

Picture Fuzzy Multisets with Application in Career Determination

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How To Cite: Feng, Y.; Sangodapo, T.O.; Mufutau, S.A. Picture Fuzzy Multisets with Application in Career Determination. *Intelligence & Control* **2026**, 2(3), 7. <https://doi.org/10.53941/ic.2026.100014>

Received: 10 April 2026

Revised: 18 August 2026

Accepted: 14 September 2026

Published: 30 September 2026

Abstract: Career determination is a vital decision-making process that often involves significant uncertainty and indecision. This paper introduces a new distance measure and a similarity measure based on picture fuzzy multisets. Also, it is shown that the distance measure satisfies the metric properties and the new similarity measure satisfies the conditions for the similarity measure. Lastly, using a hypothetical dataset, the new similarity measure was applied to decision-making problem in career determination. The new similarity measure evaluates the similarities between students and potential career paths by computing the highest similarity values between their respective picture fuzzy multisets. The results demonstrate the effectiveness of the proposed measure in identifying optimal career choices, thereby providing a more robust framework for decision-making under uncertainty.

Keywords: multiset; fuzzy multiset; picture fuzzy set; picture fuzzy multiset

2010 AMS Classification: 03E72; 08A72

1. Introduction

Career determination is a vital decision-making process that shapes an individual's trajectory for effective performance. Career assessment can not be overemphasised in carry out career determination because they play a vital role in career development. These career assessments help students or individuals to know how a set of personal features (i.e; skills, interests, values and motivations) impact their potential successes and satisfactions with different career choices and work environments. Some career counsellors, guidance counsellors, corporate human organisations etc use some or all of these features in their decision-making process to enable personalised recommendations that are both accurate and adaptable to individual uncertainties [1].

Career assessment are developed to discover the skills or talents of an individual, enhancing choosing a career that is in line with his or her goals. At times career assessment can be difficult because the complexity of these assessments is often compounded by uncertainty, ambiguity and incomplete information. To handle this, Zadeh [2] introduced the idea of fuzzy sets (FSs) generalising classical set to bridge the gap between whole and nothing. In [3], Atanassov extended FSs to intuitionistic fuzzy sets (IFSs) to accommodate the non-membership degree which was not captured in FSs. Thereafter, Wang and Xin [4] developed the distance measure between IFSs.

In [5], Cuong and Kreinovich pioneered picture fuzzy sets (PFSs) theory as a generalisation of IFSs and FSs to incorporate the neutrality degree which is crucial in decision-making problem. This degree of neutrality is relevant in variety of situations like voting system, medical diagnosis, personal selection etc. Cuong [6] extended the work in [5] to Pythagorean picture fuzzy set. Cuong and Pham [7] investigated main fuzzy logic operators: negations, conjunctions, disjunctions and implications on picture fuzzy sets. The concept of PFS was also studied by Cuong et al. [8,9] to establish more relevant properties and to obtain a classification of representable t -norm operators. Hasan et al. [10,11], extended PFS to picture fuzzy relation (PFR) and applied it to medical diagnosis. This theory

has been applied to multi-attribute decision-making via some measures of picture fuzzy sets [12], Clustering problem [13], medical diagnosis [14, 15], building material and minerals field recognitions [16].

Introduction of fuzzy multisets (FMs) was done by Yager [17], as an extension of FSs to allow membership function occurring more than once. Applications of this concept has been done in database systems by Kim and Miyamoto [18], medical records by Onasanya and Sholabomi [19] etc. Intuitionistic fuzzy multisets (IFMs) was initiated by Shinoj and John [20]. Some researchers have applied this notion in various aspects such as medical diagnosis [21, 22], appointment process [23], allocation of courses in higher education [24] student's admission process [25], accuracy in collaborative robotics [26], radiological finding and climate change [27]. Picture fuzzy multisets (PFMSs) theory was introduced by Cao et al. [28] as an extension of PFSs allowing the positive, neutral and negative membership degrees occurring more than once. Feng and Sangodapo [29], applied the concept of picture fuzzy multisets to medical diagnosis.

Decision-making problems in real life are often characterised by uncertainty, hesitation, and conflicting opinions. Traditional set theories provide limited means of handling such complexities, while fuzzy set theory and its extensions have introduced more flexible tools for representation. Picture fuzzy sets, with their ability to capture positive, neutral, negative, and refusal degrees, have emerged as a powerful model for complex decision environments. When combined with the concept of multisets, which allow multiple occurrences of elements, a new theory is emerged called the theory of picture fuzzy multiset (PFMS), a richer and more effective framework for modelling uncertain data.

Many researchers have worked on decision making problems involving career determination. Ejegwa [30] applied the concept of Pythagorean fuzzy set developed by Yager [31, 32] to career placements using max-min-max composition with respect to academic performance, Johnson [33] studied intuitionistic fuzzy set and applied it to academic career of the students, intuitionistic fuzzy set and intuitionistic fuzzy multisets were applied to career determination in [34, 35].

This paper, explores an application of the new similarity measure on career determination using a hypothetical dataset. Career determination is a domain where students, parents, and educators often face challenges due to multiple influencing factors such as interests, aptitudes, societal expectations, and employment opportunities. It is shown that, the proposed similarity measure takes into consideration the idea of neutrality degree (the average performance of the students) which was neglected in fuzzy-based and intuitionistic fuzzy-based model. The case study demonstrates that the picture fuzzy multisets-based model improves the accuracy and transparency of career choices recommendations.

The key contributions of this paper are:

- i. We have proposed a new normalized Hamming distance between PFMSs which is an extension of the one proposed in [5].
- ii. It is shown that the new normalized Hamming distance satisfies the metric properties.
- iii. We have also proposed a new similarity measure between PFMSs which is an extension of the one proposed in [15].
- iv. Numerical examples were used to validate the proposed measures.
- v. An application of the proposed similarity measure on career determination was carried out using a hypothetical data.

The paper is organised as follows. Section 2 gives the background information on the basic definitions of fuzzy sets, intuitionistic fuzzy sets, picture fuzzy sets and picture fuzzy multisets. Section 3 introduces the new distance and similarity measures. Section 4 discusses the application of the picture fuzzy multisets on career determination.

2. Preliminaries

Here, some basic definitions are recalled. Throughout, $\mathbb{I}$ denotes the closed interval $[0, 1]$.

Definition 1 ([2]). Let X be a nonempty set. Fuzzy set (FS) A of X is an object of the form

$$A = \{ \langle x, \mu_A(x) : x \in X \rangle \}$$

where $\mu_A : X \rightarrow \mathbb{I}$ is called the membership function of $x \in X$.

Definition 2 ([17]). Let X be a nonempty set. Fuzzy multiset (FMS) A drawn from X is characterised by a count membership function CM_A such that $CM_A : X \rightarrow Q$, where Q is the set of all crisp multisets drawn from $\mathbb{I}$. Then, for any $x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from $\mathbb{I}$. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by $\mu_A^1(x), \mu_A^2(x), \dots, \mu_A^n(x)$ where $\mu_A^1(x) \geq \mu_A^2(x) \geq \dots \geq \mu_A^n(x)$.

