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Photon Sphere for a Dyonic Black Hole Corresponding to A_2 Toda Chain

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Abstract: We consider a dilatonic dyon black hole solution with a gravitational radius of 2μ and two charges, Q_1 (electric) and Q_2 (magnetic), within the framework of a four-dimensional gravitational model comprising one scalar field and one 2-form. The dilaton coupling constant λ is fixed by $\lambda^2 = \frac{3}{2}$. This solution is related to the A_2 Toda chain. The circular orbits of null geodesics are explored, and the fifth-order polynomial master equation governing the photon sphere radius R_0 is analyzed. It is shown to admit a unique solution satisfying $R_0 > 2\mu$. The circular null geodesics are proven to be unstable. Finally, the black hole shadow is examined, yielding relations for the shadow angle and the critical impact parameter.

Keywords: black hole; photon sphere; dyon; Toda chain; black hole shadow

1. Introduction

Interest in black holes, initially motivated by evidence for a supermassive black hole at the center of our Galaxy [1], has been further strengthened by the recent detection of gravitational waves from a black hole merger [2]. Consequently, a large body of current literature is devoted to investigating black holes in modified theories of gravity (see [3] and references therein). In this work, we consider one such theory - namely, a dilatonic scalar-tensor model comprising a single scalar field and one Abelian gauge field.

Here we deal with photon spheres which control several observable and theoretical features of black holes. In the eikonal limit the real part of the quasinormal-mode frequency is set by the orbital frequency of the unstable circular null geodesic, while the imaginary part is set by the Lyapunov exponent that measures the instability of that orbit [4,5]. The same geodesic determines the critical impact parameter and therefore the silhouette of the black hole as seen by a distant observer; the resulting shadow radius is now constrained by Event Horizon Telescope images of $Sgr A^*$ and $M87^*$ [3,6]. Photon spheres also govern the high-frequency absorption cross-section and, for a radiation fluid, coincide with Bondi's sonic horizon [7]. These applications have stimulated a systematic study of circular null geodesics in modified-gravity and supergravity black holes.

This paper builds upon previous research [8,9] devoted to dilatonic dyonic and dyon-like black hole solutions. The present results may be regarded as a modest contribution to the extensive body of work addressing spherically symmetric configurations, including black holes and black branes, as reviewed in [10–29] and the references therein. Such solutions naturally arise in gravitational models that incorporate scalar fields and antisymmetric forms.

Here we consider a dilatonic dyon black hole solution [8] with electric and magnetic charges Q_1 and Q_2 , respectively, within the framework of a four-dimensional model comprising a metric g , a scalar field φ , and a 2-form F . The corresponding dilatonic coupling constant is denoted by λ . Here we put

$$\lambda = \pm\sqrt{3/2}. \quad (1)$$

and deal with a dyon configuration on an oriented manifold



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$\mathcal{M} = (2\mu, +\infty) \times S^2 \times \mathbb{R}$ ($\mu > 0$), with the 2-form field

$$F = Q_1 f * \tau + Q_2 \tau, \quad (2)$$

where $\tau = \text{vol}[S^2]$ is the volume form on the sphere, $*$ is $*[g]$ denotes the Hodge operator associated with $(\mathcal{M}, g)$, and f is a scalar function on $\mathcal{M}$.

In Ref. [9], the term “dyon-like” configuration was introduced to distinguish the non-composite ansatz

$$F^{(1)} = Q_1 f * \tau, \quad F^{(2)} = Q_2 \tau, \quad (3)$$

where $F^{(s)}$, $s = 1, 2$, are two 2-forms, from the composite dyonic one given in Equation (2).

In previous studies, specific dyon-like solutions for two Abelian 2-form fields were investigated for a range of Lie algebras and dilatonic couplings λ_1, λ_2 (see Refs. [9, 15, 17, 30–32]). The particular case of the $A_1 + A_1$ Lie algebra, which corresponds to the condition $\lambda_1 \lambda_2 = \frac{1}{2}$, was examined in Refs. [9, 15, 17, 32, 33]. When the couplings are equal, $\lambda_1 = \lambda_2 = \lambda$, one is led to relation $\lambda = \pm \sqrt{1/2}$, which appears in certain four-dimensional supergravity and string-induced models [7, 15, 17]. Dyonic (composite) solutions have also attracted considerable attention, as referenced in [8, 18, 19, 21, 22, 34] and related works. In fact, the dyonic solution from Ref. [18] (see also [22]) is a non-composite version of the solutions originally found in Refs. [15, 17].

This paper examines circular null geodesics that exist at a fixed radius $R = R_0$, which is equivalent to the photon-sphere radius. The characteristics of photon spheres play a key role in numerous studies involving black hole solutions, ranging from the eikonal-limit spectra of quasinormal modes (QNM) and black hole shadow radii to circular orbits for massive test particles, among other phenomena (see Refs. [3, 5–7] and the bibliography therein).

Photon spheres of static spherically symmetric supergravity black holes were analysed by Cvetič, Gibbons and Pope [7]. In particular, in subsection 3.4 they considered (for zero gauge coupling g) the same non-extremal Einstein–Maxwell–dilaton dyon black hole (with dilaton coupling equivalent to $\lambda^2 = 3/2$) in the Lü–Pang–Pope parametrization [35] of the moduli functions H_s (by fluxbrane polynomials [29] depending upon $f_0 = 1 - \frac{2\mu}{R}$), and proved that the first-order (differential) condition for a circular null geodesic admits exactly one root outside the horizon which is necessarily the radius of (unstable) photon sphere. The present paper treats the same non-extremal A_2 Toda dyon black hole in parametrization of moduli functions $H_s = 1 + P_s/R + P_s^{(2)}/R^2$ (by black brane polynomials [27]). We derive the explicit fifth-order master polynomial, give an elementary uniqueness proof based on the strict concavity of an auxiliary polynomial $Q(z)$, prove instability by a direct computation of the second derivative of the effective potential $U''(R_0)$, obtain several exact roots and a table of numerical values, and compute the shadow observables for this metric.

A companion analysis for the complementary coupling $\lambda^2 = 1/2$ with moduli functions $H_s = 1 + P_s/R$ and cubic master equation has recently appeared in Ref. [34] devoted to photon sphere for non-extremal $A_1 + A_1$ Toda dyon black hole. The two papers therefore treat two distinct members of the same family with different algebraic degrees of the master equations and the explicit formulae.

