

Galactic Dark-Matter Halo Profiles

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Abstract: Spherically averaged density profiles are indispensable compressed descriptions of galactic dark-matter halos, but profiles with superficially similar rotation curves need not have the same physical status. The Navarro–Frenk–White (NFW), Moore, and Einasto forms summarize collisionless cosmological simulations; Burkert and pseudo-isothermal laws are economical cored fits to galaxy kinematics; the generalized NFW/Zhao and Dehnen families interpolate between asymptotic slopes; and Hernquist and Jaffe models were devised primarily for stellar systems but provide useful finite-mass analytic potentials. Perfect-fluid dark matter (PFDM), scalar-field halos, and wave-dark-matter solitons instead introduce assumptions about the stress-energy or microscopic dynamics, and therefore cannot be interpreted as merely another choice of fitting function. Here we derive the principal density, mass, circular-speed, and slope relations; compare central and outer asymptotics; explain which fitted parameters carry physical information; and discuss the observational degeneracies that limit cusp–core inference. In a short final part, we address central black holes and emphasize that a galactic density law plus a Newtonian rotation curve does not uniquely determine a relativistic spacetime: the pressure components, equation of state, near-horizon response, and boundary conditions must also satisfy the Einstein equations. Our purpose is astrophysical model selection rather than an exhaustive review of black-hole phenomenology.

Keywords: dark matter; galactic halos; galaxy kinematics; halo density profiles; black-hole environments

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1. Introduction

Dark matter is inferred rather than directly imaged, and nowhere is that inference more spatially resolved than in galaxies. The discrepancy between luminous matter and gravitationally inferred mass became compelling through the extended rotation curves of spiral galaxies and is now supported by the dynamics of pressure-supported systems, gravitational lensing, the temperature and distribution of hot gas, galaxy groups and clusters, the cosmic microwave background, and the growth of large-scale structure [1–4]. Galaxies are therefore not merely one item in the evidence for dark matter. They are laboratories in which the dark component can be mapped over many dynamical times and, in favorable cases, from sub-kiloparsec scales to a substantial fraction of the virial radius.

The radial density profile is the central language of this mapping. A function $\rho_h(x)$, or more commonly its spherical average $\rho(r)$, connects an assumed mass distribution to rotation curves, stellar velocity dispersions, lensing deflections, satellite motions, tidal streams, dynamical friction, and indirect-detection signals. It also provides the bridge between observations and simulations: cosmological calculations predict ensembles of halo profiles, while galaxy data constrain the gravitational acceleration generated by the combined dark and baryonic mass. The profile consequently enters problems ranging from dwarf-galaxy dynamics and disk stability to satellite disruption, strong lensing, and the environment of central black holes.

The quantity $\rho_h(x)$ denotes the dark-halo mass density, x is a three-dimensional spatial position, and $r = |x|$ is its spherical radial coordinate.



This compression of a galaxy into a few profile parameters is extraordinarily useful, but it conceals important distinctions. A fitted density law is not a direct image of the underlying phase-space distribution, and it is not by itself a measurement of dark-matter microphysics. Real halos are triaxial, clumpy, time dependent, and correlated with disks, bulges, gas, satellites, and large-scale environment. Even under spherical symmetry, the observations usually constrain an integrated quantity such as $M(r)$, the projected surface density, or a luminosity-weighted velocity moment rather than $\rho(r)$ point by point. Recovering the density requires assumptions about stellar mass-to-light ratios, velocity anisotropy, inclination, pressure support, geometry, equilibrium, and the radial continuation of the halo beyond the data.

The same warning applies to the asymptotic language of cusps and cores. The central exponent in $\rho \propto r^{-\gamma}$ is formally defined as $r \rightarrow 0$, whereas every observation and simulation has a finite resolution scale. A profile may have finite $\rho(0)$ but no resolved constant-density plateau, as can occur for an Einasto law with gradual curvature. Conversely, a galaxy can show a central density deficit relative to a collisionless NFW baseline without being described exactly by a uniform-density core. The astrophysically meaningful questions are therefore local and comparative: what logarithmic slope is resolved, how rapidly does that slope vary, where is the transition radius, how much mass is enclosed over the measured interval, and how strongly do those conclusions depend on the baryonic model?

These questions lie at the center of the cusp–core problem and of the observed diversity of inner galaxy rotation curves [5–7]. Collisionless hierarchical assembly produces NFW- or Einasto-like structures in dark-matter-only simulations [8–10]. Baryonic condensation can steepen an inner halo, whereas repeated fluctuations driven by gas inflow and feedback can transfer energy to dark matter and produce a shallower distribution [11–13]. Self-interacting dark matter can generate a thermally relaxed central region [14, 15], and ultralight scalar fields can support a solitonic core through wave gradients [16, 17]. A core-like fit is thus a phenomenological result with several possible physical explanations; a successful interpretation must reproduce the profile together with its scaling relations, scatter, time evolution, and other observables.

The mathematical forms we review also have different intellectual origins. NFW, Moore, and Einasto are principally summaries of collisionless numerical halos. Burkert and pseudo-isothermal profiles are economical descriptions of slowly rising rotation curves. The generalized NFW/Zhao family is a flexible interpolation between independently chosen inner and outer slopes. Dehnen, Hernquist, and Jaffe models were constructed mainly as analytically tractable models of spherical stellar systems, although they are often repurposed as dark-halo components or finite-mass surrogates. Perfect-fluid dark matter and scalar-field dark matter make a stronger claim: they specify, at least in principle, properties of the stress-energy tensor or field dynamics and therefore cannot be assessed only by comparing the shapes of their density curves.

We use the distinction between a *profile*, a *dynamical model*, and a *matter model* as a central organizing principle. Two density laws can be nearly indistinguishable within an observed radial window yet imply very different total masses, outer circular speeds, phase-space distributions, or annihilation rates. Conversely, different microscopic mechanisms can produce similar core-like profiles. The physical meaning of a fit must therefore be judged through its provenance, parameter relations, asymptotic behavior, and domain of validity, not solely through a minimum χ^2 .

Central black holes sharpen the same conceptual issue. A kiloparsec-scale density law cannot simply be extrapolated to an event horizon, and a Newtonian rotation curve does not determine a relativistic metric. A consistent black-hole embedding additionally requires the radial and tangential stresses or distribution function, near-horizon behavior, boundary conditions, and the full Einstein equations. Exact anisotropic-halo constructions and relativistic spike calculations provide controlled approaches [18–20], whereas metrics obtained by imposing an extra relation between metric functions may fail to reproduce the density that motivated them [21]. We include black holes to clarify this boundary of halo modeling, but keep them secondary to the galactic astrophysics.

Accordingly, here we separate the commonly used profiles into four categories:

- (1) simulation-motivated summaries: NFW, Moore, and Einasto;
- (2) flexible or analytic dynamical families: generalized NFW/Zhao, Dehnen, Hernquist, and Jaffe;
- (3) cored phenomenological fits: Burkert and pseudo-isothermal;
- (4) matter models: PFDM constructions and scalar-field/solitonic dark matter.

These categories overlap in mathematical form but not in interpretation. We first establish common diagnostics—enclosed mass, circular speed, logarithmic slope, potential, and virial convention—and then derive the principal properties of each profile. We next examine what the fitted parameters mean astrophysically, how baryons and alternative dark-matter dynamics alter the inner halo, and how model selection is limited by the observed radial

window. In the final sections, we summarize representative black-hole embeddings and formulate consistency criteria for using a galactic halo in relativistic calculations.

An illustrative comparison of five frequently used halo laws is depicted in Figure 1.