Definition 3 ([3]). Let X be a nonempty set. An intuitionistic fuzzy set (IFS) A of X is an object of the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) : x \in X \rangle \}$$

where the functions $\mu_A, \nu_A : X \rightarrow \mathbb{I}$ are called the membership degree and non membership degree of the element $x \in X$ to the set A and for every element $x \in X$. $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 4 ([5]). Let X be a nonempty set. A picture fuzzy set (PFS) A of X is an object of the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x), \gamma_A(x) : x \in X \rangle \}$$

where the functions $\mu_A, \nu_A, \gamma_A : X \rightarrow \mathbb{I}$ are called the positive, neutral and negative membership degrees of the element $x \in X$ to the set A and for every element $x \in X$. $0 \leq \mu_A(x) + \nu_A(x) + \gamma_A(x) \leq 1$. For each PFS A of X ,

$$\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x) + \gamma_A(x))$$

is the refusal membership degree of $x \in X$.

Example 1.

$$\mathcal{D} = \{ \langle a, (0.5, 0.4, 0.1), (b, 0.7, 0.2, 0.0), (c, 0.6, 0.2, 0.2) \rangle \}$$

Definition 5 ([28]). Let $\mathcal{C}$ be a nonempty set. A picture fuzzy multiset (PFMS) $\mathcal{D}$ of $\mathcal{C}$ is an object of the form

$$\mathcal{D} = \{ \langle x, \sigma_{\mathcal{D}}^k(z), \tau_{\mathcal{D}}^k(z), \eta_{\mathcal{D}}^k(z) : z \in \mathcal{C} \rangle \}$$

where the functions $\sigma_{\mathcal{D}}^k, \tau_{\mathcal{D}}^k, \eta_{\mathcal{D}}^k : \mathcal{C} \rightarrow \mathbb{I}$ are called the positive, neutral and negative membership degrees of the element $z \in \mathcal{C}$ to the set $\mathcal{D}$ such that $z \in \mathcal{C}$

$$0 \leq \sigma_{\mathcal{D}}^k(z) + \tau_{\mathcal{D}}^k(z) + \eta_{\mathcal{D}}^k(z) \leq 1, \quad k = 1, 2, \dots, n.$$

The positive membership sequence is defined as a decreasing sequence where

$$\sigma_{\mathcal{D}}^1(z) \geq \sigma_{\mathcal{D}}^2(z) \geq \dots \geq \sigma_{\mathcal{D}}^n(z)$$

The neutral and negative membership degrees can either be decreasing or increasing sequence.

For each PFMS $\mathcal{D}$ of $\mathcal{C}$,

$$\pi_{\mathcal{D}}^k(z) = 1 - (\sigma_{\mathcal{D}}^k(z) + \tau_{\mathcal{D}}^k(z) + \eta_{\mathcal{D}}^k(z))$$

is the refusal membership degree of $\mathcal{D} \in \mathcal{C}$.

The set of all picture fuzzy multisets over $\mathcal{C}$, is denoted as PFMS($\mathcal{C}$).

Example 2. Let $\mathcal{C} = \{x_1, x_2, x_3\}$

$$\begin{aligned} \mathcal{D} = & \{ \langle x_1, (0.5, 0.4, 0.1), (0.45, 0.40, 0.05), (0.55, 0.25, 0.10) \rangle, \\ & \langle x_2, (0.65, 0.15, 0.1), (0.6, 0.30, 0.1), (0.40, 0.40, 0.20) \rangle, \\ & \langle x_3, (0.90, 0.05, 0.05), (0.80, 0.10, 0.05), (0.75, 0.05, 0.10) \rangle \} \end{aligned}$$

Definition 6 ([28]). Let $\mathcal{D} = \{ \langle z, (\sigma_{\mathcal{D}}^k(z)), (\tau_{\mathcal{D}}^k(z)), (\eta_{\mathcal{D}}^k(z)) \rangle | z \in \mathcal{C} \}$, $k = 1, 2, \dots, n$ be a PFMS, $z \in \mathcal{D}$. Then, order of $z \in \mathcal{D}$, β is defined as the cardinality of $PCM_{\mathcal{D}}(z)$ or $N_eCM_{\mathcal{D}}(z)$ or $NCM_{\mathcal{D}}(z)$ for $0 \leq \sigma_{\mathcal{D}}^k(z) + \tau_{\mathcal{D}}^k(z) + \eta_{\mathcal{D}}^k(z) \leq 1$. That is;

$$\beta = |PCM_{\mathcal{D}}(z)| = |N_eCM_{\mathcal{D}}(z)| = |NCM_{\mathcal{D}}(z)|,$$

where PCM , N_eCM and NCM are positive membership count function, neutral membership count function and negative membership count function, respectively.

Example 3. Let $\mathcal{C} = \{x_1, x_2, x_3\}$ and

$$\mathcal{D} = \{ \langle x_1, (0.5, 0.4, 0.1), (x_2, 0.7, 0.2, 0.0), (x_3, 0.6, 0.2, 0.2) \rangle \}.$$

Thus, $\beta(\mathcal{D})(a) = 3$. $\beta(\mathcal{D})(b) = 3$, and $\beta(\mathcal{D})(c) = 3$ Then, the order of $\mathcal{D}$ is $\beta(\mathcal{D}) = 3$.

Definition 7 ([28]). Given two PFMSs $\mathcal{D}_1$ and $\mathcal{D}_2$ drawn from $\mathcal{C}$. Then, the size of $\mathcal{D}_1$ and $\mathcal{D}_2$ is defined as

$$\beta = \beta(\mathcal{D}_1) \vee \beta(\mathcal{D}_2).$$

Example 4. Let $\mathcal{C} = \{x_1, x_2, x_3\}$,

$$\begin{aligned} \mathcal{D} = & \{ \langle x_1, (0.50, 0.40, 0.10), (0.45, 0.40, 0.05), (0.55, 0.25, 0.10) \rangle, \\ & \langle x_2, (0.65, 0.15, 0.10), (0.60, 0.30, 0.10), (0.40, 0.40, 0.20) \rangle, \} \end{aligned}$$

and

$$\begin{aligned} \mathcal{E} = & \{ \langle x_1, (0.55, 0.25, 0.20), (0.80, 0.10, 0.10), (0.70, 0.20, 0.00) \rangle, \\ & \langle x_3, (0.95, 0.05, 0.00), (0.60, 0.20, 0.20), (0.65, 0.30, 0.05) \rangle \} \end{aligned}$$

Thus,

$$\begin{aligned} \beta(\mathcal{D})(x_1) &= 3, \beta(\mathcal{D})(x_2) = 3, \text{ and } \beta(\mathcal{D})(x_3) = 0 \\ \beta(\mathcal{E})(x_1) &= 3, \beta(\mathcal{E})(x_2) = 0, \text{ and } \beta(\mathcal{E})(x_3) = 3 \end{aligned}$$

Then,

$$\beta(\mathcal{D} \vee \mathcal{E})(x_1) = 3, \beta(\mathcal{D} \vee \mathcal{E})(x_2) = 3, \text{ and } \beta(\mathcal{D} \vee \mathcal{E})(x_3) = 3$$

Hence, $\beta(\mathcal{D} \vee \mathcal{E}) = 3$.

Definition 8 ([28]). Let $\mathcal{D}, \mathcal{E} \in \text{PFMS}(\mathcal{C})$. That is;

$$\mathcal{D} = \{ \langle z, \sigma_{\mathcal{D}}^k(z), \tau_{\mathcal{D}}^k(z), \eta_{\mathcal{D}}^k(z) \rangle \mid z \in \mathcal{C} \}$$

and

$$\mathcal{E} = \{ \langle z, \sigma_{\mathcal{E}}^k(z), \tau_{\mathcal{E}}^k(z), \eta_{\mathcal{E}}^k(z) \rangle \mid z \in \mathcal{C} \},$$

where $k = 1, 2, \dots, n$.