The organization of the present paper is as follows. The gravitational setup and the explicit dyon black-hole solution are introduced in Section 2, while a compilation of its key physical parameters is provided in Section 3. Section 4 contains the core dynamical analysis: we derive the governing master equation for the null circular orbit radius R_0 , rigorously prove the uniqueness of its root beyond the horizon $R_0 > 2\mu$ and verify the instability of these photon orbits. Subsequent to this, Section 5 supplies several illustrative exact solutions to the derived polynomial equation. We close the analytical treatment in Section 6 with study of the black hole shadow.

2. Black Hole Dyon Solution

Let us consider a model governed by the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \left\{ R[g] - g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} e^{2\lambda\varphi} F_{\mu\nu} F^{\mu\nu} \right\}, \quad (4)$$

where $g = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ is metric, φ is the scalar field, $F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ is the 2-form with $A = A_\mu dx^\mu$; G is the gravitational constant; λ is coupling constant obeying (1) and $|g| = |\det(g_{\mu\nu})|$.

Here we deal with the case

$$\lambda^2 = \frac{3}{2} \quad (5)$$

which corresponds to Kaluza-Klein reduction from 5D gravitational model [12].

We consider dyon black hole solution [8] to the field equations corresponding to the action (4) which is defined on the manifold

$$\mathcal{M} = (2\mu, +\infty) \times S^2 \times \mathbb{R}, \quad (6)$$

and has the following form

$$ds^2 = (H_1 H_2)^{1/2} \left\{ -H_1^{-1} H_2^{-1} \left(1 - \frac{2\mu}{R} \right) dt^2 + \frac{dR^2}{1 - \frac{2\mu}{R}} + R^2 d\Omega_2^2 \right\}, \quad (7)$$

$$\exp(\varphi) = H_1^{\lambda/2} H_2^{-\lambda/2}, \quad (8)$$

$$F = \frac{Q_1}{R^2} H_1^{-2} H_2 dt \wedge dR + Q_2 \tau. \quad (9)$$

Here Q_1 is electric charge and Q_2 is magnetic charge, $\mu > 0$, $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric on the unit sphere S^2 ($0 < \theta < \pi$, $0 < \phi < 2\pi$) and $\tau = \sin \theta d\theta \wedge d\phi$ is the volume form on S^2 . For the speed of light we put $c = 1$.

The functions H_s are given by

$$H_s = 1 + \frac{P_s}{R} + \frac{P_s^{(2)}}{R^2}, \quad (10)$$

where

$$2Q_s^2 = \frac{P_s(P_s + 2\mu)(P_s + 4\mu)}{P_1 + P_2 + 4\mu}, \quad (11)$$

$$P_s^{(2)} = \frac{P_1 P_2 (P_s + 2\mu)}{2(P_1 + P_2 + 4\mu)}, \quad (12)$$

$s = 1, 2$.

The (moduli) functions H_s are generated by a Toda chain associated with the Lie algebra A_2 [8]. (For a pioneering spherically symmetric dyon solution governed by A_2 Toda chain see Ref. [12].) They obey the following boundary conditions:

$$H_s \rightarrow H_{s0} = 1 + \frac{P_s}{2\mu} + \frac{P_s^{(2)}}{(2\mu)^2} > 0 \quad (13)$$

for $R \rightarrow 2\mu$, and

$$H_s \rightarrow 1 \quad (14)$$

for $R \rightarrow +\infty$, $s = 1, 2$.

The solution is a well-defined one for $R > 2\mu$. It is a composite brane solution describing a configuration of two intersecting non-extremal black 0-branes (electric one and a magnetic one) in case when the intersection rule corresponds to the Lie algebra A_2 ($A_2 = sl(3)$) [27].

Due to the first boundary condition (13), we are led to a horizon at $R = 2\mu$ while the second condition (14) implies the asymptotic flatness of the metric as $R \rightarrow +\infty$.

For a global extension of the metric (7), initially defined for $R > 2\mu$, we get two horizons at $R = 2\mu$ and $R = 0$, and a singularity at certain $R_* < 0$. (Here $R_* < 0$ is the maximal (negative) root of the equation $H_1(R) = 0$ for $P_1 > P_2$ and it is maximal (negative) root of another quadratic equation $H_2(R) = 0$ for $P_1 < P_2$. For $P_1 = P_2 = P > 0$ we have $R_* = -\frac{P}{2}$ which obeys both equations $H_s(R) = 0$, $s = 1, 2$.)

In a special case when $P_1 = P_2 = P$ the metric (7) coincides with the Reissner-Nordström metric. Indeed, in this case we get

$$ds^2 = H \left\{ -H^{-2} \left(1 - \frac{2\mu}{R} \right) dt^2 + \frac{dR^2}{1 - \frac{2\mu}{R}} + R^2 d\Omega_2^2 \right\}, \quad (15)$$

where $H = H_1 = H_2 = \left(1 + \frac{P}{2R} \right)^2$ and $\varphi = 0$. By changing the radial variable $r = R + \frac{P}{2}$ we obtain

$$ds^2 = -A(r) dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega_2^2 \quad (16)$$

where

$$A(r) = 1 - \frac{2GM}{r} + \frac{Q_{RN}^2}{r^2}, \quad (17)$$

$$2GM = P + 2\mu, \quad Q_{RN}^2 = \frac{1}{4}P(P + 4\mu) = Q_1^2 = Q_2^2. \quad (18)$$

3. Physical Parameters

Here, we outline some physical parameters associated with the black hole solution [8].

3.1. Gravitational Mass and Scalar Charge

The ADM gravitational mass M may be found from (7) as

$$GM = \mu + \frac{1}{4}(P_1 + P_2), \quad (19)$$

where G is the gravitational constant.

For the scalar charge we find from (8)

$$Q_\varphi = \frac{1}{2}\lambda(P_1 - P_2). \quad (20)$$

By using relations (5), (19), and (20), we obtain the following identity [8]

$$2(GM)^2 + Q_\varphi^2 = Q_1^2 + Q_2^2 + 2\mu^2. \quad (21)$$

(For the extremal case $\mu = +0$ see also Ref. [21].)

3.2. The Hawking Temperature and Entropy

The Hawking temperature, calculated from the metric (7), is [8]:

$$T_H = \frac{1}{8\pi\mu}(H_{10}H_{20})^{-1/2}. \quad (22)$$

Here H_{s0} are defined in (13) and relations $c = \hbar = k_B = 1$ are adopted.

For the Bekenstein-Hawking (area) entropy $S = A/(4G)$ (A is the horizon area), corresponding to the horizon at $R = 2\mu$, we obtain

$$S_{BH} = \frac{4\pi\mu^2}{G}(H_{10}H_{20})^{1/2}. \quad (23)$$

We note that relations (22) and (23) imply $T_H S_{BH} = \frac{\mu}{2G}$.

