



Nonlinear Quantum Waves in Semiconductor Superlattices

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Abstract: Semiconductor superlattices constitute an ideal laboratory for investigating coherent electron dynamics and nonlinear charge transport in periodic nanostructures. Their miniband electronic structure gives rise to Bloch oscillations, Wannier-Stark localization, resonant tunneling, electric-field domain formation, and self-sustained current oscillations over a wide range of temporal and spatial scales. This article reviews the multiscale theoretical framework developed to describe these phenomena, from semiclassical Boltzmann-Poisson-BGK kinetic equations to fully quantum Wigner-Poisson formulations. It shows how systematic asymptotic analysis bridges microscopic quantum kinetics and macroscopic nonlinear transport. Systematic hydrodynamic reductions yield generalized and quantum drift-diffusion equations that retain the essential effects of scattering, self-consistent electrostatics, coherent tunneling, and quantum interference while remaining amenable to analytical investigation and efficient numerical simulation. These reduced descriptions account for Gunn-type oscillations mediated by traveling waves, the coexistence of Bloch and Gunn oscillations under appropriate conditions, and nonlinear transport involving multiple minibands. The review concludes by discussing how the methodology developed for semiconductor superlattices may be extended to emerging quantum materials, including moiré superlattices, twisted bilayer graphene, Floquet-engineered systems, and topological materials, where Berry curvature, quantum geometry, orbital magnetic moments, and strong electronic correlations enrich the dynamics of coherent quantum waves.

Keywords: bloch oscillations; semiclassical kinetic equation; quantum kinetic equation; self-sustained oscillations; nonlinear waves

1. Introduction

In a periodic potential, the wave nature of electrons gives rise to Bloch states whose coherent evolution under an applied electric field produces phenomena such as Bloch oscillations (BOs), Wannier-Stark localization [1–4], and sequential resonant tunneling [5]. To observe BOs at experimentally accessible frequencies, Esaki and Tsu proposed an artificial crystal consisting of alternating semiconductor layers with different band gaps but closely matched lattice constants [6]. Owing to the much larger lateral dimensions of the layers compared with the superlattice period, the conduction-band profile forms an effectively one-dimensional sequence of quantum wells separated by barriers; see Figure 1. The corresponding Bloch states form energy minibands that support coherent Bloch oscillations. Such oscillations can be observed when the Bloch frequency exceeds the inverse scattering time, typically a few picoseconds, corresponding to electric fields of approximately 10 to 100 kV/cm for a superlattice period of about 10 nm. More than two decades elapsed before Bloch oscillations were experimentally observed in semiconductor superlattices (SSLs), and the oscillations persisted for only a few periods before scattering destroyed their coherence [7]; the experimental work on Bloch oscillations in strongly coupled semiconductor superlattices has been reviewed in [8]. Nevertheless, the subsequent development of both theory and experiment established SSLs as prototypical systems for studying nonlinear electron transport and coherent quantum dynamics.



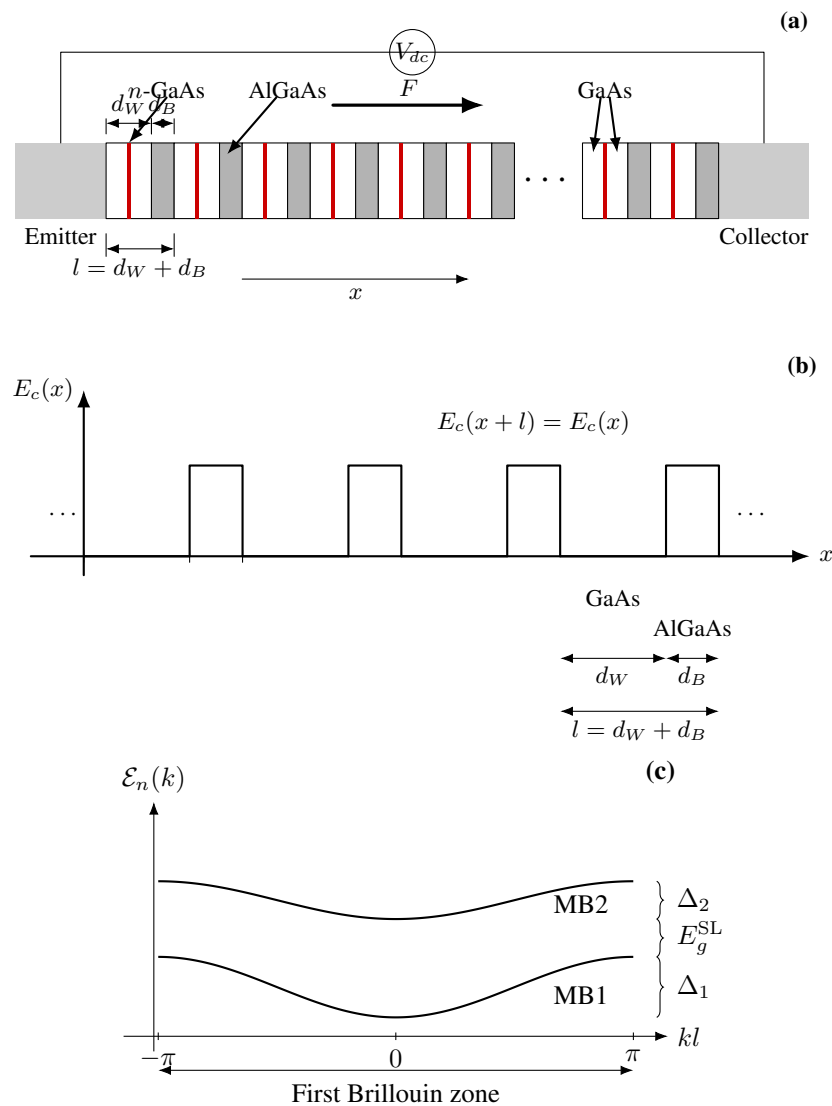


Figure 1. Hierarchy of physical descriptions of a semiconductor superlattice. (a) Schematic of a finite, δ -doped GaAs/AlGaAs superlattice at zero applied electric field, showing the conduction-band edge $E_c(x)$, the superlattice period $l = d_W + d_B$, and the well and barrier widths d_W and d_B . (b) Conduction-band profile of an ideal infinite superlattice, illustrating the periodicity $E_c(x + l) = E_c(x)$. (c) Miniband dispersion relation $E_n(k)$ resulting from the periodic potential. The sequence of panels illustrates the progression from the physical structure of the device to its quantum-mechanical description in terms of minibands.

Although scattering suppresses long-lived terahertz Bloch oscillations, semiconductor superlattices exhibit a rich variety of nonlinear transport phenomena at microwave and gigahertz frequencies. These include electric-field domain formation, self-sustained current oscillations mediated by traveling waves, and deterministic chaos [9–12]. Throughout this review we consider strongly coupled SSLs, in which the electronic states are described by extended miniband wave functions. Even in the absence of sustained Bloch oscillations, these structures can exhibit fast current oscillations generated by the repeated nucleation of nonlinear charge waves at the injecting contact and their propagation toward the collector, where they eventually disappear. This mechanism is closely analogous to the Gunn effect in bulk semiconductors [13,14]. The experimental work on nonlinear phenomena including oscillations and comparison with theory for both strongly and weakly coupled superlattices is reviewed in [10].

The purpose of this article is to review the multiscale theoretical framework that has been developed to describe nonlinear quantum waves in semiconductor superlattices. It shows how systematic asymptotic analysis bridges microscopic quantum kinetics and macroscopic nonlinear transport. Starting from semiclassical and quantum kinetic equations, systematic hydrodynamic reductions lead to reduced transport models that retain the essential physics while remaining suitable for mathematical analysis and numerical simulation. The same methodology applies to both Boltzmann-Poisson equations with a Bhatnagar-Gross-Krook (BGK) collision operator [14–16] and Wigner-Poisson-BGK kinetic equations describing one- and two-miniband superlattices [17–20]. For appropriate parameter regimes, the semiclassical theory predicts the coexistence of Bloch and Gunn-type oscillations [21],

whereas the quantum formulation yields spatially nonlocal drift-diffusion equations that consistently incorporate coherent tunneling, quantum interference, scattering, and self-consistent electrostatics.