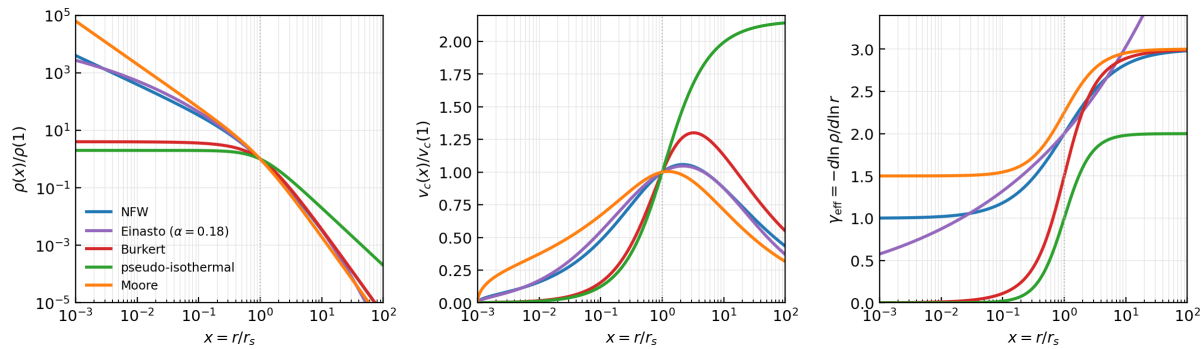


Figure 1. Dimensionless comparison of five frequently used halo laws. Each density and circular-speed curve is normalized at $x = 1$. The horizontal coordinate is labeled $x = r/r_s$ for compactness, but the named scale radii are not physically identical: the plotted Einasto scale is r_{-2} , while other conventions follow their defining equations. The right panel shows how a profile's local slope, not only its formal $r \rightarrow 0$ limit, changes across the observable region.

2. What a Spherical Halo Profile Determines

2.1. Mass, Acceleration, Circular Speed, and Potential

Throughout the Newtonian halo discussion, r is the spherical radius, $\rho(r)$ is the three-dimensional mass density, $M(r)$ is the mass enclosed within r , and G is Newton's gravitational constant. In the equations below, s is a dummy radial integration variable, $g(r)$ is the magnitude of the radial gravitational acceleration, $v_c(r)$ is the circular speed, and $\Phi(r)$ is the Newtonian gravitational potential.

For a spherical density distribution,

$$M(r) = 4\pi \int_0^r \rho(s) s^2 ds, \quad (1)$$

$$g(r) = \frac{GM(r)}{r^2}, \quad v_c^2(r) = \frac{GM(r)}{r}, \quad (2)$$

$$\frac{d\Phi}{dr} = \frac{GM(r)}{r^2}. \quad (3)$$

For a disk galaxy the actual mass decomposition is closer to

$$v_{\text{obs}}^2(R) = \Upsilon_{*,d} v_{*,d}^2 + \Upsilon_{*,b} v_{*,b}^2 + v_{\text{gas}}^2 + v_h^2, \quad (4)$$

with stellar mass-to-light ratios Υ_* . The terms in Equation (4) are covariant in a fit: increasing the stellar contribution can reduce the inferred central halo density. Non-circular motions, pressure support, beam smearing, inclination, distance, and gas geometry add further correlations.

Here R is the cylindrical radius in the disk plane, v_{obs} is the measured rotation speed, and the subscripts d, b, gas, and h denote the stellar disk, stellar bulge, gas, and dark halo. The functions $v_{*,d}$ and $v_{*,b}$ are conventionally computed for unit stellar mass-to-light ratio, so multiplication by $\Upsilon_{*,d}$ or $\Upsilon_{*,b}$ supplies the physical normalization.

Lensing and projected dynamics depend on the surface density

$$\Sigma(R) = 2 \int_R^\infty \frac{\rho(r) r dr}{\sqrt{r^2 - R^2}}, \quad (5)$$

whereas annihilation signals involve $\int \rho^2 d\ell$ and decay signals involve $\int \rho d\ell$. No single preferred profile is therefore independent of the observable and radial domain.

In this projection, $\Sigma(R)$ is the mass per unit projected area at projected radius R , and ℓ denotes distance along the line of sight.

2.2. Slopes and Asymptotic Tests

A useful local diagnostic is the positive logarithmic slope

$$\gamma_{\text{eff}}(r) \equiv -\frac{d \ln \rho}{d \ln r}. \quad (6)$$

If $\rho \sim r^{-\gamma}$ near the center, then

$$M(r) \sim r^{3-\gamma}, \quad v_c(r) \sim r^{1-\gamma/2} \quad (\gamma < 3). \quad (7)$$

A constant-density core has $\gamma = 0$ and solid-body rise $v_c \propto r$; an NFW cusp has $\gamma = 1$ and $v_c \propto r^{1/2}$; a Moore cusp has $\gamma = 3/2$ and $v_c \propto r^{1/4}$. A $\gamma = 2$ cusp, such as the inner Jaffe law, has an asymptotically constant central circular speed. The central Newtonian potential is finite for $\gamma < 2$, logarithmically divergent at $\gamma = 2$, and more strongly divergent for $\gamma > 2$. The unresolved annihilation luminosity $\int_0 \rho^2 r^2 dr$ converges only for $\gamma < 3/2$.

If $\rho \sim r^{-\beta}$ at large radius, the total mass is finite only for $\beta > 3$. It diverges logarithmically for $\beta = 3$ and as $r^{3-\beta}$ for $\beta < 3$. Such divergences are not automatically fatal: NFW and Burkert laws are normally defined only inside a virial radius or a tidal/truncation radius. They do mean that a formal profile cannot be extended to infinity without additional physics.

2.3. Virial Quantities and Scale Conventions

Cosmological halo parameters are commonly defined by

$$M_\Delta = \frac{4\pi}{3} \Delta \rho_{\text{ref}}(z) r_\Delta^3, \quad c_\Delta = \frac{r_\Delta}{r_{-2}}, \quad (8)$$

where ρ_{ref} may be the critical or mean matter density. Authors using M_{200} can therefore mean $200\rho_{\text{crit}}$ or $200\rho_m$; the convention must be stated. The radius r_{-2} is where $\gamma_{\text{eff}} = 2$. It equals the NFW scale radius and is built into the standard Einasto parametrization, but it does not equal every quantity called r_s . Many apparent normalization discrepancies are only changes of scale-radius convention.

Here z is redshift, Δ is the chosen dimensionless overdensity threshold, r_Δ is the radius within which the mean density is $\Delta\rho_{\text{ref}}(z)$, $M_\Delta = M(r_\Delta)$, and c_Δ is the corresponding concentration. The symbols ρ_{crit} and ρ_m denote the critical density and the cosmic mean matter density, respectively.

For NFW, defining

$$f(c) = \ln(1+c) - \frac{c}{1+c} \quad (9)$$

gives

$$\rho_s = \frac{\Delta\rho_{\text{ref}}}{3} \frac{c_\Delta^3}{f(c_\Delta)}. \quad (10)$$

The concentration is not a free piece of microphysics: its distribution depends on halo mass, redshift, cosmology, definition, relaxation criterion, and assembly history [22,23].

The argument c of $f(c)$ is a generic dimensionless concentration and is set to c_Δ in the normalization above. It should not be confused with the speed of light, denoted by c only in the later relativistic discussion.