4. Circular Geodesic Solutions

In this section we study the geodesic equations corresponding to the metric (7), restricting our attention to circular solutions.

4.1. Geodesic Equations

The geodesic equations are equivalent to the Euler-Lagrange equations

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0. \quad (24)$$

following from the Lagrangian

$$\mathcal{L} = \frac{1}{2}g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta. \quad (25)$$

Here $\dot{x}^\alpha = dx^\alpha/d\tau = u^\alpha$ denotes the 4-velocity vector, with $x^\alpha = x^\alpha(\tau)$ ($\alpha = 0, 1, 2, 3$). The parameter τ represents the proper time for a massive point particle moving along a timelike geodesic, and serves as the affine parameter in the null case. We adopt the normalization

$$g_{\alpha\beta}(x)u^\alpha u^\beta = -k = 2\mathcal{E}, \quad (26)$$

where $k = 0, 1$ corresponds to null and timelike geodesics, respectively. The quantity $\mathcal{E}$ is the conserved energy integral obtained from the Lagrange equations for the Lagrangian (25).

In what follows we use the relations for the redshift function $A(R)$ and central function $C(R)$:

$$A = A(R) = (H_1 H_2)^{-1/2} \left(1 - \frac{2\mu}{R} \right), \quad (27)$$

$$C = C(R) = (H_1 H_2)^{1/2} R^2. \quad (28)$$

In this subsection, we restrict our analysis to null and time-like geodesics confined to the equatorial plane: $\theta = \pi/2$. The symmetry then reduces the system to an effective three-dimensional problem, for which the Lagrangian, derived from the metric in Equation (7), is given by

$$\mathcal{L}_* = -A(R)\dot{t}^2 + (A(R))^{-1}\dot{R}^2 + C(R)\dot{\phi}^2. \quad (29)$$

The Lagrangian $\mathcal{L}_*$ governs the geodesic equations which correspond to the variables t , R and ϕ . The geodesic equation for the variable θ is satisfied identically due to our choice of $\theta = \pi/2$.

For the cyclic coordinates t and ϕ we find the following integrals of motion:

$$\hat{E} = A(R)\dot{t}, \quad \hat{L} = C(R)\dot{\phi}. \quad (30)$$

For $k = 1$, they describe the total energy $E = \hat{E}m$ and angular momentum $L = \hat{L}m$ of a test (neutral point-like) particle with mass $m > 0$.

Equation (26) takes the following form for the metric (7)

$$-A(R)\dot{t}^2 + (A(R))^{-1}\dot{R}^2 + C(R)\dot{\phi}^2 = -k. \quad (31)$$

By using (30), we are led to relation

$$-\frac{\hat{E}^2}{A} + \frac{\dot{R}^2}{A} + \frac{\hat{L}^2}{C} = -k, \quad (32)$$

which can be rewritten as follows

$$\dot{R}^2 + U(R) = \hat{E}^2. \quad (33)$$

Here

$$\begin{aligned} U = U(R) &= A(R) \left(k + \frac{\hat{L}^2}{C(R)} \right) \\ &= (H_1 H_2)^{-1} \left(1 - \frac{2\mu}{R} \right) \left(k(H_1 H_2)^{1/2} + \frac{\hat{L}^2}{R^2} \right) \end{aligned} \quad (34)$$

is the effective potential.

The Lagrange equation for the radial coordinate R reads

$$2\ddot{R} + \frac{dU}{dR} = 0. \quad (35)$$

4.2. Circular Photon Orbits: Non-Extremal Case

Now we consider null geodesics. For the effective potential we get

$$U = A(R) \frac{\hat{L}^2}{C(R)} = (H_1 H_2)^{-1} \left(1 - \frac{2\mu}{R} \right) \frac{\hat{L}^2}{R^2}, \quad (36)$$

or

$$U = \frac{\hat{L}^2}{(2\mu)^2} u, \quad (37)$$

where

$$u = u(x) = \frac{x-1}{H_1(x)H_2(x)x^3}, \quad (38)$$

and

$$x = R/(2\mu), \quad p_s = P_s/(2\mu), \quad (39)$$

are dimensionless parameters, $s = 1, 2$. See Figure 1. Here

$$H_s(x) = 1 + \frac{p_s}{x} + \frac{q_s}{x^2}, \quad (40)$$

where

$$q_s = p_s^{(2)} = \frac{p_1 p_2 (p_s + 1)}{2(p_1 + p_2 + 2)}, \quad (41)$$

$s = 1, 2$.

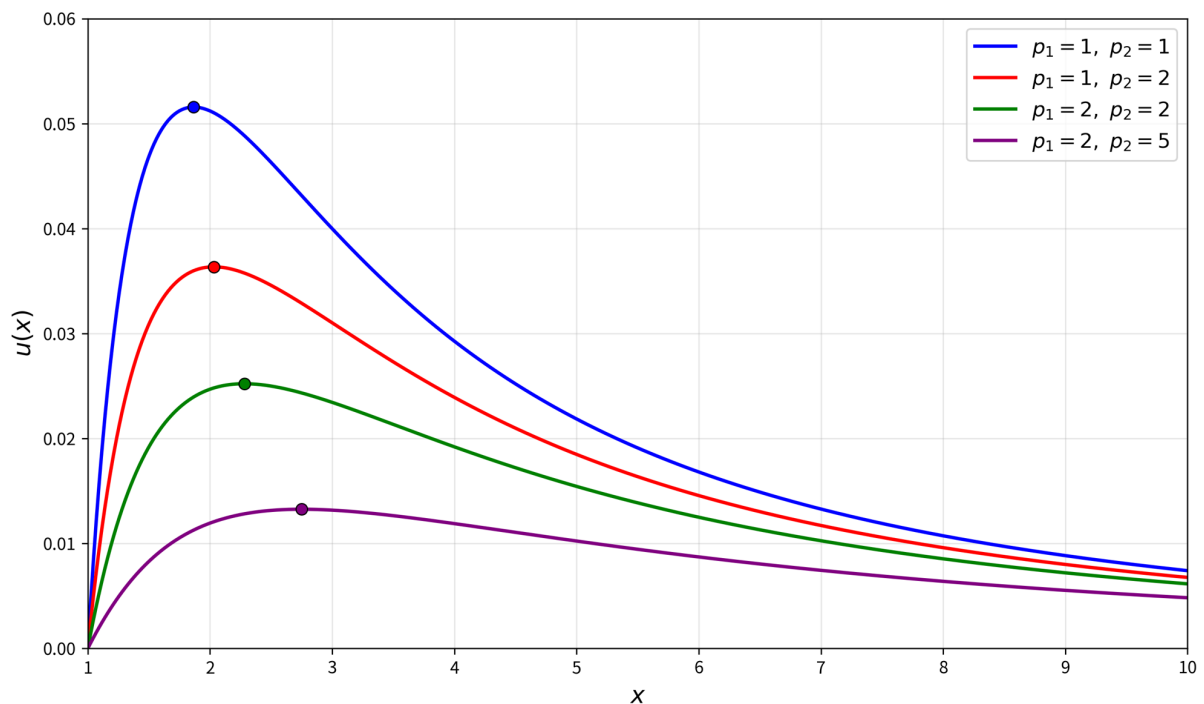


Figure 1. The reduced effective potential $u(x)$ from (38) as a function of x for particular sets of the parameters p_1, p_2 .

