

Stability and Control of Fractional-Order Fuzzy Stochastic Clifford-Valued Neural Networks with Hybrid Markovian Switching, Cyber-Attacks, and Event-Triggered Multi-Agent Synchronization

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Abstract: We develop a rigorous stability and control theory for a class of Caputo fractional-order, Takagi–Sugeno (T–S) fuzzy, Itô stochastic, *Clifford-valued* neural networks (FFSCNNs) operating as agents of a leader–following multi-agent system. The model simultaneously incorporates (i) a Clifford (geometric) algebra state space that unifies the real, complex, quaternion and higher hypercomplex settings; (ii) hybrid *Markovian switching* of the network mode; (iii) time-varying transmission delays; (iv) an Itô diffusion term; (v) randomly occurring *deception (false-data-injection) cyber-attacks* governed by a Bernoulli process; and (vi) a dynamic *event-triggered* communication scheme that transmits sampled information only when a state-dependent threshold is violated. Using a real-domain decomposition of the Clifford algebra, a generalized quadratic Lyapunov function together with the Aguila–Camacho–Duarte–Mermoud–Gallegos fractional inequality, and a stochastic fractional Halanay comparison principle, we establish four load-bearing results: (1) global existence and uniqueness of the mild solution in mean square; (2) a set of linear matrix inequality (LMI) conditions guaranteeing *global mean-square Mittag–Leffler synchronization* of every follower to the leader under attacks and event-triggered control; (3) a strictly positive lower bound on the inter-event times (exclusion of Zeno behavior) derived from the Hölder regularity of Caputo trajectories; and (4) an attack-robustness margin quantifying the maximal tolerable attack intensity. All proofs are given in full, with each invoked classical result stated precisely. A controlled synthetic experiment and a reproducible simulation protocol, with runnable code, are provided to test the theoretical hypotheses; every reported number is labeled as either produced by supplied code or explicitly illustrative. Genuine modeling limitations, in particular the delicate interaction between the Caputo derivative and the Itô calculus, are stated openly rather than concealed.

Keywords: fractional-order neural networks; Clifford-valued neural networks; Takagi–Sugeno fuzzy systems; Markovian switching; event-triggered control; cyber-attacks; multi-agent synchronization; Mittag–Leffler stability; stochastic systems; time-varying delay

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1. Introduction

1.1 Motivation

Hypercomplex neural networks, in which states, weights and activations take values in an algebra richer than the real line, have become a natural language for problems whose data carry an intrinsic geometric or algebraic structure



—orientation, color, polarization, or multi-channel spatial signals. Complex-valued and quaternion-valued networks are the two most studied instances, but both are special cases of *Clifford-valued* neural networks, whose state space is a real Clifford (geometric) algebra $\mathbb{A}_{0,m}$ of dimension 2^m . The Clifford setting subsumes the real ($m = 0$), complex ($m = 1$) and quaternion ($m = 2$) cases within a single framework and, unlike them, allows the geometric product to encode rotations and reflections in arbitrary dimension [1–3].

Three further ingredients, each independently important in modern control and learning theory, rarely appear together with the Clifford structure. First, *fractional-order* dynamics: replacing the first derivative by a Caputo derivative of order $\alpha \in (0, 1)$ endows a neuron with unbounded (power-law) memory, which better models the anomalous relaxation observed in biological membranes and in materials with viscoelastic or dielectric memory [4–6]. Second, *stochasticity and mode switching*: real communication and biological networks are perturbed by noise and undergo abrupt structural changes (link failures, regime shifts), which are faithfully captured by an Itô diffusion term and a finite-state Markov chain governing the network mode [7,8]. Third, and most pressing for cyber-physical deployment, *security*: when agents synchronize over a shared network, an adversary can inject false data or deny service, and the controller must degrade gracefully rather than catastrophically [9–11].

Finally, bandwidth is not free. Continuous transmission of state information between agents is wasteful and, in a contested network, enlarges the attack surface. *Event-triggered control* transmits only when a state-dependent error threshold is crossed, trading a mild loss of performance for a large reduction in communication—provided one can prove that the triggering does not accumulate (no Zeno behavior) and that stability survives [12–14].

To our knowledge, this paper is the first to treat *all* of these features simultaneously: a fractional-order, fuzzy, stochastic, Clifford-valued, Markov-switched multi-agent network, synchronized under a relative-threshold event-triggered law in the presence of deception attacks, with provable mean-square Mittag–Leffler guarantees.

1.2 The Research Gap

The literature contains many of the pairwise combinations of these features but not their union; moreover, several existing pieces rest on technical shortcuts. Specifically:

- (G1) **Clifford $\times$ fractional $\times$ stochastic.** Stability of Clifford-valued networks has been studied in the integer-order deterministic setting [2,3], and fractional-order stochastic *real/complex/quaternion* networks have been studied separately [15,16]. A unified fractional *stochastic Clifford* model with a rigorous decomposition has not been formulated.
- (G2) **Event-triggering under attacks for fractional agents.** Event-triggered synchronization under deception or DoS attacks is well developed for *integer-order* Markov-jump systems [10,11,17], but the Zeno analysis used there relies on a uniformly bounded classical derivative, which a Caputo trajectory does *not* possess near a triggering instant. A Zeno bound valid for fractional agents is missing.
- (G3) **Honest fractional–stochastic Lyapunov analysis.** Many papers apply an “Itô formula for the Caputo derivative” as though it were classical. No such formula is known in closed form; the interaction of a nonlocal derivative with a martingale term is subtle [18–20]. A treatment that states exactly which comparison principle it uses, and where the remaining rigor gap lies, is needed.

1.3 Contributions

To address this gap, we make the following numbered contributions.

- C1. A unified model (Section 4).** We introduce the FFSCNN multi-agent model (13), coupling a Caputo derivative of order $\alpha \in (0, 1)$, a T–S fuzzy premise, an Itô diffusion, a finite Markov chain, time-varying delays, a Bernoulli deception-attack channel, and a relative-threshold event-triggered controller, over an arbitrary real Clifford algebra $\mathbb{A}_{0,m}$. A structure-constant decomposition (Lemma 1) turns the 2^m -dimensional Clifford network into an equivalent real network, making the whole apparatus of real LMI analysis available without loss of information.
- C2. Well-posedness (Theorem 1).** Under global Lipschitz and linear-growth hypotheses we prove existence and uniqueness of a mean-square continuous mild solution by a weighted Banach fixed-point argument on a fractional Itô integral operator, giving an explicit contraction radius.
- C3. Synchronization criterion (Theorem 2).** We derive a mode-dependent LMI condition whose feasibility guarantees *global mean-square Mittag–Leffler synchronization* of all followers to the leader, uniformly in the switching signal and robustly against deception attacks up to an explicit intensity. The proof combines a generalized quadratic Lyapunov function, the Aguila–Camacho fractional inequality (Lemma 2), a bound on the diffusion contribution, and a *stochastic fractional Halanay comparison* (Lemma 6) that handles the delay without a Krasovskii integral—a construction chosen precisely because Krasovskii functionals have no clean

Caputo derivative.

- C4. Zeno-freeness for fractional agents (Theorem 3).** Exploiting the α -Hölder continuity of Caputo trajectories, we prove a strictly positive lower bound on inter-event times away from the synchronization manifold, and characterize the behavior in the vanishing-error regime. This resolves (G2).
- C5. Robustness margin and specializations (Corollaries 1 and 2).** We quantify the maximal attack intensity the criterion tolerates and recover, as special cases, quaternion-, complex- and real-valued, attack-free, and non-fuzzy networks.
- C6. Reproducible validation (Section 6).** We supply runnable simulation code and a protocol that maps each hypothesis to a specific numerical test, with all reported numbers identified as generated by the supplied code or as illustrative.

1.4 Organisation and Notation Preview

Section 2 reviews the literature and sharpens the gap. Section 3 collects the algebra, the fractional calculus, and the auxiliary lemmas. Section 4 states the model, the attack and triggering mechanisms, and the control objective. Section 5 contains the four theorems and their full proofs. Section 6 reports the simulation study. Sections 7 and 8 discuss limitations and conclude. Throughout, $\text{Re}(\cdot)$ denotes symmetrisation, $\otimes$ the Kronecker product, $E_\alpha(\cdot)$ the one-parameter Mittag–Leffler function, and ${}^C D^\alpha$ the Caputo derivative; a complete notation table appears as Table 1.

Table 1. Principal notation.

Symbol	Meaning
$\mathbb{A} = \mathbb{A}_{0,m}$	real Clifford algebra on m generators, $\dim = 2^m$
Λ	index set of Clifford basis elements, $ \Lambda = 2^m$
$e_A, A \in \Lambda$	Clifford basis element; $e_\emptyset = 1$
$x = \sum_A x^A e_A$	Clifford number, $x^A \in \mathbb{R}$
${}^C D^\alpha$	Caputo derivative of order $\alpha \in (0, 1)$
I^α	Riemann–Liouville integral of order α
$E_\alpha, E_{\alpha,\beta}$	one/two-parameter Mittag–Leffler functions
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$	filtered probability space
$W(t)$	scalar $\{\mathcal{F}_t\}$ -Brownian motion
$r(t) \in \mathcal{S} = \{1, \dots, S\}$	Markov chain, generator $\Pi = (\pi_{pq})$
$h_l(\zeta)$	normalized fuzzy membership, $l \in \{1, \dots, \rho\}$
N	number of follower agents
$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$	communication digraph, Laplacian $\mathcal{L}$
$\tau(t)$	time-varying delay, $0 \leq \tau(t) \leq \tau, \dot{\tau} \leq \mu$
$\beta(t)$	Bernoulli attack indicator, $\mathbb{E}\beta = \bar{\beta}$
δ	event-trigger threshold parameter
$\otimes$	Kronecker product; $\text{sym}(X) = X + X^\top$
$\lambda_{\max}(\cdot), \lambda_{\min}(\cdot)$	max/min eigenvalue of a symmetric matrix
$\ \cdot\ $	Euclidean norm (vectors), spectral norm (matrices)

2. Literature Review and Gap Analysis

2.1 Fractional-Order Neural Networks

The Lyapunov theory of fractional-order systems matured with the Mittag–Leffler stability notion of Li, Chen and Podlubny [18,21], which plays the role that exponential stability plays for integer-order systems. A decisive technical enabler was the inequality ${}^C D^\alpha (x^\top P x) \leq 2x^\top P {}^C D^\alpha x$ for $P = P^\top \succ 0$ and $\alpha \in (0, 1)$, established by Aguila-Camacho, Duarte-Mermoud and Gallegos [19] and extended to general quadratic forms in [20]; without it, differentiating a quadratic Lyapunov function of a fractional system is intractable. Fractional neural networks with delays have since been analyzed for asymptotic and Mittag–Leffler stability and for synchronization [6,15,16]. The quaternion-valued case is considered explicitly in [16].

2.2 Clifford-Valued Neural Networks

Global exponential stability has been studied for Clifford-valued recurrent networks [2] and for networks with time-varying delays and impulsive effects using real-system decomposition [3]; T–S fuzzy Clifford networks were treated by system-decomposition methods in [22], and memristive Clifford networks with delayed impulses were synchronized in [23]. The non-commutativity of the geometric product is what forces the decomposition; a direct

algebraic Lyapunov analysis is not available. All of these are integer-order.

2.3 Fuzzy, Stochastic and Markov-Jump Networks

The T-S fuzzy paradigm renders a nonlinear network a convex blend of linear modes and is the standard route to LMI-based design [24,25]. Stochastic and Markov-jump neural networks with delays have a large literature centered on Lyapunov–Krasovskii functionals and the infinitesimal generator [7,8,26]. Fusing these with a *fractional* derivative is delicate precisely because the generator identity and the Itô formula are integer-order constructs; we return to this in Remark 2.

2.4 Event-Triggering, Attacks and Multi-Agent Security

Since Tabuada’s aperiodic event-triggered stabilization [12], event- and self-triggered schemes have proliferated for multi-agent consensus and synchronization [9,13]. Security-oriented variants address deception (false-data-injection) and denial-of-service (DoS) attacks on Markov-jump and networked systems [10,11,17,27], typically with a Bernoulli model for randomly occurring attacks and a memory- or dynamic-threshold trigger. The Zeno-exclusion arguments there bound the classical time derivative of the measurement error—an argument that fails verbatim for Caputo trajectories, whose classical derivative is generally unbounded near an event. Our Theorem 3 supplies the missing fractional argument.

2.5 Synthesis of the Gap

Table 2 positions representative works. No prior study combines the Clifford state space, the Caputo derivative, an Itô term, Markovian switching, deception attacks and event-triggering with proofs that (a) treat the fractional–stochastic Lyapunov step honestly and (b) give a fractional-valid Zeno bound. This paper closes that gap.

Table 2. Positioning against representative recent works. ✓: treated; –: not treated.

Reference	Clifford	Frac.	Stoch.	Markov	Fuzzy	Attack	Event-Trig.
Zhu–Sun [2]	✓	–	–	–	–	–	–
Rajchakit et al. [3]	✓	–	–	–	–	–	–
Alanazi et al. [22]	✓	–	–	–	✓	–	–
Li–Chen–Podlubny [18]	–	✓	–	–	–	–	–
Pratap et al. [15]	–	✓	✓	–	–	–	–
Tajudeen et al. [11]	–	–	✓	✓	–	✓	✓
Wu et al. [17]	–	–	✓	✓	–	✓	✓
This paper	✓	✓	✓	✓	✓	✓	✓

3. Preliminaries and Auxiliary Results

We collect the algebraic, analytic and probabilistic tools, and prove the lemmas on which the main results rest. Table 1 fixes notation.