Taken together, these developments establish a unified hierarchy of models connecting microscopic Bloch-state and Wigner-function descriptions with kinetic, hydrodynamic, and drift-diffusion theories of nonlinear transport, as sketched in Figure 2. Beyond their application to semiconductor superlattices, these multiscale methods provide a systematic route for deriving macroscopic transport equations from quantum kinetic theory while consistently incorporating scattering and self-consistent electrostatic interactions. Recent developments in moiré superlattices, twisted bilayer graphene [22, 23], Floquet-engineered systems [24], and topological materials suggest a much broader range of applications. In these systems, Berry curvature [25–28], orbital magnetic moments, quantum geometry [29, 30], and strong electronic correlations enrich coherent transport and nonlinear wave dynamics, opening new directions for extending the theoretical framework reviewed here.

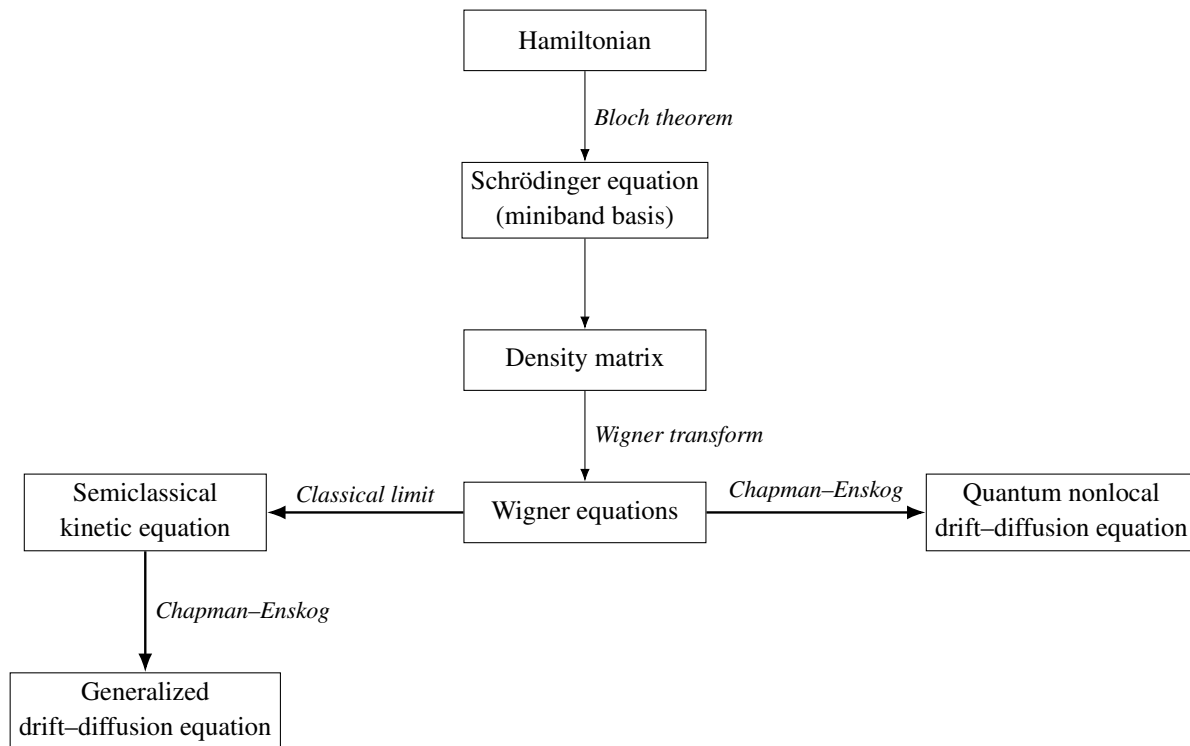


Figure 2. Hierarchy of theoretical descriptions employed in this review. The central vertical sequence represents mathematically equivalent descriptions of the same transport problem, beginning with the Hamiltonian and proceeding through Bloch’s theorem, the miniband representation of the Schrödinger equation, the density-matrix formalism, and the Wigner representation. The labeled arrows denote systematic asymptotic procedures. The classical limit leads to the semiclassical kinetic equation, whose Chapman-Enskog reduction yields generalized drift-diffusion equations. A Chapman-Enskog reduction can also be performed directly at the Wigner level, producing quantum nonlocal drift-diffusion equations.

This review is organized according to the hierarchy of theoretical descriptions used to model charge transport in semiconductor superlattices. Rather than presenting the different models independently, emphasis is placed on the mathematical connections between them and on the systematic asymptotic procedures that relate microscopic quantum dynamics to macroscopic transport equations. Figure 1 introduces the physical structure of semiconductor superlattices and the origin of the miniband description, whereas Figure 2 summarizes the logical organization of the theoretical framework discussed throughout the review.

Together, Figures 1 and 2 provide the physical and theoretical roadmap for the remainder of the review. Section 2 introduces the semiclassical kinetic description and its hydrodynamic reduction leading to generalized drift-diffusion equations. Section 3 presents the corresponding quantum formulation based on Wigner functions and the derivation of nonlocal quantum drift-diffusion equations. The final section summarizes the main achievements of this multiscale approach and discusses possible extensions to emerging quantum materials.

2. Semiclassical Formulation and Reduced Dynamics

We begin with the semiclassical description of electron dynamics within a single miniband. Let $\mathcal{E}(k)$ be the energy dispersion relation of a miniband with 1D wave vector k . In the semiclassical approximation, an electron with charge $-e$ ($e > 0$) subject to a constant electric field $-F$ evolves according to

$$\frac{dx}{dt} = \frac{1}{\hbar} \frac{d\mathcal{E}}{dk}, \quad \frac{dk}{dt} = \frac{eF}{\hbar}. \quad (1)$$

Since the group velocity is $v(k) = \mathcal{E}'(k)/\hbar$, substituting the accelerated wave vector $k = k_0 + eFt/\hbar$ into the periodic dispersion relation shows that the electron wave packet undergoes periodic motion in real space. This periodic motion constitutes the Bloch oscillation.

2.1. Boltzmann-Poisson-BGK Kinetic Equation

To describe transport in a finite voltage-biased superlattice, electron-electron interactions and scattering by impurities, phonons, and interface roughness must be incorporated. Within the Hartree approximation, a simple description is provided by the Boltzmann-Poisson equation with a BGK collision operator,

$$\frac{\partial f}{\partial t} + v(k) \frac{\partial f}{\partial x} + \frac{eF}{\hbar} \frac{\partial f}{\partial k} = -\nu_e (f - f^{FD}), \quad (2a)$$

$$\varepsilon \frac{\partial F}{\partial x} = \frac{e}{l} (n - N_D), \quad (2b)$$

$$n = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} f(x, k, t) dk = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} f^{FD}(k; n) dk, \quad (2c)$$

$$f^{FD}(k; n) = \frac{m^* k_B T}{\pi \hbar^2} \ln \left[1 + \exp \left(\frac{\mu - \mathcal{E}(k)}{k_B T} \right) \right]. \quad (2d)$$

Here l , ε , f , n , N_D , μ , k_B , T , and m^* are the SSL period, the dielectric constant, the one-particle distribution function, the 2D electron density, the 2D doping density, the chemical potential, the Boltzmann constant, the lattice temperature, and the effective mass of the electron, respectively. The collision term describes relaxation towards a 1D effective Fermi-Dirac distribution $f^{FD}(k; n)$ (local equilibrium) [9,10] with collision frequency ν_e taken to be constant throughout. For a tight binding dispersion relation in the high temperature (Boltzmann) limit, Equation (2d) becomes

$$f^B(k; n) = \frac{n e^{\frac{\Delta \cos kl}{2k_B T}}}{I_0\left(\frac{\Delta}{2k_B T}\right)}, \quad \mathcal{E}(k) = \frac{\Delta}{2} (1 - \cos kl). \quad (2e)$$

The global stability of the local equilibrium (2d) can be proved for an infinitely long superlattice and for a finite SSL with zero-flux boundary conditions under zero voltage bias [16]. Recent studies continue to show that microscopic scattering mechanisms, particularly interface-roughness and momentum relaxation, can have a strong influence on nonlinear miniband transport in semiconductor superlattices [31,32].

