\
& \quad \times \left((\bar{\mathcal{A}}_{jmn}\eta(k) + \bar{\mathcal{E}}_{jmn}\omega(k) + \bar{a}_{mn})^T P_{\bar{t}j}(\bar{\mathcal{A}}_{jmn}\eta(k) \right. \\
& \quad \left. + \bar{\mathcal{E}}_{jmn}\omega(k) + \bar{a}_{mn}) + (\mathcal{C}\eta(k) + F\omega(k))^T \right. \\
& \quad \times \alpha_{\bar{t}n} \tilde{\Upsilon}(\mathcal{C}\eta(k) + F\omega(k)) - \eta^T(k)P_{tj}\eta(k) \\
& \quad \left. + \lambda_{tj} \begin{bmatrix} \eta(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{I}_1^T Q_j^T Q_j \bar{I}_1 & * \\ q_j^T Q_j \bar{I}_1 & q_j^T q_j - 1 \end{bmatrix} \begin{bmatrix} \eta(k) \\ 1 \end{bmatrix} \right).
\end{aligned} \tag{27}$$

First, let $\omega(k) = 0$ for the stability analysis. From (27), we have

$$\begin{aligned} & \mathbb{E}\{\Delta V(k)|\eta(k)\} \\ & < \sum_{m \in \mathbb{I}(t)} \sum_{n \in \mathbb{I}(j)} h_m(f(x(k)))h_n(f(\hat{x}(k))) \\ & \quad \times \left((\bar{\mathcal{A}}_{jmn}\eta(k) + \bar{a}_{mn})^T P_{\bar{t}j} (\bar{\mathcal{A}}_{jmn}\eta(k) + \bar{a}_{mn}) \right. \\ & \quad + (\mathcal{C}\eta(k) + F\omega(k))^T \alpha_{\bar{t}n} \Upsilon (\mathcal{C}\eta(k) + F\omega(k)) \\ & \quad + \lambda_{tj} \begin{bmatrix} \eta(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{I}_1^T Q_j^T Q_j \bar{I}_1 & * \\ q_j^T Q_j \bar{I}_1 & q_j^T q_j - 1 \end{bmatrix} \begin{bmatrix} \eta(k) \\ 1 \end{bmatrix} \\ & \quad \left. - \eta^T(k) P_{tj} \eta(k) \right). \end{aligned} \quad (28)$$

Application of the Schur Complement Lemma reveals that condition (15) directly implies

$$\mathbb{E}\{\Delta V(k)|\eta(k)\} < 0. \quad (29)$$

It follows from (29) that there exists a sufficiently small scalar $\mu > 0$ satisfying

$$\mathbb{E}\{\Delta V(k)|\eta(k)\} \leq \eta^T(k)(-\mu I)\eta(k). \quad (30)$$

By taking expectation on both sides of (30), the following is obtained:

$$\mathbb{E}\{\Delta V(k)\} \leq -\mu \mathbb{E}\{\|\eta(k)\|\}. \quad (31)$$

By summing up both sides of (31) from 0 to N for any integer $N > 1$ with respect to k , one can obtain

$$\mathbb{E}\{V(N+1)\} - \mathbb{E}\{V(0)\} \leq -\mu \sum_{k=0}^N \mathbb{E}\{\|\eta(k)\|\}. \quad (32)$$

Considering $\mathbb{E}\{V(N+1)\} \geq 0$, one also has

$$\sum_{k=0}^N \mathbb{E}\{\|\eta(k)\|\} \leq \frac{1}{\mu} \mathbb{E}\{V(0)\} \leq \frac{c}{\mu} \mathbb{E}\{\|\eta(0)\|^2\} \quad (33)$$

where

$$c \triangleq \max\{\lambda_{\max}(P_{11}), \lambda_{\max}(P_{22}), \dots, \lambda_{\max}(P_{ss})\}.$$

Finally, taking $N \rightarrow \infty$, from Definition 1, we can conclude that the estimation error system (11) is stochastically stable.