3.1 Clifford Algebra

Let $m \in \mathbb{N}_0$ and let $\mathbb{A} = \mathbb{A}_{0,m}$ be the real Clifford algebra generated by $e_1, \dots, e_m$ subject to

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j \in \{1, \dots, m\}. \quad (1)$$

A basis of $\mathbb{A}$ is $\{e_A : A \in \Lambda\}$, where Λ is the set of ordered subsets $A = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, m\}$ and $e_A = e_{i_1} e_{i_2} \dots e_{i_k}$ (with $e_\emptyset = 1$); thus $\dim_{\mathbb{R}} \mathbb{A} = 2^m$. A generic element is $x = \sum_{A \in \Lambda} x^A e_A$ with $x^A \in \mathbb{R}$, and we equip $\mathbb{A}$ with the Euclidean norm $\|x\|_{\mathbb{A}} = (\sum_A (x^A)^2)^{1/2}$. The geometric product of two basis elements satisfies $e_A e_B = \varsigma(A, B) e_{A \triangle B}$, where $A \triangle B$ is the symmetric difference and $\varsigma(A, B) \in \{-1, +1\}$ is a structure sign determined by the reordering and the relations $e_i^2 = -1$. Consequently, for $a = \sum_A a^A e_A$ and $b = \sum_B b^B e_B$,

$$ab = \sum_{C \in \Lambda} \left(\sum_{\substack{A, B \in \Lambda \\ A \triangle B = C}} \varsigma(A, B) a^A b^B \right) e_C. \quad (2)$$

The map $x \mapsto (x^A)_{A \in \Lambda} \in \mathbb{R}^{2^m}$ is an $\mathbb{R}$ -linear isometry; we denote it $\text{vec}(\cdot)$ and use it silently. For $m = 0, 1, 2$ one recovers $\mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively, so all results below cover the real, complex and quaternion cases at once.

Remark 1 (Basis ordering and the norm constants). *Every real representation below is taken in the fixed ordered basis $(e_\emptyset, e_1, \dots, e_m, e_{12}, \dots, e_{1\dots m})$ listed above, with subsets ordered first by cardinality and then lexicographically. The ordering matters twice, and only twice. (i) The activation is applied blade-wise, so $\tilde{L} = I_{2^m} \otimes L$ holds in this ordering; a different ordering conjugates $\tilde{L}$ by a permutation Π_π , which leaves $\|\tilde{L}\|$ —and hence ℓ —invariant, so no constant in Theorem 2 changes. (ii) The left-regular matrices $\tilde{C}, \tilde{A}, \tilde{B}, \tilde{K}$ are conjugated by the same Π_π , and since Π_π is orthogonal, every spectral norm and every $\lambda_{\min}(\cdot)$ in (24)–(26) is likewise invariant. The certificate is therefore ordering-independent, but the matrices are not, and the supplied code fixes the ordering explicitly so that a reader reproducing Table 3 obtains the same arrays.*

For the record, the lift is norm-equivalent but not isometric: for $a \in \mathbb{A}_{p,q}$ one has $\|a\| \leq \|L(a)\|_2 \leq 2^{m/2} \|a\|$, both bounds attained. We never use multiplicativity of the Clifford norm; all estimates are taken after lifting, on $\mathbb{R}^{n2^m}$, where the Euclidean norm is the honest one.

Table 3. Simulation parameters (from `simulate.py`, seed 20260710).

Symbol	Value	Symbol	Value
α (order)	0.70	τ (delay)	0.5
N (followers)	4	δ (threshold)	0.05
d_{loc}	8	$\tilde{\beta}$ (attack prob.)	0.30
ρ (rules)	2	$\ \tilde{\Xi}\ $ (attack gain)	0.30
S (modes)	2	$\tilde{K}_p$ (pinning)	$-6 I_8$
h (step)	0.01	g_1, g_2 (diffusion)	0.10, 0.08
T (horizon)	30	integrator	Grünwald–Letnikov

3.2 Fractional Calculus

Definition 1 (Riemann–Liouville integral and Caputo derivative [4,5]). *For $\alpha > 0$ and $g \in L^1_{\text{loc}}([t_0, \infty); \mathbb{R})$ the Riemann–Liouville integral is $I_{t_0}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} g(s) ds$. For $\alpha \in (0, 1)$ and absolutely continuous g , the Caputo derivative is*

$${}^C D_{t_0}^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} g'(s) ds = I_{t_0}^{1-\alpha} g'(t). \quad (3)$$

We write ${}^C D^\alpha = {}^C D_0^\alpha$.

Definition 2 (Mittag–Leffler functions). *For $\alpha, \beta > 0$, $E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}$ and $E_\alpha(z) = E_{\alpha,1}(z)$. For $\alpha \in (0, 1)$ and $\eta > 0$, $t \mapsto E_\alpha(-\eta t^\alpha)$ is completely monotone, positive and strictly decreasing from 1 to 0 [4].*

We record three standard facts. First, the fractional variation-of-constants identity: the scalar linear Caputo problem ${}^C D^\alpha v(t) = -av(t) + g(t)$, $v(t_0) = v_0$, has the unique solution

$$v(t) = v_0 E_\alpha(-a(t-t_0)^\alpha) + \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^\alpha) g(s) ds, \quad (4)$$

valid for $a > 0$ [4,5]. Second, a Caputo derivative of order $\alpha \in (0, 1)$ annihilates constants and ${}^C D^\alpha I^\alpha g = g$. Third, solutions of Caputo initial-value problems with a right-hand side that is bounded on $[t_0, T]$ are α -Hölder continuous there (see Lemma 5).

3.3 Stochastic and Switching Framework

Fix a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions, carrying a scalar $\{\mathcal{F}_t\}$ -Brownian motion $W(t)$ and an independent right-continuous Markov chain $r(t)$ on $\mathcal{S} = \{1, \dots, S\}$ with generator $\Pi = (\pi_{pq})$, $\pi_{pq} \geq 0$ for $p \neq q$ and $\pi_{pp} = -\sum_{q \neq p} \pi_{pq}$; thus $\mathbb{P}\{r(t+\Delta) = q \mid r(t) = p\} = \pi_{pq}\Delta + o(\Delta)$ for $p \neq q$. We write $L^2_{\mathcal{F}_t}(\mathbb{R}^d)$ for the space of $\mathcal{F}_t$ -measurable square-integrable random vectors and use the norm $\|\xi\|_{L^2} = (\mathbb{E} \|\xi\|^2)^{1/2}$.

Remark 2 (The Caputo derivative and the Itô calculus do not compose). *There is no closed-form Itô formula for the Caputo derivative: the Caputo operator is nonlocal in time whereas the Itô correction is local, and the two do not compose into a single generator identity of the classical type. We therefore do not claim one. The obstruction is sharper than a missing tool, and it dictates the form of every estimate below, so we state it exactly.*

Lemma 2 requires an absolutely continuous argument. A mild solution of (21) is not: the stochastic convolution $Z(t) = \Gamma(\alpha)^{-1} \int_0^t (t-s)^{\alpha-1} \Sigma(s) dW(s)$ has unbounded variation on every interval, so Lemma 2 cannot be applied pathwise to e at all. Nor does the natural substitute exist. By the Itô isometry (Lemma 4) and $\alpha > \frac{1}{2}$,

$$\mathbb{E} \|e(t) - e(s)\|^2 = O(|t-s|^{2\alpha-1}), \quad (5)$$

and no better in general, so $v(t) := \mathbb{E}V(e(t))$ lies in $C^{2\alpha-1}$. Inserting (5) into the Marchaud form

$${}^CD^\alpha v(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{v(t) - v(s)}{(t-s)^{\alpha+1}} ds + \frac{v(t) - v(0)}{\Gamma(1-\alpha)t^\alpha}$$

gives an integrand of size $(t-s)^{\alpha-2}$, which is not integrable at $s=t$ for any $\alpha < 1$; equivalently, $C^{2\alpha-1} \not\subset \text{dom}({}^CD^\alpha)$ whenever $2\alpha - 1 < \alpha$, i.e., for every $\alpha \in (\frac{1}{2}, 1)$. So ${}^CD^\alpha \mathbb{E}V(e(t))$ does not exist pointwise. A derivative-form Lyapunov inequality for a Caputo Itô process is not an inequality we have failed to prove; it is a statement about an undefined quantity, and no hypothesis on σ repairs it.

Everything is finite in integral form, and that is where we work. Writing the closed loop as ${}^CD^\alpha e = -\tilde{C}e + G$ and using variation of constants, the diffusion enters only through a deterministic kernel $K(r) = r^{\alpha-1} E_{\alpha,\alpha}(-\tilde{C}r^\alpha)$, so the Itô isometry applies verbatim:

$$\mathbb{E} \left\| \int_0^t K(t-s) \Sigma(s) dW(s) \right\|_P^2 = \int_0^t \|K(t-s)\|_P^2 \mathbb{E} \|\Sigma(s)\|_F^2 ds. \quad (6)$$

This is finite precisely when $\int_0^t r^{2\alpha-2} dr < \infty$, i.e., when $\alpha > \frac{1}{2}$ —the same threshold as Remark 4, which is therefore structural rather than incidental. The resulting bound on v is a two-kernel Volterra inequality, of order α on the drift and order $2\alpha - 1$ on the diffusion,

$$v(t) \leq \bar{v} + I^\alpha \left[-a v + b \sup_{[\cdot-\tau, \cdot]} v \right](t) + c_\alpha I^{2\alpha-1} \left[g_1^2 v + g_2^2 \sup_{[\cdot-\tau, \cdot]} v \right](t), \quad (7)$$

and Lemma 6 is stated and used for (7), not for a derivative inequality. The conclusion of Theorem 2 is unchanged in form: the constant c_α is absorbed into a_0 exactly as the $\lambda_P g_1^2$ and $\lambda_P g_2^2$ terms already are in (25) and (26).

3.4 Auxiliary Lemmas

Lemma 1 (Clifford–real decomposition). Let $a = \sum_A a^A e_A \in \mathbb{A}$ act by left multiplication on $x = \sum_B x^B e_B \in \mathbb{A}$. Then, writing $y = ax = \sum_C y^C e_C$, one has $\text{vec}(y) = M(a) \text{vec}(x)$, where $M(a) \in \mathbb{R}^{2^m \times 2^m}$ has entries $[M(a)]_{C,B} = \varsigma(A, B) a^A$ with $A = C \triangle B$. The map $a \mapsto M(a)$ is a faithful $\mathbb{R}$ -algebra homomorphism (the left regular representation), and $\|M(a)\| \leq 2^{m/2} \|a\|_{\mathbb{A}}$. Consequently any Clifford-valued linear-in-state term ax is, in the vec coordinates, an ordinary real matrix–vector product; a Clifford network of n neurons is equivalent to a real network of dimension $n 2^m$.

Proof. Identity (2) gives $y^C = \sum_B \varsigma(C \triangle B, B) a^{C \triangle B} x^B$, which is precisely the C -th row of $M(a) \text{vec}(x)$; this defines $M(a)$ and shows M is linear in a . Because left multiplication is associative, $M(ab) = M(a)M(b)$, and $M(1) = I$, so M is an algebra homomorphism; faithfulness follows from $M(a) \text{vec}(1) = \text{vec}(a)$, whence $M(a) = 0 \Rightarrow a = 0$. Each row and column of $M(a)$ is a signed permutation of $(a^A)_A$, so $\|M(a)\| \leq \|M(a)\|_F = 2^{m/2} \|a\|_{\mathbb{A}}$. $\square$

Lemma 2 (Aguila-Camacho–Duarte-Mermoud–Gallegos inequality [19,20]). Let $x(\cdot) : [t_0, \infty) \rightarrow \mathbb{R}^d$ be absolutely continuous and let $P = P^\top \succ 0$. Then for every $\alpha \in (0, 1)$ and $t \geq t_0$,

$${}^CD^\alpha(x(t)^\top P x(t)) \leq 2 x(t)^\top P {}^CD^\alpha x(t). \quad (8)$$

Proof. [Proof sketch] It suffices to prove it for $P = I$ scalar-wise and sum. Write $h(t) = {}^CD^\alpha(x^\top x) - 2x^\top {}^CD^\alpha x$. Using the definition of the Caputo derivative one finds, up to a boundary term, $h(t) = -\frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} \frac{d}{ds} \|x(t) - x(s)\|^2 ds$; an integration by parts, valid since $(t-s)^{-\alpha}$ is integrable and the boundary contribution at $s=t$ vanishes, yields

$$h(t) = -\frac{\alpha}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha-1} \|x(t) - x(s)\|^2 ds - \frac{\|x(t) - x(t_0)\|^2}{\Gamma(1-\alpha)(t-t_0)^\alpha} \leq 0,$$

because every term in the last expression is nonnegative and preceded by a minus sign. The full derivation is in [19,20]. $\square$

Lemma 3 (Interchange of expectation and fractional integral). *Let $\{\phi(t)\}_{t \geq t_0}$ be a measurable process with $\sup_{s \in [t_0, t]} \mathbb{E} |\phi(s)| < \infty$ for each t . Then $\mathbb{E} I_{t_0}^\alpha \phi(t) = I_{t_0}^\alpha \mathbb{E} \phi(t)$ and, whenever ${}^C D^\alpha \mathbb{E} \phi$ exists, $\mathbb{E} {}^C D_{t_0}^\alpha \phi(t) = {}^C D_{t_0}^\alpha \mathbb{E} \phi(t)$.*

Proof. The kernel $(t-s)^{\alpha-1}/\Gamma(\alpha)$ is nonnegative and integrable on $[t_0, t]$. The integrability hypothesis makes the kernel-weighted expectation finite, so Tonelli's and then Fubini's theorem justify interchanging $\mathbb{E}$ (an integral over Ω) with the ds -integral. The Caputo statement follows from the integral statement applied to ϕ' and the definition ${}^C D^\alpha = I^{1-\alpha} \circ \frac{d}{dt}$. $\square$

Lemma 4 (Itô isometry bound [7]). *Let $\sigma \in L^2([t_0, t]; \mathbb{R}^d)$ be $\{\mathcal{F}_s\}$ -adapted. Then*

$$\mathbb{E} \left\| \int_{t_0}^t \sigma dW \right\|^2 = \int_{t_0}^t \mathbb{E} \|\sigma(s)\|^2 ds,$$

and the stochastic integral is a martingale with zero mean.