A reduced hydrodynamic description is obtained through an asymptotic expansion in the hyperbolic limit, in which the electric-field term and the collision term dominate the transport dynamics, whereas temporal and spatial variations act as higher-order perturbations. To leading order,

$$\begin{aligned} \frac{eF}{\hbar} \frac{\partial f^{(0)}}{\partial k} &= -\nu_e (f^{(0)} - f^{FD}) \implies \\ f^{(0)} &= \sum_{j=-\infty}^{\infty} e^{ijkl} \frac{\nu_e - ij\omega_B}{j^2 \omega_B^2 + \nu_e^2} f_j^{FD}, \quad \omega_B = \frac{eFl}{\hbar}, \end{aligned} \quad (3)$$

where $f_j^{FD}(n)$ is the j th Fourier coefficient of the $2\pi/l$ -periodic local equilibrium distribution function and ω_B is the Bloch frequency. After nondimensionalization, the temporal and spatial derivatives in Equation (2a) are multiplied by a small dimensionless parameter $\delta = ev_M N_D / (\varepsilon F_M l \nu_e)$, which measures the ratio between the macroscopic transport time scale and the microscopic collision time $1/\nu_e$; see Equations (7) below. Setting $\delta = 0$ produces Equation (3).

To derive macroscopic transport equations avoiding nondimensionalization, we formally reintroduce the small parameter δ multiplying the temporal and spatial derivatives and seek an asymptotic solution using the Chapman-Enskog method [14]. At the end of the calculation the physical value $\delta = 1$ is restored. Accordingly, we expand both the distribution function and the electron-density evolution equation as

$$f(x, k, t; \epsilon) = f^{(0)}(k; n) + \sum_{m=1}^{\infty} f^{(m)}(k; n) \delta^m, \quad (4a)$$

$$\frac{\partial n}{\partial t} = \sum_{m=0}^{\infty} N^{(m)}(n) \delta^m. \quad (4b)$$

The higher-order corrections $f^{(m)}$ depend on the slow variables x and t only through the electron density and the electric field, the latter being determined self-consistently by Poisson's equation. The functionals $N^{(m)}(n)$ are chosen so that every correction remains bounded and $2\pi/l$ -periodic in wave-vector space. In addition, the conditions $f_0^{(m)}/l = 0$, $m \geq 1$ ensure that the $f^{(m)}$ do not contain contributions proportional to the zero-order term $f^{(0)}$. $N^{(m)}(n)$ can be found by integrating (2a) over k , using (2b), and inserting (4a) in the result: $N^{(m)}(n) = -l (\partial/\partial x) \int_{-\pi/l}^{\pi/l} v(k) f^{(m)} dk / (2\pi)$. Then, integration of (2b) over x yields the Ampère law

$$\epsilon \frac{\partial F}{\partial t} + \frac{e}{2\pi} \sum_{m=0}^{\infty} \delta^m \int_{-\pi/l}^{\pi/l} v(k) f^{(m)}(k; n) dk = J(t), \quad (5)$$

where $J(t)$ is the total current density. The terms on the left hand side of this equation are the displacement current and the electron current.

Substituting the Chapman-Enskog expansions into the kinetic equation and collecting equal powers of δ yields

$$\mathcal{L}f^{(1)} = - \left(\frac{\partial}{\partial t} + v(k) \frac{\partial}{\partial x} \right) f^{(0)} \Big|_0, \quad (6a)$$

$$\mathcal{L}f^{(2)} = - \left(\frac{\partial}{\partial t} + v(k) \frac{\partial}{\partial x} \right) f^{(1)} \Big|_0 - \frac{\partial}{\partial t} f^{(0)} \Big|_1, \quad (6b)$$

and so on. We have defined $\mathcal{L}u(k) \equiv eF\hbar^{-1} du(k)/dk + \nu_e u(k)$, and the subscripts 0 and 1 mean that $\partial n/\partial t$ is replaced by $N^{(0)}(n)$ and by $N^{(1)}(n)$, respectively. The linear problem $\mathcal{L}u = S$ admits a bounded $2\pi/l$ -periodic solution provided that $\int_{-\pi/l}^{\pi/l} S dk = 0$. This solvability condition applied to Equations (6) yields the previously found $N^{(m)}$ and the reduced Equation (5).

Retaining terms up to first order in δ (and subsequently setting $\delta = 1$) yields the generalized drift-diffusion equation (GDDE) for the electric field, $F(x, t)$, which is first order in time and second order in x [15]. In the Boltzmann limit, of Equation (2c) the simplified form of this equation is [14, 15]

$$\begin{aligned} \epsilon \frac{\partial F}{\partial t} + \mathcal{V} \left(F, \frac{\partial F}{\partial x} \right) \frac{eN_D}{l} \left(1 + \frac{\epsilon l}{eN_D} \frac{\partial F}{\partial x} \right) \\ = D \left(F, \frac{\partial F}{\partial x} \right) \epsilon \frac{\partial^2 F}{\partial x^2} + A \left(F, \frac{\partial F}{\partial x} \right) J(t), \end{aligned} \quad (7a)$$

$$A = 1 + \frac{2ev_M F_M^3 [F_M^2 - 3F^2]}{\epsilon l \nu_e (F_M^2 + F^2)^3} n, \quad (7b)$$

$$\mathcal{V} = v_M V(F) \left(A - \frac{\Delta B}{2e} \frac{\partial F}{\partial x} \right), \quad (7c)$$

$$V(F) = \frac{2F_M F}{F_M^2 + F^2}, \quad v_M = \frac{\Delta l I_1 \left(\frac{\Delta}{2k_B T} \right)}{4\hbar I_0 \left(\frac{\Delta}{2k_B T} \right)}, \quad (7d)$$

$$D = \frac{\Delta^2 l F_M}{8\hbar e (F_M^2 + F^2)} \left(1 - \frac{4\hbar v_M C}{\Delta l} \right), \quad (7e)$$

$$B = \frac{(5F_M^2 - 4F^2) I_2 \left(\frac{\Delta}{2k_B T} \right)}{(F_M^2 + 4F^2)^2 I_0 \left(\frac{\Delta}{2k_B T} \right)} - \frac{8\hbar v_M F_M^2 (F_M^2 - F^2)}{\Delta l (F_M^2 + F^2)^3}, \quad (7f)$$

$$C = \frac{(F_M^2 - 2F^2) I_2 \left(\frac{\Delta}{2k_B T} \right)}{(F_M^2 + 4F^2) I_0 \left(\frac{\Delta}{2k_B T} \right)} + \frac{8\hbar v_M F^2 F_M^2}{\Delta l (F_M^2 + F^2)^2}. \quad (7g)$$

An often-used DDE consists of Equation (7a) with $\mathcal{V} = v_M V(F)$, $A = 1$, and a diffusion term obeying the Einstein relation

$$\varepsilon \frac{\partial F}{\partial t} + \frac{ev_M n}{l} V(F) = J(t) + \frac{k_B T v_M}{Fl} V(F) \frac{\partial n}{\partial x}. \quad (7h)$$

Equation (7d) corresponds to the tight-binding dispersion relation (2e). To solve either the GDDE (7a) or the DDE (7h), we have to impose an initial condition for $F(x, 0)$ and boundary conditions at the injecting contact $x = 0$ (typically Ohm's law linearly relating electric field and electron current) and at the collecting contact (charge neutrality $n = N_D$ is convenient). The total current density is obtained by adding a voltage bias condition: the area under the electric field is the dc voltage, $V_{dc} = \int_0^L F(x, t) dx$ ($L = Nl$, where N is the number of SSL periods), that acts as a control parameter.