- $\mathcal{D} \subseteq \mathcal{E}$ if and only if, $\sigma_{\mathcal{D}}^k(z) \leq \sigma_{\mathcal{E}}^k(z)$, $\tau_{\mathcal{D}}^k(z) \leq \tau_{\mathcal{E}}^k(z)$ and $\eta_{\mathcal{D}}^k(z) \geq \eta_{\mathcal{E}}^k(z)$; $k = 1, 2, \dots, n$.
- $\mathcal{D} = \mathcal{E}$ if and only if $\mathcal{D} \subseteq \mathcal{E}$ and $\mathcal{E} \subseteq \mathcal{D}$.
- $\mathcal{D} \cup \mathcal{E} = \{ \langle z, \max(\sigma_{\mathcal{D}}^k(z), \sigma_{\mathcal{E}}^k(z)), \min(\tau_{\mathcal{D}}^k(z), \tau_{\mathcal{E}}^k(z)), \min(\eta_{\mathcal{D}}^k(z), \eta_{\mathcal{E}}^k(z)) \rangle \mid z \in \mathcal{C} \}$, where $k = 1, 2, \dots, n$.
- $\mathcal{D} \cap \mathcal{E} = \{ \langle z, \min(\sigma_{\mathcal{D}}^k(z), \sigma_{\mathcal{E}}^k(z)), \min(\tau_{\mathcal{D}}^k(z), \tau_{\mathcal{E}}^k(z)), \max(\eta_{\mathcal{D}}^k(z), \eta_{\mathcal{E}}^k(z)) \rangle \mid z \in \mathcal{C} \}$, where $k = 1, 2, \dots, n$.
- $\overline{\mathcal{D}} = \{ \langle z, \eta_{\mathcal{D}}^k(z), \tau_{\mathcal{D}}^k(z), \sigma_{\mathcal{D}}^k(z) \rangle \mid z \in \mathcal{C} \}$, $k = 1, 2, \dots, n$.

Definition 9 ([28]). Let

$$\begin{aligned} \mathcal{D} = & \{ \langle z, (\sigma_{\mathcal{D}}^1(z), \sigma_{\mathcal{D}}^2(z), \dots, \sigma_{\mathcal{D}}^d(z)), (\tau_{\mathcal{D}}^1(z), \tau_{\mathcal{D}}^2(z), \dots, \tau_{\mathcal{D}}^d(z)), \\ & (\eta_{\mathcal{D}}^1(z), \eta_{\mathcal{D}}^2(z), \dots, \eta_{\mathcal{D}}^d(z)) \rangle \mid z \in \mathcal{C} \} \end{aligned}$$

and

$$\begin{aligned} \mathcal{E} = & \{ \langle z, (\sigma_{\mathcal{E}}^1(z), \sigma_{\mathcal{E}}^2(z), \dots, \sigma_{\mathcal{E}}^d(z)), (\tau_{\mathcal{E}}^1(z), \tau_{\mathcal{E}}^2(z), \dots, \tau_{\mathcal{E}}^d(z)), \\ & (\eta_{\mathcal{E}}^1(z), \eta_{\mathcal{E}}^2(z), \dots, \eta_{\mathcal{E}}^d(z)) \rangle \mid z \in \mathcal{C} \} \end{aligned}$$

be two PFMSs in $\text{PFMS}(\mathcal{C})$.

Then, the Cartesian product of $\mathcal{D}$ and $\mathcal{E}$, denoted by $\mathcal{D} \times \mathcal{E}$ is defined as

$$\begin{aligned} \mathcal{D} \times \mathcal{E} = & \{ \langle (x, y), (\sigma_{\mathcal{D} \times \mathcal{E}}^1(x, y), \sigma_{\mathcal{D} \times \mathcal{E}}^2(x, y), \dots, \sigma_{\mathcal{D} \times \mathcal{E}}^d(x, y)), \\ & (\tau_{\mathcal{D} \times \mathcal{E}}^1(x, y), \tau_{\mathcal{D} \times \mathcal{E}}^2(x, y), \dots, \tau_{\mathcal{D} \times \mathcal{E}}^d(x, y)), \\ & (\eta_{\mathcal{D} \times \mathcal{E}}^1(x, y), \eta_{\mathcal{D} \times \mathcal{E}}^2(x, y), \dots, \eta_{\mathcal{D} \times \mathcal{E}}^d(x, y)) \rangle \mid x, y \in \mathcal{C} \}. \end{aligned}$$

where

$$\sigma_{\mathcal{D} \times \mathcal{E}}^k(x, y) = \sigma_{\mathcal{D}}^k(x) \wedge \sigma_{\mathcal{E}}^k(y),$$

$$\tau_{\mathcal{D} \times \mathcal{E}}^k(x, y) = \tau_{\mathcal{D}}^k(x) \wedge \tau_{\mathcal{E}}^k(y)$$

and

$$\eta_{\mathcal{D} \times \mathcal{E}}^k(x, y) = \eta_{\mathcal{D}}^k(x) \vee \eta_{\mathcal{E}}^k(y)$$

with $k = 1, 2, \dots, d$.

Example 5. Let $\mathcal{C} = \{x_1, x_2, x_3\}$,

$$\begin{aligned} \mathcal{D} = & \{ \langle x_1, (0.50, 0.40, 0.10), (0.45, 0.40, 0.05), (0.55, 0.25, 0.10) \rangle, \\ & \langle x_2, (0.65, 0.15, 0.10), (0.60, 0.30, 0.10), (0.40, 0.40, 0.20) \rangle \} \\ & \langle x_3, (0.90, 0.05, 0.05), (0.80, 0.10, 0.05), (0.75, 0.05, 0.10) \rangle \} \end{aligned}$$

and

$$\begin{aligned} \mathcal{E} = & \{ \langle x_1, (0.55, 0.25, 0.20), (0.80, 0.10, 0.10), (0.70, 0.20, 0.00) \rangle, \\ & \langle x_2, (0.75, 0.15, 0.10), (0.50, 0.30, 0.00), (0.45, 0.30, 0.20) \rangle, \\ & \langle x_3, (0.95, 0.05, 0.00), (0.60, 0.20, 0.20), (0.65, 0.30, 0.05) \rangle \} \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{D} \times \mathcal{E} = & \{ \langle x_1, (0.50, 0.25, 0.20), (0.45, 0.10, 0.10), (0.55, 0.20, 0.10) \rangle, \\ & \langle x_2, (0.65, 0.15, 0.10), (0.50, 0.30, 0.10), (0.40, 0.30, 0.20) \rangle \} \\ & \langle x_3, (0.90, 0.05, 0.05), (0.60, 0.10, 0.20), (0.65, 0.05, 0.10) \rangle \} \end{aligned}$$

Definition 10 ([29]). Let $\mathcal{C}$ be a nonempty set such that $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \in PFMS(\mathcal{C})$. Then, $d : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{I}$, is called a distance measure if the following properties are satisfied;

- (i) $d(\mathcal{D}_1, \mathcal{D}_2) \in \mathbb{I}$.
- (ii) $d(\mathcal{D}_1, \mathcal{D}_2) = 0$ if and only if $\mathcal{D}_1 = \mathcal{D}_2$.
- (iii) $d(\mathcal{D}_1, \mathcal{D}_2) = d(\mathcal{D}_2, \mathcal{D}_1)$.
- (iv) $d(\mathcal{D}_1, \mathcal{D}_3) \leq d(\mathcal{D}_1, \mathcal{D}_2) + d(\mathcal{D}_2, \mathcal{D}_3)$.
- (v) If $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathcal{D}_3$, then $d(\mathcal{D}_1, \mathcal{D}_2) \leq d(\mathcal{D}_1, \mathcal{D}_3)$ and $d(\mathcal{D}_2, \mathcal{D}_3) \leq d(\mathcal{D}_1, \mathcal{D}_3)$

Then, $d(\mathcal{D}_1, \mathcal{D}_2)$ measures the distance between PFMSs $\mathcal{D}_1$ and $\mathcal{D}_2$.