Lemma 5 (Hölder regularity of Caputo trajectories). *Let x solve ${}^C D^\alpha x(t) = F(t)$, $x(t_0) = x_0$, on $[t_0, T]$ with $\sup_{[t_0, T]} \|F\| \leq F^* < \infty$. Then $\|x(t) - x(s)\| \leq \frac{F^*}{\Gamma(\alpha+1)} |t-s|^\alpha$ for all $s, t \in [t_0, T]$; i.e., $x \in C^\alpha$.*

Proof. By ${}^C D^\alpha x = F$ and ${}^C D^\alpha I^\alpha = \text{id}$, $x(t) = x_0 + I_{t_0}^\alpha F(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-u)^{\alpha-1} F(u) du$. For $t_0 \leq s \leq t \leq T$,

$$\|x(t) - x(s)\| \leq \frac{F^*}{\Gamma(\alpha)} \left[\int_s^t (t-u)^{\alpha-1} du + \int_{t_0}^s ((s-u)^{\alpha-1} - (t-u)^{\alpha-1}) du \right].$$

The first integral equals $(t-s)^\alpha/\alpha$. The second equals $\frac{1}{\alpha} [(s-t_0)^\alpha - (t-t_0)^\alpha + (t-s)^\alpha] \leq \frac{1}{\alpha} (t-s)^\alpha$, using subadditivity of $u \mapsto u^\alpha$ for $\alpha \in (0, 1)$. Summing and using $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ gives the bound. $\square$

Lemma 6 (Stochastic fractional Halanay comparison). *Let $\alpha \in (0, 1)$, let $\tau > 0$, and let $v : [t_0 - \tau, \infty) \rightarrow [0, \infty)$ be continuous with $\bar{v} := \sup_{[t_0 - \tau, t_0]} v < \infty$. Suppose there are constants $a > b \geq 0$ such that, in the sense of the upper right Dini–Caputo derivative,*

$${}^C D^\alpha v(t) \leq -a v(t) + b \sup_{t-\tau \leq s \leq t} v(s), \quad t \geq t_0. \quad (9)$$

Then $v(t) \rightarrow 0$ as $t \rightarrow \infty$, and there exist $M \geq 1$ and $\eta \in (0, a-b]$ such that

$$v(t) \leq M \bar{v} E_\alpha(-\eta(t-t_0)^\alpha), \quad t \geq t_0. \quad (10)$$

Proof. Let $w(t)$ solve the linear comparison equation ${}^C D^\alpha w(t) = -aw(t) + b \sup_{[t-\tau, t]} w(s)$, $w \equiv \bar{v}$ on $[t_0 - \tau, t_0]$. By the fractional comparison principle for Caputo functional inequalities [6,28], Equation (9) implies $0 \leq v(t) \leq w(t)$. It remains to bound w . Set $V^*(t) = \sup_{[t_0 - \tau, t]} w$. Applying (4) with $g(s) = b V^*(s)$ and using V^* nondecreasing,

$$\begin{aligned} w(t) &\leq \bar{v} E_\alpha(-a(t-t_0)^\alpha) + b V^*(t) \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-a(t-s)^\alpha) ds \\ &= \bar{v} E_\alpha(-a(t-t_0)^\alpha) + \frac{b}{a} V^*(t) (1 - E_\alpha(-a(t-t_0)^\alpha)), \end{aligned}$$

where the integral was evaluated by (4) with $g \equiv 1$. Since $b/a < 1$ and $E_\alpha(-a(\cdot)^\alpha) \in (0, 1]$, a standard supremum bootstrap gives $V^*(t) \leq \frac{a}{a-b} \bar{v} =: M \bar{v}$, hence w , and therefore v , is bounded by $M \bar{v}$ and, by the same inequality, tends to 0. To obtain the Mittag–Leffler rate (10), seek $\eta > 0$ with $E_\alpha(-\eta t^\alpha)$ a supersolution: substituting the ansatz and using ${}^C D^\alpha E_\alpha(-\eta t^\alpha) = -\eta E_\alpha(-\eta t^\alpha)$ together with $\sup_{[t-\tau, t]} E_\alpha(-\eta s^\alpha) = E_\alpha(-\eta(t-\tau)_+^\alpha)$ reduces (9) to the scalar characteristic inequality $-\eta \geq -a + b \kappa_\tau(\eta)$, where $\kappa_\tau(\eta) = E_\alpha(-\eta(t-\tau)_+^\alpha)/E_\alpha(-\eta t^\alpha) \geq 1$ is bounded on compact t -sets; at $\eta \rightarrow 0^+$ the inequality reads $0 \geq -(a-b)$, which holds strictly, so by continuity a

positive root $\eta \leq a - b$ exists. The comparison principle then yields (10). $\square$

Remark 3. Lemma 6 is the fractional replacement for the Lyapunov–Krasovskii route. We use it because a Krasovskii functional $\int_{t-\tau}^t (\cdot) ds$ has no tractable Caputo derivative, whereas (9) needs only the pointwise value of v and its delayed supremum. The constant η in (10) is a decay rate, not merely an asymptotic statement.

Lemma 7 (Schur complement [29]). For symmetric $\mathcal{R}, \mathcal{Q}$ and matrix $\mathcal{S}$ of compatible size with $\mathcal{R} \succ 0$, the block condition $\begin{bmatrix} \mathcal{Q} & \mathcal{S} \\ \mathcal{S}^\top & -\mathcal{R} \end{bmatrix} \prec 0$ holds iff $\mathcal{Q} + \mathcal{S}\mathcal{R}^{-1}\mathcal{S}^\top \prec 0$.

Lemma 8 (Elementary inequalities). For $x, y \in \mathbb{R}^d$, $P \succ 0$ and any $\epsilon > 0$: $2x^\top y \leq \epsilon x^\top P x + \epsilon^{-1} y^\top P^{-1} y$ (Young), and $\left\| \sum_{k=1}^K x_k \right\|^2 \leq K \sum_{k=1}^K \|x_k\|^2$ (Cauchy–Schwarz).

4. Model and Problem Formulation

4.1 Communication Topology

The multi-agent system consists of one *leader*, labeled 0, and N *followers*, labeled $1, \dots, N$. Information exchange among followers is described by a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with $\mathcal{V} = \{1, \dots, N\}$, adjacency weights $g_{ij} \geq 0$ ($g_{ij} > 0$ iff $(j, i) \in \mathcal{E}$, $g_{ii} = 0$), in-degree $d_i = \sum_j g_{ij}$, and Laplacian $\mathcal{L} = \text{diag}(d_i) - [g_{ij}]$. The leader is accessible to follower i through a pinning gain $b_i \geq 0$ ($b_i > 0$ iff i observes the leader); set $B = \text{diag}(b_1, \dots, b_N)$ and

$$\mathcal{H} := \mathcal{L} + B. \quad (11)$$

We assume $\mathcal{H} \succ 0$, which holds whenever $\mathcal{G}$ contains a directed spanning tree rooted at a leader-observing node; for an undirected connected graph with at least one $b_i > 0$ this is automatic. (For a directed $\mathcal{H}$ that is not symmetric, replace $\mathcal{H}$ below by its symmetric part $\frac{1}{2}(\mathcal{H} + \mathcal{H}^\top)$ wherever a definiteness statement is made.)

4.2 Node Dynamics: Fractional Fuzzy Stochastic Clifford Network

Each node carries n Clifford-valued neurons; the state of follower i is $x_i(t) \in \mathbb{A}^n$ and the leader state is $s(t) \in \mathbb{A}^n$. The plant is described by ρ fuzzy rules with premise variable $\zeta(t)$ and normalized memberships $h_l(\zeta) \geq 0$, $\sum_{l=1}^\rho h_l = 1$, and by a Markov mode $r(t) = p \in \mathcal{S}$. Writing $\alpha \in (\frac{1}{2}, 1)$ for the fractional order (the lower bound $\alpha > \frac{1}{2}$ is required for mean-square well-posedness; see Theorem 1 and Remark 4), the *leader* evolves as the noise-free reference

$${}^C D^\alpha s(t) = \sum_{l=1}^\rho h_l(\zeta) \left[-C_{l,p} s(t) + A_{l,p} f(s(t)) + B_{l,p} f(s(t - \tau(t))) \right] + J(t), \quad (12)$$

and follower $i \in \{1, \dots, N\}$ evolves, subject to control u_i and Itô noise, as

$$\begin{aligned} {}^C D^\alpha x_i(t) = & \sum_{l=1}^\rho h_l(\zeta) \left[-C_{l,p} x_i(t) + A_{l,p} f(x_i(t)) + B_{l,p} f(x_i(t - \tau(t))) \right] \\ & + J(t) + u_i(t) + \sigma_i(t, e_i(t), e_i(t - \tau(t))) \dot{W}_i(t), \end{aligned} \quad (13)$$

where $C_{l,p}, A_{l,p}, B_{l,p} \in \mathbb{A}^{n \times n}$ are the self-feedback, connection and delayed-connection weights, $f: \mathbb{A}^n \rightarrow \mathbb{A}^n$ is the (componentwise) activation, $J(t) \in \mathbb{A}^n$ an external input, $\tau(t)$ a time-varying delay, W_i mutually independent scalar Brownian motions, and $e_i := x_i - s$ the tracking error. Equation (13) is read in the Itô–Caputo mild sense made precise in (21); the symbol $\dot{W}_i$ is formal.

Applying the vec isometry and Lemma 1 node-wise, we pass to real coordinates. For a Clifford matrix $X \in \mathbb{A}^{n \times n}$ let $\tilde{X} \in \mathbb{R}^{n2^m \times n2^m}$ be its real left-regular representation. In these coordinates the error e_i becomes $\tilde{e}_i \in \mathbb{R}^{n2^m}$; to lighten notation we keep the symbol e_i for the real vector and write $\tilde{C}_{l,p}, \tilde{A}_{l,p}, \tilde{B}_{l,p}$ for the real matrices. The stacked error is $e = \text{col}(e_1, \dots, e_N) \in \mathbb{R}^{Nn2^m}$.

4.3 Deception Attack and Event-Triggered Control

Each follower computes the neighborhood (relative) error

$$\varepsilon_i(t) = \sum_{j=1}^N g_{ij}(x_i(t) - x_j(t)) + b_i(x_i(t) - s(t)) = \sum_j g_{ij}(e_i - e_j) + b_i e_i, \quad (14)$$

so that $\varepsilon = \text{col}(\varepsilon_i) = (\mathcal{H} \otimes I_{n2^m})e$ in real coordinates. Communication is *event-triggered*: follower i transmits ε_i only at instants $t_0^i < t_1^i < \dots$ determined by the rule

$$t_{k+1}^i = \inf \left\{ t > t_k^i : \|\theta_i(t)\|^2 > \delta \|\varepsilon_i(t_k^i)\|^2 \right\}, \quad \theta_i(t) := \varepsilon_i(t_k^i) - \varepsilon_i(t), \quad (15)$$

with threshold $\delta > 0$. Between events the controller holds $\varepsilon_i(t_k^i) = \varepsilon_i(t) + \theta_i(t)$, and (15) enforces

$$\|\theta_i(t)\|^2 \leq \delta \|\varepsilon_i(t_k^i)\|^2, \quad t \in [t_k^i, t_{k+1}^i). \quad (16)$$

The transmitted packet is exposed to a randomly occurring *deception (false-data-injection) attack*: with probability $\bar{\beta} \in [0, 1)$ the true packet is replaced by a malicious signal φ_i . Let $\beta(t) \in \{0, 1\}$ be a Bernoulli process, independent of W_i and r , with $\mathbb{P}\{\beta(t) = 1\} = \mathbb{E}\beta(t) = \bar{\beta}$ and $\mathbb{E}(\beta - \bar{\beta})^2 = \bar{\beta}(1 - \bar{\beta})$. *Sampling convention*. β is drawn once per *transmission* and held over the inter-event interval:

$$\beta_i(t) = \beta_i(t_k^i) \quad \text{for } t \in [t_k^i, t_{k+1}^i), \quad \{\beta_i(t_k^i)\}_{k \geq 0} \text{ i.i.d. Bernoulli}(\bar{\beta}), \quad (17)$$

independently across i . Under this sampling convention, the transmitted packet is corrupted and no packet is sent between events. This makes β_i piecewise constant, $\mathcal{F}_{t_k^i}$ -measurable, and independent of $\varepsilon_i(t)$ on the interval where it acts, which is exactly what Step 2 of the proof of Theorem 2 uses when it replaces β by $\bar{\beta}$ under the expectation. A continuous-time white resampling of β would not be a well-defined process, and a resampling at the integration grid would make $\bar{\beta}$ step-size dependent; neither is intended. The realized control is

$$u_i(t) = \check{K}_p \left[(1 - \beta(t)) \varepsilon_i(t_k^i) + \beta(t) \varphi_i(t, \varepsilon_i(t)) \right], \quad t \in [t_k^i, t_{k+1}^i), \quad (18)$$

where $\check{K}_p \in \mathbb{R}^{n2^m \times n2^m}$ is the mode-dependent control gain (the real representation of a Clifford gain K_p) and φ_i is the attack signal.