The GDDE [or the DDE (7h)] is solved subject to an initial electric-field profile together with boundary conditions at the injecting and collecting contacts. The total current density is determined by imposing the voltage-bias constraint $V_{dc} = \int_0^L F(x, t) dx$ ($L = Nl$, where N is the number of SSL periods), that acts as a control parameter [15].

Solving the initial-boundary value problem for the GDDE yields a constant current $J(t)$ for V_{dc} on the interval $(0, V_c)$. The nonzero mean current reflects the average transport through the superlattice under the applied dc voltage, while the oscillatory component results from the repeated nucleation and propagation of the charge pulse. Beyond the critical voltage V_c , the stationary field profile with constant current through the SSL is no longer stable and the current becomes periodic in time as shown in Figure 3a. The current self-sustained oscillations are due to the repeated nucleation of a triangular pulse wave at $x = 0$ that moves on a stationary field profile that contributes a voltage below V_c . The wave carries the excess voltage towards the collecting contact at $x = L$, and disappears there as a new wave is nucleated. Figure 3b shows the pulse when it is far from the contacts. These Gunn-type oscillations and their stability can be further studied by asymptotic methods when the size of the pulse is small compared to L ; see chapter 6 of [14] and references cited therein. Table 3 of [10] compares the frequencies predicted by the GDDE, the DDE and the asymptotic theory to those measured by Schomburg et al. in different strongly coupled SSLs [33]. The GDDE predictions are closest to observations. See also Table 4 of [10] that includes further experimental data.

The Chapman-Enskog reduction thus provides a systematic route from the microscopic Boltzmann-Poisson-BGK equation to a macroscopic generalized drift-diffusion equation. Despite its reduced form, the GDDE retains the essential nonlinear transport mechanisms responsible for charge-domain dynamics and self-sustained current oscillations. We next examine how this framework must be modified to describe coherent Bloch oscillations and their coexistence with Gunn-type dynamics.

The kinetic model considered thus far assumes that Bloch oscillations are completely suppressed by scattering. The next question is whether coherent Bloch oscillations and Gunn-type oscillations can coexist within a suitable kinetic description. If we include BOs among the solutions of the kinetic equation with the simple local equilibrium distribution of Equation (2d), they are rapidly damped on the time scale of the charge density [21]. This observation indicates that the local equilibrium must be generalized to include the average velocity and average energy as additional hydrodynamic variables. While the collisions preserve charge density to yield charge conservation, they dissipate energy and momentum. Within a BGK format, we need the local equilibrium to depend on the chemical potential μ_α , the average velocity u_α and the average temperature T_α as

$$f^{1D\alpha}(k; \mu_\alpha, u_\alpha, T_\alpha) = \frac{m^* k_B T_\alpha}{\pi \hbar^2} \ln \left[1 + \exp \left(\frac{\mu_\alpha + \hbar k u_\alpha - \mathcal{E}(k)}{k_B T_\alpha} \right) \right]. \quad (8)$$

The functions μ_α , u_α and T_α depend on the distribution function through the equations

$$\frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} f^{1D\alpha} dk = n, \quad (9a)$$

$$\frac{e}{2\pi} \int_{-\pi/l}^{\pi/l} v(k) f^{1D\alpha} dk = (1 - \alpha_j) J_n, \quad (9b)$$

$$\frac{l}{2\pi n} \int_{-\pi/l}^{\pi/l} \mathcal{E}(k) f^{1D\alpha} dk = \alpha_e E_0 + (1 - \alpha_e) E. \quad (9c)$$

Here

$$J_n(x, t) = \frac{e}{2\pi} \int_{-\pi/l}^{\pi/l} v(k) f(x, k, t) dk, \quad (9d)$$

$$E(x, t) = \frac{l}{2\pi n(x, t)} \int_{-\pi/l}^{\pi/l} \mathcal{E}(k) f(x, k, t) dk, \quad (9e)$$

$$E^{1D}(x, t) = \frac{l}{2\pi n(x, t)} \int_{-\pi/l}^{\pi/l} \mathcal{E}(k) f^{1D}(x, k, t) dk, \quad (9f)$$

are electronic current and energy densities, and E_0 is the mean energy at the lattice temperature of the global equilibrium reached by a finite SSL with insulating contacts and zero voltage bias. When the restitution coefficients $\alpha_{j,e} \in [0, 1]$ are zero, the corresponding collisions are elastic and preserve current ($\alpha_j = 0$) and energy ($\alpha_e = 0$).

The derivation of the corresponding hydrodynamic equations proceeds by introducing the Bloch phase as a fast variable and assuming restitution coefficients of order δ . A Chapman-Enskog expansion then yields evolution equations for the electron density, current density, energy density, and the complex amplitude of the Bloch oscillations. See the details for a particular tight binding dispersion relation in [21]. The total current density displays a fast Bloch oscillation modulated by a slow Gunn-type oscillation; see Figure 1 of [21]. The calculations illustrate the coexistence of BOs of 0.36 THz frequency with Gunn-type oscillations of 13.8 GHz frequency. The current density averaged over the fast Bloch phase is depicted in Figure 4a, whereas Figure 4b shows the field profiles corresponding to the four times marked in panel (a). Figure 4 shows that the Gunn-type oscillations originate from the repeated emission of a charge pulse at the injecting contact, its propagation across the superlattice, and its eventual disappearance at the collector. Figure 4c displays the amplitude of the BOs that evolves on the slower scale of the Gunn-type oscillations [21].

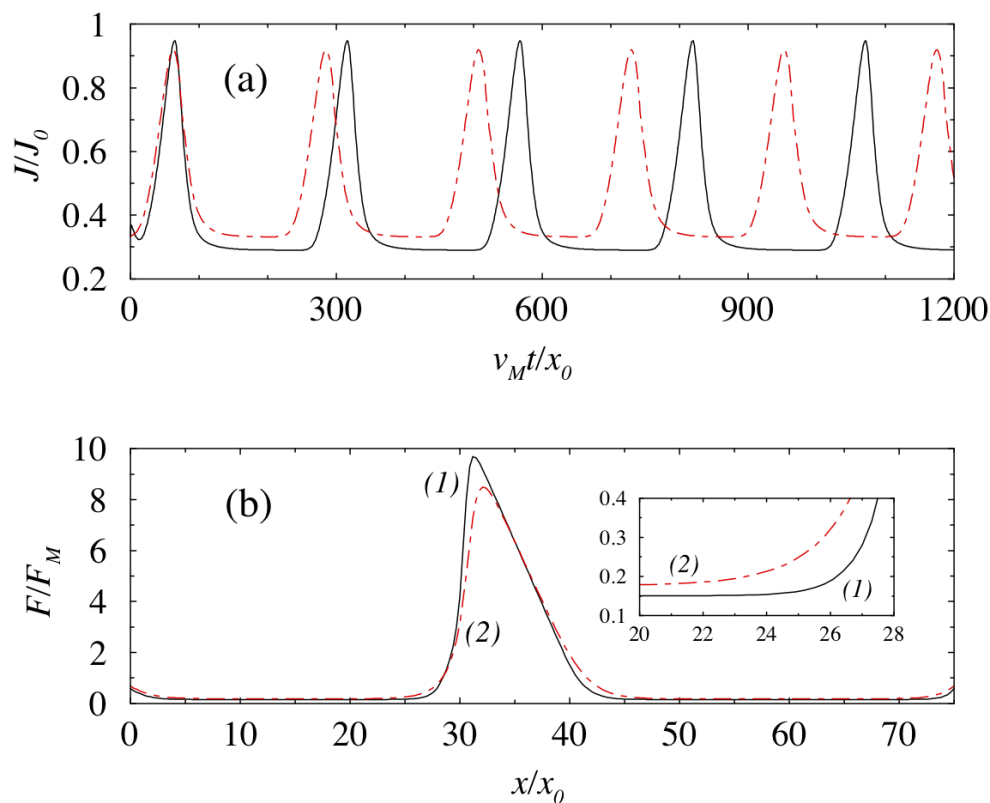


Figure 3. (a) Current ($J_0 = ev_M N_D/l$) vs. time during self-oscillations for a 100-period SL at 300 K, as described by the GDDE in the Boltzmann limit (solid line) and by the approximate equation whose diffusion satisfies the Einstein relation (DDE, dashed line). (b) Comparison between the pulse wave for the GDDE, (1), and the pulse wave for the DDE, (2). Here $F_M = \hbar\nu_e/(el)$ and $x_0 = 0.75l$. Reprinted with permission from Ref. [15]. Copyright 2003, American Physical Society.

