

Article

Control of Energy Flow from Branch Circuit and Synchronization between Neural Circuits

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Abstract: Neural firing activity is fundamentally governed by the nonlinear coupling between electric field energy and magnetic field energy in neural circuits, and the introduction of different ion channel shunting devices (capacitor, inductor, memristor) enables energy shunting (redistribution), thereby achieving modulation of firing patterns. In this paper, a neuronal circuit model with rich firing modalities is constructed by incorporating a nonharmonic excitation source (a voltage source in series with a nonlinear resistor) and an ion channel shunting mechanism. We systematically examine the effects of shunting control on neuronal firing and synchronization from three aspects: nonlinear dynamics, Hamilton energy response, and coupling-modulated synchronization. The results show that both internal parameters and external excitations effectively modulate firing patterns, with different parameters exhibiting distinct energy-regulation trends. Moderate noise optimize firing regularity and promote energy accumulation via stochastic resonance (SR). An energy-based adaptive control strategy is further introduced, which actively steers the system between among dynamical regimes, confirming energy as a valid criterion for state regulation. For synchronization, we construct multiple coupling strategies are constructed using three coupling elements and two coupling topologies. The results consistently show that complete synchronization is achievable across all strategies; however, the shunting coupling consistently requires higher critical coupling strengths, implying richer regulatory flexibility at the expense of higher coupling intensity. These findings systematically elucidate the regulatory mechanisms of ion channel shunting on neuronal firing and synchronization, and provide theoretical guidance for energy-efficient optimization and synchronization-control design in neuromorphic circuits.

Keywords: neural circuits; ion channels; Hamilton energy; stochastic resonance; adaptive control strategy

1. Introduction

Neural firing is essentially a physical process governed by the nonlinear interplay between electric and magnetic fields in neural circuits. Equivalent circuit models thus provide a natural platform for investigating the dynamical mechanisms underlying complex firing patterns. The Hodgkin–Huxley (HH) model, based on experimental data from the squid giant axon, describes the conductance changes of sodium and potassium ion channels via system of nonlinear differential equations and successfully reproduces the generation and propagation of action potentials; corresponding HH-type neural circuits have also been designed [1,2]. Chua et al. constructed neural circuits using memristors, which resolved certain paradoxes inherent in the classical HH circuit [3]. FitzHugh later proposed the FitzHugh–Nagumo (FHN) model and implemented it via an equivalent circuit [4–6].



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Building on this foundation, researchers have introduced memristors [7,8], thermistors [9,10], and photoelectric elements [11,12] to develop various modified FHN neuron-based with richer dynamical behaviors, enabling the simulation of neuronal firing under different physical-field coupling conditions.

Neural firing exhibits multiple patterns, such as periodic spiking, bursting, and chaotic firing [13–16]. These firing patterns constitute the physical basis for information encoding and transmission among neurons. Different firing patterns correspond to distinct signal processing strategies and information transfer efficiencies, a fact that is of fundamental theoretical significance for understanding how the nervous system perceives, integrates, and responds to the external environment [17–20]. Various factors can induce or modulate firing pattern transitions, including external current stimulation, synaptic input, temperature changes, and noise perturbations, and the dynamical state of ion channels is particularly critical [21–23]. Analyzing the bifurcation structure of firing patterns as a function of parameters via circuit experiments and numerical simulations helps reveal the critical conditions and stability boundaries of neuronal state transitions, thereby providing a theoretical basis for designing neuronal regulation strategies [24]. Especially in cases where noise is generally considered harmful, it can enhance signal detection and improve firing regularity through stochastic resonance (SR). This phenomenon is widely observed in nonlinear systems including neural circuits [25–28].

The memristor is a nonlinear circuit element with memory characteristics, whose resistance depends on the history of charge or flux, exhibiting a unique memristive effect [29,30]. Since the physical realization of a memristor by Hewlett-Packard Laboratories in 2008, this element has attracted considerable attention due to its inherent state-dependent properties in nonlinear dynamics [31]. Memristors can be classified into charge-controlled memristors (CCM) [32–34] and magnetic flux-controlled memristors (MFCM) [35–37] according to their operating characteristics. MFCM provide more physically realistic circuit elements for studying neuronal firing in electromagnetic-field environments [38–42]. In logic gates, logic functions can be achieved based on memristors [43,44]. Moreover, memristors are often integrated with CMOS devices, leveraging the flexible control capabilities of CMOS to optimize read and write operations [45–47]. Equivalent circuit designs based on operational transconductance amplifiers, capacitance integration, or voltage followers can emulate memristor characteristics when physical devices are unavailable [48,49]. In terms of synaptic plasticity, memristors can precisely simulate various plasticity mechanisms of biological synapses, including spike-timing-dependent plasticity, long-term potentiation/depression, and frequency-dependent plasticity, via appropriately designed pulse sequences [50–53]. Leveraging these plasticity characteristics, memristors have been widely used to construct neuromorphic computing systems such as spiking neural networks, demonstrating significant application potential in efficient, low-power brain-inspired computing [54–57].

In parallel with circuit modeling and bifurcation analysis, energy-based approaches offer a physically insightful framework for understanding neuronal dynamics. For a given neural circuit, the Hamilton energy function can be rigorously derived via Helmholtz decomposition, which separates the system's vector field into conservative (rotational) and dissipative (gradient) components [58–60]. This decomposition validates the uniqueness of the energy function and establishes a quantitative link between energy evolution and dynamical transitions—different firing patterns (periodic spiking, bursting, and chaos) are typically associated with distinct energy levels. Consequently, Hamilton energy serves not only as a real-time indicator of bifurcations and pattern switches but also as a natural control variable. This motivates the design of energy-based adaptive control strategies, which dynamically adjust system parameters in response to instantaneous energy feedback, thereby actively driving the neuron toward desired dynamical states [61,62]. This energy-centered perspective thus complements traditional bifurcation analysis and provides a unifying framework for understanding how parameter variations, external stimuli, and coupling topology collectively shape neuronal firing and synchronization dynamics.

Based on the classical neuron circuit model and the extended research on memristors, researchers have recently turned to the active regulatory function of ion channels themselves. Lei and Wang et al. proposed a class of ion channel shunt control strategies [63,64] that achieve active intervention on ion channel currents by introducing controllable shunt branches into the neuron circuit. Ma et al. further explored the influence of energy injection through ion channels on neuronal firing behavior [65], demonstrating that such regulation actively modulates ion-channel currents and, more importantly, redistributes the electromagnetic energy within the system. For more recent progress on ion channel shunting, please refer to the literature [66,67]. On this basis, Lei et al. further extended the shunting idea to the field of neuron coupling, proposing a new connection mode different from the traditional direct coupling—voltage divider coupling based on ion channels (also known as ion channel coupling) [68]. This coupling mode achieves functional connections between neural circuits through the voltage divider mechanism. Studies have shown that under this condition, neurons can achieve various synchronization states, including complete synchronization and lag synchronization, providing a new physical implementation approach for synchronization regulation in multi-neuron systems.

Although the above studies have revealed the regulatory effects of ion-channel shunting on neuronal firing and network synchronization from various perspectives, a systematic comparative investigation of synchronization performance across different coupling elements (linear resistor, nonlinear resistor, memristor) and coupling methods (direct coupling versus ion-channel shunting coupling) remains lacking. In this paper, by connecting a voltage source in series with a nonlinear resistor to construct a non-harmonic excitation source and incorporating an ion-channel shunting mechanism, we design a neuronal circuit model capable of generating rich firing modalities. We systematically examine the effects of shunting control on neuronal firing behavior and synchronization from three aspects: nonlinear dynamical evolution, Hamilton energy response, and coupling-modulated synchronization. The main contributions of this work are as follows: (1) We construct a modified FHN neuronal circuit with ion-channel shunting, revealing the synergistic regulatory effects of internal parameters and external excitations on firing patterns; (2) We establish an energy-based adaptive control framework, confirming the feasibility of using energy as a real-time criterion for state regulation; (3) We systematically compare the synchronization performance of six coupling strategies (three elements and two topologies) and find that ion-channel coupling offers richer regulatory flexibility at the expense of a higher critical coupling strength. These results provide a theoretical basis for the dynamical design and energy-optimized control of ion-channel shunting neural circuits.

2. Model Description and Solution

Neural firing is fundamentally governed by the nonlinear interplay between electric and magnetic fields within neural circuits. To achieve rich firing dynamics, a neuronal model must integrate electric-field components (capacitors or CCM) and magnetic-field components (inductors or MFCM). A branch composed of a nonlinear resistor in series with a voltage source serves as a non-harmonic excitation source. In contrast to conventional periodic sinusoidal or square-wave forcing, this configuration significantly broadens the system's frequency bandwidth and induces complex amplitude-frequency modulation characteristics. In this work, we reconstruct the classical FHN neuron by introducing a voltage source into the nonlinear resistive branch. Furthermore, in conjunction with the ion channel shunting mechanism, a dedicated shunting control circuit is developed. The primary objective is to systematically investigate the regulatory effects of shunting control on neuronal firing patterns and the underlying synchronization dynamics.