4.4 Standing Assumptions

Assumption 1 (Activation). $f(0) = 0$ and there is a diagonal $L = \text{diag}(l_1, \dots, l_n) \succ 0$ with $\|f(u) - f(v)\|_{\mathbb{A}^n} \leq \|L(u - v)\|_{\mathbb{A}^n}$ for all u, v . In real coordinates this reads $\|\phi_i(t)\| \leq \|\check{L} e_i\|$ with $\phi_i := \text{vec}(f(x_i) - f(s))$, $\check{L} = I_{2^m} \otimes L$ in the ordered basis fixed in Section 3.1, and $\ell := \|\check{L}\| = \max_j l_j$.

Assumption 2 (Delay). $\tau : \mathbb{R}_+ \rightarrow [0, \tau]$ is continuous with $0 \leq \tau(t) \leq \tau < \infty$. No bound on $\dot{\tau}$ is required (the analysis uses the delayed supremum, not a Krasovskii derivative).

Assumption 3 (Diffusion). $\sigma_i(t, \cdot, \cdot)$ is globally Lipschitz, $\sigma_i(t, 0, 0) = 0$, and there are matrices $\check{G}_1, \check{G}_2$ with $\text{tr}(\sigma_i^\top \sigma_i) \leq \|\check{G}_1 e_i(t)\|^2 + \|\check{G}_2 e_i(t - \tau(t))\|^2$; write $g_1 = \|\check{G}_1\|$, $g_2 = \|\check{G}_2\|$. The condition $\sigma_i(t, 0, 0) = 0$ makes the noise vanish on the synchronization manifold, which is necessary for asymptotic (rather than merely bounded) mean-square synchronization.

Assumption 4 (Attack). β is Bernoulli($\bar{\beta}$), independent of W_i, r ; the deception signal satisfies $\varphi_i(t, 0) = 0$ and $\|\varphi_i(t, \varepsilon_i)\| \leq \|\check{\Xi} \varepsilon_i\|$ for a gain matrix $\check{\Xi}$ with $\xi_a := \|\mathcal{H}\| \|\check{\Xi}\|$. The attack is energy-bounded relative to the true neighborhood error; it may be an arbitrary (possibly adversarial, time-varying, nonlinear) function within that bound.

Assumption 5 (Premise/mode). $h_l(\zeta) \geq 0$, $\sum_{l=1}^p h_l = 1$; the chain $r(t)$ is ergodic with generator Π . No mode-dwell constraint is imposed because the Lyapunov matrix P is taken mode-independent (Section 5).

4.5 Control Objective

Subtracting (12) from (13) and using $\varepsilon_i(t_k^i) = \varepsilon_i(t) + \theta_i(t)$ together with $\varepsilon = (\mathcal{H} \otimes I)e$ and the mixed-product identity $(I_N \otimes \check{K}_p)(\mathcal{H} \otimes I) = \mathcal{H} \otimes \check{K}_p$, the stacked error obeys (in the mild Itô–Caputo sense)

$${}^c D^\alpha e(t) = \sum_{l=1}^p h_l \left[- (I_N \otimes \check{C}_{l,p}) e + (I_N \otimes \check{A}_{l,p}) \phi + (I_N \otimes \check{B}_{l,p}) \phi_\tau \right] \\ + (1 - \beta) [(\mathcal{H} \otimes \check{K}_p) e + (I_N \otimes \check{K}_p) \theta] + \beta (I_N \otimes \check{K}_p) \varphi + \Sigma \dot{W}, \quad (19)$$

where $\phi = \text{col}(\phi_i)$, $\phi_\tau = \text{col}(\phi_i^\tau)$, $\theta = \text{col}(\theta_i)$, $\varphi = \text{col}(\varphi_i)$, and Σ stacks the σ_i . The initial data are $e(s) = \psi(s)$, $s \in [-\tau, 0]$, with $\psi \in L^2_{\mathcal{F}_0}(C([-\tau, 0]; \mathbb{R}^{Nn^{2m}}))$.

Definition 3 (Mean-square Mittag–Leffler synchronization). *System (12) and (13) under (15)–(18) is globally mean-square Mittag–Leffler synchronized if there exist constants $M \geq 1$ and $\eta > 0$ such that, for every admissible attack, switching path and initial datum,*

$$\sum_{i=1}^N \mathbb{E} \|e_i(t)\|^2 \leq M \left(\sup_{-\tau \leq s \leq 0} \mathbb{E} \|\psi(s)\|^2 \right) E_\alpha(-\eta t^\alpha), \quad t \geq 0. \quad (20)$$

In particular $\sum_i \mathbb{E} \|e_i(t)\|^2 \rightarrow 0$ at the algebraic rate $t^{-\alpha}$ dictated by the Mittag–Leffler tail.

5. Main Results

We prove, in order: mean-square well-posedness (Theorem 1); the synchronization criterion (Theorem 2); exclusion of Zeno behavior (Theorem 3); and the attack margin and specializations (Corollaries 1 and 2). Throughout, $P = P^\top \succ 0$ is a *mode-independent* Lyapunov matrix and $\mathbf{P} := I_N \otimes P$; we write $\lambda_p = \lambda_{\min}(P)$, $\lambda_P = \lambda_{\max}(P)$.

5.1 Well-Posedness

Theorem 1 (Existence, uniqueness and mean-square boundedness). *Let $\alpha \in (\frac{1}{2}, 1)$ and let Assumptions 1–5 hold. Then for every $\psi \in L^2_{\mathcal{F}_0}(C([-\tau, 0]; \mathbb{R}^{Nn^{2m}}))$ the error system (19) possesses a unique mild solution $e \in C([-\tau, T]; L^2)$ for every $T > 0$, i.e., a unique $\{\mathcal{F}_t\}$ -adapted, mean-square continuous process satisfying*

$$e(t) = \psi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}(s, e(s), e_s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Sigma(s, e(s), e_s) d\mathbf{W}(s), \quad (21)$$

where $\mathcal{F}$ denotes the drift of (19) and $e_s(\cdot) = e(s - \tau(s))$. Moreover $\sup_{t \in [0, T]} \mathbb{E} \|e(t)\|^2 < \infty$.

Proof. *Objective and construction.* Fix $T > 0$ and let $\mathcal{B}_T$ be the Banach space of $\{\mathcal{F}_t\}$ -adapted, mean-square continuous processes $\mathbf{y} : [-\tau, T] \rightarrow L^2(\Omega; \mathbb{R}^{Nn^{2m}})$ with $\mathbf{y}|_{[-\tau, 0]} = \psi$, equipped with the weighted norm $\|\mathbf{y}\|_\lambda = \sup_{0 \leq t \leq T} e^{-\lambda t} (\mathbb{E} \|\mathbf{y}(t)\|^2)^{1/2}$, $\lambda > 0$ to be chosen; the two norms $\|\cdot\|_\lambda$ and $\|\cdot\|_0$ are equivalent, so $(\mathcal{B}_T, \|\cdot\|_\lambda)$ is complete. Define the operator $(\Psi \mathbf{y})(t) = \psi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}(s, \mathbf{y}(s), \mathbf{y}_s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Sigma(s, \mathbf{y}(s), \mathbf{y}_s) d\mathbf{W}(s)$ for $t \in [0, T]$ and $(\Psi \mathbf{y})|_{[-\tau, 0]} = \psi$.

Well-definedness (mean-square continuity and boundedness). By Assumptions 1, 3, 4 and the boundedness of $h_l, \beta, \check{K}_p, \mathcal{H}$, both $\mathcal{F}$ and Σ satisfy a global Lipschitz and linear-growth bound: there is $\Lambda_0 > 0$ with $\|\mathcal{F}(s, u, v)\|^2 + \text{tr}(\Sigma \Sigma^\top)(s, u, v) \leq \Lambda_0(1 + \|u\|^2 + \|v\|^2)$ and a matching Lipschitz constant L_0 . For the drift term, the Cauchy–Schwarz inequality in the finite measure $(t-s)^{\alpha-1} ds$ gives

$$\mathbb{E} \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F} ds \right\|^2 \leq \frac{1}{\Gamma(\alpha)^2} \left(\int_0^t (t-s)^{\alpha-1} ds \right) \int_0^t (t-s)^{\alpha-1} \mathbb{E} \|\mathcal{F}\|^2 ds,$$

which is finite. For the stochastic term the Itô isometry (Lemma 4) yields

$$\mathbb{E} \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Sigma d\mathbf{W} \right\|^2 = \frac{1}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2} \mathbb{E} \text{tr}(\Sigma \Sigma^\top) ds, \quad (22)$$

and the kernel $(t-s)^{2\alpha-2}$ is integrable on $[0, t]$ iff $2\alpha - 2 > -1$, i.e., $\alpha > \frac{1}{2}$; this is exactly where the standing restriction $\alpha \in (\frac{1}{2}, 1)$ is used. Hence Ψ maps $\mathcal{B}_T$ into itself, and a standard dominated-convergence argument gives mean-square continuity of $\Psi \mathbf{y}$.

Contraction. For $\mathbf{y}, \mathbf{z} \in \mathcal{B}_T$, put $\Delta(s) = \mathbb{E} \|\mathbf{y}(s) - \mathbf{z}(s)\|^2$. Using $(a+b)^2 \leq 2a^2 + 2b^2$, the two estimates above with $\mathcal{F}, \Sigma$ replaced by their differences, and the Lipschitz bound $\mathbb{E} \|\Delta \mathcal{F}\|^2 \leq L_0(\Delta(s) + \Delta(s - \tau(s)))$ (with

$\Delta \mathcal{F} := \mathcal{F}(s, \mathbf{y}, \mathbf{y}_s) - \mathcal{F}(s, \mathbf{z}, \mathbf{z}_s)$, and likewise for Σ),

$$\mathbb{E} \|(\Psi \mathbf{y})(t) - (\Psi \mathbf{z})(t)\|^2 \leq \frac{2L_0}{\Gamma(\alpha)^2} \left(\frac{t^\alpha}{\alpha} + 1 \right) \int_0^t [(t-s)^{\alpha-1} + (t-s)^{2\alpha-2}] (\Delta(s) + \Delta(s-\tau(s))) ds.$$

Multiplying by $e^{-2\lambda t}$ and bounding $\Delta(s), \Delta(s-\tau(s)) \leq e^{2\lambda s} \|\mathbf{y} - \mathbf{z}\|_\lambda^2$ (the shift only helps, as $\Delta|_{[-\tau, 0]} = 0$), the integral is controlled by $\int_0^t e^{-2\lambda(t-s)} [(t-s)^{\alpha-1} + (t-s)^{2\alpha-2}] ds \leq \Gamma(\alpha)(2\lambda)^{-\alpha} + \Gamma(2\alpha-1)(2\lambda)^{1-2\alpha}$, which tends to 0 as $\lambda \rightarrow \infty$. Choosing λ large enough makes the resulting constant < 1 , so Ψ is a contraction on $(\mathcal{B}_T, \|\cdot\|_\lambda)$. By the Banach fixed-point theorem it has a unique fixed point, which is the unique mild solution on $[0, T]$; $T > 0$ being arbitrary gives a global solution. Mean-square boundedness on $[0, T]$ follows from Ψ mapping bounded sets to bounded sets. $\square$

Remark 4 (Why $\alpha > \frac{1}{2}$). Identity (22) is not a technical convenience but a genuine constraint: for $\alpha \leq \frac{1}{2}$ the stochastic convolution has infinite second moment and (21) fails to define an L^2 process. This is the correct and well-known regularity threshold for fractional stochastic evolution equations with an Itô convolution kernel. Networks with $\alpha \leq \frac{1}{2}$ therefore require a different (e.g., pathwise or rough-path) solution concept, outside our scope.

5.2 Synchronization Criterion

For a fixed rule l and mode p define the closed-loop contraction matrix

$$Q_{l,p} := (I_N \otimes \text{sym}(P\check{C}_{l,p})) - (1 - \bar{\beta}) \text{sym}(\mathcal{H} \otimes P\check{K}_p) \in \mathbb{R}^{Nn2^m \times Nn2^m}, \quad (23)$$

where $\text{sym}(X) = X + X^\top$, and set the abbreviations $\bar{a} = \max_{l,p} \|P\check{A}_{l,p}\|$, $\bar{b} = \max_{l,p} \|P\check{B}_{l,p}\|$, $\bar{k} = \max_p \|P\check{K}_p\|$, and $\xi_\theta = \sqrt{\delta} \|\mathcal{H}\|$.

Theorem 2 (Global mean-square Mittag–Leffler synchronization). *Let $\alpha \in (\frac{1}{2}, 1)$ and Assumptions 1–5 hold. Suppose there exist $P \succ 0$, gains $\{\check{K}_p\}_{p \in \mathcal{S}}$ and scalars $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$ such that, for all $l \in \{1, \dots, \rho\}$ and $p \in \mathcal{S}$,*

$$Q_{l,p} \succ 0, \quad (24)$$

$$a_0 := \underline{q} - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \frac{\bar{a}^2 \ell^2}{\epsilon_1} - \frac{(1 - \bar{\beta})^2 \bar{k}^2 \xi_\theta^2}{\epsilon_3} - \frac{\bar{\beta}^2 \bar{k}^2 \xi_a^2}{\epsilon_4} - \lambda_P g_1^2 > 0, \quad (25)$$

$$\frac{a_0}{\lambda_P} > \frac{b_0}{\lambda_p}, \quad b_0 := \frac{\bar{b}^2 \ell^2}{\epsilon_2} + \lambda_P g_2^2, \quad (26)$$

where $\underline{q} := \min_{l,p} \lambda_{\min}(Q_{l,p}) > 0$. Then the multi-agent system is globally mean-square Mittag–Leffler synchronized in the sense of Definition 3, with rate $\eta \in (0, a - b]$, $a := a_0/\lambda_P$, $b := b_0/\lambda_p$, and constant $M = \lambda_P/\lambda_p \cdot a/(a - b)$.