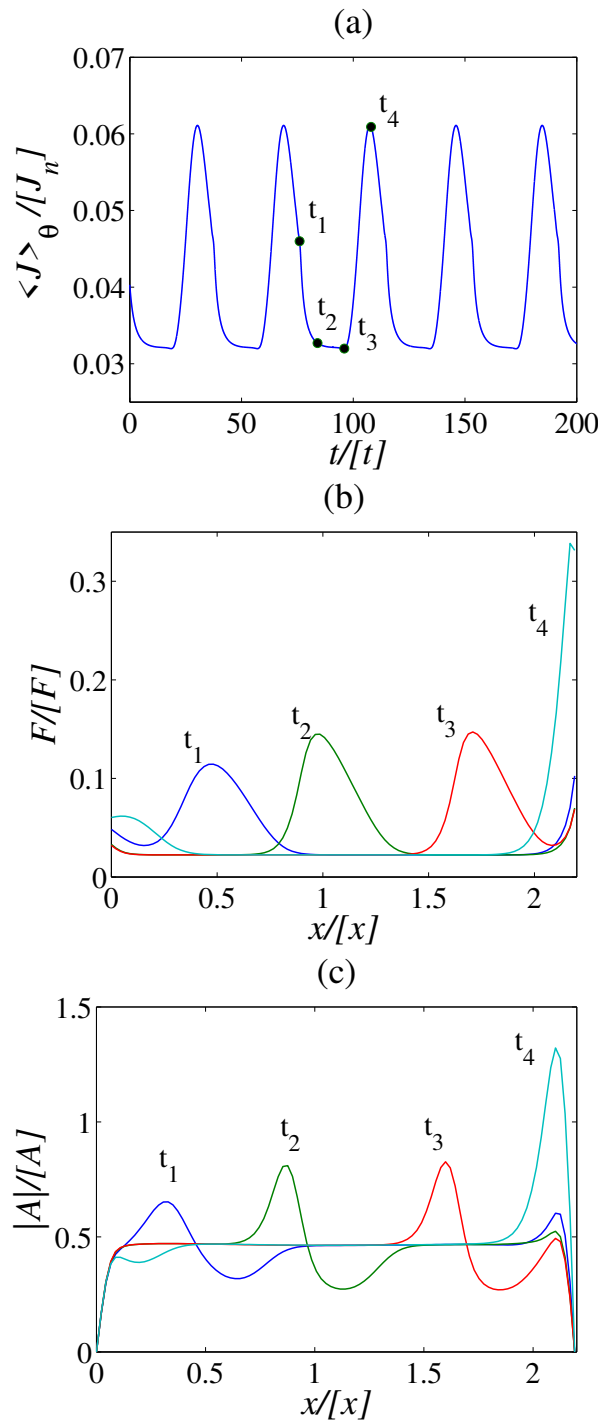


Figure 4. (a) Phase-averaged total current density vs time during coexisting Bloch and Gunn-type oscillations at 70 K in a 50-period SSL. (b) Field profile vs space at the times t_1 to t_4 marked in (a). (c) Same for the complex BO amplitude profile. Here $[A] = N_D = 4.08 \times 10^{10} \text{ cm}^{-2}$, $[F] = 130 \text{ kV/cm}$, $[J_n] = 78.8 \text{ kA/cm}^2$, $[t] = 1.88 \text{ ps}$, $[x] = 116 \text{ nm}$, $l = 5.06 \text{ nm}$, $V_{dc} = 0.133 \text{ V}$. Reprinted with permission from Ref. [21]. Copyright 2011, American Physical Society.

3. Quantum Kinetic Equation

We now turn to a fully quantum-mechanical description based on the Wigner-function formalism. Unlike the semiclassical Boltzmann approach, the Wigner representation naturally incorporates coherent tunneling, quantum interference, and nonlocal transport while preserving a phase-space description that is well suited for hydrodynamic reductions. As in the semiclassical case, our objective is not to solve the kinetic equation directly but to derive reduced macroscopic transport equations through a systematic asymptotic procedure. In the absence of an externally applied electric field, the superlattice Hamiltonian is written as

$$H = H_0 + H_{e-e} + H_{sc}, \quad (10a)$$

$$H_0 = -\frac{\hbar^2}{2m^*} \nabla^2 + v(x), \quad v(x+l) = v(x), \quad (10b)$$

H_{e-e} and H_{sc} contain the contributions of electron-electron interaction and scattering, respectively. The wave function and the dispersion relation are

$$\varphi_\nu(\mathbf{x}, \mathbf{k}) = \frac{1}{\sqrt{S}} e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} \psi_\nu(x, k), \quad \nu = 1, 2, \dots \quad (10c)$$

$$E_\nu(k, \mathbf{k}_\perp) = \frac{\hbar^2 \mathbf{k}_\perp^2}{2m^*} + \mathcal{E}_\nu(k), \quad \mathcal{E}_\nu\left(k + \frac{2\pi}{l}\right) = \mathcal{E}_\nu(k), \quad (10d)$$

Here S denotes the cross-sectional area, cross-sectional area of the superlattice, $\mathbf{x}_\perp = (y, z)$ and $\mathbf{k}_\perp = (k_y, k_z)$ are the transverse coordinates and wave vector, respectively, and ν labels the minibands. The derivation of a quantum DDE proceeds through the following sequence of steps [17]:

- (1) Find the Schrödinger equation for miniband wave functions including an external potential that solves the Poisson equation.
- (2) Define the miniband density matrix and derive its governing equation.
- (3) Find the Wigner transform of the band density matrix.
- (4) Assume only one miniband is populated and find its Wigner equation.
- (5) Model the collision term so as to facilitate study of hydrodynamic limit.
- (a) Schrödinger equation in the miniband representation.

We define the band wave functions $\psi_\nu(\mathbf{x}, t)$ by

$$\psi(\mathbf{x}, t) = \sum_{\nu, \mathbf{k}} a_{\nu, \mathbf{k}} \varphi_\nu(\mathbf{x}, \mathbf{k}) \equiv \sum_{\nu} \psi_\nu(\mathbf{x}, t). \quad (11)$$

From the Schrödinger equation for $H_0 + H_{e-e}$, we obtain

$$i\hbar \frac{\partial \psi_\nu}{\partial t} = -\frac{\hbar^2}{2m^*} \frac{\partial^2 \psi_\nu}{\partial \mathbf{x}_\perp^2} + \sum_{m=-\infty}^{\infty} E_\nu(m) \psi_\nu(x + ml, \mathbf{x}_\perp, t) - e \sum_{\nu'} \int \Phi_\nu(\mathbf{x}, \mathbf{x}') W(\mathbf{x}') \psi_{\nu'}(\mathbf{x}', t) d\mathbf{x}', \quad (12a)$$

$$\mathcal{E}_\nu(k) = \sum_{m=-\infty}^{\infty} E_\nu(m) e^{imkl}, \quad (12b)$$

$$\Phi_\nu(\mathbf{x}, \mathbf{x}') = \sum_{\mathbf{k}} \varphi_\nu(\mathbf{x}, \mathbf{k}) \varphi_\nu^*(\mathbf{x}', \mathbf{k}). \quad (12c)$$

At this stage the electrostatic interaction is described within the Hartree approximation through the self-consistent potential $W(\mathbf{x})$, whereas scattering is temporarily neglected. The latter will be incorporated later through an appropriate collision operator.