Let us consider circular solutions for the null geodesic equations obeying

$$R(\tau) = R_0 = \text{const}, \quad R_0 > 2\mu. \quad (42)$$

Here R_0 is the radius of the photon sphere which is covered by circular photon orbits.

In this case $\hat{L} \neq 0$ and the radial Equation (35) for $R = R_0$ with $k = 0$ reads

$$\frac{dU}{dR} = 0, \quad (43)$$

where

$$\frac{dU}{dR} = \frac{1}{2\mu} \frac{dU}{dx} = \frac{\hat{L}^2}{(2\mu)^3} \frac{du}{dx}. \quad (44)$$

The calculations gives us

$$\frac{du}{dx} = -\frac{F(x)}{4(p_1 + p_2 + 2)^2 x^8 (H_1 H_2)^2}, \quad (45)$$

where

$$F(x) = 8(p_1 + p_2 + 2)^2 x^5 + 4(p_1 + p_2 - 3)(p_1 + p_2 + 2)^2 x^4 - 8(p_1 + p_2)(p_1 + p_2 + 2)^2 x^3 - 4p_1 p_2 (p_1 + p_2 + 2)(p_1 p_2 + 2p_1 + 2p_2 + 3)x^2 - 2p_1^2 p_2^2 (p_1 + 1)(p_2 + 1)x + p_1^2 p_2^2 (p_1 + 1)(p_2 + 1). \quad (46)$$

The extremality condition (43) is equivalent to the master equation

$$F(x) = 0. \quad (47)$$

For future applications it is worth to rewrite dimensionless master Equation (47) in terms of radial variable $R = x(2\mu)$, where $\mu > 0$. By multiplying (47) on $(2\mu)^7$ we obtain

$$8(P_1 + P_2 + 4\mu)^2 R^5 + 4(P_1 + P_2 - 6\mu)(P_1 + P_2 + 4\mu)^2 R^4 - 16\mu(P_1 + P_2)(P_1 + P_2 + 4\mu)^2 R^3 - 4P_1 P_2 (P_1 + P_2 + 4\mu)(P_1 P_2 + 4(P_1 + P_2)\mu + 12\mu^2) R^2 - 2P_1^2 P_2^2 (P_1 + 2\mu)(P_2 + 2\mu) R + 2\mu P_1^2 P_2^2 (P_1 + 2\mu)(P_2 + 2\mu) = 0. \quad (48)$$

We remind that $P_s = 2\mu p_s$, $s = 1, 2$.

We note that relation (45) just follows from identity

$$\frac{d \ln u}{dx} = \frac{1}{u} \frac{du}{dx} = \Phi(x), \quad (49)$$

where

$$\Phi(x) = \frac{1}{x-1} - \frac{3}{x} - \frac{d}{dx} \ln(H_1(x)H_2(x)). \quad (50)$$

The master Equation (47) may be rewritten as

$$\Phi(x) = 0. \quad (51)$$

Proposition 1. For all $\mu > 0$, $P_1 > 0$ and $P_2 > 0$ the Equation (43) has one and only one real solution R_0 which satisfies the inequality $R_0 > 2\mu$.

Proposition 1 states the existence and uniqueness of photon sphere (outside the event horizon) for the metric (7) for any proper set of parameters. It is valid due to the following proposition.

Proposition 2. For all $p_1 > 0$, $p_2 > 0$ the fifth order polynomial (reduced master) Equation (47) has one and only one real solution $x = x_0$ which satisfies the inequality $x_0 > 1$.

Proof. Let us set

$$z = \frac{1}{x}, \quad x > 1 \iff z \in (0, 1). \quad (52)$$

We get

$$H_s(x) = 1 + \frac{p_s}{x} + \frac{q_s}{x^2} = 1 + p_s z + q_s z^2, \quad (53)$$

$s = 1, 2$. The equation $\Phi(x) = 0$ from (51) can be rewritten as

$$\Phi\left(\frac{1}{z}\right) = 0. \quad (54)$$

A direct calculation yields

$$\Phi\left(\frac{1}{z}\right) = \frac{z Q(z)}{(z-1) H_1(1/z) H_2(1/z)}, \quad (55)$$

where

$$Q(z) = q_1 q_2 z^5 - 2q_1 q_2 z^4 - (p_1 p_2 + p_1 q_2 + p_2 q_1 + q_1 + q_2) z^3 - 2(p_1 + p_2) z^2 + (p_1 + p_2 - 3) z + 2. \quad (56)$$

Due to $z \in (0, 1)$ we have

$$z > 0, \quad z - 1 < 0, \quad H_s = 1 + p_s z + q_s z^2 > 0, \quad (57)$$

$s = 1, 2$. The inequalities $H_s > 0$ in (57) just follow from

$$p_s > 0, \quad q_s > 0, \quad (58)$$

$s = 1, 2$. Due to relations (41), (45) and (55) we obtain

$$z^5 F(1/z) = 4(p_1 + p_2 + 2)^2 Q(z) \quad (59)$$

for $z \in (0, 1)$.

Thus, the polynomial master Equation (47) (or equivalently $\Phi(x) = 0$ for $x \in (1, +\infty)$) is equivalent to another polynomial equation

$$Q(z) = 0 \quad (60)$$

for $z \in (0, 1)$ ($z = 1/x$).