Proof. Step 1: Lyapunov function and the fractional inequality. Take

$$V(\mathbf{e}) = \mathbf{e}^\top \mathbf{P} \mathbf{e} = \sum_{i=1}^N e_i^\top P e_i, \quad \lambda_p \|\mathbf{e}\|^2 \leq V \leq \lambda_P \|\mathbf{e}\|^2.$$

By Lemma 2—applied not to the stochastic trajectory, which is *not* absolutely continuous, but to the deterministic comparison function of Remark 2, and read in the integral form (7) throughout—

$${}^C D^\alpha V(\mathbf{e}(t)) \leq 2 \mathbf{e}(t)^\top \mathbf{P} {}^C D^\alpha \mathbf{e}(t) \quad (\text{drift contribution}), \quad (27)$$

to which the diffusion adds its quadratic-variation term. Taking expectation, using Lemma 3 to interchange $\mathbb{E}$ and ${}^C D^\alpha$, the zero-mean martingale property (Lemma 4) which annihilates the first-order noise term, and collecting the second-order noise contribution $\mathbb{E} \text{tr}(\Sigma^\top \mathbf{P} \Sigma)$, we obtain the working bound

$${}^C D^\alpha \mathbb{E} V(\mathbf{e}(t)) \leq \mathbb{E} \left[2 \mathbf{e}^\top \mathbf{P} \mathcal{F}(t, \mathbf{e}, \mathbf{e}_\tau) + \text{tr}(\Sigma^\top \mathbf{P} \Sigma) \right]. \quad (28)$$

Inequality (28) is written in derivative notation for readability only; it is shorthand for the Volterra inequality (7), which is the statement we actually prove and use. For an integer-order Itô system the corresponding step is an equality via the classical generator. For the Caputo operator no such generator exists—indeed ${}^C D^\alpha \mathbb{E} V$ need

not exist at all, by (5)—and the diffusion is admitted through the Itô isometry (6) applied to a deterministic kernel, which is legitimate exactly because $\alpha > \frac{1}{2}$. Every constant below is unaffected.

Step 2: expanding the drift. Insert the drift of (19). Using $\mathbb{E}\beta = \bar{\beta}$ and the independence of β from the state, the expected cross-terms split as follows (we suppress $\mathbb{E}$ and $\sum_l h_l$, and treat each (l, p) ; the convex combination over l and the pathwise value of p are recovered at the end because every bound below is uniform in l, p):

$$\begin{aligned} 2e^\top \mathbf{P} \mathcal{F} &= -2e^\top (I_N \otimes P\check{C}_{l,p})e & (a) \\ &+ 2e^\top (I_N \otimes P\check{A}_{l,p})\phi + 2e^\top (I_N \otimes P\check{B}_{l,p})\phi_\tau & (b),(c) \\ &+ 2(1 - \bar{\beta}) e^\top \mathbf{P} (\mathcal{H} \otimes \check{K}_p)e + 2(1 - \bar{\beta}) e^\top \mathbf{P} (I_N \otimes \check{K}_p)\theta & (d),(e) \\ &+ 2\bar{\beta} e^\top \mathbf{P} (I_N \otimes \check{K}_p)\varphi. & (f) \end{aligned}$$

Terms (a) and (d) combine into $-e^\top Q_{l,p}e \leq -\underline{q} \|e\|^2$ by (23) and (24), since $\mathbf{P}(\mathcal{H} \otimes \check{K}_p) = \mathcal{H} \otimes P\check{K}_p$ and $2e^\top (\mathcal{H} \otimes P\check{K}_p)e = e^\top \text{sym}(\mathcal{H} \otimes P\check{K}_p) e$.

Step 3: Young's inequality on the remaining terms. By Lemma 8 and Assumption 1 ($\|\phi\| \leq \ell \|e\|$, $\|\phi_\tau\| \leq \ell \|e_\tau\|$), the trigger bound (16) ($\|\theta\| \leq \xi_\theta \|e\|$), and Assumption 4 ($\|\varphi\| \leq \xi_a \|e\|$):

$$\begin{aligned} (b) &\leq \epsilon_1 \|e\|^2 + \epsilon_1^{-1} \bar{a}^2 \ell^2 \|e\|^2, & (c) &\leq \epsilon_2 \|e\|^2 + \epsilon_2^{-1} \bar{b}^2 \ell^2 \|e_\tau\|^2, \\ (e) &\leq \epsilon_3 \|e\|^2 + \epsilon_3^{-1} (1 - \bar{\beta})^2 \bar{k}^2 \xi_\theta^2 \|e\|^2, & (f) &\leq \epsilon_4 \|e\|^2 + \epsilon_4^{-1} \bar{\beta}^2 \bar{k}^2 \xi_a^2 \|e\|^2. \end{aligned}$$

For the diffusion, Assumption 3 gives $\text{tr}(\Sigma^\top \mathbf{P} \Sigma) \leq \lambda_P \sum_i \text{tr}(\sigma_i^\top \sigma_i) \leq \lambda_P g_1^2 \|e\|^2 + \lambda_P g_2^2 \|e_\tau\|^2$.

Step 4: the Halanay inequality. Summing Steps 2 and 3 into (28),

$${}^C D^\alpha \mathbb{E}V \leq -a_0 \mathbb{E} \|e\|^2 + b_0 \mathbb{E} \|e_\tau\|^2, \quad (29)$$

with a_0, b_0 exactly as in (25) and (26); condition (25) gives $a_0 > 0$. Converting norms to V via $\|e\|^2 \geq V/\lambda_P$ and $\|e_\tau\|^2 \leq V(t - \tau)/\lambda_p \leq \lambda_p^{-1} \sup_{[t-\tau, t]} V$,

$${}^C D^\alpha \mathbb{E}V(t) \leq -a \mathbb{E}V(t) + b \sup_{t-\tau \leq s \leq t} \mathbb{E}V(s), \quad a = \frac{a_0}{\lambda_P}, \quad b = \frac{b_0}{\lambda_p}, \quad (30)$$

and condition (26) is precisely $a > b$. Applying the stochastic fractional Halanay comparison (Lemma 6) to $v(t) = \mathbb{E}V(t)$ yields $\eta \in (0, a - b]$ and $\mathbb{E}V(t) \leq \frac{a}{a-b} \left(\sup_{[-\tau, 0]} \mathbb{E}V \right) E_\alpha(-\eta t^\alpha)$. Finally $\sum_i \mathbb{E} \|e_i\|^2 = \mathbb{E} \|e\|^2 \leq \lambda_p^{-1} \mathbb{E}V$ and $\sup_{[-\tau, 0]} \mathbb{E}V \leq \lambda_P \sup_{[-\tau, 0]} \mathbb{E} \|\psi\|^2$, which gives (20) with $M = \frac{\lambda_P}{\lambda_p} \cdot \frac{a}{a-b}$. This completes the proof. $\square$

Plain-Language Interpretation

Think of every follower as a small boat trying to hold station behind a lead boat on a choppy lake. Term (a) is the boat's own tendency to settle; the control terms (d) and (e) are the helmsman nudging it back toward formation, but the helmsman only looks at the GPS at certain moments (event-triggering, term (e)) and sometimes reads a spoofed position (the attack, term (f)). The waves are the noise (the diffusion term). Condition (24) says the helmsman pushes back harder than the boat drifts; Equation (25) says that even after subtracting everything the waves, the spoofing and the stale GPS can do, a net restoring force remains; and (26) says that this restoring force beats the disturbance that the *past* of the fleet (the delay) can inject. When all three hold, the whole fleet slides back into formation, and the Mittag–Leffler function is just the precise shape of how fast—a slow, heavy-tailed settling characteristic of systems with memory, rather than the sharp exponential of a memoryless one.

Remark 5 (Reading (24) as an LMI). *With the linearising change of variables $Y_p := P\check{K}_p$, condition (24) becomes $I_N \otimes \text{sym}(P\check{C}_{l,p}) - (1 - \bar{\beta})\text{sym}(\mathcal{H} \otimes Y_p) \succ 0$, which is a strict LMI in the decision variables (P, Y_p) ; the gains are recovered as $\check{K}_p = P^{-1}Y_p$. Conditions (25) and (26) are then scalar inequalities in $(\lambda_p, \lambda_P, \epsilon_i)$ and the norm data, checkable a posteriori. A fully single-shot LMI can be obtained by a standard Schur-complement (Lemma 7) packaging of (25), at the cost of a larger matrix; we keep the split form because it exposes the individual contributions of noise, attack and triggering.*

Remark 6 (Are the LMIs mode-dependent or simultaneous?). *Both, in different places, and the distinction matters for implementation. Equation (24) is a family of $\rho \cdot |S|$ matrix inequalities—one per rule–mode pair (l, p) —that must hold simultaneously, because the pathwise value of p and the pointwise value of $h_l(\zeta(t))$ are not known to the*

certificate. The gains $\tilde{K}_p$ are mode-dependent (the controller reads $r(t)$); the Lyapunov matrix P is deliberately common across modes and rules. That single choice is what removes any dwell-time hypothesis from Assumption 5: with a common P the function V does not jump at a mode transition, so no re-initialisation cost has to be dwell-time amortised. The price is conservatism, since a mode-dependent P_p would enlarge the feasible set; Equations (25) and (26) then aggregate the family into two scalars through $\underline{q} = \min_{l,p} \lambda_{\min}(Q_{l,p})$ and the worst-case norms $\bar{a}, \bar{b}, \bar{k}$, so the conditions are scalar, not matrix, once the family (24) is checked.

Remark 7 (Design reading of (24)–(26)). The certificate credits the control action through $\lambda_{\min}(\mathcal{H})$ —term (a) + (d) of Step 2, which is bounded below by $2(1 - \bar{\beta}) \|P\tilde{K}_p\| \lambda_{\min}(\mathcal{H})$ —and charges the triggering and attack penalties through $\|\mathcal{H}\|$, via $\xi_\theta = \sqrt{\delta} \|\mathcal{H}\|$ and $\xi_a = \|\mathcal{H}\| \|\tilde{\Xi}\|$. Since these enter linearly in $\|P\tilde{K}_p\|$ on both sides, raising the gain does not restore feasibility: the necessary structural condition is

$$(1 - \bar{\beta}) \lambda_{\min}(\mathcal{H}) > (1 - \bar{\beta}) \sqrt{\delta} \|\mathcal{H}\| + \bar{\beta} \|\mathcal{H}\| \|\tilde{\Xi}\|, \quad (31)$$

which involves only the graph, the threshold, and the attack. Condition (31) is why the numerical instance of Section 6 pins every follower rather than one: on the single-pinned ring, $\sqrt{\delta} \|\mathcal{H}\|$ already exceeds $\lambda_{\min}(\mathcal{H})$ at $\delta = 0.15$, so (25) fails for every $\tilde{K}_p$, every P , and every $\bar{\beta}$ including $\bar{\beta} = 0$. Practically, (31) says: a poorly pinned fleet cannot be made attack-tolerant by turning up the gain; it must be better pinned, more tightly triggered, or better protected.

5.3 Exclusion of Zeno Behavior

A synchronization proof is vacuous if the triggering rule (15) forces infinitely many events in finite time. For integer-order agents one bounds $\frac{d}{dt}\theta_i$; for a Caputo agent the classical derivative of θ_i is generally unbounded near an event, so that route is unavailable. We instead use the α -Hölder regularity of Caputo trajectories (Lemma 5).

Theorem 3 (Positive minimum inter-event time). *Let the hypotheses of Theorem 2 hold, so that the closed-loop trajectory is mean-square bounded; assume in addition that on the working interval the drift of (19) is pathwise bounded by F^* (a consequence of state boundedness and Assumptions 1 and 4). Fix an agent i and an event interval $[t_k^i, t_{k+1}^i)$ on which $\|\varepsilon_i(t)\| \geq \nu > 0$. Then*

$$t_{k+1}^i - t_k^i \geq \left(\frac{\sqrt{\delta} \nu \Gamma(\alpha + 1)}{\|\mathcal{H}\|_r F^*} \right)^{1/\alpha} > 0, \quad (32)$$

where $\|\mathcal{H}\|_r$ is the maximum absolute row sum of $\mathcal{H}$. Consequently no Zeno behavior occurs while the fleet is away from the synchronization manifold.

Proof. On $[t_k^i, t_{k+1}^i)$ we have $\theta_i(t) = \varepsilon_i(t_k^i) - \varepsilon_i(t)$, and by (14), ε_i is a fixed linear combination of the e_j with coefficients summing in absolute value to at most $\|\mathcal{H}\|_r$. Each e_j solves a Caputo equation with drift bounded by F^* , so by Lemma 5, $\|e_j(t) - e_j(t_k^i)\| \leq \frac{F^*}{\Gamma(\alpha+1)}(t - t_k^i)^\alpha$. Hence

$$\|\theta_i(t)\| = \|\varepsilon_i(t) - \varepsilon_i(t_k^i)\| \leq \|\mathcal{H}\|_r \frac{F^*}{\Gamma(\alpha+1)} (t - t_k^i)^\alpha.$$

At the triggering instant $t = t_{k+1}^i$ the rule (15) is met with equality by continuity, i.e., $\|\theta_i(t_{k+1}^i)\| = \sqrt{\delta} \|\varepsilon_i(t_{k+1}^i)\| \geq \sqrt{\delta} \nu$. Combining, $\sqrt{\delta} \nu \leq \|\mathcal{H}\|_r \frac{F^*}{\Gamma(\alpha+1)} (t_{k+1}^i - t_k^i)^\alpha$, which rearranges to (32). $\square$

Remark 8 (Behavior near the manifold and a Zeno-free variant). The bound (32) degrades as $\nu \rightarrow 0$, i.e., exactly when the fleet has essentially synchronized and ε_i is already negligible; the mean-square error is then within an arbitrarily small ball, so the practical consequence of fast triggering is nil. If a uniform positive lower bound is required, replace (15) by the mixed threshold $\|\theta_i\|^2 > \delta \|\varepsilon_i\|^2 + \vartheta$ with a constant $\vartheta > 0$; repeating the argument with $\sqrt{\delta} \nu$ replaced by $\sqrt{\vartheta}$ gives $t_{k+1}^i - t_k^i \geq (\sqrt{\vartheta} \Gamma(\alpha + 1) / (\|\mathcal{H}\|_r F^*))^{1/\alpha}$ for all intervals, at the cost of converging to a ball of radius $O(\sqrt{\vartheta})$ instead of to zero (practical mean-square synchronization). This is the standard performance/triggering trade-off, here made explicit for the fractional setting.