3. Collisionless Simulation Profiles

3.1. Navarro–Frenk–White

The NFW profile emerged from dissipationless cold-dark-matter simulations [8,9]:

$$\rho_{\text{NFW}}(r) = \frac{\rho_s}{x(1+x)^2}, \quad x = \frac{r}{r_s}. \quad (11)$$

The NFW parameters ρ_s and r_s are the characteristic density and scale radius, respectively, while x is the dimensionless radius. Its enclosed mass and circular speed are

$$M_{\text{NFW}}(r) = 4\pi\rho_s r_s^3 \left[\ln(1+x) - \frac{x}{1+x} \right], \quad (12)$$

$$v_{\text{NFW}}^2(r) = 4\pi G \rho_s r_s^2 \frac{\ln(1+x) - x/(1+x)}{x}. \quad (13)$$

The logarithmic slope is

$$\gamma_{\text{eff}} = 1 + \frac{2x}{1+x}, \quad (14)$$

so it changes from 1 to 3 and equals 2 at r_s . The mass diverges logarithmically, while the potential normalized to vanish at infinity,

$$\Phi_{\text{NFW}}(r) = -4\pi G \rho_s r_s^2 \frac{\ln(1+x)}{x}, \quad (15)$$

is finite at the center.

3.1.1. Physical Meaning

NFW is a remarkably useful empirical regularity of hierarchical, approximately collisionless collapse. In a cosmological application the parameter pair (M_Δ, c_Δ) is more interpretable than (ρ_s, r_s) : mass fixes the halo scale, while concentration correlates statistically with assembly history. Universality should not be read literally. Resolved halos exhibit scatter, non-sphericity, substructure, transient departures during mergers, and systematic changes with accretion history. Baryons can contract or expand the central distribution. Moreover, numerical claims inside the convergence radius require relaxation and force-resolution tests [24].

3.1.2. Strengths and Limitations

NFW has only two shape/scale parameters, analytic mass and potential, and a direct connection to collisionless cosmological simulations. It is consequently a strong baseline and a useful prior. Its fixed inner cusp can be too restrictive for slowly rising dwarf and low-surface-brightness rotation curves, and its fixed curvature can fit resolved simulated halos less accurately than Einasto. A formally excellent NFW fit does not demonstrate collisionless cold-dark-matter microphysics, because baryonic and alternative-dark-matter halos can look NFW-like across a limited radial interval.

3.2. Moore Profile

Early high-resolution simulations motivated a steeper central form [25],

$$\rho_M(r) = \frac{\rho_s}{x^{3/2} (1+x^{3/2})}. \quad (16)$$

In Equation (16), $x = r/r_s$, while ρ_s and r_s are Moore-profile normalization parameters; they need not equal the NFW parameters carrying the same conventional symbols. With this common convention,

$$M_M(r) = \frac{8\pi}{3} \rho_s r_s^3 \ln(1+x^{3/2}), \quad (17)$$

$$\gamma_{\text{eff}}(r) = \frac{3}{2} + \frac{3}{2} \frac{x^{3/2}}{1+x^{3/2}}. \quad (18)$$

The profile has an $r^{-3/2}$ cusp, an r^{-3} exterior, and logarithmically divergent total mass. Alternative Moore-like normalizations occur; one must quote the exact equation.

Physical Meaning

The Moore law records an important stage in the numerical cusp debate: increased resolution initially appeared to reveal a cusp steeper than NFW. Later simulations showed continuously varying slopes better described by Einasto-like curvature rather than a universal $-3/2$ asymptote [10, 26]. The Moore profile remains useful as a deliberately cuspy benchmark, particularly when testing sensitivity of annihilation or strong-field extrapolations. Because $\gamma = 3/2$, its formal central annihilation luminosity is logarithmically divergent and requires a physical cutoff, phase-space limit, annihilation plateau, or resolved inner model.

3.3. Einasto Profile

The Einasto form, introduced originally as a model for stellar systems [27], has a logarithmic slope that varies as a power of radius:

$$\rho_{\text{Ein}}(r) = \rho_{-2} \exp \left[-\frac{2}{\alpha} \left\{ \left(\frac{r}{r_{-2}} \right)^\alpha - 1 \right\} \right], \quad (19)$$

where $\rho_{-2} = \rho(r_{-2})$ and

$$\gamma_{\text{eff}}(r) = 2 \left(\frac{r}{r_{-2}} \right)^\alpha. \quad (20)$$

The dimensionless parameter $\alpha > 0$ controls the curvature of the Einasto profile, and r_{-2} and ρ_{-2} are its slope—2 radius and the density at that radius. The symbol e denotes the base of the natural logarithm. Some literature uses the index $n = 1/\alpha$. The enclosed mass is

$$M_{\text{Ein}}(r) = \frac{4\pi\rho_{-2}r_{-2}^3e^{2/\alpha}}{\alpha} \left(\frac{\alpha}{2} \right)^{3/\alpha} \gamma\left(\frac{3}{\alpha}, \frac{2}{\alpha}x^\alpha\right), \quad (21)$$

where $x = r/r_{-2}$. The function $\gamma(q, y)$ is the lower incomplete gamma function [28]. Replacing it by $\Gamma(3/\alpha)$ gives the finite total mass.

Explicitly, $\gamma(q, y) = \int_0^y t^{q-1}e^{-t}dt$, where q and y are the function arguments and t is a dummy integration variable; $\Gamma(q) = \gamma(q, \infty)$ is the complete gamma function. These dummy arguments are unrelated to the halo cusp slope γ and the cosmological redshift z .

Einasto has finite central density, $\rho(0) = \rho_{-2}e^{2/\alpha}$, and no nonzero asymptotic inner power-law slope. It is therefore mathematically cored in the strict $r \rightarrow 0$ sense, but for the small α values typical of simulated halos the slope changes so slowly that there need not be an observable constant-density plateau at resolved radii. This is why classifying Einasto simply as a cored halo is often misleading.

Physical Meaning

Einasto is a high-accuracy description of the gradual curvature seen in simulated collisionless halos [10,29,30]. The third parameter α is a shape parameter, not a core radius: smaller α gives more gradual variation of logarithmic slope. Simulations find systematic, though definition-dependent, trends of α with peak height and assembly state. The additional flexibility can be physically informative in high-quality data, but it also increases covariance among mass, concentration, and shape when the radial lever arm is short.

4. Double-Power-Law and Finite-Mass Analytic Families

4.1. Generalized NFW/Zhao (α, β, γ) Profile

A broad double-power-law family is

$$\rho_{\alpha\beta\gamma}(r) = \frac{\rho_s}{x^\gamma (1+x^\alpha)^{(\beta-\gamma)/\alpha}}, \quad x = \frac{r}{r_s}. \quad (22)$$

Here γ is the inner slope, β the outer slope, and $\alpha > 0$ controls the sharpness of the transition. This parameterization is widely associated with the Zhao family [31]; conventions that exchange symbols or introduce an extra scale normalization are common. The local slope is

$$\gamma_{\text{eff}}(r) = \gamma + (\beta - \gamma) \frac{x^\alpha}{1 + x^\alpha}, \quad (23)$$

and, when the expression is meaningful,

$$r_{-2} = r_s \left(\frac{2-\gamma}{\beta-2} \right)^{1/\alpha}. \quad (24)$$

The mass can be written

$$M(r) = \frac{4\pi\rho_s r_s^3 x^{3-\gamma}}{3-\gamma} {}_2F_1\left(\frac{3-\gamma}{\alpha}, \frac{\beta-\gamma}{\alpha}; 1 + \frac{3-\gamma}{\alpha}; -x^\alpha\right), \quad (25)$$

for $\gamma < 3$, which ensures convergence of the mass integral at the origin. For $\gamma \geq 3$, the enclosed mass diverges unless an inner cutoff or a modified central profile is specified.

The symbol ${}_2F_1$ in Equation (25) denotes the Gauss hypergeometric function. The expression for r_{-2} in Equation (24) assumes the usual case $\gamma < 2 < \beta$, so that a positive finite slope—2 radius exists.

The commonly used gNFW is the restricted case $\alpha = 1$, $\beta = 3$, leaving the central slope γ free. NFW is (1, 3, 1). The Moore convention of Equation (16) is (3/2, 3, 3/2). Hernquist is (1, 4, 1), and Jaffe is (1, 4, 2), after choosing the corresponding normalization.