Let us prove that there exists at least one solution to (60) (or, to (47)). We have

$$Q(0) = 2 > 0, \quad (61)$$

while

$$\begin{aligned} Q(1) &= q_1 q_2 - 2q_1 q_2 - (p_1 p_2 + p_1 q_2 + p_2 q_1 + q_1 + q_2) - 2(p_1 + p_2) + (p_1 + p_2 - 3) + 2 \\ &= -(p_1 p_2 + p_1 q_2 + p_2 q_1 + q_1 q_2 + p_1 + p_2 + q_1 + q_2 + 1) < 0. \end{aligned} \quad (62)$$

Function $Q(z)$ is continuous on $[0, 1]$ and satisfies the boundary conditions (61) and (62). The Intermediate Value (Bolzano) theorem implies the existence of at least one point $z_0 \in (0, 1)$ which obeys

$$Q(z_0) = 0. \quad (63)$$

Now compute the second derivative:

$$Q''(z) = 20q_1 q_2 z^3 - 24q_1 q_2 z^2 - 6(p_1 p_2 + p_1 q_2 + p_2 q_1 + q_1 + q_2)z - 4(p_1 + p_2), \quad (64)$$

(Here and below $A' = \frac{dA}{dz}$, $A'' = \frac{d^2 A}{dz^2}$.) Rewrite it as

$$Q''(z) = 4q_1 q_2 z^2 (5z - 6) - 6(p_1 p_2 + p_1 q_2 + p_2 q_1 + q_1 + q_2)z - 4(p_1 + p_2). \quad (65)$$

For $z \in (0, 1)$, we have $5z - 6 < 0$. Since all coefficients p_s, q_s are positive we obtain

$$Q''(z) < 0, \quad (66)$$

for all $z \in (0, 1)$. See Figure 2.

Hence the function Q is strictly concave on $(0, 1)$, i.e.

$$Q((1-t)u_1 + tu_2) > (1-t)Q(u_1) + tQ(u_2) \quad (67)$$

for all $t \in (0, 1)$ and $0 < u_1 < u_2 < 1$. This is the two-point case of Jensen's inequality for a concave function. Geometrically, since Q is strictly concave, its graph lies strictly above the chord connecting the points $(u_1, Q(u_1))$ and $(u_2, Q(u_2))$.

Now we prove the uniqueness of the root z_0 . Assume the contrary: suppose there exists another root $z_1 \in (0, 1)$, which obeys

$$Q(z_1) = 0. \quad (68)$$

without loss of generality we put $z_1 < z_0$. Let $z_2 > 0$ is small enough such that $0 < z_2 < z_1$ and $Q(z_2) > 1$. This is valid due to $Q(z) \rightarrow Q(0) = 2$ for $z \rightarrow 0$ since the function $Q(z)$ is continuous on $[0, 1]$. Due to concavity relation (67) we get

$$Q((1-t)z_2 + tz_0) > (1-t)Q(z_2) + tQ(z_0) \quad (69)$$

for $t \in (0, 1)$. Since $0 < z_2 < z_1 < z_0$ there exists $t_1 \in (0, 1)$ obeying

$$(1-t_1)z_2 + t_1z_0 = z_1. \quad (70)$$

Then, due to (69) and $Q(z_0) = 0$ we obtain the relation

$$Q(z_1) > (1-t_1)Q(z_2) > 0, \quad (71)$$

which contradicts to the equality $Q(z_1) = 0$. The obtained contradiction completes the proof of the Proposition 2. $\square$

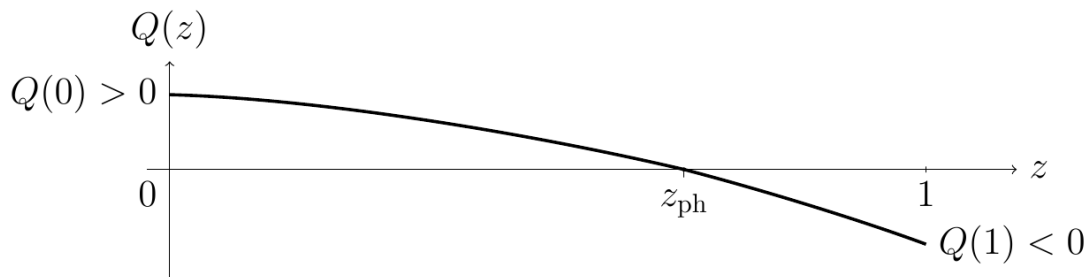


Figure 2. Schematic graph of the function $Q(z)$ on the interval $(0, 1)$: $Q''(z) < 0$, $Q(0) > 0$, and $Q(1) < 0$. Here $z_0 = z_{ph}$ corresponds to photonic sphere.

Remark 1. It should be noted that the Proposition 2 was proved for general $q_s > 0$, $s = 1, 2$, without using the relations (41). That means that the Proposition 1 is valid for a class of black holes (e.g. with anisotropic fluid matter source) described by metric (7) and moduli functions H_s given by (10) with arbitrary parameters $P_s > 0$ and $P_s^{(2)} > 0$, $s = 1, 2$.

Lemma 1. In notations of the proof of the Proposition 2 the following relation is valid: $Q'(z_0) < 0$.

Proof. For strictly concave function we have well-known inequality

$$Q(z) < Q'(z_0)(z - z_0) + Q(z_0) \quad (72)$$

for all $z \neq z_0$. Let us set $z = z_2 < z_0$ with $2 > Q(z_2) > 1$. (Such real number does exist due to $Q(0) = 2$, $Q(z_0) = 0$ and Intermediate Value theorem). We obtain from (72) and $Q(z_0) = 0$:

$$1 < Q(z_2) < Q'(z_0)(z_2 - z_0) \quad (73)$$

which implies $Q'(z_0) < 0$. $\square$

Proposition 3. For all $\mu > 0$, $P_1 > 0$, $P_2 > 0$ the circular photonic geodesics given by unique solution to master Equation (47) (or, equivalently $(\frac{dU}{dR})_{R=R_0} = 0$) with $R_0 = 2\mu x_0 > 2\mu$ are unstable due to inequality

$$\left(\frac{d^2U}{dR^2}\right)_{R=R_0} < 0. \quad (74)$$

Proof. It follows from (44), (45) and $R = 2\mu x$ that

$$\left(\frac{d^2U}{dR^2}\right)_{R=R_0} = \frac{\hat{L}^2}{(2\mu)^4} \left(\frac{d^2u}{dx^2}\right)_{x=x_0}, \quad (75)$$

where

$$\left(\frac{d^2u}{dx^2}\right)_{x=x_0} = -\frac{\left(\frac{dF}{dx}\right)_{x=x_0}}{4(p_1 + p_2 + 2)^2 x^8 (H_1 H_2)^2}, \quad (76)$$

where $F(x)$ is given by (47) and x_0 is the unique root of fifth order polynomial $F(x)$, i.e. $F(x_0) = 0$, which obeys $x_0 > 1$ (see Proposition 2).