In the circuit depicted in Figure 1, NR denotes a nonlinear resistor, with its current denoted by i_{NR} . The current-voltage characteristic of the nonlinear resistor is described by Equation (1).

$$\begin{cases} i_{NR} = -\frac{1}{\rho} \left(V - \frac{V^3}{3V_0^2} \right), \\ V = -E - V_1. \end{cases} \quad (1)$$

where, ρ and V_0 are the characteristic parameters of the nonlinear resistor. E can be set in either a constant form or a periodic form; in this paper, we use the periodic function form. The current-voltage relationship for the linear resistance R in the ion channel branch is shown in Equation (2).

$$\begin{cases} V_R = V_2 - E_2, i_R = \frac{V_R}{R}. \end{cases} \quad (2)$$

Applying Kirchhoff's current law yields the circuit equations corresponding to Figure 1, as presented in Equation (3).

$$\begin{cases} C_1 \frac{dV_1}{dt} = i_{NR} - i_L, \\ C_2 \frac{dV_2}{dt} = i_L - i_R, \\ L \frac{di_L}{dt} = V_{C1} - V_{C2}. \end{cases} \quad (3)$$

To eliminate the influence of units and highlight the dominant dynamic behavior, the Equation (3) is subjected to dimensionless processing.

$$\begin{cases} x = \frac{V_1}{V_0}, y = \frac{V_2}{V_0}, z = \frac{\rho}{V_0} i_L, \tau = \frac{t}{\rho C_1}, \\ \mu = \frac{E}{V_0}, b = \frac{E_2}{V_0}, c = \frac{C_1}{C_2}, d = \frac{\rho}{R}, \alpha = \frac{\rho^2 C_1}{L}. \end{cases} \quad (4)$$

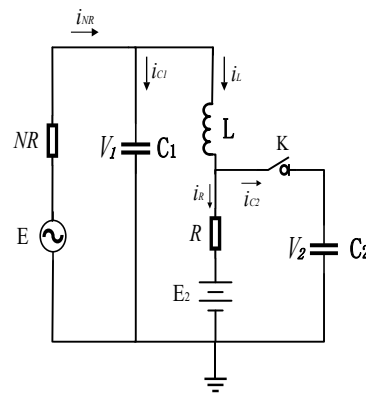


Figure 1. Schematic diagram of the neural circuit for ion channel shunting. C_1 and C_2 are capacitors, L is inductor, NR is nonlinear resistor, E represents the voltage source, E_2 is a constant voltage source, R is the resistance value of a fixed resistor.

In this article, E selects a periodic function, so μ is set as $\mu = a \cdot \sin(\omega \cdot \tau)$, where a is the amplitude and ω is the frequency. By introducing the scaling relations defined in Equation (4) and substituting them into Equation (3), after algebraic manipulation we obtain the dimensionless neuron system in Equation (5).

$$\begin{cases} \frac{dx}{d\tau} = a \sin(\omega\tau) + x - \frac{(a \sin(\omega\tau) + x)^3}{3} - z, \\ \frac{dy}{d\tau} = c[z - d(y - b)], \\ \frac{dz}{d\tau} = \alpha(x - y). \end{cases} \quad (5)$$

In the neural circuit system, the instantaneous energy W stored in the reactive elements (capacitors and inductors) and the corresponding Hamilton energy function H are defined as follows

$$\begin{cases} W = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2 + \frac{1}{2}Li_L^2, \\ H_x = \frac{1}{2}x^2, H_y = \frac{1}{2c}y^2, H_z = \frac{1}{2\alpha}z^2, \\ H = \frac{1}{2}x^2 + \frac{1}{2c}y^2 + \frac{1}{2\alpha}z^2. \end{cases} \quad (6)$$

The nonlinear dynamical system described by Equation (5) can be equivalently transformed into a matrix form. The detailed derivation is as follows

$$\begin{aligned}
 \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} a \sin(\omega \tau) + x - \frac{(a \sin(\omega \tau) + x)^3}{3} - z \\ c[z - d(y - b)] \\ \alpha(x - y) \end{bmatrix} = F_c + F_d \\
 &= \begin{bmatrix} -z \\ cz \\ \alpha(x - y) \end{bmatrix} + \begin{bmatrix} a \sin(\omega \tau) + x - \frac{(a \sin(\omega \tau) + x)^3}{3} \\ -cd(y - b) \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & \alpha c \\ \alpha & -\alpha c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \alpha \end{bmatrix} + \begin{bmatrix} a \sin(\omega \tau) + x - \frac{(a \sin(\omega \tau) + x)^3}{3} \\ x \\ 0 \\ 0 \end{bmatrix} - \frac{c^2 d(y - b)}{y} \begin{bmatrix} 0 \\ 0 \\ z \\ \alpha \end{bmatrix},
 \end{aligned}
 \tag{7}$$

Here, F_c and F_d denote the solenoidal (rotational) component and the irrotational (gradient) component, respectively. Substituting Equations (6) and (7) into Helmholtz's decomposition theorem yields Equation (8).

$$\begin{cases} \nabla H^\top F_c(x, y, z) = (-z) \frac{\partial H}{\partial x} + (cz) \frac{\partial H}{\partial y} + \alpha(x - y) \frac{\partial H}{\partial z} = 0, \\ \nabla H^\top F_d(x, y, z) = \frac{dH}{d\tau} = x \left[a \sin(\omega \tau) + x - \frac{(a \sin(\omega \tau) + x)^3}{3} \right] - dy(y - b). \end{cases} \quad (8)$$

This result mathematically validates both the uniqueness and consistency of the proposed Hamilton energy function, while also confirming its physical fidelity in energy characterization. Consequently, it becomes feasible to reinterpret the dynamical behaviors of the neural circuit from an energy perspective, thereby establishing a quantitative link between energy evolution and firing activities. This framework provides a theoretical foundation for understanding the intrinsic energy mechanisms underlying neuronal firing patterns.

Most neural circuits exhibit nonlinear resonance phenomena under noise excitation when the noise intensity is properly tuned, and at an appropriate noise level, the temporal regularity of neural activity can be optimized. To characterize the noise-induced resonance phenomena, we compute the average energy $\langle H \rangle$ and coefficient of variation (CV) of interspike intervals using Equation (9), which serve as measures of the energy response and the firing regularity, respectively.

$$\begin{cases} \langle H \rangle = \frac{1}{\tau - \tau_0} \int_{\tau_0}^{\tau} H(\tau) \cdot d\tau \approx \frac{1}{N} \sum_{i=1}^N H_i, \\ CV = \frac{\sqrt{\langle ISI^2 \rangle - \langle ISI \rangle^2}}{\langle ISI \rangle}. \end{cases} \quad (9)$$

Let N denote the total number of iteration steps, and ISI the interspike interval extracted from the sampled membrane potential time series. Conventionally, the signal-to-noise ratio (SNR) is widely used to identify SR, but its calculation requires Fourier transformation, which is computationally intensive and can be problematic for non-stationary or short spike trains. In contrast, both CV and $\langle H \rangle$ can be evaluated directly from the time series without spectral analysis, offering a simpler and more efficient alternative. It quantifies the temporal regularity of the firing sequence and effectively captures the frequency-dependent interplay between the periodic stimulus and noise, which is a hallmark of SR. The $\langle H \rangle$, on the other hand, measures the mean power of the system's oscillatory activity and reflects the intensity of energy exchange associated with ion-channel dynamics. Its role is analogous to that of SNR in assessing the system's response to external forcing. When CV attains a minimum and $\langle H \rangle$ attains a maximum at the same noise intensity, the system is considered to be in a resonance state, as the noise optimally regularizes the firing pattern while simultaneously maximizing the energy response. Therefore, the combined use of CV and $\langle H \rangle$ provides a robust and physically meaningful criterion for detecting SR, without the need for computationally demanding SNR calculations. To investigate the adaptive regulation mechanism under shunting control, we further examine the effect of the adaptively varying parameter c on the selection of neural firing patterns.

$$\begin{cases} \frac{dc}{d\tau} = \sigma \cdot c \cdot \Theta\left(\left|\frac{H_L}{H}\right| - \gamma\right) = \sigma \cdot c \cdot \Theta\left(\left|\frac{y^2}{2cH}\right| - \gamma\right), \\ \Theta(P) = 1, P \geq 0, \Theta(P) = 0, P < 0. \end{cases} \quad (10)$$

In this control scheme, the ratio threshold γ determines the proportion of C_2 channel energy to the total neuronal energy, the gain parameter σ governs the evolution rate of the shunting element c , and Θ denotes the Heaviside step function.

In practice, two or more neural circuits can be coupled via resistors, nonlinear resistors, or memristors to realize electrical or chemical synaptic interactions. These interactions may give rise to diverse firing regimes including complete synchronization, partial synchronization, anti-phase synchronization, and lag synchronization, accompanied by inter-circuit energy exchange. To systematically investigate the influence of coupling configurations on synchronization performance, we consider two identical neural circuits and employ resistors, nonlinear resistors, and memristors as coupling devices. Furthermore, both direct coupling and ion channel shunting coupling (also known as voltage shunting coupling [43]) are adopted, yielding a total of six coupling strategies comprehensively examine the synchronization dynamics between the two neurons.

When a linear resistor R_c is employed as the coupling element, the dimensionless model based on Figures 2 and 3 can be expressed as follows:

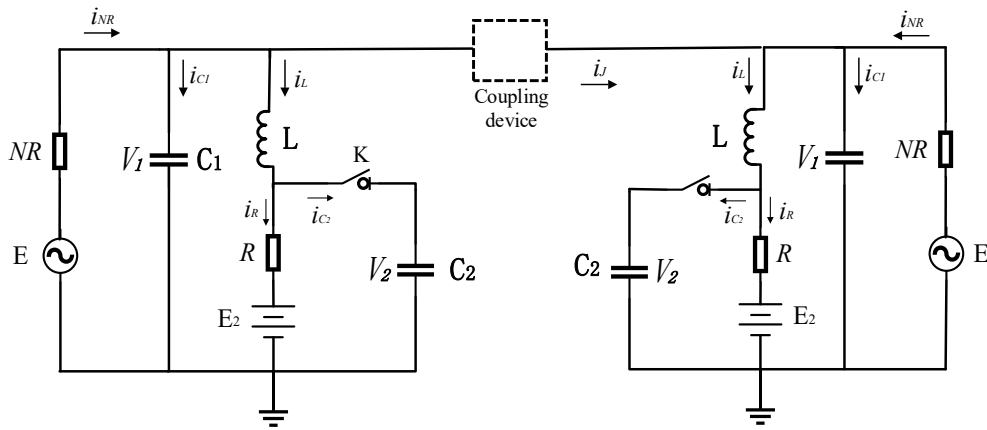


Figure 2. Schematic diagram of the direct coupling of the neural circuit with ion channel shunting.