Physical Meaning

The principal value of gNFW is controlled flexibility. It asks whether the data prefer a core-like or cuspy inner behavior without abandoning an NFW-like exterior. But γ inferred from data that do not resolve $r \ll r_s$ is an extrapolated asymptote. Strong covariance between γ , r_s , ρ_s , and the stellar mass-to-light ratio is unavoidable. Broad priors on all five Zhao parameters can make the fit weakly identifiable; physically motivated restrictions or hierarchical population inference are usually preferable.

4.2. Dehnen γ Models

The finite-mass Dehnen family is

$$\rho_D(r) = \frac{(3-\gamma)M}{4\pi} \frac{a}{r^\gamma(r+a)^{4-\gamma}}, \quad 0 \leq \gamma < 3, \quad (26)$$

For the Dehnen family, M is the finite total mass, $a > 0$ is the scale length, and γ is the dimensionless inner logarithmic slope. The enclosed mass is

$$M(r) = M \left(\frac{r}{r+a} \right)^{3-\gamma}. \quad (27)$$

All members have outer slope -4 , finite total mass M , and inner slope $-\gamma$. Their potential, with $\Phi(\infty) = 0$, is

$$\Phi(r) = -\frac{GM}{a(2-\gamma)} \left[1 - \left(\frac{r}{r+a} \right)^{2-\gamma} \right], \quad \gamma \neq 2, \quad (28)$$

while $\gamma = 2$ has a logarithmic limit. Related models were developed independently as the η -models [32,33].

The equivalent η -model index is $\eta = 3 - \gamma$.

Physical Meaning

These are analytic spherical distribution models, initially aimed at galaxies and bulges rather than at a universal cosmological dark halo. Their finite mass, simple force law, and tractable distribution functions make them especially valuable for orbit libraries, N -body initial conditions, and controlled black-hole embedding. A Dehnen fit is therefore often a choice of dynamical basis rather than a claim about halo formation.

The conventional term N -body denotes a dynamical calculation with N mutually gravitating simulation particles.

4.3. Hernquist Profile

The $\gamma = 1$ Dehnen member is the Hernquist model [34]:

$$\rho_H(r) = \frac{M}{2\pi} \frac{a}{r(r+a)^3}, \quad (29)$$

$$M_H(r) = M \frac{r^2}{(r+a)^2}, \quad (30)$$

$$\Phi_H(r) = -\frac{GM}{r+a}, \quad (31)$$

$$v_H^2(r) = \frac{GMr}{(r+a)^2}. \quad (32)$$

It has inner slope -1 , outer slope -4 , and $r_{-2} = a/2$. Its projected light profile approximates the de Vaucouleurs $R^{1/4}$ law, and its analytic potential and isotropic distribution function are major practical advantages.

In the projected $R^{1/4}$ law, R denotes distance on the plane of the sky rather than the three-dimensional radius r .

Hernquist and NFW share an r^{-1} cusp but are not interchangeable: Hernquist has finite total mass and a steeper exterior. In galaxy models it is most naturally a bulge or elliptical-galaxy component, although a suitably scaled Hernquist sphere is also used as a finite surrogate for an NFW halo over a chosen radial range.

4.4. Jaffe Profile

The $\gamma = 2$ Dehnen member is the Jaffe model [35]:

$$\rho_J(r) = \frac{M}{4\pi} \frac{a}{r^2(r+a)^2}, \quad (33)$$

$$M_J(r) = M \frac{r}{r+a}, \quad (34)$$

$$\Phi_J(r) = \frac{GM}{a} \ln \left(\frac{r}{r+a} \right), \quad (35)$$

$$v_J^2(r) = \frac{GM}{r+a}. \quad (36)$$

It has an r^{-2} central cusp, an r^{-4} exterior, and finite mass. The central potential diverges logarithmically and the circular speed approaches a constant. Like Hernquist, it was designed to describe the stellar luminosity and dynamics of spherical galaxies. Its steep cusp is generally not a default description of the dark halo in a late-type galaxy, but it is useful for analytic work and for compact spheroidal components.

5. Cored Phenomenological Profiles

5.1. Pseudo-Isothermal Profile

The pseudo-isothermal sphere is

$$\rho_{\text{ISO}}(r) = \frac{\rho_0}{1+x^2}, \quad x = \frac{r}{r_c}. \quad (37)$$

Here $\rho_0 = \rho_{\text{ISO}}(0)$ is the central density, r_c is the core radius at which $\rho_{\text{ISO}} = \rho_0/2$, and x is dimensionless. Its mass and circular speed are

$$M_{\text{ISO}}(r) = 4\pi\rho_0 r_c^3 (x - \arctan x), \quad (38)$$

$$v_{\text{ISO}}^2(r) = 4\pi G\rho_0 r_c^2 \left(1 - \frac{\arctan x}{x} \right). \quad (39)$$

The profile has a constant-density center and an r^{-2} exterior, so $M(r) \propto r$ and $v_c^2 \rightarrow 4\pi G\rho_0 r_c^2$. This asymptotically flat rotation curve is the reason for its enduring utility. The price is linearly divergent total mass and a logarithmically divergent potential unless the halo is truncated.

Physical Meaning

The adjective isothermal refers to the close relation between an r^{-2} envelope and a roughly constant velocity-dispersion scale. The pseudo-isothermal law is a convenient nonsingular analytic approximation, not the unique self-consistent solution of an isothermal distribution function. Its fitted combination $4\pi G\rho_0 r_c^2$ sets the outer speed, so a rotation curve that does not reach the flat regime constrains ρ_0 and r_c strongly along a degeneracy direction [36].

5.2. Burkert Profile

The Burkert law was introduced to fit dwarf-galaxy halos [37]:

$$\rho_B(r) = \frac{\rho_0}{(1+x)(1+x^2)}, \quad x = \frac{r}{r_0}. \quad (40)$$

For the Burkert law, $\rho_0 = \rho_B(0)$ is the central density and r_0 is the scale radius at which $\rho_B(r_0) = \rho_0/4$. Its mass is

$$M_B(r) = 2\pi\rho_0 r_0^3 \left[\ln(1+x) + \frac{1}{2} \ln(1+x^2) - \arctan x \right]. \quad (41)$$

The Burkert profile has a constant-density center and an r^{-3} exterior. Its mass therefore diverges only logarithmically, as for NFW. Unlike the pseudo-isothermal law, its circular speed ultimately declines, although the transition can lie beyond the measured disk.

Physical Meaning

Burkert is an empirical core-plus-envelope law. The product $\mu_{0D} = \rho_0 r_0$ is a characteristic central surface-density scale, and approximate regularities in this product have been reported across galaxy samples [38,39]. The strength and universality of such relations depend on sample selection, baryonic modeling, distance, and profile choice; they should not be imposed as exact identities. A Burkert fit establishes that a smooth core-like mass model

is effective over the observed range, not whether the core arose from feedback, self-interactions, wave pressure, or primordial phase-space structure.

5.3. Burkert versus Pseudo-Isothermal

Both laws have two parameters and finite $\rho(0)$, but their outer physics differs:

$$\rho_{\text{ISO}} \sim r^{-2}, \quad M_{\text{ISO}} \sim r, \quad v_{\text{ISO}} \rightarrow \text{constant}, \quad (42)$$

whereas

$$\rho_{\text{B}} \sim r^{-3}, \quad M_{\text{B}} \sim \ln r, \quad v_{\text{B}} \sim \sqrt{\frac{\ln r}{r}}. \quad (43)$$

If data span only the inner few core radii the two can be nearly degenerate. Extrapolated virial masses may nevertheless differ greatly. It is unsafe to infer a cosmological halo mass from either profile without an outer-halo prior or additional tracers.