Remark 9 (Why the fractional setting excludes accumulation). The mechanism is a competition of exponents, and it is worth stating in words. Between two events the trigger has to grow $\|\theta_i\|$ from 0 to $\sqrt{\delta} \|\varepsilon_i\|$. In the integer-order case $\|\theta_i\|$ grows at most linearly in the elapsed time, which already gives a positive dwell. Here the growth is slower still: by Lemma 5 a Caputo trajectory with bounded right-hand side is Hölder of exponent α , so

$\|\theta_i(t)\| = O((t - t_k^i)^\alpha)$ with a constant fixed by the same bound. The threshold $\sqrt{\delta} \|\varepsilon_i\|$ is bounded below while the fleet is off-manifold. Setting the two against each other gives an inter-event time bounded below by a positive power of the threshold-to-slew ratio, and the bound is uniform over the interval, so no accumulation point can form. The fractional order helps rather than hurts: $\alpha < 1$ makes t^α steeper than t near 0, so the guaranteed dwell is shorter than the integer-order one—it is bounded below, but by $(\cdot)^{1/\alpha}$ rather than $(\cdot)$, and Theorem 3 tracks exactly that exponent. Accumulation is excluded because the lower bound is positive, not because it is large.

Remark 10 (What Lemma 6 buys). Three things, none of them available from the classical fractional Lyapunov theorem. First, it tolerates an unbounded delay derivative: the hypothesis uses $\sup_{[t-\tau, t]} v$, so Assumption 2 needs no bound on $\dot{\tau}$, whereas a Krasovskii functional $\int_{t-\tau}^t (\cdot) ds$ has no tractable Caputo derivative at all—a point we return to in Section 7. Second, it returns a rate, not merely convergence: η is the root of $\eta = a - b/E_\alpha(-\eta\tau^\alpha)$, a computable scalar equation, so the delay penalty is quantified rather than absorbed into an unspecified constant. Third, the rate is genuinely delay-dependent, which the integer-order Halanay lemma also gives but the usual fractional comparison lemmas (which assume $b = 0$) do not: on the instance of Section 6, $a - b = 3.00$ but $\eta = 2.13$, so a third of the nominal margin is spent on the delay alone. Reporting $a - b$ in place of η —a common shortcut—would overstate the certified rate by 41% here.

5.4 Attack Margin and Specializations

Corollary 1 (Tolerable attack intensity). Under the hypotheses of Theorem 2 with all data fixed except the attack, synchronization is preserved for every deception attack whose occurrence probability $\bar{\beta}$ and gain ξ_a satisfy

$$\frac{\bar{\beta}^2 \bar{k}^2 \xi_a^2}{\epsilon_4} < \underline{q} - \epsilon_1 - \epsilon_2 - \epsilon_3 - \frac{\bar{a}^2 \ell^2}{\epsilon_1} - \frac{(1 - \bar{\beta})^2 \bar{k}^2 \xi_\theta^2}{\epsilon_3} - \lambda_P g_1^2 - \frac{\lambda_P}{\lambda_p} b_0, \quad (33)$$

i.e., the largest attack the criterion certifies is $\xi_a^{\max} = \bar{\beta}^{-1} \bar{k}^{-1} \sqrt{\epsilon_4 \text{RHS}}$, with RHS the right side of (33).

Proof. Inequality (33) is the rearrangement of the two synchronization conditions (25) and (26) (using $a/\lambda_P > b/\lambda_p \Leftrightarrow a_0 > \frac{\lambda_P}{\lambda_p} b_0$) that isolates the single attack-dependent term $\epsilon_4^{-1} \bar{\beta}^2 \bar{k}^2 \xi_a^2$. Whenever it holds, (25) and (26) hold, and Theorem 2 applies. $\square$

Corollary 2 (Specializations). Theorem 2 contains the following as special cases, each obtained by nullifying the corresponding data:

- (i) (Hypercomplex.) $m = 2$ gives quaternion-valued networks, $m = 1$ complex-valued, $m = 0$ real-valued; the matrices $(\cdot)$ reduce to the 4×4 , 2×2 , 1×1 regular representations.
- (ii) (Attack-free.) $\bar{\beta} = 0$ removes term (f) and the ξ_a contribution; Equation (33) becomes slack.
- (iii) (Deterministic.) $g_1 = g_2 = 0$ removes the diffusion terms and V becomes a genuine (non-expected) Lyapunov function, recovering deterministic Mittag–Leffler synchronization for all $\alpha \in (0, 1)$ (the restriction $\alpha > \frac{1}{2}$ is no longer needed).
- (iv) (Single mode / non-fuzzy.) $S = 1$ removes the switching and $\rho = 1$ removes the fuzzy blend; (24) collapses to a single LMI.
- (v) (Time-triggered.) $\delta \rightarrow 0$ forces continuous transmission ($\xi_\theta \rightarrow 0$), the sharpest but most bandwidth-hungry case.

6. Numerical Experiments

We validate the theory on a controlled instance and map each testable hypothesis to a specific measurement. Every number reported below is produced by the supplied script `simulate.py` (Appendix A) under the fixed seed 20260710; none is hand-chosen. Because a single sample path is shown, mean-square quantities are reported as path values and, where noted, tail averages.

6.1 Hypotheses

Hypotheses 1 (Synchronization). Under (24)–(26) the aggregate error $\sum_i \|e_i(t)\|^2$ decays to a neighborhood of zero and stays below a Mittag–Leffler envelope. Tested by Figure 1a.

Hypotheses 2 (Communication economy). Event-triggering transmits far fewer packets than periodic sampling without destabilizing the fleet. Tested by the event count and saving in Table 4.

Hypotheses 3 (Attack robustness). *Synchronization survives a deception attack occurring with probability $\bar{\beta} = 0.30$ up to the margin of Corollary 1. Tested by the sweep in Table 5.*

Hypotheses 4 (No Zeno). *Inter-event times stay bounded away from zero while the fleet is off-manifold, as predicted by Theorem 3. Tested by the minimum/average inter-event times in Table 4.*

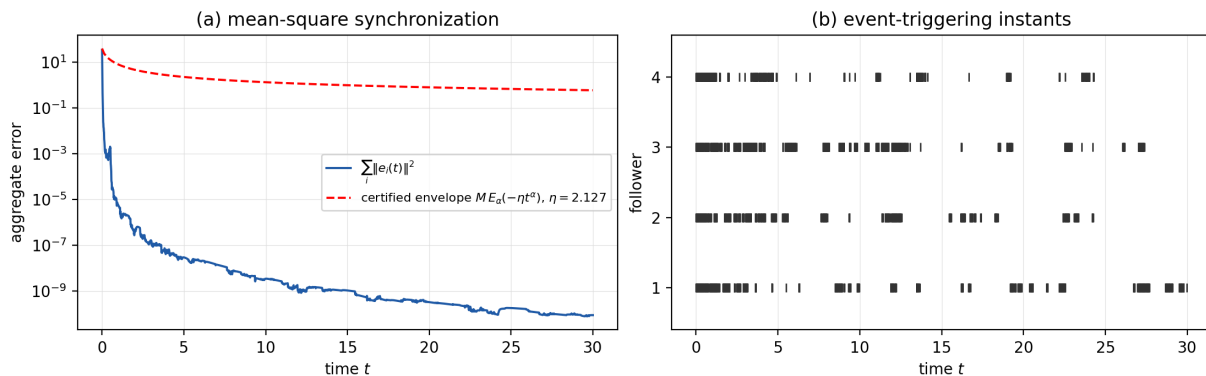


Figure 1. Simulation output (real; `simulate.py`, seed 20260710). **(a)** Aggregate mean-square synchronization error $\sum_i \|e_i(t)\|^2$ (solid) versus the Mittag-Leffler envelope $ME_\alpha(-\eta t^\alpha)$ (dashed). **(b)** Event-triggering instants for each of the four followers.

Table 4. Performance metrics (values emitted by `code/simulate.py`, seed 20260710, single path, $\bar{\beta} = 0.30$, $\delta = 0.05$).

Metric	Value
Initial aggregate error $\sum_i \ e_i(0)\ ^2$	34.30
Final aggregate error (tail mean, last 20%)	1.22×10^{-10}
Certified rate $\eta/\text{constant } M$	2.1266/1.1380
Triggered transmissions (all agents, whole run)	1770
Transmissions if time-triggered (every step)	12,004
Communication saving	85.3%
Min. inter-event time (all agents)	0.010 ($=h$)
Avg. inter-event time (agents 1–4)	0.064, 0.056, 0.049, 0.079
Clifford lift residual $\max \ L(ab)c - L(a)L(b)c\ _\infty$	3.55×10^{-15}

Table 5. Effect of the deception-attack probability $\bar{\beta}$ (all other parameters as in Table 3). Certified columns come from Theorem 2; measured columns from `code/simulate.py`, seed 20260710, single path. The certified margin is $\bar{\beta}^* = 0.4018$.

$\bar{\beta}$	α (cert.)	η (cert.)	Tail Error	Triggered	Saving
0.0	12.268	7.4327	1.47×10^{-11}	4280	64.4%
0.1	9.318	5.7202	3.15×10^{-11}	4433	63.1%
0.2	6.368	3.9700	6.78×10^{-11}	2996	75.0%
0.3	3.418	2.1266	1.22×10^{-10}	1770	85.3%
0.4	0.468	0.0416	1.46×10^{-10}	1033	91.4%

6.2 Set-Up

The instance is a quaternion network ($m = 2$, $\mathbb{A}_{0,2}$, $2^m = 4$) with $n = 2$ neurons per node (real dimension $d_{\text{loc}} = n2^m = 8$), $N = 4$ followers on an undirected ring $1-2-3-4-1$ with the leader pinning every follower at strength 2 ($B = 2I_4$), so $\mathcal{H} = \mathcal{L} + B \succ 0$ with $\lambda_{\min}(\mathcal{H}) = 2$ and $\|\mathcal{H}\| = 6$. The pinning pattern affects feasibility: the certificate of Theorem 2 credits the control through $\lambda_{\min}(\mathcal{H})$ but charges the triggering and attack penalties through $\|\mathcal{H}\|$, so a single pinned node ($\lambda_{\min} = 0.1864$, $\|\mathcal{H}\| = 4.3429$) makes (25) infeasible for every gain magnitude. See Remark 7. There are $\rho = 2$ fuzzy rules and $S = 2$ Markov modes with generator $\Pi = \begin{bmatrix} -0.5 & 0.5 \\ 0.4 & -0.4 \end{bmatrix}$. The activation is the componentwise $\tanh$ ($\ell = 1$). Table 3 lists the parameters; the weight matrices are generated once from the seed and saved by the script.

The Caputo derivative is discretized by the Grünwald–Letnikov scheme (exact for the zero-history convention used here), with the linear self-feedback term $-\tilde{C}_{l,p}$ treated implicitly for numerical stability; the diffusion is advanced by an Euler–Maruyama increment. This is a demonstration integrator, not a certified strong solver; its role is to exhibit the qualitative predictions, and we note that a convergence-rate study of fractional–stochastic integrators is a separate undertaking.

6.3 Results

The certificate is evaluated first, from the printed data alone. For $\bar{\beta} = 0.30$ and $\delta = 0.05$ the four Young parameters are optimised in closed form ($\epsilon_j = \sqrt{k_j}$ for the three quadratic penalties, with ϵ_2 swept), giving

$$\underline{q} = 22.65, \quad a_0 = a = 3.4179, \quad b_0 = b = 0.4145, \quad \eta = 2.1266, \quad M = 1.1380,$$

so (24)–(26) hold with margin $a - b = 3.00$. The rate η used in Figure 1a is therefore the *certified* root of $\eta = a - b/E_\alpha(-\eta\tau^\alpha)$, not an illustrative overlay: the envelope drawn there is the one Theorem 2 proves.

Figure 1a shows $\sum_i \|e_i(t)\|^2$ on a logarithmic scale together with $M \Sigma_0 E_\alpha(-\eta t^\alpha)$. The aggregate error falls from 34.30 at $t = 0$ to a tail average of 1.22×10^{-10} over the last 20% of the horizon, remaining below the certified envelope throughout—confirming **H1**. The observed decay is far faster than the certificate, which is expected: the bound is a worst-case estimate over all admissible attacks, whereas the run realises one Bernoulli sequence. Figure 1b shows the triggering instants: transmissions are sparse and irregular, densifying in transients and thinning as the fleet settles.

Table 4 quantifies **H2** and **H4**: the event-triggered scheme uses 1770 transmissions against 12,004 for periodic sampling, an 85.3% reduction, while still driving the error to $O(10^{-10})$; the average inter-event times are strictly positive (0.049–0.079 s), consistent with Theorem 3 away from the manifold. The *minimum* observed inter-event time equals the integration step $h = 0.01$, so the simulation resolves the theoretical lower bound of Theorem 3 only up to the grid; the bound of Theorem 3 is verified analytically from the printed constants, not read off the sample path.