The analysis of the $l_2 - l_\infty$ performance under the zero initial condition with $\omega(k) \neq 0$ is now in order. For this purpose, the following term is calculated:

$$\begin{aligned} & \mathbb{E}\{\Delta V(k) - \gamma^2 \omega^T(k) \omega(k) | \eta(k)\} \\ & = (\mathcal{A}_{tj}\eta(k) + \mathcal{E}_{tj}\omega(k) + \hat{a}_{tj})^T P_{\bar{t}j} (\mathcal{A}_{tj}\eta(k) + \mathcal{E}_{tj}\omega(k) \\ & \quad + \hat{a}_{tj}) + (\mathcal{C}\eta(k) + F\omega(k))^T (\bar{\Upsilon} \circ (\mathcal{B}_j^T P_{\bar{t}j} \mathcal{B}_j)) \\ & \quad \times (\mathcal{C}\eta(k) + F\omega(k)) - \eta^T(k) P_{tj} \eta(k) - \gamma^2 \omega^T(k) \omega(k). \end{aligned} \quad (34)$$

By following the similar process of the stability analysis, and using the inequalities (24) and (26) again, we get

$$\begin{aligned}
& \mathbb{E}\{\Delta V(k) - \gamma^2 \omega^T(k) \omega(k) | \eta(k)\} \\
& < \sum_{m \in \mathbb{I}(t)} \sum_{n \in \mathbb{I}(j)} h_m(f(x(k))) h_n(f(\hat{x}(k))) \\
& \quad \times \left((\bar{\mathcal{A}}_{jmn} \eta(k) + \bar{\mathcal{E}}_{jmn} \omega(k) + \bar{a}_{mn})^T P_{tj} (\bar{\mathcal{A}}_{jmn} \eta(k) \right. \\
& \quad + \bar{\mathcal{E}}_{jmn} \omega(k) + \bar{a}_{mn}) + (\mathcal{C} \eta(k) + F \omega(k))^T \\
& \quad \times \alpha_{tn} \Upsilon(\mathcal{C} \eta(k) + F \omega(k)) - \eta^T(k) P_{tj} \eta(k) \\
& \quad - \gamma^2 \omega^T(k) \omega(k) \\
& \quad \left. + \lambda_{tj} \begin{bmatrix} \eta(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{I}_1^T Q_j^T Q_j \bar{I}_1 & * \\ q_j^T Q_j \bar{I}_1 & q_j^T q_j - 1 \end{bmatrix} \begin{bmatrix} \eta(k) \\ 1 \end{bmatrix} \right). \tag{35}
\end{aligned}$$

In terms of the Schur Complement Lemma, condition (15) implies

$$\mathbb{E}\{\Delta V(k) - \gamma^2 \omega^T(k) \omega(k) | \eta(k)\} < 0. \tag{36}$$

The expectation applied to both sides of the above inequality, for $0 < k \leq T$ with T being an integral, we have

$$\begin{aligned}
& \sum_{\tau=0}^{k-1} \mathbb{E}\{\Delta V(\tau) - \gamma^2 \omega^T(\tau) \omega(\tau)\} \\
& = \mathbb{E}\{V(k)\} - \sum_{\tau=0}^{k-1} \mathbb{E}\{\gamma^2 \omega^T(\tau) \omega(\tau)\} < 0. \tag{37}
\end{aligned}$$

It is deduced based on (37) that

$$\begin{aligned}
\mathbb{E}\{V(k)\} & = \mathbb{E}\{\eta^T(k) P_{tj} \eta(k)\} \\
& < \sum_{\tau=0}^{k-1} \mathbb{E}\{\gamma^2 \omega^T(\tau) \omega(\tau)\} \\
& \leq \sum_{\tau=0}^T \mathbb{E}\{\gamma^2 \omega^T(\tau) \omega(\tau)\}, \quad 0 < k \leq T. \tag{38}
\end{aligned}$$

Furthermore, one has

$$\begin{aligned}
& \mathbb{E}\{\tilde{z}^T(k) \tilde{z}(k)\} - \mathbb{E}\{V(k)\} \\
& \leq \sum_{m \in \mathbb{I}(t)} \sum_{n \in \mathbb{I}(j)} \mathbb{E}\left\{h_m(f(x(k))) h_n(f(\hat{x}(k))) \right. \\
& \quad \left. \times \left(\eta^T(k) (\bar{\mathcal{G}}_{mn}^T \bar{\mathcal{G}}_{mn} - P_{tj}) \eta(k) \right) \right\}. \tag{39}
\end{aligned}$$