So, we need to prove only that

$$\left(\frac{dF}{dx}\right)_{x=x_0} > 0. \quad (77)$$

Due to relation (59), $F(x_0) = 0$ and $x = 1/z$

$$z_0^5(-1/z_0^2)\left(\frac{dF}{dx}\right)_{x=x_0} = 4(p_1 + p_2 + 2)^2 Q'(z_0), \quad (78)$$

where $0 < z_0 = 1/x_0 < 1$. But according to Lemma 1: $Q'(z_0) < 0$. Hence the relation (77) is valid. Thus, the proposition is proved. $\square$

5. Examples

Here we consider certain examples of exact solutions to master equation $F(x) = 0$ with fifth order polynomial $F(x)$ defined in (47). Some numerical roots of the reduced master equation are presented in Table 1.

Example 1. Let us put

$$p_1 = p_2 = p > 0. \quad (79)$$

Then the dimensionless (reduced) master Equation (47) reads

$$(p+1)^2(2x+p)^3(4x^2-2px-6x+p)=0. \quad (80)$$

After excluding the roots $x_1 = -p/2 < 0$, we are led to quadratic equation

$$x^2 - \frac{1}{2}(p+3)x + \frac{1}{4}p = 0 \quad (81)$$

which yields other two solutions

$$x_{\pm} = \frac{1}{4}(p+3) \pm \frac{1}{4}\sqrt{d}, \quad (82)$$

$$d = p^2 + 2p + 9 > 0. \quad (83)$$

It could be readily verified that

$$x_3 = x_+ > 1 > x_- = x_2 > 0 \quad (84)$$

for $p > 0$. Thus, our unique solution $x_0 > 1$ reads

$$x_0 = x_+ = \frac{1}{4}(p+3) + \frac{1}{4}\sqrt{p^2 + 2p + 9}. \quad (85)$$

In this case our metric is coinciding with the Reissner-Nordström metric (16) written in radial variable $r = R + \frac{p}{2}$. The relation (85) for $x_0 = R_0/(2\mu)$ together with formulae (18) lead us to well-known relation for the radius of RN photon sphere $r_0 = R_0 + \frac{p}{2}$ in terms physical variables $2GM$ and Q_{RN}^2 :

$$r_0 = \frac{1}{2}\left(3GM + \sqrt{9(GM)^2 - 8Q_{RN}^2}\right). \quad (86)$$

Example 2. For $p_1 = 2$, $p_2 = 1$, the given equation simplifies to

$$25x^5 - 75x^3 - 55x^2 - 6x + 3 = 0, \quad (87)$$

which factors as

$$(5x^2 + 5x - 1)(5x^3 - 5x^2 - 9x - 3) = 0. \quad (88)$$

Hence the solutions are the roots of the quadratic and the cubic equations. The roots of quadratic equation $5x^2 + 5x - 1 = 0$ are real but less than 1. The cubic equation $5x^3 - 5x^2 - 9x - 3 = 0$ gives only one real root

$$x_0 = \frac{1}{3} + \sqrt[3]{\frac{430 + 18\sqrt{65}}{675}} + \sqrt[3]{\frac{430 - 18\sqrt{65}}{675}} \approx 2.031, \quad (89)$$

which obeys $x > 1$. Two other roots are essentially complex.

Table 1. Examples of roots of the reduced master Equation (47) for several values of p_1 and p_2 .

p_1	p_2	x_1	x_2	x_3	x_4	x_5
0.03	0.01	−0.019	−0.005 − 0.003 <i>i</i>	−0.005 + 0.003 <i>i</i>	0.003	1.507
0.1	0.1	−0.050	−0.050	−0.050	0.016	1.534
0.3	0.1	−0.187	−0.053 − 0.027 <i>i</i>	−0.053 + 0.027 <i>i</i>	0.027	1.565
1	1	−0.500	−0.500	−0.500	0.134	1.866
2	1	−1.171	−0.516 − 0.171 <i>i</i>	−0.516 + 0.171 <i>i</i>	0.171	2.031
2	2	−1.000	−1.000	−1.000	0.219	2.281
3	1	−1.760	−0.541 − 0.229 <i>i</i>	−0.541 + 0.229 <i>i</i>	0.191	2.152
3	2	−1.708	−1.005 − 0.231 <i>i</i>	−1.005 + 0.231 <i>i</i>	0.246	2.472
3	3	−1.500	−1.500	−1.500	0.275	2.725
4	1	−2.330	−0.560 − 0.264 <i>i</i>	−0.560 + 0.264 <i>i</i>	0.204	2.246
4	2	−2.313	−1.037 − 0.320 <i>i</i>	−1.037 + 0.320 <i>i</i>	0.262	2.624
4	3	−2.238	−1.494 − 0.272 <i>i</i>	−1.494 + 0.272 <i>i</i>	0.294	2.931
4	4	−2.000	−2.000	−2.000	0.314	3.186
5	1	−2.886	−0.574 − 0.287 <i>i</i>	−0.574 + 0.287 <i>i</i>	0.213	2.320
5	2	−2.897	−1.063 − 0.378 <i>i</i>	−1.063 + 0.378 <i>i</i>	0.274	2.749
5	3	−2.856	−1.527 − 0.386 <i>i</i>	−1.527 + 0.386 <i>i</i>	0.307	3.104
5	4	−2.763	−1.983 − 0.304 <i>i</i>	−1.983 + 0.304 <i>i</i>	0.328	3.402
5	5	−2.500	−2.500	−2.500	0.342	3.658
10	5	−5.722	−2.601 − 0.740 <i>i</i>	−2.601 + 0.740 <i>i</i>	0.375	4.548
10	10	−5.000	−5.000	−5.000	0.411	6.089

Example 3. Let us put $p_1 = k_1 p$, $p_2 = k_2 p$, where $k_1 > 0$, $k_2 > 0$ are fixed constants obeying $k_1 + k_2 = 1$ and $p > 0$. We obtain two asymptotic relations for the root x_0

$$x_0 = \frac{3}{2} + \frac{p}{3} + O(p^2), \text{ as } p \rightarrow +0, \quad (90)$$

$$x_0 \sim Xp, \text{ as } p \rightarrow +\infty. \quad (91)$$

In (91) $X > 0$ is a solution to the quartic equation

$$P(X) = X^4 + \frac{1}{2}X^3 - \frac{1}{2}K^2X - \frac{1}{4}K^3 = 0, \quad (92)$$

where $K = k_1 k_2$, $k_1 + k_2 = 1$ ($k_1 > 0$, $k_2 > 0$). It is evident that $0 < K = k_1 k_2 \leq \frac{1}{4}$. The following proposition is valid.