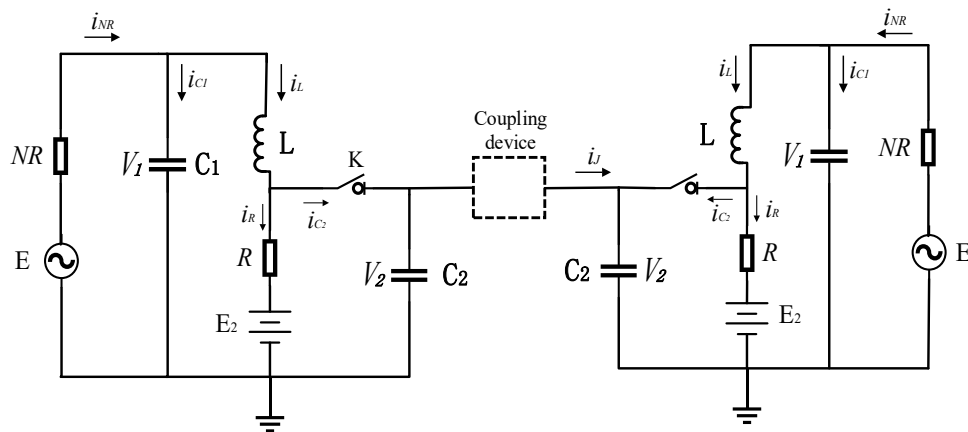


Figure 3. Schematic diagram of ion channel shunting coupling (voltage shunting coupling) in the neural circuits with hybrid ion channels.

$$\begin{cases} \frac{dx_1}{d\tau} = a \sin(\omega\tau) + x_1 - \frac{(a \sin(\omega\tau) + x_1)^3}{3} - z_1 - i_{JR1}, \\ \frac{dy_1}{d\tau} = c[z_1 - d(y_1 - b)], \\ \frac{dz_1}{d\tau} = \alpha(x_1 - y_1), \\ \frac{dx_2}{d\tau} = a \sin(\omega\tau) + x_2 - \frac{(a \sin(\omega\tau) + x_2)^3}{3} - z_2 + i_{JR1}, \\ \frac{dy_2}{d\tau} = c[z_2 - d(y_2 - b)], \\ \frac{dz_2}{d\tau} = \alpha(x_2 - y_2), \end{cases} \quad (11a)$$

$$\begin{cases} \frac{dx_1}{d\tau} = a \sin(\omega\tau) + x_1 - \frac{(a \sin(\omega\tau) + x_1)^3}{3} - z_1, \\ \frac{dy_1}{d\tau} = c[z_1 - d(y_1 - b) - i_{JR2}], \\ \frac{dz_1}{d\tau} = \alpha(x_1 - y_1), \\ \frac{dx_2}{d\tau} = a \sin(\omega\tau) + x_2 - \frac{(a \sin(\omega\tau) + x_2)^3}{3} - z_2, \\ \frac{dy_2}{d\tau} = c[z_2 - d(y_2 - b) + i_{JR2}], \\ \frac{dz_2}{d\tau} = \alpha(x_2 - y_2). \end{cases} \quad (11b)$$

where i_{JR1} and i_{JR2} are the linear coupling functions, defined as follows:

$$\begin{cases} i_{JR1} = k_R(x_1 - x_2), \\ i_{JR2} = k_R(y_1 - y_2). \end{cases} \quad (12)$$

where k_R is the dimensionless coupling strength, defined as $k_R = \rho/R_s$. When a nonlinear resistor NR is adopted as the coupling element, the dimensionless model based on Figures 2 and 3 can be written as:

$$\begin{cases} \frac{dx_1}{d\tau} = a \sin(\omega\tau) + x_1 - \frac{(a \sin(\omega\tau) + x_1)^3}{3} - z_1 - i_{JNR1}, \\ \frac{dy_1}{d\tau} = c[z_1 - d(y_1 - b)], \\ \frac{dz_1}{d\tau} = \alpha(x_1 - y_1), \\ \frac{dx_2}{d\tau} = a \sin(\omega\tau) + x_2 - \frac{(a \sin(\omega\tau) + x_2)^3}{3} - z_2 + i_{JNR1}, \\ \frac{dy_2}{d\tau} = c[z_2 - d(y_2 - b)], \\ \frac{dz_2}{d\tau} = \alpha(x_2 - y_2), \end{cases} \quad (13a)$$

$$\begin{cases} \frac{dx_1}{d\tau} = a \sin(\omega\tau) + x_1 - \frac{(a \sin(\omega\tau) + x_1)^3}{3} - z_1, \\ \frac{dy_1}{d\tau} = c[z_1 - d(y_1 - b) - i_{JNR2}], \\ \frac{dz_1}{d\tau} = \alpha(x_1 - y_1), \\ \frac{dx_2}{d\tau} = a \sin(\omega\tau) + x_2 - \frac{(a \sin(\omega\tau) + x_2)^3}{3} - z_2, \\ \frac{dy_2}{d\tau} = c[z_2 - d(y_2 - b) + i_{JNR2}], \\ \frac{dz_2}{d\tau} = \alpha(x_2 - y_2). \end{cases} \quad (13b)$$

The current–voltage characteristic of the nonlinear resistor in the coupling channel is described by the first line of Equation (1). In this context, i_{JNR1} and i_{JNR2} are the nonlinear coupling functions, defined as follows:

$$\begin{cases} i_{JNR1} = k_{NR} \left[(x_1 - x_2) - \frac{(x_1 - x_2)^3}{3} \right], \\ i_{JNR2} = k_{NR} \left[(y_1 - y_2) - \frac{(y_1 - y_2)^3}{3} \right]. \end{cases} \quad (14)$$

Here k_{NR} is the dimensionless coupling strength, defined as $k_{NR} = \rho/\rho_c$. When a MFCM $M(\varphi)$ is employed as the coupling element, the dimensionless model based on Figures 2 and 3 can be written as:

$$\begin{cases} \frac{dx_1}{d\tau} = a \sin(\omega\tau) + x_1 - \frac{(a \sin(\omega\tau) + x_1)^3}{3} - z_1 - i_{JM1}, \\ \frac{dy_1}{d\tau} = c[z_1 - d(y_1 - b)], \\ \frac{dz_1}{d\tau} = \alpha(x_1 - y_1), \\ \frac{dx_2}{d\tau} = a \sin(\omega\tau) + x_2 - \frac{(a \sin(\omega\tau) + x_2)^3}{3} - z_2 + i_{JM1}, \\ \frac{dy_2}{d\tau} = c[z_2 - d(y_2 - b)], \\ \frac{dz_2}{d\tau} = \alpha(x_2 - y_2), \\ \frac{d\psi}{d\tau} = x_1 - x_2, \end{cases} \quad (15a)$$

$$\begin{cases}
\frac{dx_1}{d\tau} = a \sin(\omega\tau) + x_1 - \frac{(a \sin(\omega\tau) + x_1)^3}{3} - z_1, \\
\frac{dy_1}{d\tau} = c[z_1 - d(y_1 - b) - i_{JM1}], \\
\frac{dz_1}{d\tau} = \alpha(x_1 - y_1), \\
\frac{dx_2}{d\tau} = a \sin(\omega\tau) + x_2 - \frac{(a \sin(\omega\tau) + x_2)^3}{3} - z_2, \\
\frac{dy_2}{d\tau} = c[z_2 - d(y_2 - b) + i_{JM2}], \\
\frac{dz_2}{d\tau} = \alpha(x_2 - y_2), \\
\frac{d\psi}{d\tau} = y_1 - y_2.
\end{cases} \quad (15b)$$

The memductance of the memristor in the coupling channel is a smooth nonlinear function, given by $M(\varphi) = a_m + 3b_m\varphi^2$. Here, i_{JM1} and i_{JM2} are the memristive coupling functions, defined as follows:

$$\begin{cases}
i_{JM1} = k_M (A_m + 3B_m\varphi^2)(x_1 - x_2), \\
i_{JM2} = k_M (A_m + 3B_m\varphi^2)(y_1 - y_2).
\end{cases} \quad (16)$$

where k_M is the dimensionless coupling strength, which for the memristive coupling is defined as

$$\begin{cases}
\varphi = \frac{\phi}{V_0\rho C_1}, A_m = \rho a_m, B_m = \rho b_m (V_0\rho C_1)^2.
\end{cases} \quad (17)$$

Here, ϕ represents magnetic flux, while a_m and b_m are physical parameters of the memristor, and A_m and B_m are the corresponding dimensionless parameters. Equation (18) provides the energy definition for each neuron and the energy difference for the two neurons.

$$\begin{cases}
H_1 = \frac{1}{2}x_1^2 + \frac{1}{2c}y_1^2 + \frac{1}{2\alpha}z_1^2, \\
H_2 = \frac{1}{2}x_2^2 + \frac{1}{2c}y_2^2 + \frac{1}{2\alpha}z_2^2, \\
\Delta H = |H_1 - H_2|.
\end{cases} \quad (18)$$

To describe the synchronization between two neurons, an error function is defined in Equation (19) when all variables are available.

$$ER = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}. \quad (19)$$

The two systems will first achieve phase locking before achieving complete synchronization. Usually, using Hilbert transformation, the phase sequence corresponding to the time series for variables can be obtained.

$$\begin{cases}
\tilde{x}_1(t) = -\frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{x_1(\tau)}{t - \tau} d\tau, \quad \varphi(t) = \arctan \frac{\tilde{x}_1(t)}{x_1(t)}, \\
\tilde{x}_2(t) = -\frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{x_2(\tau)}{t - \tau} d\tau, \quad \varphi'(t) = \arctan \frac{\tilde{x}_2(t)}{x_2(t)}, \\
\Delta\theta = \varphi - \varphi'.
\end{cases} \quad (20)$$

3. Numerical Results and Discussion

Using the ODE45 function in MATLAB, output a calculation result at a time interval of 0.01, and conduct numerical simulation for the neuron model. Figure 4 presents the bifurcation diagram and the corresponding largest Lyapunov exponent (LLE) spectrum as functions of the parameter a . Subsequently, Figure 5 illustrates the phase portraits (attractors) and the corresponding time series of membrane potential for different firing patterns.