From the condition (18), we derive

$$\mathbb{E}\{\tilde{z}^T(k) \tilde{z}(k)\} < \mathbb{E}\{V(k)\} < \sum_{\tau=0}^T \gamma^2 \omega^T(k) \omega(k), \quad 0 < k \leq T. \tag{40}$$

By letting $T \rightarrow \infty$, it is observed that the prescribed l_2 – l_∞ performance in (13) is satisfied. By conducting the similar analysis, the condition (14) for Case 1 and the condition (16) for Case 2 can be easily obtained, respectively. For space considerations, the details are not presented here, and the proof is hereby completed. $\square$

Theorem 1 establishes a sufficient condition, grounded in Lyapunov stability theory and stochastic analysis, for assessing the performance of the estimation error dynamics. Next, on the basis of Theorem 1, we will give the following theorem to show how to design estimator gains under the given performance requirements.

Theorem 2. Let the disturbance attenuation level $\gamma > 0$ be given. If there exist positive definite matrices $P_{tj} > 0$ ($t \in \mathbb{M}$), nonsingular matrix Y , matrices M_{jn} , N_{jn} ($j \in \mathbb{M}$, $n \in \mathbb{I}(j)$) and scalars $\alpha_{\bar{t}n} > 0$ ($\bar{t} \in \mathbb{M}$, $n \in \mathbb{I}(\bar{t})$), $\lambda_{tj} < 0$ ($t, j \in \mathbb{M}$), such that the following linear matrix inequalities hold:

$$\begin{bmatrix} -P_{tj} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \Omega_{1jmn} & \Omega_{2jmn} & \Omega_{3\bar{t}\bar{j}} & * \\ \alpha_{\bar{t}n} \mathcal{C} & \alpha_{\bar{t}n} F & 0 & -\alpha_{\bar{t}n} \Upsilon^{-1} \end{bmatrix} < 0, \quad (41)$$

$$\begin{bmatrix} \Pi_{11tj}^a & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ \lambda_{tj} q_j^T Q_j \bar{I}_1 & 0 & \Pi_{33tj}^a & * & * \\ \Omega_{1jmn} & \Omega_{2jmn} & \bar{a}_{mn} & \Omega_{3\bar{t}\bar{j}} & * \\ \alpha_{\bar{t}n} \mathcal{C} & \alpha_{\bar{t}n} F & 0 & 0 & -\alpha_{\bar{t}n} \Upsilon^{-1} \end{bmatrix} < 0, \quad (42)$$

$$\begin{bmatrix} \Pi_{11tj}^b & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ \lambda_{tj} q_t^T Q_t \bar{I}_2 & 0 & \Pi_{33tj}^b & * & * \\ \Omega_{1jmn} & \Omega_{2jmn} & \bar{a}_{mn} & \Omega_{3\bar{t}\bar{j}} & * \\ \alpha_{\bar{t}n} \mathcal{C} & \alpha_{\bar{t}n} F & 0 & 0 & -\alpha_{\bar{t}n} \Upsilon^{-1} \end{bmatrix} < 0, \quad (43)$$

$$t \in \mathbb{M}_1, j \in \mathbb{M}_0, \bar{t} \in \mathbb{M}, \bar{j} \in \mathbb{M}, m \in \mathbb{I}(t), n \in \mathbb{I}(j),$$

$$\begin{bmatrix} -\alpha_{\bar{t}n} I & * \\ -M_{jn} + Y \tilde{I}_3 & P_{\bar{t}\bar{j}} - Y - Y^T \end{bmatrix} < 0 \quad (44)$$

$$j \in \mathbb{M}, \bar{t} \in \mathbb{M}, \bar{j} \in \mathbb{M}, n \in \mathbb{I}(j),$$

$$\bar{\mathcal{G}}_{mn}^T \bar{\mathcal{G}}_{mn} - P_{tj} < 0, \quad (45)$$

$$t \in \mathbb{M}, j \in \mathbb{M}, m \in \mathbb{I}(j), n \in \mathbb{I}(j)$$

where

$$\begin{aligned} \Omega_{1jmn} &\triangleq Y \vec{A}_{mn} - M_{jn} \vec{C} - N_{jn} \tilde{I}_2, \\ \Omega_{2jmn} &\triangleq Y \vec{E}_m - M_{jn} \vec{\Phi} F, \\ \Omega_{3\bar{t}\bar{j}} &\triangleq P_{\bar{t}\bar{j}} - Y - Y^T, \end{aligned}$$