Definition 11 ([29]). Let

$$\mathcal{D}_1 = \{ \langle z, \sigma_{\mathcal{D}_1}^k(z), \tau_{\mathcal{D}_1}^k(z), \eta_{\mathcal{D}_1}^k(z) \rangle \mid z \in \mathcal{C} \}$$

and

$$\mathcal{D}_2 = \{ \langle z, \sigma_{\mathcal{D}_2}^k(z), \tau_{\mathcal{D}_2}^k(z), \eta_{\mathcal{D}_2}^k(z) \rangle \mid z \in \mathcal{C} \}$$

defined on $\mathcal{C}$. Then,

- (1) the Hamming Distance is defined as

$$\begin{aligned} hd(\mathcal{D}_1, \mathcal{D}_2) = & \frac{1}{2} \sum_{k,j=1}^n \{ |\sigma_{\mathcal{D}_1}^k(z_j) - \sigma_{\mathcal{D}_2}^k(z_j)| + |\tau_{\mathcal{D}_1}^k(z_j) - \tau_{\mathcal{D}_2}^k(z_j)| \\ & + |\eta_{\mathcal{D}_1}^k(z_j) - \eta_{\mathcal{D}_2}^k(z_j)| + |\pi_{\mathcal{D}_1}^k(z_j) - \pi_{\mathcal{D}_2}^k(z_j)| \}; \end{aligned}$$

- (2) the Normalised Hamming Distance is defined as

$$\begin{aligned} nhd(\mathcal{D}_1, \mathcal{D}_2) = & \frac{1}{2n} \sum_{k,j=1}^n \{ |\sigma_{\mathcal{D}_1}^k(z_j) - \sigma_{\mathcal{D}_2}^k(z_j)| + |\tau_{\mathcal{D}_1}^k(z_j) - \tau_{\mathcal{D}_2}^k(z_j)| \\ & + |\eta_{\mathcal{D}_1}^k(z_j) - \eta_{\mathcal{D}_2}^k(z_j)| + |\pi_{\mathcal{D}_1}^k(z_j) - \pi_{\mathcal{D}_2}^k(z_j)| \}; \end{aligned}$$

- (3) the Euclidean Distance is defined as

$$\begin{aligned} ed(\mathcal{D}_1, \mathcal{D}_2) = & \left\{ \frac{1}{2} \sum_{k,j=1}^n [(\sigma_{\mathcal{D}_1}^k(z_j) - \sigma_{\mathcal{D}_2}^k(z_j))^2 + (\tau_{\mathcal{D}_1}^k(z_j) - \tau_{\mathcal{D}_2}^k(z_j))^2 \right. \\ & \left. + (\eta_{\mathcal{D}_1}^k(z_j) - \eta_{\mathcal{D}_2}^k(z_j))^2 + (\pi_{\mathcal{D}_1}^k(z_j) - \pi_{\mathcal{D}_2}^k(z_j))^2] \right\}^{\frac{1}{2}}; \end{aligned}$$

(4) the Normalised Euclidean Distance is defined as

$$\text{ned}(\mathcal{D}_1, \mathcal{D}_2) = \left\{ \frac{1}{2n} \sum_{k,j=1}^n [(\sigma_{\mathcal{D}_1}^k(z_j) - \sigma_{\mathcal{D}_2}^k(z_j))^2 + (\tau_{\mathcal{D}_1}^k(z_j) - \tau_{\mathcal{D}_2}^k(z_j))^2 + (\eta_{\mathcal{D}_1}^k(z_j) - \eta_{\mathcal{D}_2}^k(z_j))^2 + (\pi_{\mathcal{D}_1}^k(z_j) - \pi_{\mathcal{D}_2}^k(z_j))^2] \right\}^{\frac{1}{2}}.$$

Theorem 1 ([29]). *Given*

$$\mathcal{D}_1 = \{ \langle z, \sigma_{\mathcal{D}_1}^k(z), \tau_{\mathcal{D}_1}^k(z), \eta_{\mathcal{D}_1}^k(z) \rangle \mid z \in \mathcal{C} \},$$

and

$$\mathcal{D}_2 = \{ \langle z, \sigma_{\mathcal{D}_2}^k(z), \tau_{\mathcal{D}_2}^k(z), \eta_{\mathcal{D}_2}^k(z) \rangle \mid z \in \mathcal{C} \}$$

defined on $\mathcal{C}$. Then, the distance measures defined are metric.

3. Distance and Similarity Measures Between Picture Fuzzy Multisets

In this section, we first proposed a new normalized Hamming distance measure as an extension of the one in [5]. Also, we proposed a new similarity measure between the PFMSs which is an extension of the similarity measure proposed in [15]. The properties associated with both the normalized Hamming distance and similarity measures proposed were established.

3.1. Distance Measure

Here, an extension of the normalized Hamming distance measure proposed in [5] was stated and it was shown to be a metric.

Definition 12 ([15]). *Let A and B be two PFMSs in $X = \{x_1, x_2, \dots, x_n\}$. The normalized Hamming distance $d_p(A, B)$ is defined as:*

$$d(A, B) = \frac{1}{n} \sum_{i=1}^n \{ |\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \}$$

The next definition is an extension of Definition 12.

Definition 13. *Let D and E be two PFMSs in $C = \{c_1, c_2, \dots, c_n\}$. The normalized Hamming distance $d(D, E)$ is defined as:*

$$d(D, E) = \frac{1}{n} \sum_{i,k=1}^n \{ |\sigma_D^k(c_i) - \sigma_E^k(c_i)| + |\tau_D^k(c_i) - \tau_E^k(c_i)| + |\eta_D^k(c_i) - \eta_E^k(c_i)| \}$$

Example 6. *Let D , E and F be PFMSs in $C = \{c_1, c_2, c_3\}$ such that*

$$D = \{ (c_1, (0.50, 0.30, 0.10), (0.70, 0.10, 0.00)), (c_2, (0.60, 0.20, 0.10), (0.40, 0.40, 0.10)), (c_3, (0.30, 0.20, 0.40), (0.80, 0.20, 0.00)) \}$$

$$E = \{ (c_1, (0.70, 0.20, 0.10), (0.50, 0.40, 0.00)), (c_2, (0.70, 0.10, 0.00), (0.60, 0.30, 0.00)), (c_3, (0.80, 0.00, 0.00), (0.30, 0.50, 0.10)) \}$$

$$F = \{ (c_1, (0.80, 0.20, 0.00), (0.50, 0.50, 0.00)), (c_2, (0.70, 0.30, 0.00), (0.80, 0.20, 0.00)), (c_3, (0.70, 0.30, 0.00), (0.50, 0.20, 0.00)) \}$$

We employed the method in [36] to convert them to PFSSs.

Thus, D , E and F become

$$D = \{ (c_1, (0.60, 0.20, 0.05)), (c_2, (0.50, 0.30, 0.10)), (c_3, (0.55, 0.20, 0.20)) \}$$

$$E = \{ (c_1, (0.60, 0.30, 0.05)), (c_2, (0.65, 0.20, 0.00)), (c_3, (0.55, 0.25, 0.05)) \}$$

$$F = \{ (c_1, (0.65, 0.35, 0.00)), (c_2, (0.75, 0.25, 0.00)), (c_3, (0.60, 0.25, 0.00)) \}$$

Hence,

$$d(D, E) = 0.22, \quad d(E, F) = 0.13, \quad \text{and} \quad d(D, F) = 0.32.$$

Theorem 2. Let D , E and F be PFMSs in a nonempty set $C = \{c_1, c_2, \dots, c_n\}$. Then, the normalized Hamming distance measure defined is a metric.