Lemma 2. Only one root of quartic Equation (92) with $0 < K \leq \frac{1}{4}$ is positive. It obeys $0 < X = X(K) \leq 1/4$.

The proof of the Lemma 2 will be given in a separate publication devoted to extremal case ($\mu = +0$) of the black hole solution under consideration.

Example 4. Certain numerical examples of roots of the reduced master Equation (47) for several values of p_1 and p_2 are presented in Table 1.

6. BH Shadow

Following the standard procedure [6], one arrives at a particular solution which takes the form of a geodesic with a spiral-like shape and an infinitely large winding angle. This trajectory acts as a separatrix between the two distinct families of solutions—those that intersect the photon sphere and those that do not. It directly corresponds to the critical shadow angle $\vartheta_{sh} \in (0, \pi/2)$, which is geometrically determined at the observer's position (R_{obs}, ϕ_{obs}) as the angle formed by the tangent to this spiral curve and the local radial direction.

Let a light ray be emitted from (R_{obs}, ϕ_{obs}) at an angle smaller than the critical angle:

$$\vartheta < \vartheta_{sh}, \quad (93)$$

then it will traverse the photon sphere and enter the event horizon (and “fall” into the black hole). In opposite case, when the emission angle obeys

$$\vartheta > \vartheta_{sh}, \quad (94)$$

the ray will not reach the photon sphere and will escape to infinity (after a finite number of revolutions).

Let us define reduced effective potential as

$$\hat{U}(R) = U/\hat{L}^2 = \frac{A(R)}{C(R)} = (H_1 H_2)^{-1} \left(1 - \frac{2\mu}{R}\right) \frac{1}{R^2},$$

where $\hat{L} \neq 0$. R_0 corresponds to the point of maximum of the effective potential and hence

$$\hat{U}(R_0) > \hat{U}(R) > 0 \quad (95)$$

for $R > R_0$.

The black hole shadow angle ϑ_{sh} can be found by using a well-known relation [6]:

$$\vartheta_{sh} = \arcsin \sqrt{\frac{\hat{U}(R_{obs})}{\hat{U}(R_0)}}, \quad 0 < \vartheta_{sh} < \frac{\pi}{2}, \quad (96)$$

for all $R_{obs} > R_0$. Here, R_{obs} denotes the radial coordinate of a point-like source or light receiver (e.g. observer), while R_0 represents the radius of the photon sphere.

For sufficiently large values of the observer's radial coordinate R_{obs} satisfying $R_{obs} \gg P$ and $R_{obs} \gg \mu$, the shadow angle can be found by using the following asymptotic formula:

$$\vartheta_{sh} = \frac{b_c}{R_{obs}} + O(1/R_{obs}^2), \quad (97)$$

as $R_{obs} \rightarrow +\infty$. Here

$$b_c = \frac{1}{\sqrt{\hat{U}(R_0)}}. \quad (98)$$

is critical impact parameter [6].

For dyonic BH under consideration we obtain

$$b_c = R_0 (H_1(R_0) H_2(R_0))^{1/2} \left(1 - \frac{2\mu}{R_0}\right)^{-1/2}. \quad (99)$$

It is evident, that $b_c > R_0$ for all $\mu > 0$, $P_1 > 0$ and $P_2 > 0$.

Let us fix $\mu > 0$. We get

$$b_c \rightarrow 3\sqrt{3}\mu \quad (100)$$

in the (Schwarzschild) limit when $P_1 \rightarrow +0$, $P_2 \rightarrow +0$.

For $P_1 \rightarrow +\infty$, $P_2 \rightarrow +\infty$ and $P_1/P_2 = k > 0$ (k is constant) we are led to the asymptotical relation

$$b_c \sim 2\mu X p (H_{1,\infty} H_{2,\infty})^{1/2} \quad (101)$$

(see (91)) as $p = (P_1 + P_2)/(2\mu) \rightarrow +\infty$, where

$$H_{1,\infty} = 1 + \frac{K_1}{X} + \frac{K_1^2 K_2}{2X^2}, \quad H_{2,\infty} = 1 + \frac{K_2}{X} + \frac{K_1 K_2^2}{2X^2}. \quad (102)$$

Here $K_s = P_s/(P_1 + P_2) > 0$, $s = 1, 2$, are constants and $X = X(K_1 K_2) > 0$ is a solution to quartic Equation (92). The relation (101) just follows from $x_0 = R_0/(2\mu) \sim Xp$, as $p \rightarrow +\infty$ and relations for moduli functions H_s , $s = 1, 2$, from (10).

Now, let us put $P_1 = P_2 = P$. We obtain $R_0 = 2\mu x_0$, $H_1(R_0) = H_2(R_0) = \left(1 + \frac{P}{2R_0}\right)^2$ and hence

$$b_c = 2\mu x_0 \left(1 + \frac{p}{2x_0}\right)^2 \left(1 - \frac{1}{x_0}\right)^{-1/2}, \quad (103)$$

where $p = P/(2\mu)$ and $x_0 = \frac{1}{4}(p+3) + \frac{1}{4}\sqrt{p^2 + 2p + 9}$ (see (82)). Relation (103) is in an agreement with the well-known formula for critical impact parameter of the non-extremal Reissner-Nordström charged black hole [36]:

$$b_c = \frac{r_0}{\sqrt{A(r_0)}}, \quad (104)$$

where $r_0 = R_0 + \frac{P}{2} = R_0(H_1(R_0)H_2(R_0))^{1/4}$ and $A(r_0) = 1 - \frac{2GM}{r_0} + \frac{Q_{RN}^2}{r_0^2}$.

Relation (99) implies the identity for the photon capture cross section

$$\sigma_{cap} = \pi b_c^2 = R_0^2 (H_1(R_0)H_2(R_0)) \left(1 - \frac{2\mu}{R_0}\right)^{-1}. \quad (105)$$

For $P_1 = P_2 = P$ we lead to the photon capture cross section of the non-extremal Reissner-Nordström black hole $\sigma_{cap} = \frac{\pi r_0^2}{A(r_0)}$.