- (b) Miniband density matrix and electron density.

The miniband density matrix is obtained by averaging over the statistical ensemble of occupied wave functions,

$$\rho_{\mu, \nu}(\mathbf{x}, \mathbf{y}, t) = \langle \psi_\mu^\dagger(\mathbf{x}, t) \psi_\nu(\mathbf{y}, t) \rangle, \quad (13a)$$

$$n(\mathbf{x}, t) = 2l \sum_{\mu, \nu} \rho_{\mu, \nu}(\mathbf{x}, \mathbf{x}, t). \quad (13b)$$

The evolution equation for the density matrix follows directly from the Schrödinger equation and has the form

$$\begin{aligned}
& i\hbar \frac{\partial}{\partial t} \rho_{\mu,\nu}(\mathbf{x}, \mathbf{y}, t) + \frac{\hbar^2}{2m^*} \left(\frac{\partial^2}{\partial \mathbf{y}_\perp^2} - \frac{\partial^2}{\partial \mathbf{x}_\perp^2} \right) \rho_{\mu,\nu}(\mathbf{x}, \mathbf{y}, t) \\
& - \sum_{m=-\infty}^{\infty} [E_\nu(m) \rho_{\mu,\nu}(\mathbf{x}, y + ml, \mathbf{y}_\perp, t) \\
& - E_\mu(m) \rho_{\mu,\nu}(x - ml, \mathbf{x}_\perp, \mathbf{y}, t)] \\
& + e \sum_{\nu'} \int W(\mathbf{z}) [\Phi_\nu(\mathbf{y}, \mathbf{z}) \rho_{\mu,\nu'}(\mathbf{x}, \mathbf{z}, t) \\
& - \Phi_\mu(\mathbf{z}, \mathbf{x}) \rho_{\nu',\nu}(\mathbf{z}, \mathbf{y}, t)] d\mathbf{z} = Q[\rho], \\
& \varepsilon \nabla^2 W = \frac{e}{l} (n - N_D).
\end{aligned}
\tag{14a}$$

$$\tag{14b}$$

Here $Q[\rho]$ is the collision term due to scattering (considered later).

(c) Wigner transform and periodic Wigner function.

The Wigner transform of the band density matrix is

$$w_{\mu,\nu}(\mathbf{x}, \mathbf{k}, t) = \int \rho_{\mu,\nu} \left(\mathbf{x} + \frac{1}{2} \boldsymbol{\xi}, \mathbf{x} - \frac{1}{2} \boldsymbol{\xi}, t \right) e^{i\mathbf{k} \cdot \boldsymbol{\xi}} d\boldsymbol{\xi}. \tag{15a}$$

Because the crystal momentum is defined modulo the reciprocal-lattice vector, the Wigner function has to be periodic in wave-vector space, leading naturally to

$$\begin{aligned}
f_{\mu,\nu}(\mathbf{x}, \mathbf{k}, t) &= \sum_{s=-\infty}^{\infty} w_{\mu,\nu} \left(\mathbf{x}, k + \frac{2\pi s}{l}, \mathbf{k}_\perp, t \right) = \sum_{j=-\infty}^{\infty} e^{ijkl} \\
&\times \int \rho_{\mu,\nu} \left(x + \frac{jl}{2}, \mathbf{x}_\perp + \frac{1}{2} \boldsymbol{\xi}_\perp, x - \frac{jl}{2}, \mathbf{x}_\perp - \frac{1}{2} \boldsymbol{\xi}_\perp, t \right) \\
&\times e^{i\mathbf{k}_\perp \cdot \boldsymbol{\xi}_\perp} d\boldsymbol{\xi}_\perp,
\end{aligned}
\tag{15b}$$

which is periodic in k .

(d) Wigner equation for a single populated miniband.

Defining $f = \sum_{\mu,\nu} f_{\mu,\nu}$ and assuming $f \approx f_{1,1}$ yields the Wigner transport equation for the 3D Wigner function $f(\mathbf{x}, \mathbf{k}, t)$:

$$\begin{aligned}
& \frac{\partial}{\partial t} f + \frac{\hbar k}{m^*} \frac{\partial}{\partial \mathbf{x}_\perp} f + \frac{i}{\hbar} \sum_{m=-\infty}^{\infty} e^{imkl} E_1(m) \\
& \times \left[f \left(x + \frac{ml}{2}, \mathbf{x}_\perp, \mathbf{k}, t \right) - f \left(x - \frac{ml}{2}, \mathbf{x}_\perp, \mathbf{k}, t \right) \right] \\
& + \frac{ie}{\hbar} \left[W \left(x + \frac{1}{2i} \frac{\partial}{\partial k}, \mathbf{x}_\perp \right) - W \left(x - \frac{1}{2i} \frac{\partial}{\partial k}, \mathbf{x}_\perp \right) \right] f \\
& = Q[f].
\end{aligned}
\tag{16}$$

Integrating the periodic Wigner function over the Brillouin zone and the transverse wave vector yields the three-dimensional electron density

$$\begin{aligned}
n(\mathbf{x}, t) &= \frac{2l}{(2\pi)^3} \sum_{\mu,\nu} \int_{-\frac{\pi}{l}}^{\frac{\pi}{l}} \int f_{\mu,\nu}(\mathbf{x}, \mathbf{k}, t) dk d\mathbf{k}_\perp \\
&= \frac{l}{4\pi^3} \int_{-\frac{\pi}{l}}^{\frac{\pi}{l}} \int f(\mathbf{x}, \mathbf{k}, t) dk d\mathbf{k}_\perp,
\end{aligned}
\tag{17}$$

which replaces Equation (13b).

(e) Simple model of collision term.

For a simple 1D BGK-type collision model, we integrate the Wigner function (15c) over the transversal position and wave vector to obtain

$$f(x, k, t) = \frac{2l}{(2\pi)^2 S} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, \mathbf{x}_\perp, k, \mathbf{k}_\perp, t) d\mathbf{x}_\perp d\mathbf{k}_\perp, \quad (18a)$$

$$n(x, t) = \frac{l}{2\pi} \int_{-\frac{\pi}{l}}^{\frac{\pi}{l}} f(x, k, t) dk. \quad (18b)$$

To obtain a tractable one-dimensional model, we average the Wigner function over the transverse coordinates and wave vector and adopt the same BGK relaxation model employed in the semiclassical description. Carrying out the transverse averaging of Equation (16) yields the one-dimensional Wigner-Poisson-BGK system