6. Compact Comparison

Table 1 presents a comparative characterization of a number of halo profiles considered in this work, including the description of their density shape, slopes, total mass and primary role.

Table 1. Asymptotic and interpretive summary. Slopes are positive values $\gamma_{\text{eff}} = -d \ln \rho / d \ln r$.

Profile	Density Shape	Inner/Outer Slope	Total Mass	Primary Role
NFW	$\rho_s [x(1+x)^2]^{-1}$	1/3	Log divergent	Two-parameter collisionless-simulation baseline.
Einasto	$\rho_{-2} \exp[-(2/\alpha)(x^\alpha - 1)]$	0/ ∞ formally	Finite	Curved, accurate simulation fit; α controls shape.
Moore	$\rho_s [x^{3/2}(1+x^{3/2})]^{-1}$	3/2 / 3	Log divergent	Historical steep-cusp benchmark.
Burkert	$\rho_0 [(1+x)(1+x^2)]^{-1}$	0/3	Log divergent	Empirical core with NFW-like outer slope.
Pseudo-isothermal	$\rho_0 (1+x^2)^{-1}$	0/2	Linear divergent	Empirical core and asymptotically flat rotation curve.
gNFW/Zhao	$\rho_s x^{-\gamma} (1+x^\alpha)^{-(\beta-\gamma)/\alpha}$	γ/β	Finite if $\beta > 3$ and $\gamma < 3$	Flexible asymptotic family; can be weakly identifiable.
Dehnen	$\frac{(3-\gamma)M}{4\pi} \frac{a}{r^\gamma (r+a)^{4-\gamma}}$	$\gamma/4$	Finite	Analytic spherical dynamical family.
Hernquist	$\frac{M}{2\pi} \frac{a}{r(r+a)^3}$	1/4	Finite	Stellar spheroid model; finite NFW-like-cusp surrogate.
Jaffe	$\frac{M}{4\pi} \frac{a}{r^2(r+a)^2}$	2/4	Finite	Compact stellar-system model with logarithmic central potential.
Wave-DM soliton	$\rho_c [1 + 0.091(r/r_c)^2]^{-8}$	0/16 for isolated fit	Finite	Ground-state core; requires an outer halo in galaxies.

7. Profiles Derived from Matter Models

7.1. Why a Matter Model Is a Stronger Statement

A density fit specifies one scalar function. A microscopic or relativistic matter model specifies additional dynamics: a distribution function, equation of state, field equation, sound speed, anisotropic stress, stability conditions, and cosmological initial conditions. Consequently, fitting the same $\rho(r)$ does not make two matter models equivalent. The distinction is especially important for PFDM and scalar-field dark matter, labels that cover several inequivalent constructions.

7.2. Perfect-Fluid Dark Matter

An isotropic perfect fluid has

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}. \quad (44)$$

Here $T^{\mu\nu}$ is the stress-energy tensor, ρ is the fluid energy density, p is its isotropic pressure, u^μ is the four-velocity normalized by $u^\mu u_\mu = -1$, $u_\mu = g_{\mu\nu}u^\nu$, and $g^{\mu\nu}$ is the inverse spacetime metric. Greek indices μ, ν run over the four spacetime coordinates, and relativistic units with $c = 1$ are understood in this definition. This stress tensor is not itself a radial density profile; an equation of state, equilibrium equation, and boundary conditions are required. In the weak-field spherical limit, a perfectly flat circular-speed curve v_0 implies

$$M(r) = \frac{v_0^2}{G}r, \quad \rho(r) = \frac{v_0^2}{4\pi G r^2}, \quad \Phi(r) = v_0^2 \ln(r/r_\star) + \text{constant}. \quad (45)$$

The constant v_0 is the asymptotic circular speed and $r_\star$ is an arbitrary reference radius that fixes the additive zero of the logarithmic potential. The logarithmic potential and r^{-2} density are therefore generic consequences of an idealized flat rotation curve, not unique evidence for a perfect-fluid particle model.

Relativistic PFDM halo constructions have been built by combining an approximately constant tangential velocity with the Einstein equations and a fluid stress tensor [40,41]. Such models can approximate a finite observed radial interval, but a global r^{-2} envelope has infinite mass and is not asymptotically flat. It must be cut off or matched to an exterior or cosmological solution. The pressure, energy conditions, and stability should be checked rather than inferred from the name perfect fluid. Recent direct rotation-curve tests illustrate that the viability of a particular PFDM parameterization is an empirical question, not a consequence of its relativistic notation [42].

A separate literature calls the logarithmic black-hole metric term

$$f(r) = 1 - \frac{2M_\bullet}{r} + \frac{\alpha_{\text{PFDM}}}{r} \ln\left(\frac{r}{|\alpha_{\text{PFDM}}|}\right) \quad (46)$$

a PFDM background [43]. In geometrized units, the stress tensor reconstructed from this metric is convention dependent in sign, scales as r^{-3} , and is generally anisotropic; it should not be conflated with the isotropic galactic fluid of Equation (44) or with the r^{-2} flat-curve density in Equation (45). Thus PFDM profile has no unique definition. Any application must state the density, pressure components, metric signature, sign convention, and radial domain explicitly.

In Equation (46), $f(r)$ is the Schwarzschild-gauge metric function defined by $ds^2 = -f(r)dt^2 + dr^2/f(r) + r^2 d\Omega^2$, $M_\bullet$ is the black-hole mass parameter, and α_{PFDM} is a signed PFDM length scale. The formula uses geometrized units $G = c = 1$, so masses and lengths have the same dimension.

7.3. Scalar-Field and Solitonic Dark Matter

A coherently occupied ultralight scalar can have kiloparsec-scale de Broglie wavelength. In the nonrelativistic limit a minimally coupled field is described by the Schrödinger–Poisson system,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_\psi} \nabla^2 \psi + m_\psi \Phi \psi, \quad (47)$$

$$\nabla^2 \Phi = 4\pi G m_\psi |\psi|^2, \quad (48)$$

up to convention-dependent normalization of ψ [17,44].

In these equations, $i^2 = -1$, $\hbar$ is the reduced Planck constant, t is time, $\psi(\mathbf{x}, t)$ is the nonrelativistic scalar wavefunction, m_ψ is the particle mass, ∇^2 is the spatial Laplacian, and Φ is the Newtonian potential. With the normalization used here, the mass density is $\rho = m_\psi |\psi|^2$.

The gradient term supplies wave, or quantum, support. The ground-state core is usually called a soliton. A widely used fit to simulated solitons is

$$\rho_{\text{sol}}(r) = \rho_c \left[1 + 0.091 \left(\frac{r}{r_c} \right)^2 \right]^{-8}, \quad (49)$$

where the numerical factor makes r_c approximately the half-density radius [16].

The quantity $\rho_c = \rho_{\text{sol}}(0)$ is the soliton's central density, and r_c is its half-density core radius in this fitting convention.

The scaling symmetry of the Schrödinger–Poisson ground state gives

$$\rho_c \simeq 1.9 \frac{M_\odot}{\text{pc}^3} \left(\frac{m_\psi}{10^{-23} \text{ eV}} \right)^{-2} \left(\frac{r_c}{1 \text{ kpc}} \right)^{-4}, \quad (50)$$

for this fit and radius convention. Equivalently, $M_{\text{sol}} \propto m_\psi^{-2} r_c^{-1}$.

Here $M_\odot$ is the solar mass, pc and kpc are the parsec and kiloparsec, eV is the electronvolt, and $M_{\text{sol}} = 4\pi \int_0^\infty \rho_{\text{sol}}(r) r^2 dr$ denotes the integrated soliton mass.