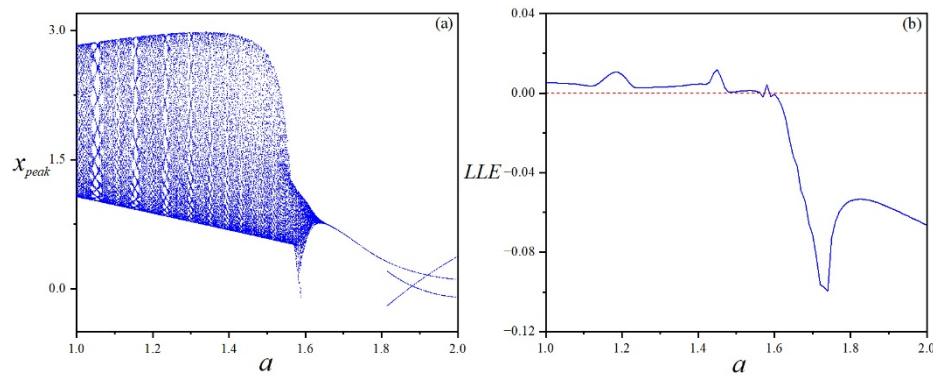


Figure 4. Bifurcation diagram (a) and LLE (b) with the parameter a . Parameter setting $b = 0.1$, $c = 0.8$, $d = 0.01$, $\alpha = 5$, $\omega = 2.8$, initial value is $(0.01, 0.01, 0.01)$.

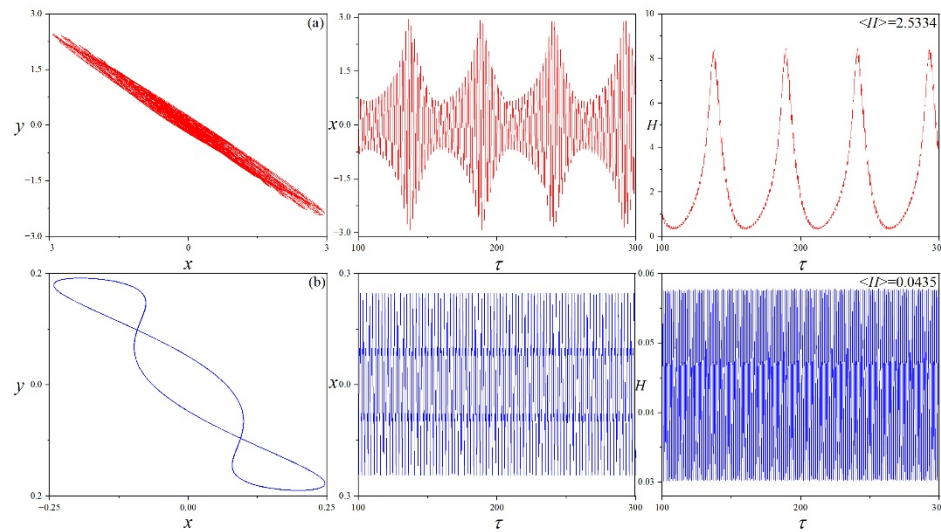


Figure 5. Phase diagrams, evolution of x and Hamilton energy H . For (a) $a = 1.4$; (b) $a = 1.85$. Parameter setting $b = 0.1$, $c = 0.8$, $d = 0.01$, $\alpha = 5$, $\omega = 2.8$, initial value is $(0.01, 0.01, 0.01)$.

The results in Figures 4 and 5 show that the system exhibits distinct phase portraits under different initial conditions. As parameter a increases, the firing pattern transitions from bursting to periodic spiking, accompanied by a decrease in Hamilton energy from a higher to a lower level. In contrast, parameter c shows an opposite energy-regulation trend. Figure 6 presents the bifurcation diagram and the corresponding LLE spectrum as functions of c .

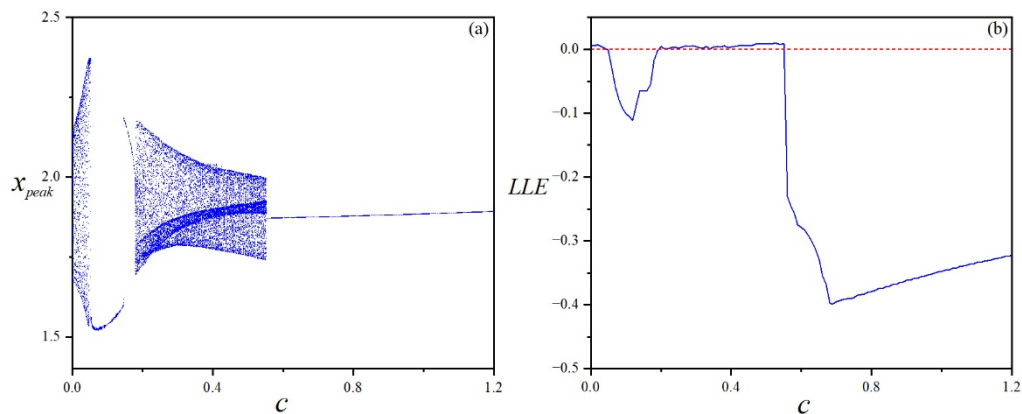


Figure 6. Bifurcation diagram (a) and LLE (b) with the parameter c . Parameter setting $a = 0.5$, $b = 0.1$, $d = 0.1$, $\alpha = 2$, $\omega = 2.9$, initial value is $(0.01, 0.01, 0.01)$.

As shown in Figures 6 and 7, the neuronal system progressively transitions from chaos to periodicity as parameter c increases. The distribution of the LLE spectrum corroborates the bifurcation analysis. Concurrently, the Hamilton energy exhibits a consistent trend: the chaotic regime has higher energy, whereas the periodic regime

corresponds to a lower energy level, implying that increasing c drives the system toward low-energy periodic states. Moreover, adjusting the parameter of the periodic voltage source effectively modulates the ion channel dynamics, leading to significant alterations in firing patterns. Figures 8 and 9 quantitatively present the response of firing patterns to the driving frequency.

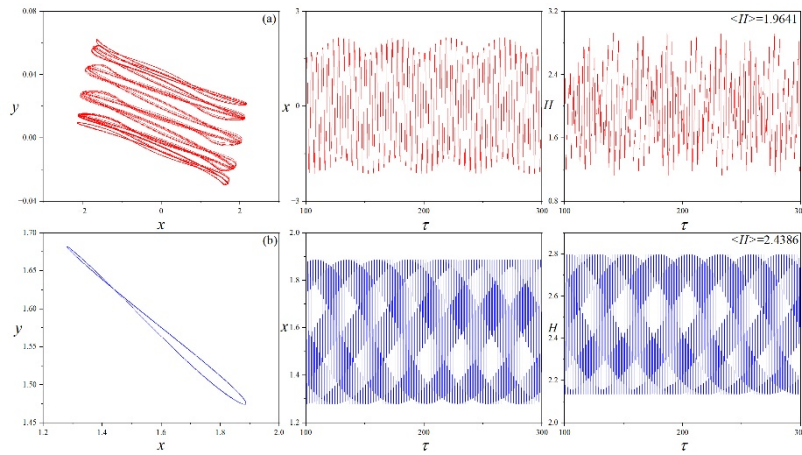


Figure 7. Phase diagrams, evolution of x and Hamilton energy H . For (a) $c = 0.01$; (b) $c = 1.08$. Parameter setting $a = 0.5$, $b = 0.1$, $d = 0.1$, $\alpha = 2$, $\omega = 2.9$, initial value is (0.01, 0.01, 0.01).

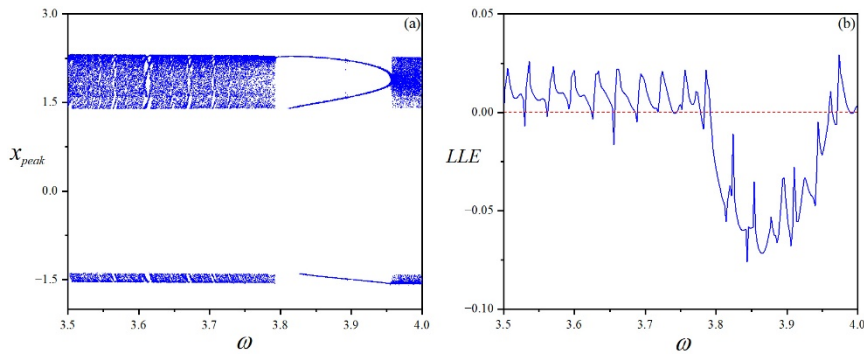


Figure 8. Bifurcation diagram (a) and LLE (b) with the parameter ω . Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, initial value is (0.01, 0.01, 0.01).

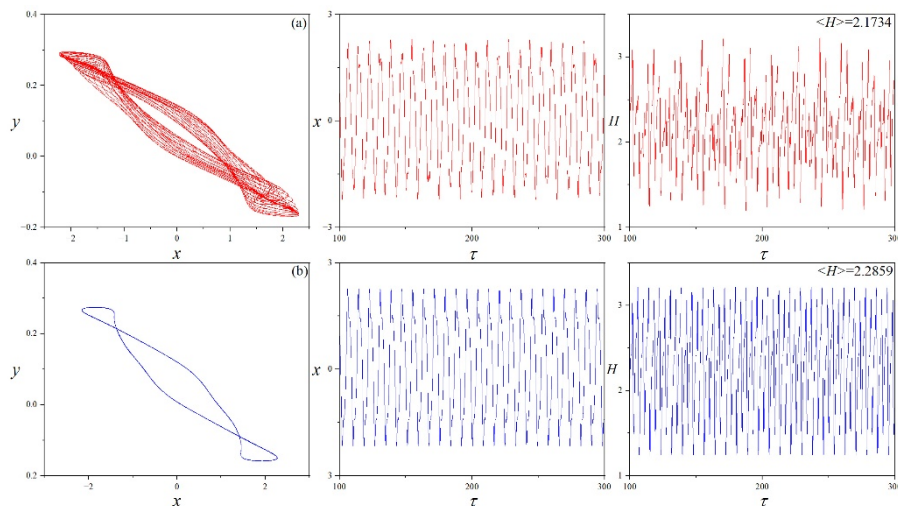


Figure 9. Phase diagrams, evolution of x and Hamilton energy H . For (a) $\omega = 3.53$; (b) $\omega = 3.84$. Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, initial value is (0.01, 0.01, 0.01).