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{i}{\hbar} \left[\mathcal{E} \left(k + \frac{1}{2i} \frac{\partial}{\partial x} \right) - \mathcal{E} \left(k - \frac{1}{2i} \frac{\partial}{\partial x} \right) \right] f \\ + \frac{ie}{\hbar} \left[W \left(x + \frac{1}{2i} \frac{\partial}{\partial k}, t \right) - W \left(x - \frac{1}{2i} \frac{\partial}{\partial k}, t \right) \right] f \\ = -\nu_e (f - f^{FD}), \end{aligned} \quad (19a)$$

$$\varepsilon \frac{\partial^2 W}{\partial x^2} = \frac{e}{l} (n - N_D), \quad (19b)$$

$$n = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} f(x, k, t) dk = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} f^{FD}(k; n) dk, \quad (19c)$$

$$\begin{aligned} f^{FD}(k; n) = \frac{m^* k_B T}{\pi \hbar^2} \left\{ \tan^{-1} \left(\frac{\Gamma}{\mathcal{E}(k)} \right) \right. \\ \left. + \int_0^\infty \frac{\Gamma \ln \left[1 + \exp \left(\frac{\mu - E}{k_B T} \right) \right]}{[E - \mathcal{E}(k)]^2 + \Gamma^2} dE \right\}, \end{aligned} \quad (19d)$$

$$f \left(x, k + \frac{2\pi}{l}, t \right) = f(x, k, t). \quad (19e)$$

In Equation (19d), the parameter Γ represents the finite broadening of the equilibrium spectral function produced by scattering. In the limit $\Gamma \rightarrow 0$, the equilibrium distribution reduces to the semiclassical expression (2d). Expressing both the Wigner function and the miniband dispersion relation as Fourier series transforms the kinetic equation into

$$\begin{aligned} \frac{\partial f}{\partial t} + \sum_{j=-\infty}^{\infty} \frac{ijl}{\hbar} e^{ijk l} \left(\mathcal{E}_j \frac{\partial}{\partial x} \langle f \rangle_j + e \langle F \rangle_j f_j \right) \\ = -\nu_e (f - f^{FD}), \end{aligned} \quad (20a)$$

$$f(x, k, t) = \sum_{j=-\infty}^{\infty} f_j(x, t) e^{ijk l}, \quad (20b)$$

$$\mathcal{E}(k) = \sum_{j=-\infty}^{\infty} \mathcal{E}_j e^{ijk l}, \quad (20c)$$

$$\langle F \rangle_j(x, t) = \frac{1}{jl} \int_{-\frac{jl}{2}}^{\frac{jl}{2}} F(x + s, t) ds, \quad (20d)$$

and a similar definition for $\langle f \rangle_j$. The operators $\langle \cdot \rangle_j$ correspond to centered spatial averages and satisfy

$$\frac{\partial}{\partial x} \langle g(x) \rangle_j = \langle \frac{\partial g}{\partial x} \rangle_j = \frac{g(x + \frac{jl}{2}) - g(x - \frac{jl}{2})}{jl}.$$

Integration of Equation (20a) over k yields the charge continuity equation. Combining it with Poisson's equation leads to the corresponding Ampère equation

$$\varepsilon \frac{\partial F}{\partial t} + \frac{2e}{\hbar} \sum_{j=1}^{\infty} j \langle \text{Im}(\mathcal{E}_{-j} f_j) \rangle_j = J(t), \quad (21)$$

where $J(t)$ is the space-independent total current density.

Applying the same hyperbolic scaling and Chapman-Enskog procedure used in the semiclassical theory leads to a quantum drift-diffusion equation (QDDE). For the tight-binding dispersion relation (2e), the resulting equation is spatially nonlocal because it involves averages extending over two superlattice periods [17]. The QDDE is solved using the same contact boundary conditions as the GDDE, applied over the two auxiliary intervals $[-2l, 0]$ and $[L, L + 2l]$ required by the nonlocal averaging operators. Figure 5 shows that the QDDE reproduces the same qualitative Gunn-type dynamics as the semiclassical GDDE while incorporating quantum broadening and nonlocal transport effects.

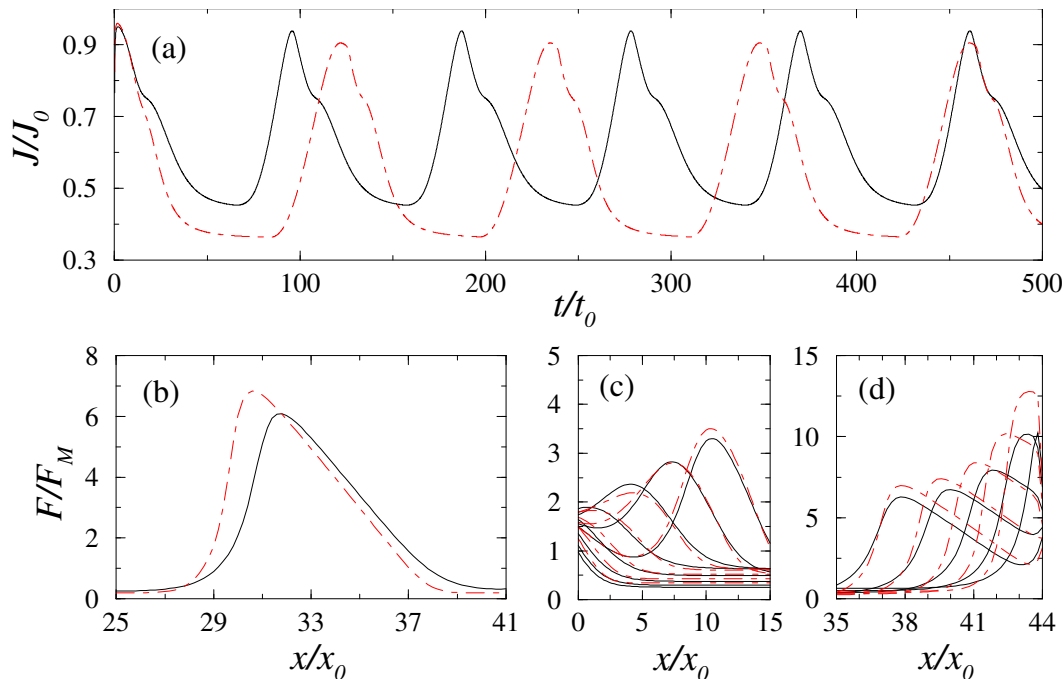


Figure 5. (a) Current ($J_0 = ev_M N_D / l$) vs. time ($t_0 = \varepsilon F_M / J_0$) during self-oscillations for a voltage-biased GaAs/AlAs SL, as described by the QDDE (solid line), the QDDE with $\Gamma = 0$ (long-dashed line) and by the GDDE (dot-dashed line). (b) Comparison between the fully developed dipole wave for the QDDE (solid line) and the dipole wave for the GDDE (dashed line). (c) Dipole wave at different times during the stage in which it is shed from the injecting contact. (d) Same as (c) for the stage in which the dipole disappears at the anode, located at $Nl/x_0 \approx 44$. Parameter values are $x_0 = v_M t_0 = 16$ nm, $t_0 = 0.43$ ps, $J_0 = 6.07 \times 10^5$ A/cm², $\phi = 1$. Reprinted with permission from Ref. [18]. Copyright 2006, Springer.

The Wigner formalism also extends naturally to systems in which two minibands coupled by the Rashba spin-orbit interaction [19] or by field-dependent tunneling [20] participate in the transport dynamics. The Hamiltonian is now a 2×2 matrix that can be written as a linear combination of the Pauli matrices. Correspondingly, the Wigner matrix is a linear combination of Pauli matrices with coefficients that take values in the real numbers and produce balance equations for the miniband populations and the electric field [20]. When two minibands are populated, self-sustained Gunn-type oscillations may be accompanied by resonant inter-miniband tunneling during part of each oscillation cycle [20]. Figure 6 illustrates the transport dynamics in the two-miniband model [20]. The current-field characteristic shown in Figure 6a exhibits a local maximum at the resonant electric field. The resulting negative differential conductivity gives rise to stable self-sustained current oscillations, illustrated in Figure 6b. The corresponding electric-field profiles are shown in Figure 6c. Figure 6d displays the evolution of the electron populations in the two minibands at the midpoint of the superlattice. Before the propagating charge domain reaches the resonant field proportional to the gap energy between minibands, essentially all electrons occupy the lower miniband. As the field exceeds the resonance, resonant tunneling transfers electrons to the upper miniband. Once the field falls below the resonant value, the upper miniband is progressively depleted and the lower miniband becomes repopulated.

The Wigner formulation therefore extends the semiclassical theory in a systematic manner. While the resulting transport equations retain the computational efficiency of drift-diffusion models, they also incorporate quantum coherence, nonlocality, and tunneling effects that are absent from semiclassical descriptions. Together with the earlier hydrodynamic reductions, these results establish a unified multiscale framework linking microscopic quantum dynamics to macroscopic nonlinear transport.