Proof. We shall check the properties stated in Definition 10.

Property i is true from the definition and it is also the sum of the non-negative terms.

Property ii. Suppose that $d(D, E) = 0$, then,

$$|\sigma_D^k(c_i) - \sigma_E^k(c_i)| = 0, \quad |\tau_D^k(c_i) - \tau_E^k(c_i)| = 0, \quad |\eta_D^k(c_i) - \eta_E^k(c_i)| = 0$$

Hence, $D = E$.

Conversely, Suppose that $D = E$ then

$$\sigma_D^k(c_i) = \sigma_E^k(c_i), \quad \tau_D^k(c_i) = \tau_E^k(c_i), \quad \text{and} \quad \eta_D^k(c_i) = \eta_E^k(c_i)$$

Hence, $d(D, E) = 0$.

Property iii.

$$\begin{aligned} d(D, E) &= \frac{1}{n} \sum_{i,k=1}^n \{|\sigma_D^k(c_i) - \sigma_E^k(c_i)| + |\tau_D^k(c_i) - \tau_E^k(c_i)| \\ &\quad + |\eta_D^k(c_i) - \eta_E^k(c_i)|\} \\ &= \frac{1}{n} \sum_{i,k=1}^n \{|\sigma_E^k(c_i) - \sigma_D^k(c_i)| + |\tau_E^k(c_i) - \tau_D^k(c_i)| \\ &\quad + |\eta_E^k(c_i) - \eta_D^k(c_i)|\} \\ &= d(E, D). \end{aligned}$$

Property iv.

$$\begin{aligned} d(D, F) &= \frac{1}{n} \sum_{i,k=1}^n \{|\sigma_D^k(c_i) - \sigma_F^k(c_i)| + |\tau_D^k(c_i) - \tau_F^k(c_i)| \\ &\quad + |\eta_D^k(c_i) - \eta_F^k(c_i)|\} \\ &= \frac{1}{n} \sum_{i,k=1}^n \{(|\sigma_D^k(c_i) - \sigma_E^k(c_i)| + |\sigma_E^k(c_i) - \sigma_F^k(c_i)|) \\ &\quad + (|\tau_D^k(c_i) - \tau_E^k(c_i)| + |\tau_E^k(c_i) - \tau_F^k(c_i)|) \\ &\quad + (|\eta_D^k(c_i) - \eta_E^k(c_i)| + |\eta_E^k(c_i) - \eta_F^k(c_i)|)\} \\ &\leq \frac{1}{n} \sum_{i,k=1}^n \{|\sigma_D^k(c_i) - \sigma_E^k(c_i)| + |\tau_D^k(c_i) - \tau_E^k(c_i)| \\ &\quad + |\eta_D^k(c_i) - \eta_E^k(c_i)|\} \\ &\quad + \frac{1}{n} \sum_{i,k=1}^n \{|\sigma_E^k(c_i) - \sigma_F^k(c_i)| + |\tau_E^k(c_i) - \tau_F^k(c_i)| \\ &\quad + |\eta_E^k(c_i) - \eta_F^k(c_i)|\} \\ &= d(D, E) + d(E, F). \end{aligned}$$

Property v. Suppose that $D \subseteq E \subseteq F$, it implies that

$$\sigma_D^k(c_i) \leq \sigma_E^k(c_i), \quad \tau_D^k(c_i) \leq \tau_E^k(c_i) \quad \text{and} \quad \eta_D^k(c_i) \leq \eta_E^k(c_i).$$

We consider two cases.

Case 1: Compare $d(D, E)$ and $d(D, F)$.

Since

$$\sigma_D^k(c_i) \leq \sigma_E^k(c_i) \leq \sigma_F^k(c_i),$$

$$\tau_D^k(c_i) \leq \tau_E^k(c_i) \leq \tau_F^k(c_i)$$

and

$$\eta_D^k(c_i) \geq \eta_E^k(c_i) \geq \eta_F^k(c_i)$$

by the assumption, we get

$$|\sigma_D^k(c_i) - \sigma_E^k(c_i)| = \sigma_E^k(c_i) - \sigma_D^k(c_i) \quad \text{and} \quad |\sigma_D^k(c_i) - \sigma_F^k(c_i)| = \sigma_F^k(c_i) - \sigma_D^k(c_i)$$

$$|\tau_D^k(c_i) - \tau_E^k(c_i)| = \tau_E^k(c_i) - \tau_D^k(c_i) \quad \text{and} \quad |\tau_D^k(c_i) - \tau_F^k(c_i)| = \tau_F^k(c_i) - \tau_D^k(c_i)$$

and

$$|\eta_D^k(c_i) - \eta_E^k(c_i)| = \eta_D^k(c_i) - \eta_E^k(c_i) \quad \text{and} \quad |\eta_D^k(c_i) - \eta_F^k(c_i)| = \eta_D^k(c_i) - \eta_F^k(c_i)$$

For,

$$\sigma_E^k(c_i) \leq \sigma_F^k(c_i), \quad \sigma_E^k(c_i) - \sigma_D^k(c_i) \leq \sigma_F^k(c_i) - \sigma_D^k(c_i),$$

$$\tau_E^k(c_i) \leq \tau_F^k(c_i), \quad \tau_E^k(c_i) - \tau_D^k(c_i) \leq \tau_F^k(c_i) - \tau_D^k(c_i)$$

and

$$\eta_E^k(c_i) \geq \eta_F^k(c_i), \quad \eta_D^k(c_i) - \eta_E^k(c_i) \geq \eta_D^k(c_i) - \eta_F^k(c_i)$$

we have the equations below,

$$|\sigma_D^k(c_i) - \sigma_E^k(c_i)| \leq |\sigma_D^k(c_i) - \sigma_F^k(c_i)| \quad (1)$$

$$|\tau_D^k(c_i) - \tau_E^k(c_i)| \leq |\tau_D^k(c_i) - \tau_F^k(c_i)| \quad (2)$$

and

$$|\eta_D^k(c_i) - \eta_E^k(c_i)| \leq |\eta_D^k(c_i) - \eta_F^k(c_i)| \quad (3)$$

Combining equations (3.1), (3.2) and (3.3), we have

$$|\sigma_D^k(c_i) - \sigma_E^k(c_i)| + |\tau_D^k(c_i) - \tau_E^k(c_i)| + |\eta_D^k(c_i) - \eta_E^k(c_i)| \leq |\sigma_D^k(c_i) - \sigma_F^k(c_i)| + |\tau_D^k(c_i) - \tau_F^k(c_i)| + |\eta_D^k(c_i) - \eta_F^k(c_i)|$$

Therefore, $d(D, E) \leq d(D, F)$.

Case 2: This can be done in a similar manner by Comparing $d(E, F)$ and $d(D, F)$.

Thus, $d(E, F) \leq d(D, F)$.