Bifurcation analysis with respect to the driving frequency ω reveals a distinct periodic window in the range $3.8 \leq \omega \leq 3.95$, consistent with the distribution of the LLE spectrum. Variation of the driving frequency effectively modulates the firing dynamics. During the state transition, the Hamilton energy shifts from a lower average level

in the chaotic regime to a higher level in the periodic regime. Moreover, at a fixed driving frequency, the external stimulus intensity also regulates the neuronal states. Figures 10 and 11 present the bifurcation diagram, LLE spectrum, and typical attractor phase portraits as functions of parameter α (which characterizes the external stimulus intensity), thereby elucidating the regulatory effect of stimulus intensity on firing patterns.

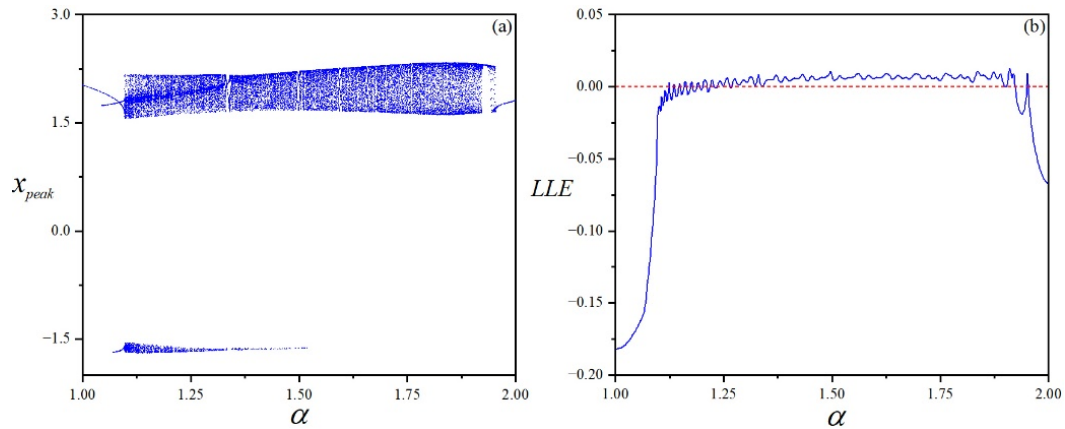


Figure 10. Bifurcation diagram (a) and LLE (b) with the parameter α . Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\omega = 2.9$, initial value is $(0.01, 0.01, 0.01)$.

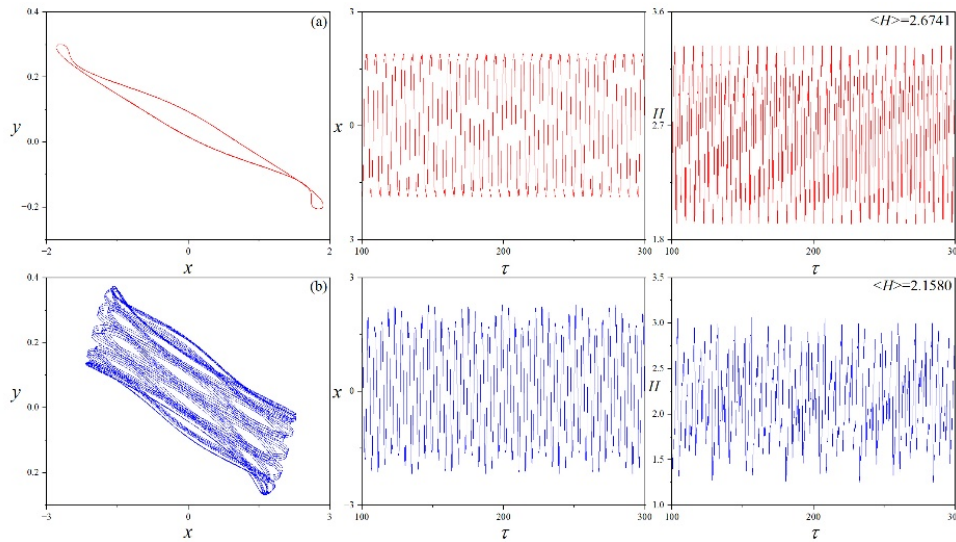


Figure 11. Phase diagrams, evolution of x and Hamilton energy H . For (a) $\alpha = 1.05$; (b) $\alpha = 1.68$. Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\omega = 2.9$, initial value is $(0.01, 0.01, 0.01)$.

The external stimulus intensity parameter α plays a significant role in regulating the firing patterns of the neuronal circuit. As α varies, the system undergoes a marked dynamical transition. Similar to the driving frequency ω , the variation in α can induce a switch between periodic and chaotic firing regimes. Specifically, increasing α progressively complicates the orbital structure, accompanied by a transition from regular periodic oscillations to irregular chaotic oscillations. This indicates that appropriate adjustment of the external stimulus intensity enables effective control over both the firing state and the energy level of the system. To further clarify the relationship between the average energy $\langle H \rangle$ and the firing mode, we plot the firing states and their corresponding $\langle H \rangle$ values as a histogram.

As shown in Figure 12, $\langle H \rangle$ can be used to roughly distinguish the firing states: with few exceptions, the energy of chaotic firing is generally lower than that of periodic firing. For instance, the periodic firing in Figure 5 has consistently small amplitudes, resulting in very low total energy; when amplitudes are comparable, chaotic firing still tend to have lower energy than periodic ones. Furthermore, Figure 13 presents the distributions of the CV and $\langle H \rangle$ under different noise intensities, aiming to reveal the nonlinear cooperative relationship between noise intensity and system response.

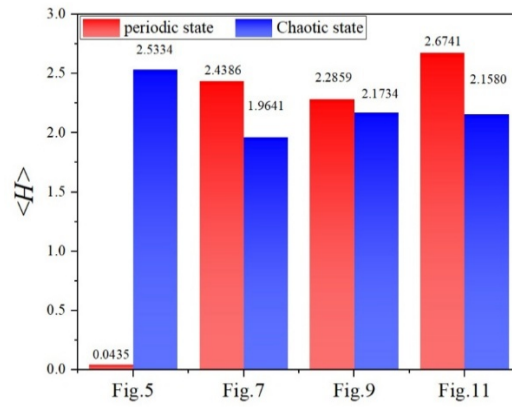


Figure 12. Histogram of the average Hamilton energy $\langle H \rangle$.

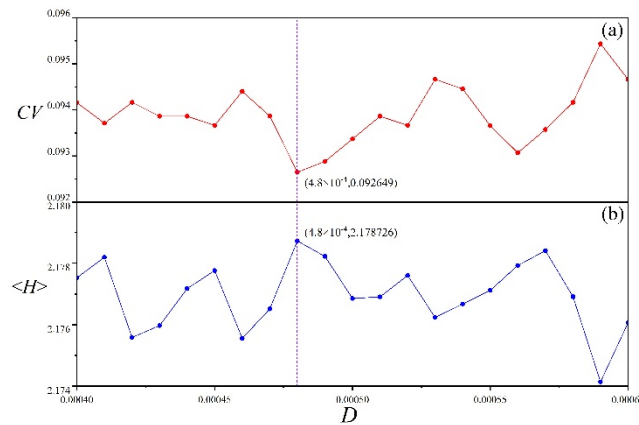


Figure 13. Dependence of CV (a) and $\langle H \rangle$ (b) on noise intensity D . Noise is applied to the variable z in Equation (5). Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.53$, initial value is (0.01, 0.01, 0.01).

The variations of the CV and $\langle H \rangle$ with noise intensity, as shown in Figure 13, indicate that an appropriate level of ion channel noise can induce SR. Specifically, at the noise intensity $D = 4.8 \times 10^{-4}$, the system attains its the maximum value of $\langle H \rangle$ and the minimum value of CV, suggesting that moderate noise can effectively modulate neuronal firing and facilitate energy accumulation. This observation further implies that the influence of noise on neuronal firing is not merely random; instead, it can induce SR at certain intensities. Given that energy is a key factor governing the evolution of firing dynamics, we adopt the energy-based adaptive control strategy defined in Equation (10) to modulate the firing patterns. The corresponding numerical results are presented in Figures 14–16.

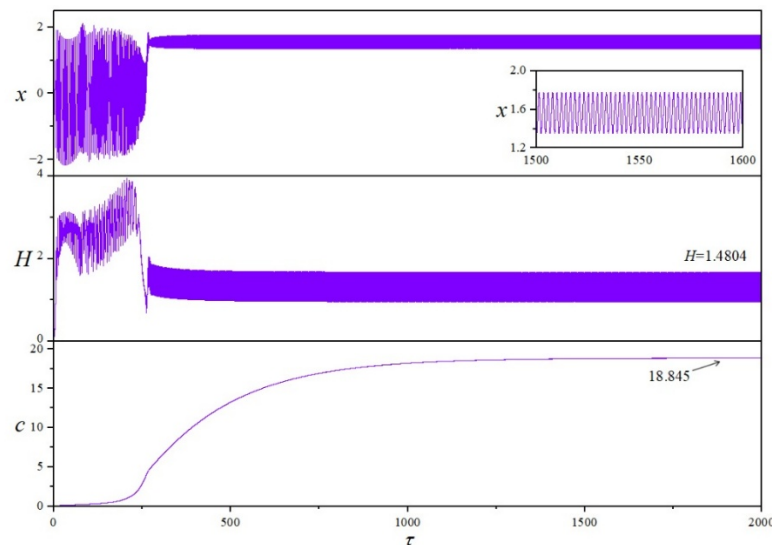


Figure 14. Evolution of x and H , growth of parameter c . Parameter setting $\sigma = 0.09$, $\gamma = 0.05$, $a = 0.5$, $b = 0.1$, $d = 0.1$, $\alpha = 2$, $\omega = 2.9$, initial value is (0.01, 0.01, 0.01, 0.1).