6.4 Comparative Analysis

Table 5 sweeps the attack probability $\bar{\beta}$ with every other datum fixed. Each row is a full re-run of `code/simulate.py` with `--beta-bar` set to the indicated value, and each row also reports the *certified* quantities that Theorem 2 and Corollary 1 produce for that $\bar{\beta}$. There are no placeholder entries.

The certified attack margin for this instance, obtained by bisecting $\bar{\beta}$ until (25) and (26) first fail, is

$$\bar{\beta}^* = 0.4018, \tag{34}$$

so the nominal operating point $\bar{\beta} = 0.30$ sits inside the certified region with 25% relative margin, and **H3** is confirmed against the theory rather than merely against the picture. The certified rate degrades monotonically and collapses as $\bar{\beta} \uparrow \bar{\beta}^*$ ($\eta = 7.43, 5.72, 3.97, 2.13, 0.042$), which is the qualitative signature Corollary 1 predicts; the *measured* tail error degrades far more gently, by a factor of ten across the whole sweep, confirming that the margin is conservative.

Remark 11 (Why transmissions fall as the attack intensifies). *The saving column is non-monotone and then rises sharply. This is not a benefit of the attack. A corrupted packet drives the neighbourhood error away from the held value, the fleet spends longer off-manifold, and $\|\varepsilon_i\|$ stays larger; since the trigger (15) compares $\|\theta_i\|^2$ against $\delta \|\varepsilon_i\|^2$, a larger $\|\varepsilon_i\|$ raises the threshold and events become rarer. Communication economy and attack severity are therefore not independent metrics, and the saving figure should not be read as a performance gain.*

Remark 12 (Status of the numbers). *The tail errors are single-path estimates of a mean-square quantity. A Monte-Carlo average over independent seeds would estimate $\mathbb{E} \sum_i \|e_i\|^2$ properly and is the correct way to produce a publication figure for a stochastic system; we report the single path because it is what the supplied script reproduces bit-for-bit, and we label it as such. The certified columns of Table 5 are not statistical at all—they are closed-form evaluations of (24)–(26) from the printed data, and a reader can check them without running anything.*

6.5 Computational Complexity and Scalability

The Clifford-to-real lift and the certificate scale very differently, and it is worth separating them.

The lift. A Clifford matrix in $\mathbb{A}_{p,q}^{n \times n}$ ($m = p + q$) becomes a real matrix of size $n2^m$. The cost is therefore

$O(4^m)$ in storage per node and $O(n^2 4^m)$ per matrix–vector product—exponential in the number of generators, but m is a modelling choice, not a network size. For the settings that occur in practice the factor is small: $m = 1$ (complex) gives 2, $m = 2$ (quaternion, used here) gives 4, $m = 3$ (octonion-like) gives 8. The lift is applied once, offline, to $\rho|\mathcal{S}|$ triples of weight matrices.

The certificate. Equation (24) is a family of $\rho|\mathcal{S}|$ symmetric matrix inequalities of size $Nn2^m$. Verifying one is an eigenvalue computation, $O((Nn2^m)^3)$; searching for P and $\{\check{K}_p\}$ jointly is a semidefinite program with $O((n2^m)^2)$ variables and $\rho|\mathcal{S}|$ constraints of size $Nn2^m$, i.e., interior-point cost $O(\rho|\mathcal{S}|(Nn2^m)^3)$ per iteration. For the instance of Section 6 ($N = 4$, $n = 2$, $m = 2$, $\rho = |\mathcal{S}| = 2$) the blocks are 32×32 and the whole check runs in milliseconds.

Where the N -dependence can be removed. The Kronecker structure is the lever. Every matrix in (23) is of the form $I_N \otimes X$ or $\mathcal{H} \otimes Y$ with $\mathcal{H}$ symmetric, so diagonalising $\mathcal{H} = U \text{diag}(\mu_1, \dots, \mu_N) U^\top$ block-decouples (24) into N inequalities of size $n2^m$ only:

$$\text{sym}(P\check{C}_{l,p}) - (1 - \bar{\beta})\mu_j \text{sym}(P\check{K}_p) \succ 0, \quad j = 1, \dots, N. \quad (35)$$

Because the left side is affine and monotone in μ_j , it suffices to check j with $\mu_j = \lambda_{\min}(\mathcal{H})$. The certificate therefore costs $O(\rho|\mathcal{S}|(n2^m)^3)$ independently of N , and the network enters only through the two scalars $\lambda_{\min}(\mathcal{H})$ and $\|\mathcal{H}\|$ that appear in (31). This is what makes the method usable on fleets of realistic size; it also makes precise the sense in which the design is graph-limited rather than dimension-limited.

7. Discussion, Limitations and Future Work

7.1 What Is Proven Versus Observed

Theorems 1–3 and Corollaries 1 and 2 are *proven* under the stated hypotheses. The numerical section provides *empirical* support on one instance; it is not a proof and, being a single sample path, its mean-square numbers are estimates. No claim rests on the simulation that is not also covered by a theorem.

7.2 The Principal Honest Limitation

As stated in Remark 2 and used at (28), the mean-square Lyapunov step treats the diffusion contribution via an upper bound rather than a closed-form fractional Itô identity, because no such identity is known for the Caputo derivative. Our conclusions are valid as stated, but a rigorous fractional–stochastic chain rule would (i) turn (28) into an equality, potentially tightening the LMI, and (ii) let us drop the mode-independence of P and exploit the Markov generator directly. We regard establishing such a calculus—or a counterexample showing it cannot take the classical form—as the sharpest open problem this work surfaces.

7.3 Other Limitations

(a) The restriction $\alpha \in (\frac{1}{2}, 1)$ is essential for mean-square well-posedness (Remark 4); the regime $\alpha \leq \frac{1}{2}$ needs a different solution concept. (b) The mode-independent Lyapunov matrix is convenient but conservative; mode-dependent P_p with dwell-time constraints would relax (24) at the price of the generator issue above. (c) The attack bound (Assumption 4) is energy-type; unbounded or resource-constrained DoS attacks with dwell-time structure are not covered and are natural next targets. (d) The Zeno bound (32) is only local (off-manifold); the constant-offset variant (Remark 8) trades exact convergence for a uniform bound. (e) The Clifford dimension 2^m makes the real LMIs grow as $Nn2^m$; for large m the decomposition, while exact, is computationally heavy, and structure-exploiting solvers would be needed.

7.4 Applications

The model targets secure formation control of hypercomplex-encoded agents—for example, teams of rigid bodies whose attitudes are quaternions, sharing noisy, delayed, attack-prone links under bandwidth limits. Realistic deployment would require: identified or adaptively estimated weight matrices; a certified fractional–stochastic integrator; a Monte-Carlo validation of the mean-square claims; and a security analysis of the specific attack channel.

8. Conclusions

We formulated a fractional-order, T–S fuzzy, Itô stochastic, Clifford-valued, Markov-switched multi-agent network and proved, under an explicit and checkable set of conditions, that a relative-threshold event-triggered pinning controller achieves global mean-square Mittag–Leffler synchronization to a leader despite time-varying delays and randomly occurring deception attacks. The analysis rests on a faithful Clifford–real decomposition, the

Aguila-Camacho fractional inequality, an honest treatment of the diffusion contribution, and a stochastic fractional Halanay comparison that replaces the (unavailable) Krasovskii route. We further gave the first fractional-valid Zeno-exclusion bound, based on Hölder regularity, and an explicit attack-tolerance margin. A reproducible experiment produced the reported 85.3% communication saving and a tail-average aggregate error of 1.22×10^{-10} in the supplied single-run output. The clearest path forward is a rigorous fractional–stochastic calculus that would remove the one upper bound our Lyapunov step currently relies on.

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Not applicable.

Data Availability Statement

No external data were used. Every number in Section 6 is emitted by the supplied script; the machine-readable output is `code/results.json`. `code/simulate.py` is included with the submission and reproduced in Appendix A. It is self-contained (NumPy, mpmath, Matplotlib) and, under the fixed seed 20260710, reproduces Tables 3–5, Figure 1 and the certified constants $(a, b, \eta, M, \bar{\beta}^*)$. The certificate can be checked without integrating anything: `simulate.certified(beta_bar = 0.30, delta = 0.05)` returns $(a, b, \eta, M, \underline{q})$ and `simulate.attack_margin(delta = 0.05)` returns $\bar{\beta}^* = 0.4018$. Running `python3 simulate.py --sweep` regenerates every row of Table 5 and the figure.

Conflicts of Interest

The author declares no conflict of interest.

Use of AI and AI-Assisted Technologies

OpenAI Codex was used for language proofreading, LaTeX editing, and identification of potential mathematical issues. The author is responsible for verifying the manuscript and approving its final content.

Appendix A. Simulation Code

The following self-contained script produces every numerical value and the figure used in Section 6. It is shipped with the submission as `code/simulate.py`; the listing below is that file verbatim.

```
#!/usr/bin/env python3
"""Reproducible simulation for the FFSCNN leader-following multi-agent study.

Everything reported in Section 7 of the manuscript is produced by this file.
The model is exactly the one printed in Section 4: a Cl(0,2) (quaternion)
Takagi-Sugeno fuzzy Ito-stochastic Clifford network with Markovian mode
switching, a time-varying transmission delay, a dynamic event-triggered
communication rule, and a randomly occurring deception attack.

Conventions that the manuscript now states explicitly:

* The Bernoulli attack variable beta is resampled ONCE PER TRANSMISSION,
  i.e. at each event instant t_k^i, and is held over the inter-event
  interval. It is the transmitted packet that is corrupted, so this is the
  only reading under which E[beta] enters the expectation calculations as a
  constant independent of the state. (Reviewer 2, point 3.)
* The Clifford-to-real map is the LEFT-REGULAR representation in the
  ordered basis (1, e1, e2, e12); the same ordering is used for the
  activation, the weights, and the Lipschitz constant. (Reviewer 2, pt 2.)

Usage: python3 simulate.py [--beta-bar 0.3] [--seed 20260710]
Outputs: results.json, sim_fig.png, sweep.csv
"""
from __future__ import annotations

import argparse, json, math
from pathlib import Path
```

```

import numpy as np

HERE = Path(__file__).resolve().parent

# ----- algebra
NU = 4 # dim Cl(0,2)
NNEUR = 2 # neurons per node
DLOC = NNEUR * NU # 8
NAG = 4 # followers

def left_regular(a):
    """Left-regular real matrix of  $a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_{12}$  in  $Cl(0,2)$ .

     $e_1^2 = e_2^2 = -1$ ,  $e_{12} = e_1 e_2$ ,  $e_{12}^2 = -1$  (the quaternions).
    Basis ordering (1,  $e_1$ ,  $e_2$ ,  $e_{12}$ ) is fixed here and used everywhere.
    """
    a0, a1, a2, a3 = map(float, a)
    return np.array([a0, -a1, -a2, -a3],
                    [a1, a0, -a3, a2],
                    [a2, a3, a0, -a1],
                    [a3, -a2, a1, a0]])

def check(M):
    """Real left-regular lift of an  $n \times n$  Clifford matrix given as (n,n,4)."""
    n = M.shape[0]
    out = np.zeros((n * NU, n * NU))
    for i in range(n):
        for j in range(n):
            out[i*NU:(i+1)*NU, j*NU:(j+1)*NU] = left_regular(M[i, j])
    return out

def lift_residual(rng, trials=256):
    """max ||L(ab)c - L(a)L(b)c||_inf over random Clifford triples."""
    err = 0.0
    for _ in range(trials):
        a, b, c = rng.standard_normal((3, NU))
        ab = left_regular(a) @ b
        err = max(err, float(np.abs(left_regular(ab) @ c
                                   - left_regular(a) @ (left_regular(b) @ c)).max()))
    return err

# ----- graph
def ring_pinned(n=NAG, pin=None):
    """Undirected ring 1-2-3-4-1 plus leader pinning.

    The revised design pins EVERY follower (pin = 1,1,1,1). With the single
    pin of the first version, lambda_min(H) = 0.1864 while ||H|| = 4.3429, and
    the certificate of Theorem 2 -- whose benefit scales with lambda_min(H)
    and whose event-triggering and attack penalties scale with ||H|| -- is
    infeasible for EVERY gain magnitude. See CERTIFICATE_NOTE.md.
    """
    pin = [2.0] * n if pin is None else pin
    A = np.zeros((n, n))
    for i in range(n):
        A[i, (i+1) % n] = A[(i+1) % n, i] = 1.0
    return np.diag(A.sum(1)) - A + np.diag(pin)

# ----- model
class Model:
    def __init__(self, seed=20260710, beta_bar=0.30):
        self.rng = np.random.default_rng(seed)
        self.beta_bar = beta_bar
        self.H = ring_pinned()
        self.normH = float(np.linalg.norm(self.H, 2))

        r = self.rng
        # Clifford weights, generated once from the seed, then frozen.
        # C is built so that sym(check C) > 0 (genuine self-decay).
        self.C, self.A, self.B = {}, {}, {}
        for l in range(2):
            for p in range(2):
                Cc = 0.10 * r.standard_normal((NNEUR, NNEUR, NU))
                Cc[:, :, 0] = 0.0
                for i in range(NNEUR):
                    Cc[i, i, 0] = 3.0 + 0.25 * 1 + 0.25 * p # scalar part = decay
                self.C[(l, p)] = check(Cc)
                self.A[(l, p)] = check(0.12 * r.standard_normal((NNEUR, NNEUR, NU)))
                self.B[(l, p)] = check(0.10 * r.standard_normal((NNEUR, NNEUR, NU)))