with the remaining variables as defined in Theorem 1, then, it is ensured that the estimation error system (11) is stochastically stable, and the $l_2 - l_\infty$ performance requirement specified in (13) is satisfied. Moreover, the estimator parameters can be calculated according to the following relation:

$$K_{jn} = (\tilde{I}_1^T Y \tilde{I}_1)^{-1} \tilde{I}_1^T M_{jn}, \quad L_{jn} = (\tilde{I}_1^T Y \tilde{I}_1)^{-1} \tilde{I}_1^T N_{jn}.$$

Proof. Pre- and postmultiplying the matrix in (14) by $\text{diag}\{I, I, Y, I\}$ and $\text{diag}\{I, I, Y^T, I\}$ respectively, let $M_{tj} = Y \tilde{I}_1 K_{jn}$ and $N_{tj} = Y \tilde{I}_1 L_{jn}$. In addition, from $Y P_{\bar{t}\bar{j}} Y^T + P_{\bar{t}\bar{j}} - Y - Y^T = (P_{\bar{t}\bar{j}} - Y) P_{\bar{t}\bar{j}}^{-1} (P_{\bar{t}\bar{j}} - Y^T) \geq 0$, we obtain $-Y P_{\bar{t}\bar{j}}^{-1} Y^T \leq P_{\bar{t}\bar{j}} - Y - Y^T$. Thus, we can obtain the condition (41). By conducting the similar transformation on (15), (16) and (17) respectively, we can obtain (42), (43) and (44) directly. Thus, the proof of this theorem is completed. $\square$

Remark 3. In summary, the problem of fuzzy PI state estimation has been investigated in this paper for a class of discrete-time affine T-S fuzzy systems experiencing channel fading. Distinct from existing fuzzy state estimation studies, the main contributions are twofold. First, a novel problem is addressed where the influence of channel fading is examined within affine T-S fuzzy systems for the first time. Second, a new region-dependent fuzzy PI estimator is developed via the state-space partition technique, leading to enhanced estimation performance. A numerical example is presented in the following section to demonstrate the effectiveness of the proposed scheme.

Remark 4. The proposed estimator design involves a piecewise-region analysis, where the fuzzy system is partitioned according to the activation status of the membership functions. Consequently, the estimator gains depend not only on the fuzzy rules but also on the operating regions. Compared with conventional single-region design methods, this strategy yields less conservative conditions by exploiting more detailed region-dependent information. On the other hand, the piecewise-region approach inevitably leads to a larger number of matrix inequalities and thus increased computational complexity. However, it should be emphasized that the system under consideration is time-invariant, so all matrix inequalities are solved offline; hence the online implementation burden remains unaffected. Moreover, the proposed method can be specialized to the single-region case by a straightforward simplification, which provides a useful trade-off between conservatism and computational cost when desired.

4. Simulation Example

Consider a three-rule affine T-S fuzzy system, which is in the form of (1) and possesses the parameters listed below.

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.1 & 0.42 \\ 0.1 & 0.95 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.4 & -0.2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.2 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0.1 \\ 0.6 \end{bmatrix}, \\ C &= [0.5 \quad 0.6], \quad F = 0.1, \quad G = [0.5, 0.5], \\ a_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, \quad x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}. \end{aligned}$$

According to the partition method detailed in Section 2, the state space is separable into three distinct regions. The membership function corresponding to the nonlinear plant is depicted in Figure 1.

$$\begin{aligned} \mathbb{S}_1 &= \{x(k) \mid -1 \leq x_2(k) \leq -0.5\}, \\ \mathbb{S}_2 &= \{x(k) \mid -0.5 < x_2(k) \leq 0.5\}, \\ \mathbb{S}_3 &= \{x(k) \mid 0.5 < x_2(k) \leq 1\}. \end{aligned}$$

Correspondingly, the parameters in (5) can be calculated as follows:

$$q_1 = 3, \quad Q_1 = \begin{bmatrix} 0 & 4 \end{bmatrix}, \quad q_3 = -3, \quad Q_3 = \begin{bmatrix} 0 & 2 \end{bmatrix}.$$

The transmission of measurements from sensors to the estimator is assumed to take place through a fading channel. The channel information in (8) is given by $\bar{\phi}_1 = 0.5$ and $\phi_{11}^* = 0.01$.