□

3.2. Similarity Measure

Definition 14. Let $d(D, E)$ be normalized Hamming distance between PFMSs D and E as defined in Definition 13. Then, the similarity measure is defined as

$$s = 1 - d(D, E)$$

Example 7. Using Example 6, the similarity measure

$$s(D, E) = 1 - 0.22 = 0.78, \quad s(E, F) = 1 - 0.13 = 0.87, \quad s(D, F) = 1 - 0.32 = 0.68.$$

Theorem 3. Let C be a nonempty set, D, E, F be PFMSs in C , and

$$s : PFMS(C) \times PFMS(C) \rightarrow \mathbb{I}$$

be a mapping. If the following properties are satisfied, then, s is a similarity measure on $PFMS(C)$.

- (i) $s(D, E) \in \mathbb{I}$.
- (ii) $s(D, E) = 1$ if and only if $D = E$.

$$(iii) \quad s(D, E) = s(E, D).$$

$$(iv) \quad \text{If } D \subseteq E \subseteq F, \text{ then } s(D, F) \leq s(D, E) \text{ and } s(D, F) \leq s(E, F).$$

Proof.

Property i. By Theorem 2 (i), $d(D, E) \in \mathbb{I}$ this means that $1 - d(D, E) \in \mathbb{I}$. Thus, $s(D, E) \in \mathbb{I}$.

Property ii. Suppose that $D = E$,

$$\Rightarrow d(D, E) = 0 \text{ by Theorem 2 (ii), thus } s(D, E) = 1 - 0 = 1.$$

Conversely, suppose that $s(D, E) = 1$

$$\Rightarrow 1 - d(D, E) = 1 \Rightarrow d(D, E) = 0 \Rightarrow D = E, \text{ by Theorem 2 (ii).}$$

Property iii. $s(D, E) = 1 - d(D, E) = 1 - d(E, D)$ by Theorem 2 (iii). Thus, $s(D, E) = s(E, D)$.

Property iv. Suppose that $D \subseteq E \subseteq F$. By Theorem 2 (v), $d(D, F) \geq d(D, E)$ and $d(E, F) \geq d(D, F)$, thus, $1 - d(D, F) \leq 1 - d(D, E)$, $\Rightarrow s(D, E) \geq s(D, F)$.

Following the same argument, we have $s(E, F) \geq s(D, F)$.

□

Definition 15. Proposed Similarity Measure

Given

$$D = \{ \langle c, \sigma_D^k(c), \tau_D^k(c), \eta_D^k(c) \rangle \mid c \in C \},$$

and

$$E = \{ \langle c, \sigma_E^k(c), \tau_E^k(c), \eta_E^k(c) \rangle \mid c \in C \}$$

defined on C . Then,

$$s(D, E) = 1 - d(D, E)$$

4. Application of Picture Fuzzy Multisets in Career Determination

This section is about an application of the proposed similarity measure on career determination using a hypothetical dataset to handle uncertainty and imprecision students encounter in choosing a suitable career. To determine the suitable career, it often times involves complexity and uncertainty as a result of lack of proper guide or lack or insufficient information. So, and adequate information is needed for the students to choose wisely their careers to enhance proper planning, preparation and proficiency.

Most of the following factors affect academic performance of the students;

- (i) Students' factors e.g interest of the students, mood of the students, background of the students, health challenge, fear, peer influence, etc.
- (ii) Teachers' factors e.g lack of teaching or learning aids, attitudes of the teachers, competency, or quality of the teachers, etc.
- (iii) Environmental factors e.g time of the day, sitting arrangement, weather condition, invigilation etc.
- (iv) Other items: exam format, duration of the exam, mood of the teacher when marking, adequate and relevancy of the exam to the syllabus, etc.

In this application, we suppose that a set of students attempt closely related examinations more than once. The picture fuzzy multiset comprises of the degree of positive membership (i.e., the score received for the student's response to the question), degree of neutral membership (i.e., the questions left unanswered) and degree of negative membership (i.e., the scores for the questions the students got wrong).

The following parameters Su , C and St were used to represent the sets of subjects, career options and students. Define two PFMRs R_1 and R_2 from Su to C and St to Su , respectively.

The procedures employed in picture fuzzy career determination are listed below:

- i. Determine Su_i , $i = 1, 2, 3, 4$.
- ii. Formulate the PFMRs R_1 from Su to C and R_2 from St to Su .
- iii. Compute the similarity between St_i and C_i $i = 1, 2, 3, 4$ using Definitions 13 and 14.
- iv. Identify the highest similarity between St_i and C_i . This indicates that the student St_i chooses a suitable career C_i $i = 1, 2, 3, 4$.

4.1. Case Study

Let $St = \{St_1, St_2, St_3, St_4\}$ be the set of students for career determination,

$$C = \{BusinessAdministration, Accounting, Marketing, Auditing\}$$

be the set of careers the students are competing for and

$$Su = \{Mathematics, English, Economics, FinancialAccounting, Commerce\}$$

be set of subjects related to the set of careers. The sets of careers and subjects can be shortened as $C = \{Acc, BA, M, Au\}$ and $Su = \{Maths, Eng, Econs, FinAcc, Comm\}$, respectively.

Table 1 illustrates an ideal dataset for careers with respect to the performance in the subject requirements.

Table 1. Careers vs. Subjects in picture fuzzy values form.

	Maths	Eng	Econs	Fin Acc	Comm
BA	(0.3, 0.4, 0.1)	(0.4, 0.1, 0.2)	(0.5, 0.3, 0.0)	(0.5, 0.2, 0.1)	(0.3, 0.4, 0.1)
Acc	(0.5, 0.4, 0.1)	(0.3, 0.4, 0.1)	(0.3, 0.4, 0.1)	(0.4, 0.2, 0.2)	(0.4, 0.1, 0.2)
M	(0.4, 0.2, 0.2)	(0.4, 0.2, 0.2)	(0.5, 0.3, 0.0)	(0.4, 0.3, 0.2)	(0.4, 0.2, 0.1)
Au	(0.4, 0.1, 0.2)	(0.4, 0.3, 0.2)	(0.5, 0.4, 0.1)	(0.5, 0.2, 0.1)	(0.3, 0.3, 0.2)

In Table 2, The following marks were obtained after four students have sat for the examination three times differently in the following subjects, Mathematics, English, Economics, Financial Accounting and Commerce. Each of their performances is described by σ , τ and η representing the positive, neutral and negative membership degrees.

Table 2. Students vs. Subjects in picture fuzzy multi-values form.

	St_1	St_2	St_3	St_4
Maths	(0.5, 0.6, 0.4)	(0.4, 0.5, 0.3)	(0.4, 0.2, 0.6)	(0.5, 0.4, 0.6)
	(0.2, 0.2, 0.2)	(0.2, 0.1, 0.3)	(0.2, 0.4, 0.0)	(0.3, 0.4, 0.2)
	(0.1, 0.0, 0.2)	(0.3, 0.2, 0.1)	(0.1, 0.2, 0.0)	(0.0, 0.0, 0.0)
Eng	(0.4, 0.4, 0.4)	(0.5, 0.4, 0.6)	(0.5, 0.5, 0.5)	(0.3, 0.4, 0.2)
	(0.3, 0.3, 0.3)	(0.1, 0.1, 0.1)	(0.2, 0.2, 0.2)	(0.4, 0.2, 0.6)
	(0.2, 0.3, 0.1)	(0.2, 0.2, 0.2)	(0.1, 0.1, 0.1)	(0.1, 0.2, 0.0)
Econs	(0.5, 0.4, 0.6)	(0.6, 0.6, 0.6)	(0.4, 0.3, 0.5)	(0.5, 0.6, 0.4)
	(0.1, 0.1, 0.1)	(0.1, 0.1, 0.1)	(0.2, 0.4, 0.0)	(0.0, 0.0, 0.0)
	(0.2, 0.2, 0.2)	(0.2, 0.2, 0.2)	(0.1, 0.1, 0.1)	(0.1, 0.1, 0.1)
Fin Acc	(0.4, 0.3, 0.5)	(0.6, 0.6, 0.6)	(0.4, 0.4, 0.4)	(0.5, 0.6, 0.4)
	(0.2, 0.3, 0.1)	(0.1, 0.2, 0.0)	(0.3, 0.3, 0.3)	(0.2, 0.0, 0.4)
	(0.3, 0.3, 0.3)	(0.2, 0.0, 0.4)	(0.2, 0.2, 0.2)	(0.1, 0.2, 0.0)
Comm	(0.4, 0.2, 0.6)	(0.4, 0.3, 0.5)	(0.3, 0.3, 0.3)	(0.4, 0.4, 0.4)
	(0.5, 0.7, 0.3)	(0.1, 0.2, 0.0)	(0.4, 0.4, 0.4)	(0.3, 0.3, 0.3)
	(0.1, 0.1, 0.1)	(0.2, 0.2, 0.2)	(0.2, 0.1, 0.3)	(0.3, 0.3, 0.3)