Physical Meaning

Unlike a Burkert core, the soliton has a field-equation origin: its density and radius are not generally independent once the particle mass and state are specified. The isolated fit falls as r^{-16} and must *not* be used as the whole galactic halo. Wave-dark-matter simulations instead show a central soliton embedded in a fluctuating, approximately CDM-like outer envelope [16]. A practical galaxy fit therefore needs a matching radius and an outer component, often NFW- or Einasto-like. The proposed core–halo relation is a statistical result whose normalization depends on halo definition, assembly, redshift, baryons, self-interactions, and numerical method.

Scalar-field dark matter is broader than the soliton ansatz. Other potentials, self-interactions, finite-temperature states, or excited configurations produce different profiles. For example, the relativistic scalar-field halo of [45] assumes an exponential scalar-field potential and is distinct from the static Bose–Einstein-condensate model considered in [46]. In the latter model, the Thomas–Fermi approximation gives

$$\rho_{\text{SF}}(r) = \rho_0 \frac{\sin(kr)}{kr}, \quad 0 \leq r \leq r_{\text{edge}} = \frac{\pi}{k}, \quad (51)$$

with the value at the origin understood by continuity. The condensate is truncated at the first zero, beyond which this expression would become negative. This finite-radius condensate profile and the soliton of Equation (49) should not be identified merely because both are called scalar-field cores. Their potentials, states, boundary conditions, and cosmological predictions differ.

In Equation (51), $\rho_0 = \rho_{\text{SF}}(0)$ is the central density, $k > 0$ is a radial wavenumber, and r_{edge} is the condensate’s outer radius.

8. What the Fitted Parameters Mean Astrophysically

8.1. Cusp, Core, and Curvature

Three logically distinct quantities are often compressed into the word core:

- (1) a finite mathematical limit $\rho(0)$;
- (2) a resolved interval over which $\gamma_{\text{eff}} \simeq 0$;
- (3) a deficit of central density relative to a collisionless-only cosmological baseline.

Einasto satisfies the first condition but may fail the second at observable radii. A feedback-modified NFW halo can satisfy the third without being exactly constant density. Robust comparisons should quote γ_{eff} at a specified resolved radius, or a density contrast across a stated interval, rather than only the formal $r \rightarrow 0$ exponent.

High-resolution H I and H α rotation curves of many dwarf and low-surface-brightness galaxies have favored slowly rising, core-like halo contributions over a fixed NFW cusp [5,6,47]. Large compilations such as SPARC provide an important common basis for testing mass models [7]. Yet diversity is itself a central observation: galaxies of similar outer speed can have markedly different inner rotation curves. A universal two-parameter density law should not be expected to absorb differences in baryonic surface density, feedback history, halo response, bars, and non-circular motions.

8.2. Baryonic Response

Slow baryonic condensation can increase central dark-matter density through adiabatic contraction [11,12]. Repeated rapid potential fluctuations driven by gas inflow and outflow can transfer energy to collisionless dark matter and generate cores [13]. Hydrodynamic simulations consequently predict a response that depends on star-formation efficiency, feedback, assembly history, and radius; flexible mass-dependent profiles are one way to summarize this behavior [48].

This makes a fitted profile a record of the combined formation history of halo and galaxy. NFW is most directly a dark-matter-only baseline. A departure in the inner galaxy may diagnose baryonic response rather than new

particle physics. Conversely, a successful feedback model must reproduce not only mean cores but their diversity, energetics, stellar histories, and behavior across halo mass.

8.3. Alternative Dark-Matter Dynamics

Self-interacting dark matter can thermalize inner phase space and produce approximately isothermal cores, while retaining a collisionless outer halo [14, 15]. Wave dark matter produces solitonic support and interference structure. These mechanisms make predictions beyond a one-dimensional density curve—including halo shapes, substructure, time variability, core scaling, merger behavior, and cosmological suppression. A Burkert-like rotation curve may be compatible with either mechanism but does not by itself select one.

8.4. The Radial Window Is Part of the Result

In a pressure-supported dwarf spheroidal, line-of-sight velocities constrain a convolution of the potential, tracer density, and velocity anisotropy. Enclosed mass near the three-dimensional half-light radius can be comparatively robust, while the inner slope remains much less so [49, 50]. In disks, the stellar mass-to-light ratio and gas pressure correction play analogous roles. At cluster scales, hydrostatic nonthermal pressure and lensing geometry matter. A profile paper should therefore report:

- the innermost and outermost constraining radii;
- the baryonic components and their priors;
- geometry, anisotropy, and pressure-support assumptions;
- whether virial quantities are data driven or prior dominated;
- the full parameter covariance, not only best fits;
- results under at least one plausible competing profile.

9. Model Comparison and Practical Selection

9.1. About the Chi-Square Statistic

The quantity $\chi^2 = \sum_i [(d_i - m_i)/\sigma_i]^2$ is the usual chi-square statistic for data values d_i , model predictions m_i , and adopted uncertainties σ_i , with the sum taken over the fitted measurements.

Interpreting this expression as a Gaussian goodness-of-fit statistic assumes independent Gaussian measurement errors. However, for correlated kinematic or lensing measurements, the corresponding statistic is

$$\chi^2 = (\mathbf{d} - \mathbf{m})^\top \mathbf{C}^{-1} (\mathbf{d} - \mathbf{m}), \quad C_{ij} = \text{Cov}(d_i, d_j).$$

Here $\mathbf{d}$ and $\mathbf{m}$ are the data and model vectors, $\mathbf{C}$ is the adopted positive-definite measurement-error covariance matrix, and $\top$ denotes transpose. The independent-error expression is recovered for $C_{ij} = \sigma_i^2 \delta_{ij}$, where δ_{ij} is the Kronecker delta. For a fixed covariance matrix, minimizing χ^2 is equivalent to maximizing the Gaussian likelihood; if the covariance depends on fitted parameters, the likelihood normalization also contributes a $\ln \det \mathbf{C}$ term.

Adding a free inner slope or Einasto curvature usually cannot worsen the maximum likelihood, but it can yield an unconstrained parameter. Information criteria or Bayesian evidence penalize some excess flexibility, yet their conclusions still depend on likelihood, systematic-error model, and prior volume. Posterior predictive checks are especially useful: a model should reproduce the observed shape and residual structure, not merely return a plausible scalar statistic.

The most meaningful comparison often concerns derived quantities inside the measured window—for example $M(r_{1/2})$, dark-matter fraction, $\gamma_{\text{eff}}(r_{\text{min}})$, or acceleration—rather than $\rho(0)$ or M_{vir} , which may be extrapolated far beyond the data.

Here $r_{1/2}$ is the three-dimensional stellar half-light radius, r_{min} is the innermost radius regarded as resolved by the data, and M_{vir} is the mass enclosed by the adopted virial boundary.

9.2. A Task-Oriented Guide

Table 2 represents a brief practical guide to using various profiles in a number of typical cases.

Table 2. A brief task-oriented guide.

Cosmological baseline	Use NFW when a compact dark-matter-only baseline or a mass–concentration prior is desired; use Einasto when the data or simulation resolution justify an extra curvature parameter
Core testing	Use Burkert and pseudo-isothermal as transparent two-parameter alternatives. Burkert is better behaved in the outskirts; pseudo-isothermal directly represents an extended flat-speed region.
Free cusp	Use gNFW if the main question is the inner power-law slope while retaining an NFW-like exterior. State priors and verify that $r \ll r_s$ is actually sampled.
Analytic dynamics	Use Dehnen, Hernquist, or Jaffe when finite mass, simple potentials, distribution functions, or orbit calculations are central. Identify whether the component represents stars, dark matter, or a surrogate.
Particle-model test	Fit the model’s linked predictions, not just a generic core. For a soliton, impose or test its field-theory scaling and include an outer halo. For PFDM, specify the actual stress tensor, equation of state, and matching.
Indirect detection	Treat the unresolved center and substructure as explicit nuisance physics. NFW versus Moore extrapolation can dominate a ρ^2 -weighted signal.