The fuzzy PI estimator in the form of (9) is sought to be designed in this example so that both requirements (R1) and (R2) on the estimation error are satisfied. The noise is configured as $\omega(k) = \frac{0.1 \sin k}{k}$, and the disturbance attenuation level is set to $\gamma = 1$. The initial conditions are given by $x(k) = [-0.04 \quad -0.1]^T$ for the system and $\hat{x}(k) = [0 \quad -0.08]^T$ for the estimator. Using the estimator gains derived from Theorem 2, the simulation outcomes are presented in Figures 2–5. Specifically, the trajectories of the original system states along with their estimates are illustrated in Figures 2 and 3. Figure 4 depicts both the signal $z(k)$ and its estimated counterpart. The estimation error $\tilde{z}(k)$ alongside the external noise is provided in Figure 5, where it can be observed that the error remains relatively small in comparison with the noise. Figure 6 illustrates the actual measurement output $y(k)$ together with the faded output $\tilde{y}(k)$, where attenuation of the measurement amplitude is observed. Moreover, a straightforward calculation leads to $\max \tilde{z}^T(k)z(k) < \gamma^2 \sum_{k=0}^T \omega^T(k)\omega(k)$ with T being the simulation length, which indicates that the required $l_2 - l_\infty$ performance criterion is satisfied.

To illustrate the effects of fading on estimation performance, we define the accumulative estimation error of $z(k)$ as follows:

$$z_{\text{sum}} \triangleq \sum_{k=0}^T \tilde{z}^T(k) \tilde{z}(k)$$

Then, Table 1 lists the obtained values of z_{sum} for both the proposed PI estimator and the widely used proportional-type (PT) estimator [28] under different fading levels, where the fading level is defined as $\check{\phi} \triangleq 1 - \bar{\phi}_1$. A higher fading level indicates poorer communication quality. As can be seen from Table 1, worse communication quality leads to a larger cumulative estimation error, implying that more severe attenuation results in degraded estimation performance. Moreover, the simulation results demonstrate that, under the same fading level, the proposed affine PI estimator outperforms the conventional PT estimator. This performance improvement is mainly attributed to the incorporation of historical estimation errors and the adoption of a piecewise-region design scheme.

Table 1. Accumulative estimation error under different fading levels.

Fading Level $\check{\phi}$	0.3	0.5	0.7
z_{sum} (fuzzy PI)	0.0314	0.0814	0.0998
z_{sum} (fuzzy PT)	0.0746	0.1532	0.1753

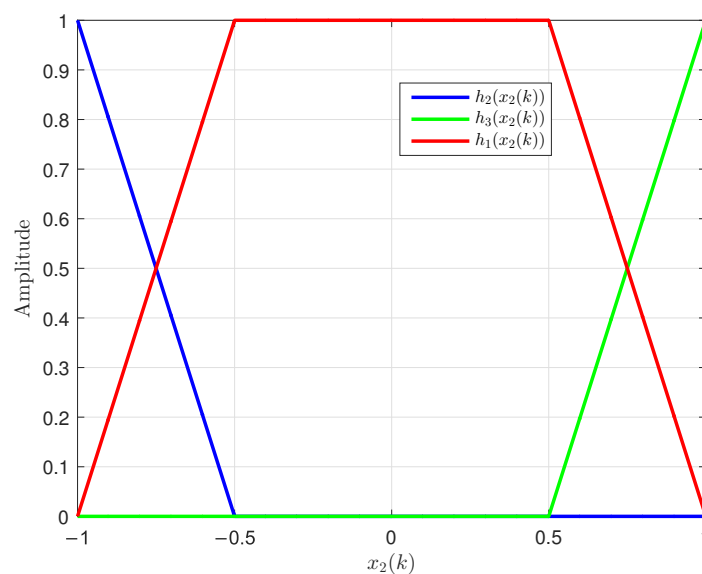


Figure 1. The membership functions of the affine fuzzy system.