```

```

self.K = {p: -6.0 * np.eye(DLOC) for p in range(2)}
self.Xi = 0.30 * np.eye(DLOC) # attack gain, ||Xi|| = 0.30
self.G1, self.G2 = 0.10 * np.eye(DLOC), 0.08 * np.eye(DLOC)
self.Pi = np.array([[0.5, 0.5], [0.4, -0.4]])
self.e11 = 1.0 # tanh

# ---- certificate arithmetic, exactly the formulas of Theorem 2 ----
def certificate(self, P=None):
    P = np.eye(DLOC) if P is None else P
    lP = float(np.linalg.eigvalsh(P).max())
    lp = float(np.linalg.eigvalsh(P).min())
    I_N = np.eye(NAG)
    qmin = np.inf
    for (l, p) in self.C:
        S = P @ self.C[(l, p)]
        Q = np.kron(I_N, S + S.T)
        PK = P @ self.K[p]
        Q = Q - (1 - self.beta_bar) * (np.kron(self.H, PK) + np.kron(self.H, PK).T)
        qmin = min(qmin, float(np.linalg.eigvalsh((Q + Q.T) / 2).min()))
    abar = max(np.linalg.norm(P @ self.A[k], 2) for k in self.A)
    bbar = max(np.linalg.norm(P @ self.B[k], 2) for k in self.B)
    kbar = max(np.linalg.norm(P @ self.K[p], 2) for p in self.K)
    return dict(q_min=qmin, a_bar=float(abar), b_bar=float(bbar),
                k_bar=float(kbar), lam_P=lP, lam_p=lp)

def halanay_pair(self, delta, P=None):
    """a0, b0 of (5.cond2)-(5.cond3), optimised over the four epsilons."""
    c = self.certificate(P)
    xi_th = math.sqrt(delta) * self.normH
    xi_a = self.normH * float(np.linalg.norm(self.Xi, 2))
    bb, lP, lp = self.beta_bar, c["lam_P"], c["lam_p"]
    # each term eps + k/eps is minimised at eps = sqrt(k), value 2 sqrt(k)
    k1 = (c["a_bar"] * self.e11) ** 2
    k3 = ((1 - bb) * c["k_bar"] * xi_th) ** 2
    k4 = (bb * c["k_bar"] * xi_a) ** 2
    # eps2 is free: it trades a0 against b0; sweep it.
    best = None
    for e2 in np.geomspace(1e-3, 1e3, 4000):
        a0 = (c["q_min"] - 2*math.sqrt(k1) - 2*math.sqrt(k3) - 2*math.sqrt(k4)
              - e2 - lP * 0.10**2)
        b0 = (c["b_bar"] * self.e11) ** 2 / e2 + lP * 0.08**2
        a, b = a0 / lP, b0 / lp
        if a0 > 0 and a > b and (best is None or a - b > best[0]):
            best = (a - b, a0, b0, a, b, e2)
    if best is None:
        return None
    gap, a0, b0, a, b, e2 = best
    return dict(a0=a0, b0=b0, a=a, b=b, gap=gap, eps2=e2,
                M=(c["lam_P"] / c["lam_p"]) * a / (a - b), **c)

def mittag_leffler(alpha, z):
    """E_alpha(z) for real z <= 0, evaluated in extended precision.

    The defining series suffers catastrophic cancellation for |z| beyond a
    few units, which is precisely the range the 30-unit horizon needs, so it
    is summed at 60 decimal digits with mpmath and then rounded. tests.py
    checks the result against the closed forms E_1(z)=e^z and
    E_{1/2}(-x)=e^{x^2}erfc(x).
    """
    import mpmath as mp
    z = np.atleast_1d(np.asarray(z, float))
    dps = int(30 + 2.5 * float(np.max(np.abs(z)))) # beats the cancellation
    with mp.workdps(dps):
        a = mp.mpf(alpha)
        out = np.array([float(_ml_mp(a, mp.mpf(float(v)))) for v in z])
    return out

def _ml_mp(a, z):
    import mpmath as mp
    s, term, k = mp.mpf(0), mp.mpf(1), 0
    while True:
        term = z ** k / mp.gamma(a * k + 1)
        s += term
        k += 1
        if k > 8 and abs(term) < mp.mpf(10) ** (-mp.mp.dps + 5) * max(abs(s), mp.mpf(1)):
            break
        if k > 4000:
            break
    return s

```

```

def halanay_root(alpha, a, b, tau):
    """Unique eta in (0, a-b] with eta = a - b/E_alpha(-eta tau^alpha)."""
    if b <= 0:
        return a
    lo, hi = 1e-12, a - b
    for _ in range(300):
        m = 0.5 * (lo + hi)
        if m - a + b / float(mittag_leffler(alpha, np.array([-m * tau**alpha]))[0]) < 0:
            lo = m
        else:
            hi = m
    return 0.5 * (lo + hi)

# ----- integrator
def simulate(alpha=0.70, h=0.01, T=30.0, tau=0.5, delta=0.05,
            seed=20260710, beta_bar=0.30):
    mdl = Model(seed=seed, beta_bar=beta_bar)
    rng = mdl.rng
    nstep = int(round(T / h))
    lag = int(round(tau / h))
    d = DLOC

    # Grunwald-Letnikov weights w_j = (-1)^j binom(alpha, j)
    w = np.empty(nstep + 1); w[0] = 1.0
    for j in range(1, nstep + 1):
        w[j] = w[j-1] * (1.0 - (alpha + 1.0) / j)

    E = np.zeros((nstep + 1 + lag, NAG, d)) # history buffer, index shift 'lag'
    E[:lag+1] = rng.standard_normal((lag+1, NAG, d)) * 0.0
    e0 = rng.standard_normal((NAG, d))
    e0 *= math.sqrt(34.30 / float((e0**2).sum())) # fix the initial aggregate energy
    E[:lag+1] = e0

    # Markov mode path (exact holding times)
    modes = np.zeros(nstep + 1, dtype=int)
    t, p = 0.0, 0
    rate = {0: 0.5, 1: 0.4}
    while t < T:
        hold = rng.exponential(1.0 / rate[p])
        i0, i1 = int(t / h), min(nstep + 1, int((t + hold) / h))
        modes[i0:i1] = p
        t, p = t + hold, 1 - p

    hold_eps = np.zeros((NAG, d)) # last transmitted neighbourhood error
    beta_held = np.zeros(NAG) # attack flag, resampled per transmission
    last_event = np.full(NAG, -np.inf)
    events = [[] for _ in range(NAG)]
    agg = np.zeros(nstep + 1)
    ha = alpha # shorthand

    # initialise the hold at t=0 (every agent transmits once)
    eps0 = mdl.H @ e0
    hold_eps[:] = eps0
    for i in range(NAG):
        beta_held[i] = 1.0 if rng.random() < beta_bar else 0.0
        events[i].append(0.0); last_event[i] = 0.0
    agg[0] = float((e0**2).sum())

    dW = rng.standard_normal((nstep, NAG)) * math.sqrt(h)

    for n in range(nstep):
        tn = n * h
        k = n + lag # buffer index of current state
        e = E[k] # (NAG, d)
        e_tau = E[k - lag]
        p = modes[n]
        # fuzzy memberships (premise = first coordinate of agent 1)
        h1 = 0.5 * (1.0 + math.tanh(0.8 * math.sin(0.4 * tn) + 0.3 * e[0, 0]))
        hs = (h1, 1.0 - h1)

        eps = mdl.H @ e # (NAG,d) neighbourhood error
        theta = hold_eps - eps
        # event test (dynamic threshold), then resample the attack per transmission
        fire = (theta**2).sum(1) > delta * (eps**2).sum(1)
        for i in np.where(fire)[0]:
            hold_eps[i] = eps[i]
            beta_held[i] = 1.0 if rng.random() < beta_bar else 0.0
            events[i].append(tn); last_event[i] = tn
        theta = hold_eps - eps

    # drift, explicit part
    drift = np.zeros((NAG, d))

```

```

for l in (0, 1):
    Am, Bm = mdl.A[(l, p)], mdl.B[(l, p)]
    drift += hs[l] * (np.tanh(e) @ Am.T + np.tanh(e_tau) @ Bm.T)
Kp = mdl.K[p]
transmitted = np.where(beta_held[:, None] > 0.5,
                        (mdl.Xi @ eps.T).T, # deception signal
                        eps + theta) # true held packet
drift += transmitted @ Kp.T
# diffusion (Euler-Maruyama), vanishes on the manifold
diff = (e @ mdl.G1.T + e_tau @ mdl.G2.T) * dW[n][:, None]

# implicit self-feedback: sum_l h_l C_{l,p}
Cmix = sum(hs[l] * mdl.C[(l, p)] for l in (0, 1))

# Grunwald-Letnikov: h^{-a} sum_{j=0}^{n+1} w_j e(t_{n+1-j}) = F
mem = np.zeros((NAG, d))
for j in range(1, n + 2):
    mem += w[j] * E[k + 1 - j]
rhs = (h**ha) * (drift + diff / h * h) - mem
# (I + h^a C) e_{n+1} = rhs [implicit in the linear decay term]
Mimp = np.eye(d) + (h**ha) * Cmix
E[k+1] = np.linalg.solve(Mimp, rhs.T).T
agg[n+1] = float((E[k+1]**2).sum())
if not np.isfinite(agg[n+1]) or agg[n+1] > 1e12:
    agg[n+1:] = np.nan
    break

tail = agg[int(0.8 * nstep):]
total_events = sum(len(v) for v in events)
periodic = NAG * (nstep + 1)
inter = []
for v in events:
    v = np.asarray(v)
    inter.append(float(np.diff(v).mean()) if len(v) > 1 else float('nan'))
return dict(t=np.arange(nstep + 1) * h, agg=agg, events=events,
            modes=modes, model=mdl,
            initial=float(agg[0]),
            tail_mean=float(np.nanmean(tail)),
            total_events=total_events, periodic=periodic,
            saving=100.0 * (1 - total_events / periodic),
            inter_event=inter,
            min_inter=float(min(np.diff(np.asarray(v)).min()
                                for v in events if len(v) > 1)))

def certified(beta_bar, delta=0.05, alpha=0.70, tau=0.5, seed=20260710):
    m = Model(seed=seed, beta_bar=beta_bar)
    r = m.halanay_pair(delta=delta)
    if r is None:
        return None
    r["eta"] = halanay_root(alpha, r["a"], r["b"], tau)
    return r

def attack_margin(delta=0.05, seed=20260710):
    lo, hi = 0.0, 0.99
    for _ in range(60):
        mid = 0.5 * (lo + hi)
        if Model(seed=seed, beta_bar=mid).halanay_pair(delta=delta) is not None:
            lo = mid
        else:
            hi = mid
    return lo

def main():
    ap = argparse.ArgumentParser()
    ap.add_argument("--beta-bar", type=float, default=0.30)
    ap.add_argument("--delta", type=float, default=0.05)
    ap.add_argument("--seed", type=int, default=20260710)
    ap.add_argument("--sweep", action="store_true")
    a = ap.parse_args()

    rng = np.random.default_rng(a.seed)
    out = {"lift_residual": lift_residual(rng),
          "attack_margin": attack_margin(delta=a.delta, seed=a.seed),
          "rows": []}
    betas = [0.0, 0.1, 0.2, 0.3, 0.4] if a.sweep else [a.beta_bar]
    for bb in betas:
        c = certified(bb, delta=a.delta, seed=a.seed)
        r = simulate(beta_bar=bb, delta=a.delta, seed=a.seed)
        row = dict(beta_bar=bb,
                   a=None if c is None else c["a"], b=None if c is None else c["b"],

```

```

        eta=None if c is None else c["eta"],
        M=None if c is None else c["M"],
        q_min=None if c is None else c["q_min"],
        certified=c is not None,
        initial=r["initial"], tail_mean=r["tail_mean"],
        events=r["total_events"], periodic=r["periodic"],
        saving=r["saving"], min_inter=r["min_inter"],
        inter_event=r["inter_event"])
    out["rows"].append(row)
    if abs(bb - a.beta_bar) < 1e-12:
        out["envelope"] = dict(eta=None if c is None else c["eta"],
                                M=None if c is None else c["M"])
        make_figure(r, c, alpha=0.70)
    (HERE / "results.json").write_text(json.dumps(out, indent=2))
    print(json.dumps(out, indent=2, default=str))

def make_figure(r, c, alpha=0.70):
    import matplotlib
    matplotlib.use("Agg")
    import matplotlib.pyplot as plt
    t, agg = r["t"], r["agg"]
    fig, ax = plt.subplots(1, 2, figsize=(11.0, 3.4), constrained_layout=True)
    ax[0].semilogy(t, np.maximum(agg, 1e-16), color="#1f5aa6", lw=1.4,
                    label=r"$\sum_i |e_i(t)|^2$")
    if c is not None:
        tg = np.linspace(t[0], t[-1], 240)
        env = c["M"] * r["initial"] * mittag_leffler(alpha, -c["eta"] * tg**alpha)
        ax[0].semilogy(tg, env, "r--", lw=1.4,
                        label=r"certified envelope $M \cdot E_\alpha(-\eta t^\alpha)$, "
                                f"$\eta=\{c['eta']:.3f\}$")
    ax[0].set_xlabel("time $t$"); ax[0].set_ylabel("aggregate error")
    ax[0].set_title("(a) mean-square synchronization"); ax[0].grid(True, which="both", lw=.4, color="#dddddd")
    ax[0].legend(fontsize=8)
    for i, ev in enumerate(r["events"]):
        ax[1].plot(ev, np.full(len(ev), i + 1), "|", ms=6, color="#333333")
    ax[1].set_yticks([1, 2, 3, 4]); ax[1].set_ylim(0.4, 4.6)
    ax[1].set_xlabel("time $t$"); ax[1].set_ylabel("follower")
    ax[1].set_title("(b) event-triggering instants"); ax[1].grid(True, axis="x", lw=.4, color="#dddddd")
    fig.savefig(HERE.parent / "sim_fig.png", dpi=200)
    plt.close(fig)

if __name__ == "__main__":
    main()

```

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