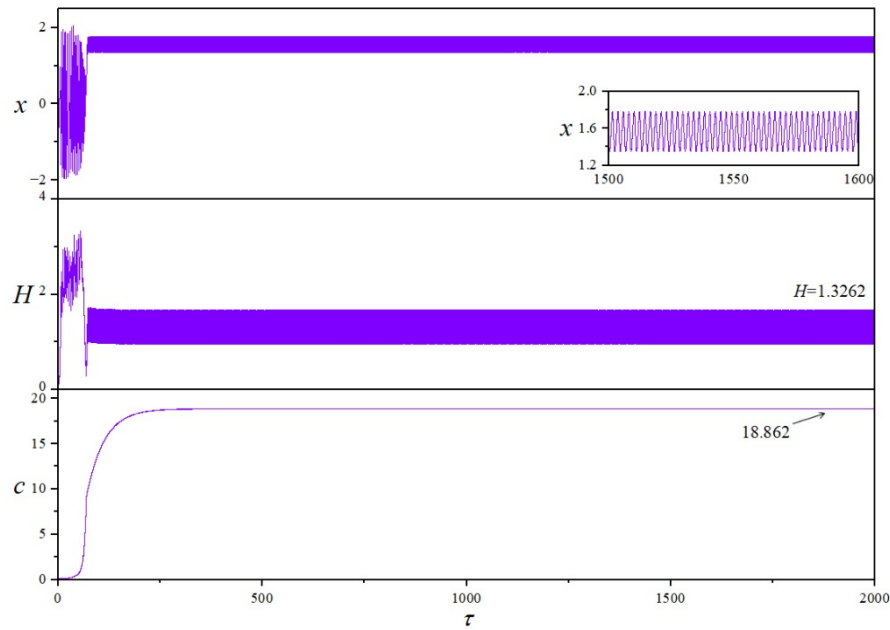


Figure 15. Evolution of x and H , growth of parameter c . Parameter setting $\sigma = 0.5$, $\gamma = 0.05$, $a = 0.5$, $b = 0.1$, $d = 0.1$, $\alpha = 2$, $\omega = 2.9$, initial value is $(0.01, 0.01, 0.01, 0.1)$.

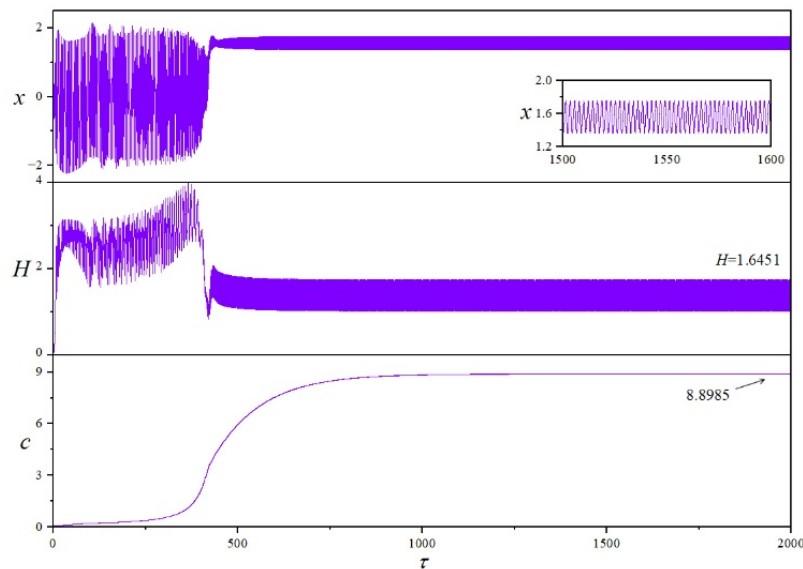


Figure 16. Evolution of x and H , growth of parameter c . Parameter setting $\sigma = 0.09$, $\gamma = 0.1$, $a = 0.5$, $b = 0.1$, $d = 0.1$, $\alpha = 2$, $\omega = 2.9$, initial value is $(0.01, 0.01, 0.01, 0.1)$.

Figures 14–16 illustrate the evolution of membrane potential x and Hamilton energy H as parameter c gradually increases and saturates over time. Once c stabilizes, the neural firing pattern transitions from chaotic to periodic oscillations. A comparison of Figures 14 and 15 reveals that a decrease in the adaptive intensity parameter σ leads to a longer adaptation time required to reach the steady state. In contrast, a comparing of Figures 15 and 16 shows that an increase in the adaptive threshold parameter γ significantly reduces the final saturation value of c . Collectively, these results indicate that both σ and γ can effectively modulate the adaptive process, thereby controlling the evolution of parameter c . This finding confirms that energy-based regulation of the adaptive parameter c enables effective switching of neuronal firing states.

We further investigate the synchronization between two neural circuits. Employing the six coupling strategies mentioned above, we evaluate the synchronization performance under various coupling elements and configurations. We first consider the case of linear resistive coupling [Equation (11)], with the corresponding results presented in Figures 17–19. These figures depict the temporal evolution of the energy difference ΔH , synchronization error ER, and phase difference $\Delta\theta$ under both direct coupling and ion channel shunting coupling.

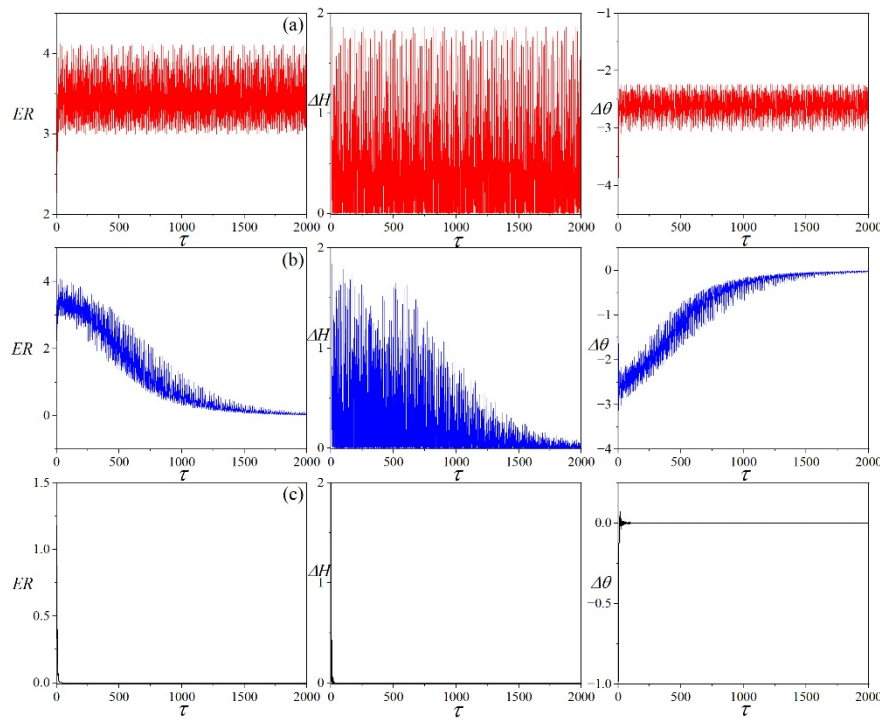


Figure 17. Evolutions of the synchronization error ER , energy difference ΔH , and phase difference $\Delta\theta$ in Equation (11a). For (a) $k_R = 0$; (b) $k_R = 0.0033$; (c) $k_R = 0.4$. Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.53$, and initial value is $(0.01, 0.01, 0.01, 0.2, 0.5, 1.2)$.

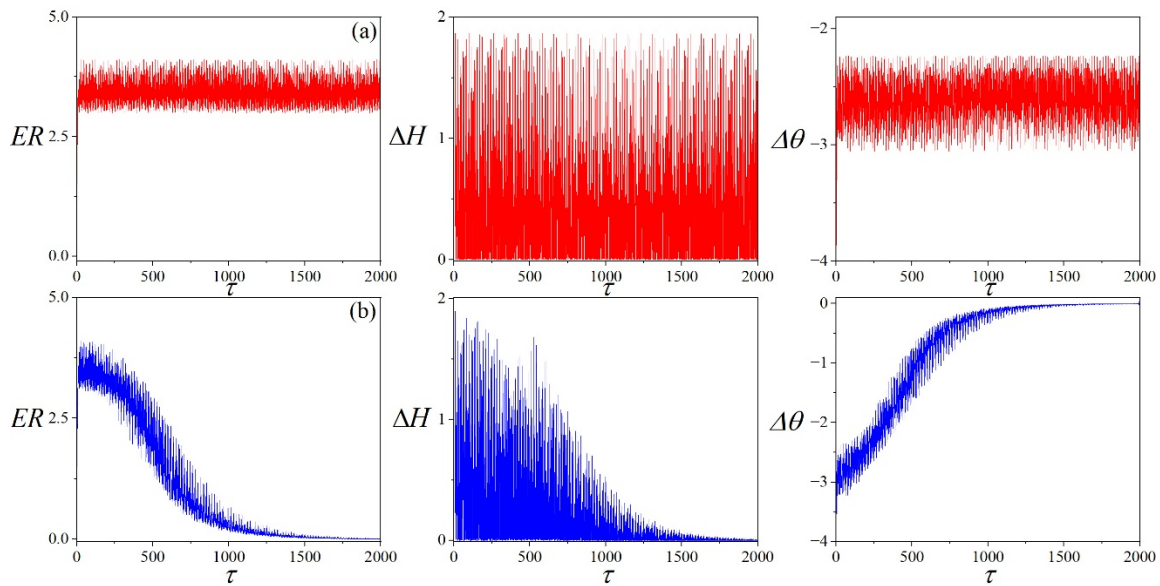


Figure 18. Evolutions of the synchronization error ER , energy difference ΔH , and phase difference $\Delta\theta$ in Equation (11b). For (a) $k_R = 0$; (b) $k_R = 0.4$. Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.53$, and initial value is $(0.01, 0.01, 0.01, 0.2, 0.5, 1.2)$.