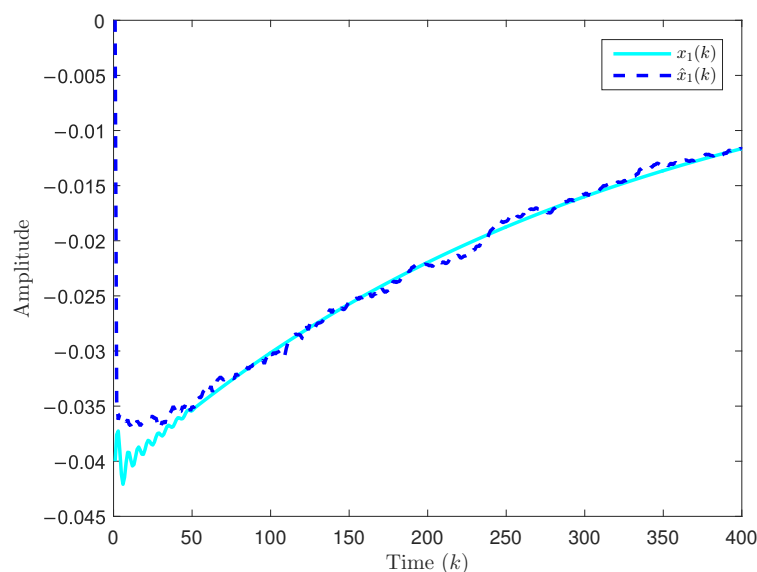


Figure 2. The original system state $x_1(k)$ and its estimate $\hat{x}_1(k)$.

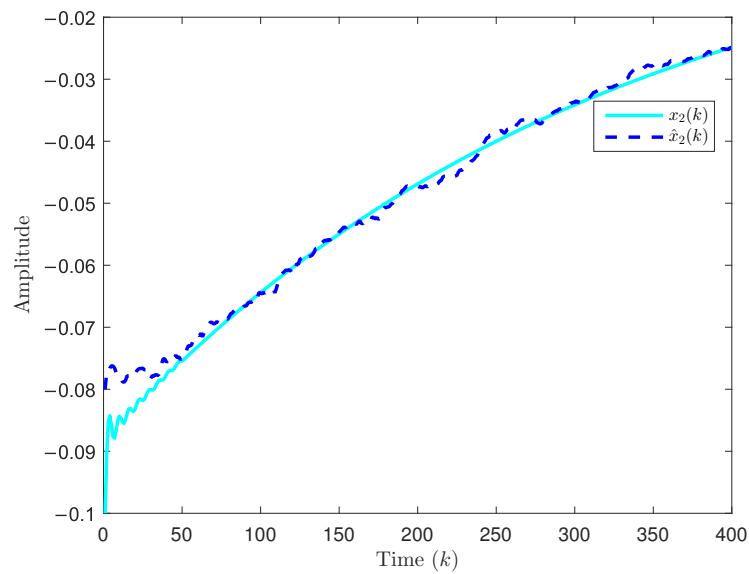


Figure 3. The original system state $x_2(k)$ and its estimate $\hat{x}_2(k)$.

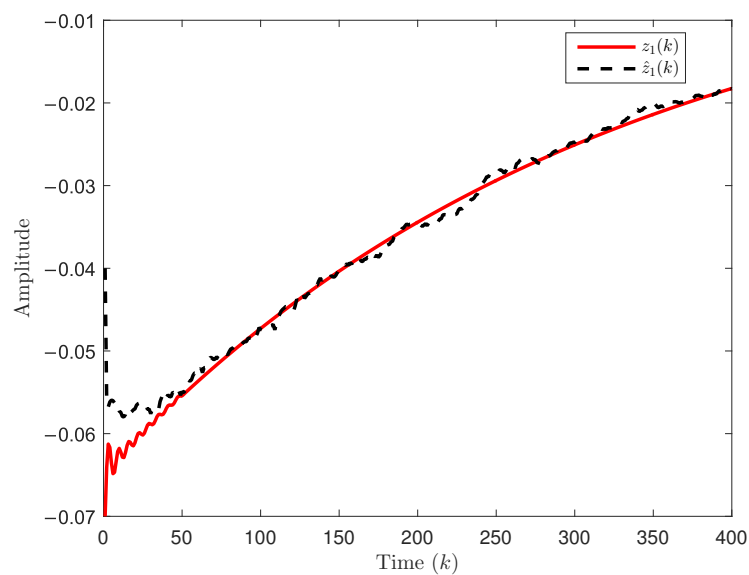


Figure 4. The signal $z(k)$ and its estimate $\hat{z}(k)$.