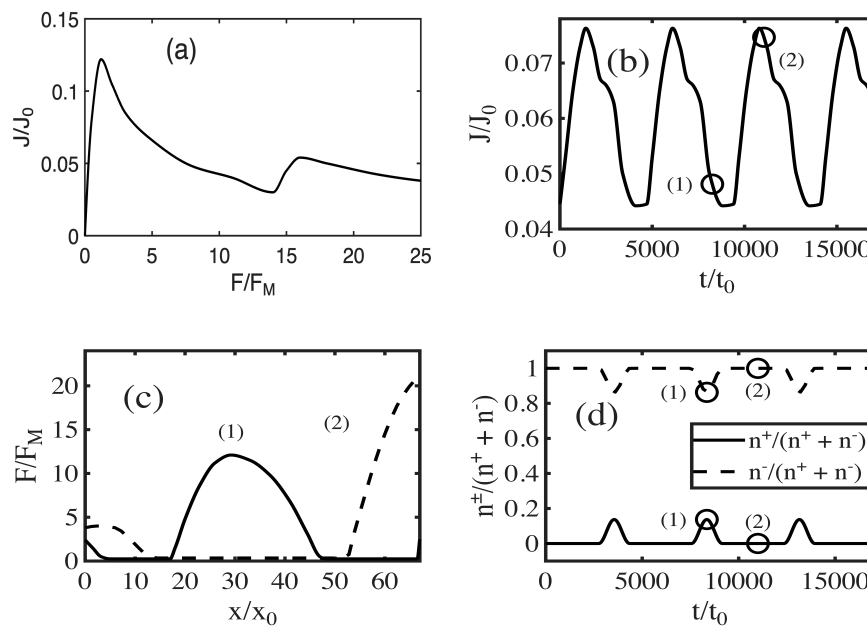


Figure 6. (a) Electron current vs field in a spatially uniform stationary state. (b) Total current density vs time. (c) Electric field profile at different times of one current self-oscillation cycle. At time (1) the field is above the resonant value for the middle SL point $x = Nl/2$. (d) Electron densities at the minibands $n^\pm/(n^+ + n^-)$ vs time for point $x = Nl/2$. When the electric field is above the resonant value (time (1)), the electron transport between minibands occurs. Reprinted with permission from Ref. [20]. Copyright 2010, American Physical Society.

4. Discussion and Final Remarks

Semiconductor superlattices have provided an exceptional setting in which to investigate the interplay between quantum coherence, scattering, and nonlinear transport. Their artificial periodicity makes it possible to observe transport phenomena that are difficult to isolate in bulk crystals, including Bloch oscillations, electric-field domains, resonant tunneling, and self-sustained current oscillations [1,2,4–7,9–21]. Beyond their direct technological interest, these systems have become paradigmatic model materials for developing and validating multiscale theories of nonequilibrium quantum transport [14–22].

The principal objective of this review is to present a unified theoretical framework connecting microscopic quantum dynamics with macroscopic transport equations. Beginning with semiclassical Boltzmann-Poisson-BGK kinetic equations [14–16] and extending to fully quantum Wigner-Poisson formulations [17–20], systematic asymptotic reductions lead to generalized drift-diffusion equations that retain the essential effects of scattering, self-consistent electrostatics, coherent tunneling, and quantum interference while remaining suitable for analytical investigation and efficient numerical simulation. This hierarchy of models illustrates how complex nonequilibrium transport phenomena can be described at progressively reduced levels without losing the underlying physical mechanisms. An alternative quantum description of superlattice transport [9] is based on nonequilibrium Green functions [35–37]. Within this formalism, the expressions for the stationary uniform current density and the chemical potential are calculated numerically for a constant electric field on an infinitely long superlattice. Then these expressions are inserted in *postulated equations of motion* for transport between quantum wells [9]. Nonequilibrium Green-function methods continue to provide a microscopic description of quantum transport in semiconductor heterostructures and have recently been used for quantitative simulations of high-performing terahertz quantum cascade lasers [38].

A central feature of our approach is the use of the Chapman-Enskog method to derive hydrodynamic equations directly from kinetic theory [15–20]. Rather than relying on phenomenological constitutive relations, the transport coefficients and nonlinear terms emerge systematically from the microscopic dynamics. In the semiclassical regime, the resulting generalized drift-diffusion equation successfully reproduces electric-field domain formation and Gunn-type oscillations mediated by traveling waves [14–16,21]. When the local equilibrium is generalized appropriately, the same kinetic framework also predicts the coexistence of Bloch and Gunn-type oscillations under suitable operating conditions [21]. Extending the analysis to the Wigner representation naturally incorporates coherent tunneling and quantum nonlocality, leading to quantum drift-diffusion equations that remain computationally tractable while capturing essential quantum-mechanical effects [17–20].

From a broader perspective, the methodology developed for semiconductor superlattices is not restricted to these systems. The central concepts are multiscale asymptotics, kinetic reductions, and hydrodynamic closures. They are sufficiently general to be applicable whenever coherent Bloch electrons evolve under external driving while experiencing scattering and self-consistent interactions. In this sense, semiconductor superlattices have served as a laboratory not only for discovering nonlinear transport phenomena but also for developing theoretical tools that may be transferred to other classes of quantum materials.

Recent developments suggest particularly promising directions for such extensions. Moiré superlattices and twisted bilayer graphene exhibit narrow electronic bands, strong electronic correlations, and long coherence lengths [22,23]. In particular, Bloch oscillations in moiré superlattices have recently been proposed as a means of probing the Berry curvature of their minibands [28]. Floquet-engineered systems introduce periodically driven band structures with tunable effective Hamiltonians [23,39], while topological materials combine Bloch dynamics with Berry curvature, orbital magnetic moments, and nontrivial quantum geometry [25–27,29,30]. In these systems, transport is influenced not only by the band dispersion but also by geometric properties of the Bloch states, suggesting that future kinetic theories should incorporate Berry-curvature corrections and quantum-geometric effects at the microscopic level before carrying out hydrodynamic reductions.

Several theoretical challenges remain open. Existing BGK collision models provide an efficient description of relaxation but neglect many-body correlations beyond the mean-field approximation. Extending the present framework to include electron-electron correlations, excitonic effects, and nonequilibrium quantum statistics constitutes an important direction for future work. Likewise, deriving hydrodynamic equations for systems with multiple coupled bands, non-Abelian Berry connections, or strong spin-orbit coupling will require new asymptotic methods capable of preserving gauge invariance while maintaining computational efficiency.

From the mathematical standpoint, further progress is also desirable. Questions concerning the rigorous justification of the hydrodynamic limit, the stability of nonlinear wave solutions, the existence and uniqueness of solutions to the resulting nonlocal drift-diffusion equations, and the asymptotic structure of quantum transport models remain largely open problems. Their resolution would strengthen the mathematical foundations of multiscale quantum transport and further clarify the range of validity of hydrodynamic reductions derived from quantum kinetic equations.

Finally, the framework reviewed here illustrates a more general principle in nonequilibrium statistical physics: microscopic quantum dynamics, kinetic theory, and macroscopic transport equations should be viewed as complementary levels of description rather than competing theories. Semiconductor superlattices have provided one of the clearest demonstrations of this hierarchy, serving both as experimental realizations of nonlinear quantum transport and as theoretical laboratories in which multiscale methods can be developed and tested. As new classes of quantum materials continue to emerge, the ideas reviewed here are likely to remain relevant well beyond semiconductor superlattices, providing a foundation for future studies of nonlinear quantum-wave dynamics in driven, coherent, and topological systems.

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Conflicts of Interest

The author declare no conflict of interest.

Use of AI and AI-Assisted Technologies

During the preparation of this work, the author used Open AI to correct the text and draw Figures 1 and 2. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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