Applying the method in [36], the picture fuzzy multi data become picture fuzzy data as shown in Table 3.

Using Definitions 13 and 14, the similarity between students and careers in Table 3 and Table 1, respectively with respect to each subject is obtained as shown in Table 4.

Using the information in Tables 1–3, we first calculate the refusal degrees using $\pi(c) = 1 - (\sigma(c) - \tau(c) - \eta(c))$ for each performance in the Tables. Applying Definition 11, the distances between students and careers with respect to each of the subjects are calculated as shown in Tables 5–8.

Table 3. Students vs. Subjects.

	St_1	St_2	St_3	St_4
Maths	(0.5, 0.2, 0.1)	(0.4, 0.2, 0.2)	(0.4, 0.2, 0.1)	(0.5, 0.3, 0.0)
Eng	(0.4, 0.3, 0.2)	(0.5, 0.1, 0.2)	(0.5, 0.2, 0.1)	(0.3, 0.4, 0.1)
Econs	(0.5, 0.1, 0.2)	(0.6, 0.1, 0.2)	(0.4, 0.2, 0.1)	(0.5, 0.0, 0.1)
Fin Acc	(0.4, 0.2, 0.3)	(0.6, 0.1, 0.2)	(0.4, 0.3, 0.2)	(0.5, 0.2, 0.1)
Comm	(0.4, 0.5, 0.1)	(0.4, 0.1, 0.2)	(0.3, 0.4, 0.2)	(0.4, 0.3, 0.3)

Table 4. Similarity between students and careers.

	BA	Acc	M	Au
St_1	0.70	0.64	0.76	0.66
St_2	0.66	0.60	0.70	0.74
St_3	0.76	0.74	0.80	0.64
St_4	0.72	0.70	0.76	0.72

Table 5. Hamming distance between students and careers.

	BA	Acc	M	Au
St_1	1.15	1.05	0.90	1.30
St_2	1.15	1.20	1.20	1.10
St_3	0.75	0.85	0.70	1.25
St_4	0.95	1.15	1.05	1.05

Table 6. N-hamming distance between students and careers.

	BA	Acc	M	Au
St_1	0.20	0.22	0.18	0.26
St_2	0.23	0.12	0.24	0.22
St_3	0.15	0.17	0.14	0.25
St_4	0.19	0.23	0.21	0.21

Table 7. Euclidean distance between students and careers.

	BA	Acc	M	Au
St_1	0.42	0.41	0.32	0.45
St_2	0.49	0.50	0.51	0.50
St_3	0.35	0.30	0.30	0.46
St_4	0.41	0.43	0.42	1.05

Table 8. N-euclidean distance between students and careers.

	BA	Acc	M	Au
St_1	0.19	0.18	0.14	0.20
St_2	0.22	0.22	0.23	0.22
St_3	0.16	0.13	0.13	0.21
St_4	0.18	0.19	0.19	0.21

4.2. Discussion

The highest similarity value gives the proper determination of each of the careers. In Table 4, according to the academic performance of the students with respect to careers are as follows; St_1 , St_3 and St_4 choose Marketing as their suitable careers while St_2 , chooses Accounting as his/her suitable career. In Tables 5–8, there is inconsistency in the choice of career determination for instance St_2 shows different career choices, there is uncertainty for St_3 both in Tables 7 and 8.

5. Conclusions

To determine a suitable career by a student, most time it involves complexity and uncertainty as a result of lack of proper guide or insufficient information. Thus, an adequate information is needed for a student to choose wisely a suitable career to enhance efficiency and high degree of expertise. In this paper, we have introduced a new distance and similarity measures between PFMSs as extensions in [5, 15], respectively, and the properties associated with them were established. Also, we have explored an application of the new similarity measure on career determination using a hypothetical dataset. In this application, we have assumed four students, five subjects and four different careers. The highest similarity values between the students and the careers according to their academic requirements was calculated using the proposed similarity measure, and the result was established. The application is significant because students will choose rightly the best career suitable for them based on the level

of their academic performances. For future work, the applications of picture fuzzy multisets can be channelled towards pattern recognition, admission procedure, recruitment exercise and even extending it to Pythagorean picture fuzzy multisets.

Author Contributions

Y.F.: Supervision ; T.O.S.: conceptualization, methodology, data curation, writing-reviewing and editing; S.A.M.: writing-original draft preparation, visualization, investigation. All authors have read and agreed to the published version of the manuscript.

Funding

This research is supported by the Science and Technology Research Program of Chongqing Municipal Education Commission (Grant Nos. KJQN202501242, KJQN202201221) and the Chongqing Natural Science Foundation (Grant Nos. CSTB2024NSCQ-LZX0083, CSTB2024NSCQ-LZX0119).

Data Availability Statement

The datasets used in this work are hypothetical datasets.

Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

References

1. Kapes, J.T.; Mastie, M.M.; Whitfield, E.A. *A Counselor's Guide to Career Assessment Instruments*; National Career Development Association: Alexandria, VA, USA, 1994.
2. Zadeh, L.A. Fuzzy Sets. *Inf. Control* **1965**, *8*, 338–353. [https://doi.org/10.1016/s0019-9958\(65\)90241-x](https://doi.org/10.1016/s0019-9958(65)90241-x).
3. Atanassov, K.T. Intuitionistic Fuzzy Sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [https://doi.org/10.1016/s0165-0114\(86\)80034-3](https://doi.org/10.1016/s0165-0114(86)80034-3).
4. Wang, W.; Xin, X. Distance Measure Between Intuitionistic Fuzzy sets. *Pattern Recognit. Lett.* **2005**, *26*, 2063–2069. <https://doi.org/10.1016/j.patrec.2005.03.018>.
5. Cuong, B.C.; Kreinovich, V. Picture Fuzzy Sets—A new concept for Computational intelligence problems. In Proceedings of the Third World Congress on Information and Communication Technologies, Hanoi, Vietnam, 15–18 December 2013; pp. 1–6. <https://doi.org/10.1109/wict.2013.7113099>.
6. Cuong, B.C. Pythagorean Picture Fuzzy Sets, part 1—Basic notions. *J. Comput. Sci. Cybern.* **2019**, *35*, 293–304. <https://doi.org/10.15625/1813-9663/35/4/13898>.
7. Cuong, B.C.; Pham, V.H. Some Fuzzy Logic Operators for Picture Fuzzy Sets. In Proceedings of the 2015 Seventh International Conference on Knowledge and Systems Engineering (KSE), Ho Chi Minh City, Vietnam, 8–10 October 2015; pp. 132–137. <https://doi.org/10.1109/kse.2015.20>.
8. Cuong, B.C.; Ngan, R.T.; Hai, B.D. An Involutive Picture Fuzzy Negator on Picture Fuzzy Sets and Some De Morgan Triples. In Proceedings of the Seventh International Conference on Knowledge and Systems Engineering, Ho Chi Minh City, Vietnam, 8–10 October 2015; pp. 126–131. <https://doi.org/10.1109/kse.2015.21>.
9. Cuong, B.C.; Kreinovich, V.; Ngan, R.T. A Classification of Representable t -norm Operators for Picture Fuzzy Sets. In Proceedings of the 2016 Eighth International Conference on Knowledge and Systems Engineering (KSE), Hanoi, Vietnam, 6–8 October 2016. <https://doi.org/10.1109/kse.2016.7758023>.
10. Hasan, M.K.; Sultana, A.; Mitra, N.K. Compositions of Picture Fuzzy Relations with Application in Decision Making. *Eur. J. Math. Stat.* **2023**, *4*, 19–28. <https://doi.org/10.24018/ejmath.2023.4.2.188>.
11. Hasan, M.K.; Sultana, A.; Mitra, N.K. Picture Fuzzy Relations Over Picture Fuzzy Sets. *Am. J. Comput. Math.* **2023**, *13*, 161–184. <https://doi.org/10.4236/ajcm.2023.131008>.
12. Nguyen, V.D.; Nguyen, X.T. Some Measures of Picture Fuzzy Sets and Their Application in Multi-attribute Decision Making. *Int. J. Math. Sci. Comput.* **2018**, *4*, 23–41. <https://doi.org/10.5815/ijmsc.2018.03.03>.
13. Son, L.H. Generalized picture distance measure and applications to picture fuzzy clustering. *Appl. Soft Comput.* **2016**, *46*, 284–295. <https://doi.org/10.1016/j.asoc.2016.05.009>.
14. Dutta, P. Medical Diagnosis Based on Distance Measures between Picture Fuzzy Sets. *Int. J. Fuzzy Syst. Appl.* **2018**, *7*, 15–36. <https://doi.org/10.4018/ijfsa.2018100102>.
15. Rozy; Kaur, G. Similarity Measure between Picture Fuzzy sets and its Applications in Medical Diagnosis. *J. Adv. Sch. Res. Allied Educ.* **2022**, *19*, 210–212.

16. Wei, G. Some Similarity Measures for Picture Fuzzy Sets and Their Applications. *Iran. J. Fuzzy Syst.* **2018**, *15*, 77–89.
17. Yager, R.R. On the Theory of Bags. *Int. J. Gen. Syst.* **1986**, *13*, 23–37. <https://doi.org/10.1080/03081078608934952>.
18. Kim, K.S.; Miyamoto, S. Application of Fuzzy Multisets to Fuzzy Database Systems. In *Soft Computing in Intelligent Systems and Information Processing, Proceedings of the 1996 Asian Fuzzy Systems Symposium, Kenting, Taiwan, 11–14 December 1996*; IEEE: New York, NY, USA, 1996; pp. 115–120. <https://doi.org/10.1109/afss.1996.583573>.
19. Onasanya, B.O.; Sholabomi, A.S. Fuzzy multiset operations in applications. *Ann. Fuzzy Math. Inform.* **2019**, *17*, 279–288. <https://doi.org/10.30948/afmi.2019.17.3.279>.
20. Shinoj, T.K.; John, S.J. Intuitionistic Fuzzy Multisets. *Int. J. Eng. Sci. Innov. Technol.* **2013**, *2*, 1–24.
21. Rajarajeswari, P.; Uma, N. Normalized Hamming Similarity Measure for Intuitionistic Fuzzy Multi Sets and Its Application in Medical Diagnosis. *Int. J. Math. Trends Technol.* **2014**, *5*, 219–225.
22. Shinoj, T.K.; John, S.J. Intuitionistic Fuzzy Multisets and its Application in Medical Diagnosis. *World Acad. Sci. Eng. Technol.* **2012**, *6*, 1418–1421.
23. Ejegwa, P.A.; Kwarkar, L.N.; Ihuoma, K.N. Application of Intuitionistic Fuzzy Multisets in Appointment Process. *Int. J. Comput. Appl.* **2016**, *135*, 1–4. <https://doi.org/10.5120/ijca2016908167>.
24. Ejegwa, P.A.; Onyeke, I.C.; Adah, V. Recognition Principle for Course Allocation in Higher Institution Based on Intuitionistic fuzzy Correlation Coefficient. *J. Fuzzy Ext. Appl.* **2025**, *6*, 94–108.
25. Ejegwa, P.A.; Onyeke, I.C. A novel Intuitionistic Fuzzy Correlation Algorithm and its Application in Pattern recognition and Student admission process. *Int. J. Fuzzy Syst. Appl.* **2022**, *11*, 1–20. <https://doi.org/10.4018/ijfsa.285984>.
26. Shinoj, T.K.; Sunil, J.J. Accuracy in Collaborative Robotics: An Intuitionistic Fuzzy Multisets Approach. *Glob. J. Sci. Front. Res. Math. Decis. Sci.* **2013**, *13*, 21–28.
27. Usman, A.; Isah, A.; Adeyinka, A.S. An Application of Intuitionistic Fuzzy Multiset Distance Measure to Radiological Finding and Climate Change. *Sci. World J.* **2025**, *20*, 1700–1705. <https://doi.org/10.4314/swj.v20i4.51>.
28. Cao, L.; Feng, Y.; Sangodapo, T.O. Picture Fuzzy Multisets. *Ital. J. Pure Appl. Math.* **2024**, *51*, 64–76.
29. Feng, Y.; Sangodapo, T.O. Distance Measures between Picture Fuzzy Multisets and Their Application to Medical Diagnosis. *Ital. J. Pure Appl. Math.* **2025**, *54*, 113–125.
30. Ejegwa, P.A. Pythagorean fuzzy set and its application in career placements based on academic performance using max–min–max composition. *Complex Intell. Syst.* **2019**, *5*, 165–175. <https://doi.org/10.1007/s40747-019-0091-6>.
31. Yager, R.R. Pythagorean fuzzy subsets. In *Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, AB, Canada, 24–28 June 2013*; pp. 57–61. <https://doi.org/10.1109/ifsa-nafips.2013.6608375>.
32. Yager, R.R. Pythagorean Membership Grades in Multicriteria Decision Making. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 958–965. <https://doi.org/10.1109/tfuzz.2013.2278989>.
33. Johnson, T. Applications of intuitionistic fuzzy sets in the Academic Career of the Students. *Indian J. Sci. Technol.* **2017**, *10*, 23–25.
34. Ejegwa, P.A.; Tyongi, B.B.; Osuwa, S.C.; et al. Career Determination Using Intuitionistic fuzzy Multisets Approach. *MAYFEB J. Math.* **2016**, *1*, 1–10.
35. Ejegwa, P.A.; Akubo, A.J.; Joshua, O.M. Intuitionistic fuzzy sets in career determination. *J. Inf. Comput. Sci.* **2014**, *9*, 285–288.
36. Sangodapo, T.O.; Kausar, N.; Chreif, M.Y. A New Decision-Making Analysis Model Based on the Transformation of Picture Fuzzy Sets into Fuzzy Sets. In *Analytical Decision Making and Data Envelopment Analysis: Advances and Challenges*; Springer Nature: Singapore, 2024; pp. 455–464. https://doi.org/10.1007/978-981-97-6972-8_20.