The results presented in Figures 17 and 18 indicate that as the coupling strength k_R increases, the two neurons gradually approach synchronization under both direct coupling and ion channel shunting coupling. Notably, at $k_R = 0.4$, direct coupling exhibits a faster convergence rate to the synchronous state. Figure 19 further reveals a marked difference in the critical coupling strength required for complete synchronization: the threshold for ion channel shunting coupling is considerably higher than that for direct coupling. This discrepancy may be attributed to the presence of lag synchronization or the specific topological features inherent to the ion channel coupling configuration. To further investigate the influence of coupling devices on neuronal synchronization, Figures 20–22 present a nonlinear resistor as the coupling element and present the temporal evolution of the energy difference ΔH , synchronization error ER , and phase difference $\Delta\theta$ under both direct and shunting coupling schemes.

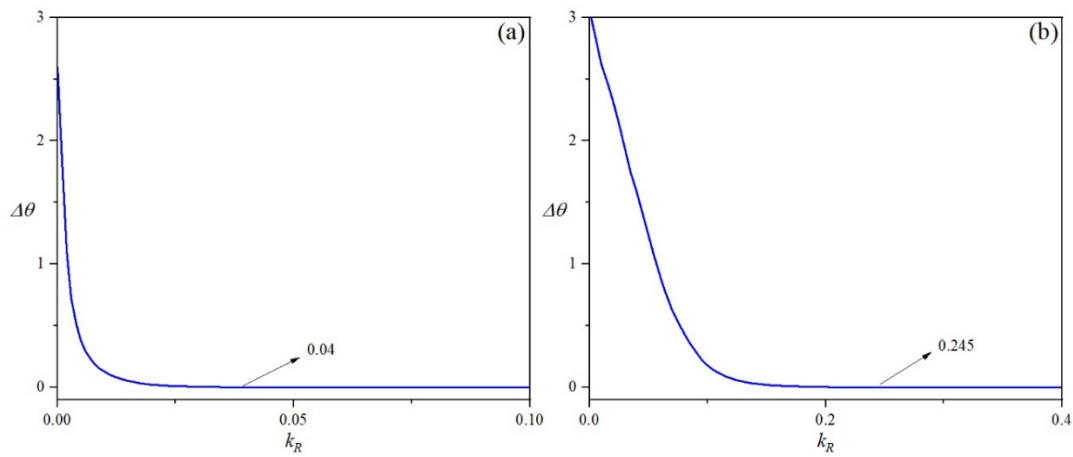


Figure 19. Distribution of average phase difference varies with the coupling strength k_R . For (a) Direct coupling in Equation (11a); (b) ion channel shunting coupling in Equation (11b). Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.53$, and initial value is (0.01, 0.01, 0.01, 0.2, 0.5, 1.2). Set the threshold for achieving synchronization to 10^{-4} , and all the subsequent thresholds used for calculating the synchronization critical value will be consistent.

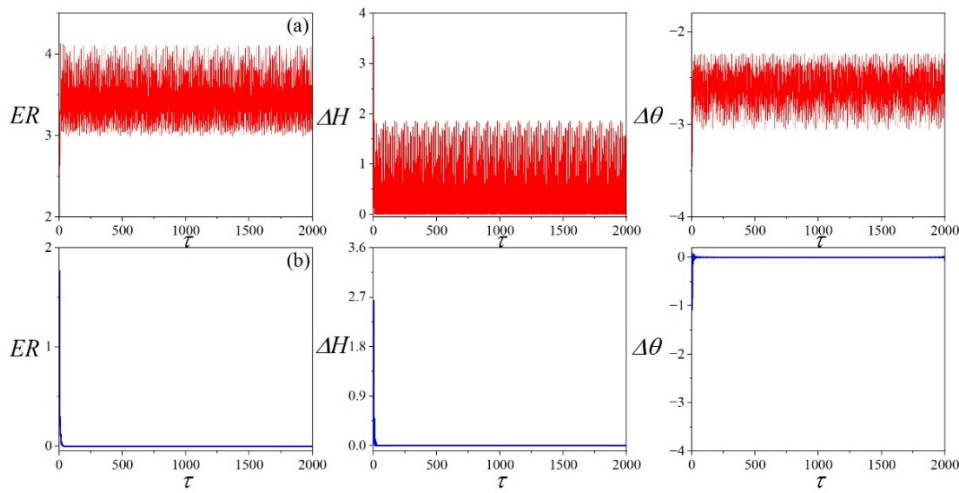


Figure 20. Evolutions of the synchronization error ER , energy difference ΔH , and phase difference $\Delta\theta$ in Equation (13a). For (a) $k_{NR} = 0$; (b) $k_{NR} = 0.4$. Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.53$, and initial value is (0.01, 0.01, 0.01, 0.2, 0.5, 1.2).

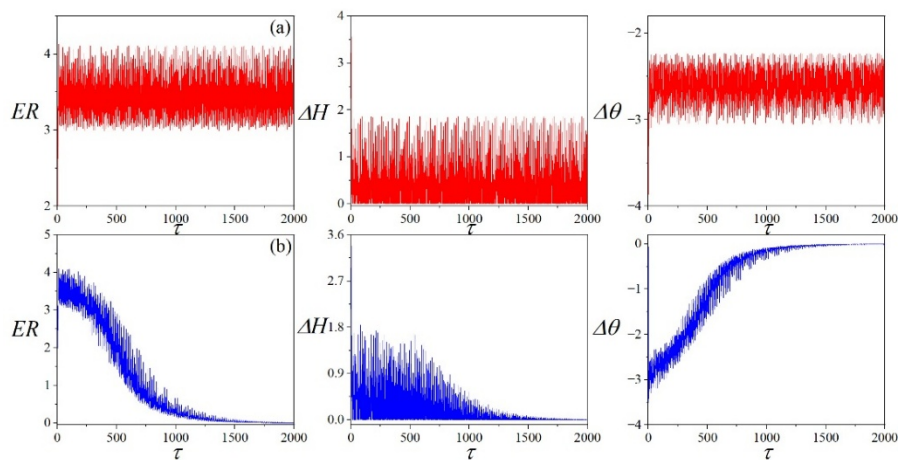


Figure 21. Evolutions of the synchronization error ER , energy difference ΔH , and phase difference $\Delta\theta$ in Equation (13b). For (a) $k_{NR} = 0$; (b) $k_{NR} = 0.4$. Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.53$, and initial value is (0.01, 0.01, 0.01, 0.2, 0.5, 1.2).

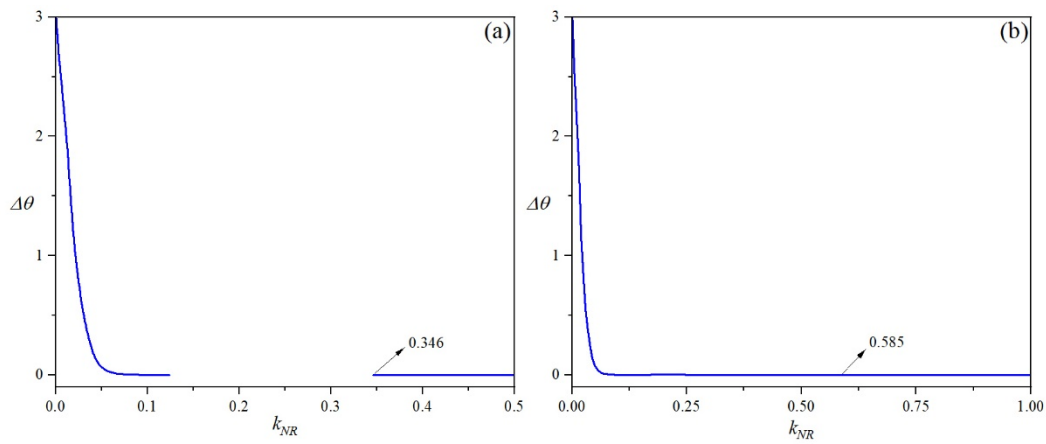


Figure 22. Distribution of average phase difference varies with the coupling strength k_{NR} . For (a) Direct coupling in Equation (13a); (b) ion channel shunting coupling in Equation (13b). Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.53$, and initial value is (0.01, 0.01, 0.01, 0.2, 0.5, 1.2).

Figures 20 and 21 demonstrate that both direct coupling and ion channel shunting coupling can achieve complete synchronization. At the coupling strength $k_{NR} = 0.4$, direct coupling rapidly attains full synchronization, whereas ion channel shunting coupling approaches synchronization but retains residual deviations. Figure 22 further reveals a divergent region in the direct coupling case (where numerical results tend to infinity), and the critical coupling strength required for synchronization under ion channel shunting coupling is considerably higher than that under direct coupling—a trend qualitatively consistent with the previous results obtained using linear resistive coupling. To further investigate the influence of coupling devices on neuronal synchronization, Figures 23–25 present the temporal evolution of the energy difference ΔH , synchronization error ER, and phase difference $\Delta\theta$ under both coupling schemes, with a memristor as the coupling element.