10. Central Black Holes in Galactic Halos

10.1. Astrophysical Scale Separation and Dark-Matter Spikes

Supermassive black holes correlate with central spheroidal components, but the black hole dominates the gravitational field only within its influence region [51]. At larger radii stars, gas, and dark matter determine the galactic potential. Conversely, a kiloparsec-scale halo fit should not be extrapolated unchanged to the event horizon, a radial jump of many orders of magnitude.

If a black hole grows slowly and centrally inside an initial cusp $\rho \propto r^{-\gamma}$, adiabatic arguments can produce a dark-matter spike with

$$\gamma_{\text{sp}} = \frac{9 - 2\gamma}{4 - \gamma} \quad (52)$$

under restrictive assumptions [52]. Mergers, off-center growth, scattering by stars, nonsphericity, annihilation, and capture can weaken or erase a spike [53]. A spike is a small-scale dynamical response, not simply the $r \rightarrow 0$ continuation of the host’s NFW, Burkert, or Einasto fit.

In Equation (52), γ is the logarithmic slope of the initial halo cusp and γ_{sp} is the logarithmic slope of the idealized adiabatically generated spike.

For a galaxy with characteristic speed v , the dimensionless halo compactness is of order

$$\epsilon_h \sim \frac{GM_h}{R_h c^2} \sim \left(\frac{v}{c}\right)^2 \ll 1. \quad (53)$$

Thus the direct strong-field effect of a diffuse halo is normally small, even though its integrated potential may cause measurable orbital, propagation, or redshift effects. Controlled models can exploit the separation $M_\bullet \ll M_h$ and $r_g \ll r_s$, but must remain self-consistent near the horizon.

Here M_h and R_h are a characteristic halo mass and radius, v is the associated orbital-speed scale, c is the speed of light, and ϵ_h is the dimensionless halo compactness. The symbol $M_\bullet$ denotes the central black-hole mass, $r_g = GM_\bullet/c^2$ its gravitational radius, and r_s the halo scale radius appropriate to the selected density profile.

10.2. Why a Density Law Does Not Determine a Black-Hole Metric

For a static spherical spacetime write

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2, \quad T^\mu{}_\nu = \text{diag}[-\rho, p_r, p_t, p_t]. \quad (54)$$

In Equation (54), t is the static time coordinate, r is the areal radius, and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the line element on the unit two-sphere. The functions $A(r)$ and $B(r)$ are dimensionless metric functions. The mixed tensor $T^\mu{}_\nu$ contains the energy density ρ , radial pressure p_r , and equal angular pressures p_t .

The spherical mass function $m(r)$ is defined through $B(r) = 1 - 2m(r)/r$. In the geometrized units $G = c = 1$, the Einstein equations include

$$m'(r) = 4\pi r^2 \rho(r), \quad (55)$$

$$\frac{A'}{A} = \frac{2[m(r) + 4\pi r^3 p_r(r)]}{r[r - 2m(r)]}, \quad (56)$$

$$p'_r = -\frac{\rho + p_r}{2} \frac{A'}{A} + \frac{2(p_t - p_r)}{r}. \quad (57)$$

The mass-function Equation (55) determines $m(r)$, and hence $B(r)$, only after a density and an integration constant have been specified. For a timelike circular geodesic in the static region, the local speed v_{circ} measured by a static observer obeys the exact relation

$$v_{\text{circ}}^2(r) = \frac{rA'(r)}{2A(r)} = \frac{m(r) + 4\pi r^3 p_r(r)}{r - 2m(r)} \simeq \frac{m(r)}{r}. \quad (58)$$

Here the speed is expressed in units of c , and the second equality follows from Equation (56). The final approximation requires both $2m(r)/r \ll 1$ and $4\pi r^3 |p_r(r)| \ll m(r)$. Even when $p_r = 0$, the exact result is $v_{\text{circ}}^2 = m/(r - 2m)$, not m/r . A circular-speed curve inferred after correcting for projection and non-circular motions therefore constrains $A'/A = 2v_{\text{circ}}^2/r$, fixing A only up to its boundary normalization. It does not independently determine m or B without assumptions about the pressure support.

Therefore $\rho(r)$ alone fixes neither $A(r)$ nor the pressure support. One must specify a distribution function, equation of state, or anisotropy prescription and impose horizon regularity and outer matching. Collisionless galactic dark matter is particularly poorly represented by an arbitrary isotropic perfect fluid.

A tempting shortcut starts from a Newtonian circular speed, constructs one redshift function, sets $A = B$, and then adds a Schwarzschild term. But $A = B$ imposes an additional stress relation—in the simple static spherical case, $p_r = -\rho$ —and is not implied by the rotation curve. The Einstein tensor of the resulting metric must be recomputed and may not reproduce the density profile that motivated it. This issue affects the construction used in [54] and many subsequent applications, and is analyzed explicitly in [21]. The problem is not merely that an approximation was made; the imposed metric relation overdetermines the matter model and can make the claimed source inconsistent.

There are self-consistent alternatives. Ref. [18] obtained an exact black-hole solution embedded in a Hernquist-like anisotropic matter distribution. Ref. [19] developed solutions for several halo profiles using an anisotropic stress model with vanishing radial pressure, suppressed the density near the event horizon, solved the remaining Einstein equations, and matched the interior halo to an exterior vacuum region. Within their stated matter model, this is conceptually different from the shortcut above. The astrophysical limitation remains that the pressure prescription is a model of the halo, not a unique consequence of the kiloparsec-scale density fit.

10.3. Frequently Used Embedded Black-Hole Solution Families

The phrase embedded black hole is used for several mathematically distinct problems. Some solutions embed a hole in an expanding cosmology, some surround it with an exact but idealized stress tensor, and others incorporate a galactic halo or a collisionless spike. Table 3 lists the principal families that recur in the literature. Only the last three rows directly address matter distributions resembling the galactic profiles reviewed above.

In Table 3, Λ is the cosmological constant, K is the Kiselev normalization constant, and w is the equation-of-state-like exponent that fixes the radial power $1 + 3w$; in geometrized units K has the dimensions required to make K/r^{1+3w} dimensionless. FLRW abbreviates the Friedmann–Lemaître–Robertson–Walker cosmological geometry. The functions A and B , the mass $M_\bullet$, and the pressures p_r, p_t retain the definitions given above.

For the galactic problem, the Einstein-cluster and generic-anisotropic-fluid approaches are currently the most direct bridge from a specified spherical density to a self-consistent static geometry. They make explicit the tangential stresses required to prevent a collisionless cloud from falling radially into the black hole. The Cardoso–Hernquist solution is a widely used closed-form example; the Konoplya–Zhidenko construction, the numerical framework of [55], and the analytic core/cusp models of [56] extend the same broad program.

For the Cardoso–Hernquist geometry, electromagnetic quasinormal ringing, late-time tails, greybody factors, and the Unruh temperature were studied in [57]; in the parameter regime that best represents galaxies, the resulting environmental corrections to classical and quantum radiation were found to be relatively small.

A distinct recent construction adopts the alternative closure $p_r = -\rho$ and shows that sufficiently dense Einasto and Dehnen-type halos can source asymptotically flat, curvature-regular black-hole geometries that satisfy the weak energy condition and are stable under axial perturbations [58]. These solutions provide a specific regular strong-field completion of the halo profiles, rather than a generic prediction for ordinary galactic halo densities.