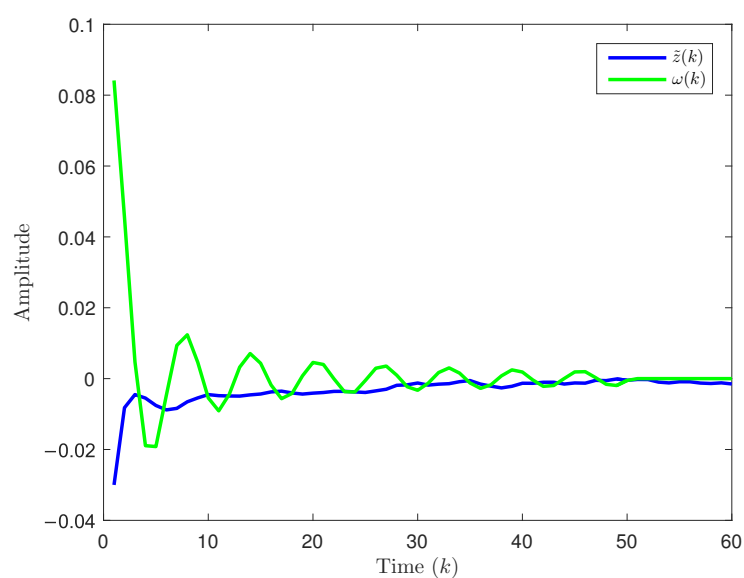


Figure 5. The estimation error $\tilde{z}(k)$ and the external noise $\omega(k)$.

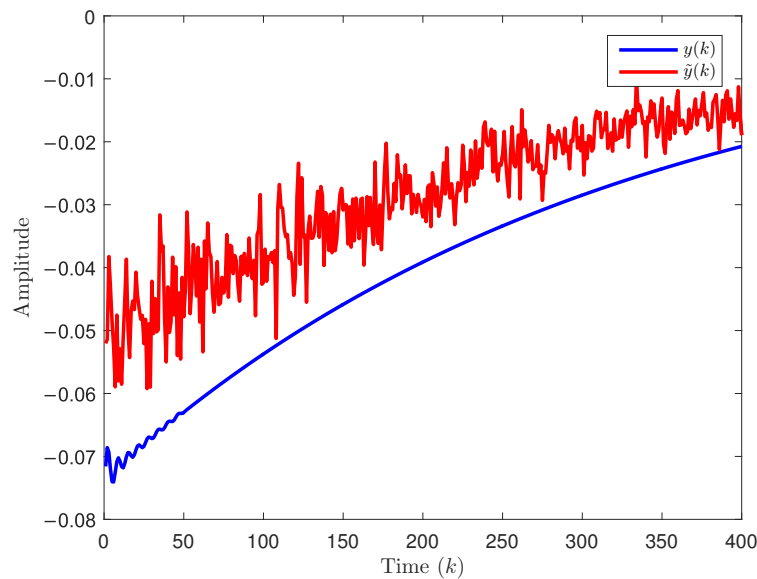


Figure 6. The measurement output $y(k)$ and the faded output $\tilde{y}(k)$.

5. Conclusions

In this paper, the PI state estimation problem has been investigated for nonlinear systems represented by affine T-S fuzzy models. Signal transmissions from sensors to the estimator have been carried out over a wireless network, where the channel fading phenomenon has been taken into account. Based on the structure of the membership functions, the state space has been divided into several regions, and a novel region-based fuzzy PI estimator has been proposed accordingly. By employing a piecewise Lyapunov function, sufficient conditions have been derived to ensure the desired estimation performance, and the estimator gains have been obtained by solving a set of linear matrix inequalities. Finally, an illustrative example has been provided to demonstrate the effectiveness of the proposed estimation scheme. Future topics include extending the proposed fuzzy estimation approach to systems with periodic disturbances, incorporating complex communication protocols (e.g., gossip protocols), and leveraging artificial-intelligence-based techniques such as deep learning to enhance estimation performance in uncertain environments.

Author Contributions

X.C.: software, investigation; Y.W.: writing—original draft preparation, conceptualization, supervision; F.W.: visualization, investigation, writing—reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

Not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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