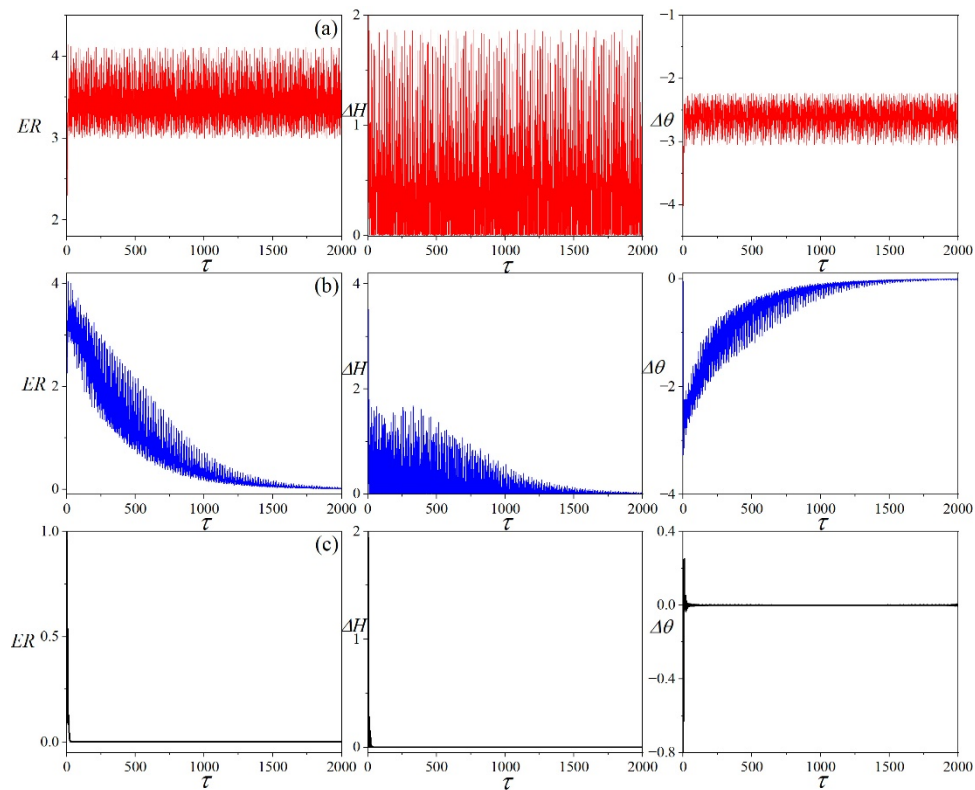


Figure 23. Time evolutions of the synchronization error ER, energy difference ΔH , and phase difference $\Delta\theta$ for the system described by Equation (15a). The coupling strength is set to (a) $k_M = 0$, (b) $k_M = 0.0009$ and (c) $k_M = 0.4$. Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.54$, $A_m = 0.001$, $B_m = 0.5$, and the initial conditions are fixed at (0.01, 0.01, 0.01, 0.2, 0.5, 1.2, 0).

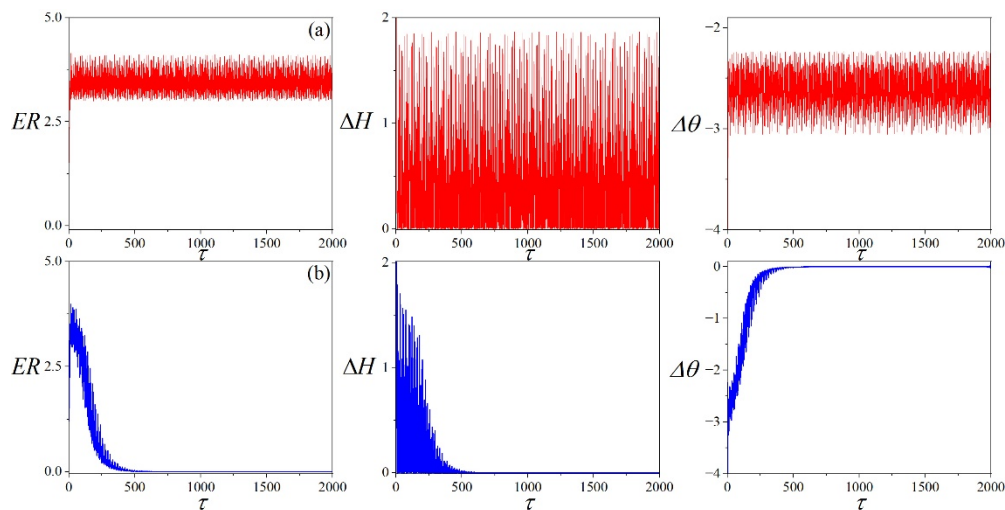


Figure 24. Time evolutions of the synchronization error ER , energy difference ΔH , and phase difference $\Delta\theta$ for the system described by Equation (15b). The coupling strength is set to (a) $k_M = 0$ and (b) $k_M = 0.4$. Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.54$, $A_m = 0.001$, $B_m = 0.5$, and the initial conditions are fixed at $(0.01, 0.01, 0.01, 0.2, 0.5, 1.2, 0)$.

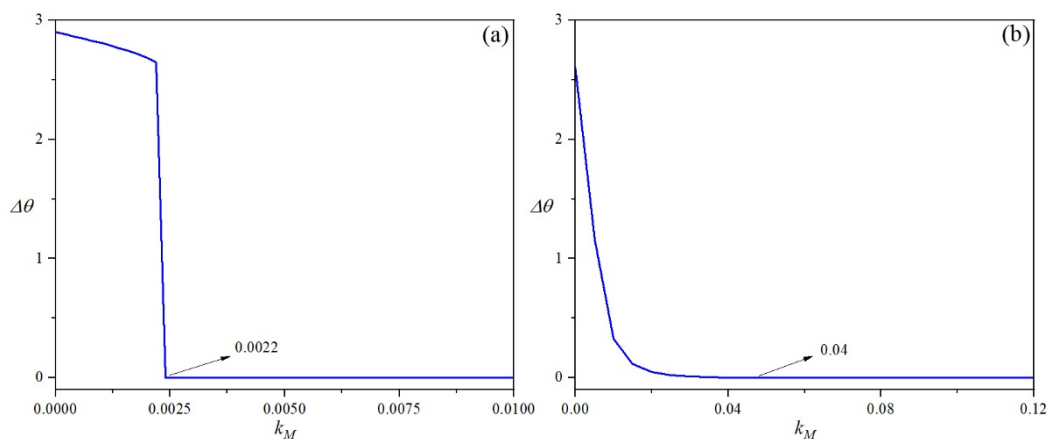


Figure 25. Distribution of average phase difference varies with the coupling strength k_M . For (a) Direct coupling in Equation (15a); (b) ion channel shunting coupling in Equation (15b). Parameter setting $a = 0.5$, $b = 0.1$, $c = 0.1$, $d = 0.8$, $\alpha = 0.6$, $\omega = 3.54$, $A_m = 0.001$, $B_m = 0.5$, and initial value is $(0.01, 0.01, 0.01, 0.2, 0.5, 1.2, 0)$.

Figures 23 and 24 demonstrate that complete synchronization can also be achieved when a memristor is employed as the coupling element. At $k_M = 0.4$, direct coupling converges significantly faster than ion channel shunting coupling. Figure 25 further reveals that the critical coupling strength required for synchronization under ion channel shunting coupling is markedly higher than that under direct coupling. We have plotted the critical coupling strengths for these six situations in a table, as shown in Table 1.

Table 1. Synchronization criticality of 6 types of coupling strategies.

Coupling Method	Direct Coupling	Ion Channel Shunting Coupling
Linear resistance R	0.04	0.245
Nonlinear resistance NR	0.346	0.585
Magnetic flux-controlled memristor MFCM	0.0022	0.04

Taken together, the results from Figures 17–25 lead to a consistent conclusion: regardless of the coupling element (linear resistor, nonlinear resistor, or memristor) and the coupling scheme (direct or ion channel shunting), the system can ultimately achieve complete synchronization; however, the ion channel shunting coupling consistently requires a higher critical coupling strength, implying a greater cost in terms of energy and synchronization. Using MFCM devices as coupling devices results in a lower synchronization cost compared to R and NR .

In summary, this paper designs a neuronal circuit with rich firing modalities by incorporating a voltage source in series with a nonlinear resistor and integrating an ion channel shunting mechanism. The firing dynamics of the resulting circuit are systematically investigated from three aspects: nonlinear dynamical analysis, energy evolution characterization, and coupling-modulated synchronization. The results demonstrate that both internal system parameters and external stimuli can effectively regulate the dynamical states of the neuronal system. Specifically, variations in internal parameters induce bifurcation-mediated firing pattern transitions; appropriate noise triggers SR; and adaptive regulation realizes self-tuning capability towards steady-state convergence. Regarding synchronization, complete synchronization is attainable for all coupling devices considered (linear resistor, nonlinear resistor, and memristor); however, the ion channel shunting coupling consistently requires a higher critical coupling strength than direct coupling. These findings provide a valuable theoretical foundation for designing synchronization control strategies based on ion channel shunting mechanisms in neuronal circuits.

Recent studies on synchronization have addressed diverse aspects relevant to this work. On the theoretical side, Jafari et al. uncovered new classes of synchronization stability in periodic systems [69], and extended the master stability function (MSF) framework to torus and stable equilibrium attractors [70], offering analytical tools that can be directly applied to assess the stability of synchronous regimes. In terms of specific neuronal systems, Qi et al. explored synchronization of bi-membrane nanofluidic memristive neuron circuits under resistive coupling [71], and examined bursting synchronization and excitability of bi-membrane neuron-like circuits under electromagnetic stimuli [72]; Chen et al. studied synchronization of starlike neural networks with electromagnetic autapse under electrical and fast chemical synapses [73]; and Shao et al. analyzed bursting dynamics and synchronization of neuromorphic systems with VO₂ memristors and Josephson junctions [74]. Together, these theoretical and experimental/numerical investigations underscore the critical roles of coupling strength, external stimulation, network topology, and MSF in modulating synchronous behavior.

4. Conclusions

In this paper, a neuronal circuit model with rich firing modalities is constructed by introducing a voltage source in series with a nonlinear resistor and incorporating an ion channel shunting mechanism. The effects of shunting control on neuronal firing behavior and synchronization are systematically investigated from three perspectives: nonlinear dynamics, Hamilton energy response, and coupling-modulated synchronization. The results demonstrate that both internal parameters (a , c) and external excitations (driving frequency ω , stimulus intensity α) can effectively modulate firing patterns. Notably, a and c exhibit opposite trends in energy regulation, while moderate noise can optimize firing regularity and promote energy accumulation via SR. An energy-based adaptive control strategy (with parameters σ and γ) can actively switch the system between chaotic and periodic regimes, confirming energy as a valid criterion for state regulation. Regarding synchronization, all six coupling schemes (three coupling elements $\times$ two topologies) lead to complete synchronization; however, the ion channel shunting coupling consistently requires higher critical coupling strengths than direct coupling, implying that it offers richer regulatory flexibility at the cost of higher coupling intensity. These findings provide a theoretical basis for the dynamical design and synchronization control of ion channel shunting neural circuits.

Author Contributions

X.D.: data curation, software, visualization, methodology; Z.L.: validation, methodology, supervision, writing—original draft preparation, reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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Informed Consent Statement

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Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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