Several complementary studies illustrate how observable predictions depend on both the halo geometry and the treatment of its matter sector. Shadows, light rings, and lensing in the exact Schwarzschild–Hernquist spacetime were analyzed in [59], including constraints from the Event Horizon Telescope. For generic matter halos, ref. [60] found a leading compactness-controlled redshift of axial quasinormal frequencies in the dilute-halo regime. A coupled analysis of tidal Love numbers and quasinormal modes further showed that different prescriptions for perturbing the Hernquist matter can produce distinct spectra [61], underscoring that the background density alone does not close the perturbation problem. The role of an explicit inner halo boundary and its relation to the energy conditions was examined in [62]. More recently, self-consistent Einstein-cluster realizations of Hernquist, NFW, and Jaffe environments were used to map changes in geodesics, light rings, ultracompact configurations, and scalar ringdown signals [63].

Table 3. Representative, widely used black-hole embedding families. The table distinguishes exact solutions from fixed-background or reconstructed models and, crucially, distinguishes cosmological media from galactic halos.

Family	Defining Construction	Intended Environment	Astrophysical Interpretation and Caveat
Kottler, or Schwarzschild–de Sitter	$A = B = 1 - 2M_\bullet/r - \Lambda r^2/3$ [64].	Uniform vacuum energy.	An exact and canonical non-vacuum background, but Λ is not a clustered galactic halo and does not reproduce a galaxy rotation curve.
McVittie and Einstein–Straus	A central mass is placed in FLRW, or a spherical FLRW region is excised and matched to a Schwarzschild vacuole [65,66].	Cosmological expansion.	Fundamental for local-versus-cosmological gravity. These models do not supply a stationary virialized dark-matter halo around the hole.
Kiselev	$A = B = 1 - 2M_\bullet/r - K/r^{1+3w}$ [67].	Prescribed anisotropic surrounding matter.	A popular exact toy metric. Except for special limits, its source is neither an isotropic perfect fluid nor cosmological quintessence in the usual sense [68]; it is not a standard halo profile.
Logarithmic PFDM	$A = B$ with the logarithmic term in Equation (46) [43].	Effective dark fluid near a static or rotating black hole.	Analytically convenient, but its reconstructed anisotropic r^{-3} source is not the r^{-2} flat-curve PFDM halo and is not NFW, Burkert, or Einasto.
Rotation-curve reconstructed metrics	A Newtonian halo rotation curve determines $A(r)$; one then commonly sets $B = A$ and adds a Schwarzschild term [54].	NFW, Burkert, universal-rotation-curve, or related galactic fits.	Popular for shadows and geodesics, but the imposed metric may fail to solve the Einstein equations for the claimed density. It must be treated as phenomenological unless its Einstein tensor is checked [21].
Einstein-cluster or generic anisotropic-halo solutions	$p_r = 0$, $m' = 4\pi r^2 \rho$, and $A'/A = 2m/[r(r - 2m)]$; tangential motion supports the halo.	Hernquist, NFW, Einasto, and generalized core/cusp halos.	The closest widely used static realization of collisionless halo support. Exact analytic and numerical examples include Hernquist and generic halo profiles [18,19,55,56,69]. The orbital closure and inner cutoff remain physical assumptions.
Relativistic dark-matter spikes	A distribution function is adiabatically transformed in an exact Schwarzschild or Kerr background rather than inserted unchanged down to the horizon [20,70].	Collisionless dark matter inside the black-hole influence region.	Physically well suited to particle redistribution and capture. In its fixed-background form it does not include the spike’s self-gravitational backreaction, which is normally a small correction.

By contrast, Kottler, McVittie, and Einstein–Straus answer the different question of how a compact mass coexists with a cosmological background. Kiselev and logarithmic PFDM metrics are useful exact anisotropic-fluid laboratories but should not be relabeled as empirical galactic halos without matching their stress tensors and weak-field mass distributions. The relativistic-spike approach is complementary: it models how particles respond to a dominant vacuum black-hole geometry rather than solving for a fully backreacting global halo.

Rotating versions require an additional warning. Newman–Janis-type transformations are often applied to a static halo metric, but the transformation alone does not prove that the rotating spacetime is sourced by the same matter model. The Einstein tensor, stress-energy conservation, horizon structure, and asymptotics must be verified anew. Consequently, a metric described as Kerr plus NFW, Burkert, or PFDM is not automatically a self-consistent rotating halo solution.

10.4. Consistency Requirements and Interpretive Limits

A black-hole–halo configuration is most naturally formulated as a relativistic boundary-value problem rather than as an algebraic superposition of two familiar metrics. The density, pressure components, or collisionless distribution function determine the source, while the Einstein equations determine the geometry subject to inner and outer boundary conditions. It is therefore essential that the complete metric reproduce the stated stress-energy tensor and that this tensor satisfy covariant conservation. Specifying only $\rho(r)$, or only the weak-field rotation curve, leaves the radial support and hence part of the relativistic problem undetermined.

The inner boundary requires particular care. Matter variables that appear regular in Schwarzschild-like coordinates need not remain regular in a horizon-penetrating frame. A viable construction must distinguish the physical black-hole singularity from coordinate behavior at the horizon and must account for capture, orbital stability, and any depletion or rearrangement of the halo within the black-hole influence region. These considerations explain why a kiloparsec-scale NFW, Burkert, or Einasto law cannot be continued unchanged to $r = 2M_\bullet$.

The outer boundary is equally important. Profiles with infinite formal mass require a truncation or matching prescription, while even finite-mass profiles must approach an exterior geometry compatible with their intended astrophysical setting. In the region where galactic observations are available, the relativistic solution should reduce to the original weak-field mass model and reproduce its circular-speed curve. This provides a necessary connection between the strong-field construction and the empirical halo from which it was motivated.

Predictions for shadows, orbital frequencies, quasinormal modes, or gravitational-wave propagation should finally be tested against reasonable variations of the unresolved inner density and stress model. If such observables change substantially when the near-horizon completion is altered, they are predictions of that completion rather than robust consequences of the measured galactic halo. For most galaxies, uncertainties in the stellar, gaseous, and dark components on parsec-to-kiloparsec scales remain more consequential than the direct relativistic influence of diffuse halo matter at the horizon.

11. Conclusions

The variety of halo profiles discussed above reflects both the complexity of galaxy formation and the different purposes for which these models were introduced. Their comparison is therefore most informative when mathematical convenience, empirical fitting power, and claims about dark-matter physics are kept conceptually distinct.

No single profile is the dark-matter halo. NFW is the standard two-parameter collisionless baseline; Einasto captures continuously varying slopes in resolved simulations; Moore is a useful historical steep-cusp limit. Burkert and pseudo-isothermal profiles are efficient descriptions of core-like galaxy kinematics but require outer truncation or matching. The Zhao/gNFW family exposes how inner, outer, and transition slopes enter a fit, while Dehnen, Hernquist, and Jaffe provide finite-mass analytic dynamical models whose original interpretation is mainly stellar.

PFDM and scalar-field models carry more physical structure than an empirical density law. PFDM is not a unique profile and can refer to inequivalent isotropic, anisotropic, galactic, or black-hole constructions. A wave-dark-matter soliton is a field-supported central state with linked density, radius, and particle-mass scalings; it requires an outer halo and should not be replaced by a generic core without testing those links.

The most defensible inference states the measured radial window, baryonic and dynamical assumptions, scale convention, priors, and parameter covariance. It compares derived mass and slope at resolved radii before extrapolating to $r = 0$, the virial boundary, or a black hole. When a central black hole is included, the density profile is only one input: pressure or phase-space support, near-horizon behavior, and the full Einstein equations are indispensable. Used with these qualifications, halo profiles remain exceptionally effective bridges between galaxy observations, simulations, and dark-matter theory.

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Conflicts of Interest

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