7. Conclusions

In this work we have studied the non-extremal dyonic black hole generated by the A_2 Toda chain for a dilatonic coupling parameter λ obeying $\lambda^2 = 3/2$. The associated moduli functions are quadratic polynomials in inverse radial variable $1/R$, and the condition for a circular null geodesic yields a fifth-order polynomial master equation for the radius of photon sphere R_0 . We have proved that this polynomial possesses exactly one real root outside the horizon $R_0 > 2\mu$ and that the corresponding photon orbits are unstable. The proof uses only elementary properties of a strictly concave auxiliary polynomial and does not rely on energy conditions.

We note that the existence of a unique unstable photon sphere for this solution is already implied by the result of Cvetič, Gibbons and Pope [7]. The new results here are the explicit fifth-order polynomial master equation, the self-contained concavity argument for the proof of the uniqueness of the photon sphere outside the horizon, the calculation of the second derivative of the effective potential $U''(R_0)$, the examples of analytical and numerical roots, an explicit expression for the shadow angle, the critical impact parameter and the capture cross-section. The complementary case with $\lambda^2 = 1/2$ and cubic master equation was treated in Ref. [34]; the two analyses are complementary rather than overlapping.

The results may be used in future studies of quasinormal modes in the eikonal limit and of innermost stable circular orbits.

Author Contributions

V.D.I.: conceptualization, supervision, investigation, visualization, validation, writing—reviewing and editing; E.E.T.: investigation, data curation, methodology, software, visualization, validation. All authors have read and agreed to the published version of the manuscript.

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References

1. Ghez, A.M.; Klein, B.L.; Morris, M.; et al. High Proper-Motion Stars in the Vicinity of Sagittarius A*: Evidence for a Supermassive Black Hole at the Center of Our Galaxy. *Astrophys. J.* **1998**, *509*, 678–686. <https://doi.org/10.1086/306528>.
2. Abbott, B.P.; Abbott, R.; Abbott, T.D.; et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* **2016**, *116*, 061102.
3. Vagnozzi, S.; Roy, R.; Tsai, Y.D.; et al. Horizon-Scale Tests of Gravity Theories and Fundamental Physics from the Event Horizon Telescope Image of Sagittarius A*. *Class. Quantum Gravity* **2023**, *40*, 165007. <https://doi.org/10.1088/1361-6382/acd97b>.
4. Cardoso, V.; Miranda, A.S.; Berti, E.; et al. Geodesic Stability, Lyapunov Exponents, and Quasinormal Modes. *Phys. Rev. D* **2009**, *79*, 064016. <https://doi.org/10.1103/physrevd.79.064016>.
5. Konoplya, R.A.; Zhidenko, A. Quasinormal Modes of Black Holes: From Astrophysics to String Theory. *Rev. Mod. Phys.* **2011**, *83*, 793–836. <https://doi.org/10.1103/revmodphys.83.793>.
6. Perlick, V.; Tsupko, O.Y. Calculating Black Hole Shadows: Review of Analytical Studies. *Phys. Rep.* **2022**, *947*, 1–39.
7. Cvetič, M.; Gibbons, G.W.; Pope, C.N. Photon Spheres and Sonic Horizons in Black Holes from Supergravity and Other Theories. *Phys. Rev. D* **2016**, *94*, 106005. <https://doi.org/10.1103/physrevd.94.106005>.
8. Abishev, M.E.; Boshkayev, K.A.; Dzhunushaliev, V.D.; et al. Dilatonic Dyon Black Hole Solutions. *Class. Quantum Gravity* **2015**, *32*, 165010. <https://doi.org/10.1088/0264-9381/32/16/165010>.
9. Abishev, M.E.; Boshkayev, K.A.; Ivashchuk, V.D. Dilatonic Dyon-like Black Hole Solutions in the Model with Two Abelian Gauge Fields. *Eur. Phys. J. C* **2017**, *77*, 180. <https://doi.org/10.1140/epjc/s10052-017-4749-1>.
10. Bronnikov, K.A.; Shikin, G.N. On Interacting Fields in General Relativity Theory. *Sov. Phys. J.* **1977**, *20*, 1138–1143.
11. Gibbons, G.W. Antigravitating Black Hole Solitons with Scalar Hair in N=4 Supergravity. *Nucl. Phys. B* **1982**, *207*, 337–349. [https://doi.org/10.1016/0550-3213\(82\)90170-5](https://doi.org/10.1016/0550-3213(82)90170-5).
12. Lee, S.C. Kaluza-Klein Dyons and the Toda Lattice. *Phys. Lett. B* **1984**, *149*, 98–99. [https://doi.org/10.1016/0370-2693\(84\)91560-0](https://doi.org/10.1016/0370-2693(84)91560-0).
13. Gibbons, G.W.; Wiltshire, D.L. Spacetime as a Membrane in Higher Dimensions. *Nucl. Phys. B* **1987**, *287*, 717–742. [https://doi.org/10.1016/0550-3213\(87\)90125-8](https://doi.org/10.1016/0550-3213(87)90125-8).
14. Heinrich, O. Charged Black Holes in Compactified Higher-Dimensional Einstein-Maxwell Theory. *Astron. Nachr.* **1988**, *309*, 249–251. <https://doi.org/10.1002/asna.2113090410>.
15. Gibbons, G.W.; Maeda, K.I. Black Holes and Membranes in Higher-Dimensional Theories with Dilaton Fields. *Nucl. Phys. B* **1988**, *298*, 741–775. [https://doi.org/10.1016/0550-3213\(88\)90006-5](https://doi.org/10.1016/0550-3213(88)90006-5).
16. Garfinkle, D.; Horowitz, G.; Strominger, A. Charged Black Holes in String Theory. *Phys. Rev. D* **1991**, *43*, 3140; Erratum in *Phys. Rev. D* **1992**, *45*, 3888.
17. Kallosh, R.; Linde, A.; Ortín, T.; et al. Supersymmetry as a Cosmic Censor. *Phys. Rev. D* **1992**, *46*, 5278–5302. <https://doi.org/10.1103/physrevd.46.5278>.
18. Cheng, G.J.; Lin, W.F.; Hsu, R.R. Dyonic Black Holes in Dilaton Gravity. *J. Math. Phys.* **1994**, *35*, 4839–4847. <https://doi.org/10.1063/1.530817>.
19. Gibbons, G.W.; Kastor, D.; London, L.A.J.; et al. Supersymmetric Self-Gravitating Solitons. *Nucl. Phys. B* **1994**, *416*, 850–880. [https://doi.org/10.1016/0550-3213\(94\)90558-4](https://doi.org/10.1016/0550-3213(94)90558-4).
20. Bleyer, U.; Melnikov, V.N.; Bronnikov, K.A.; et al. Black Hole Stability in Multidimensional Gravity Theory. *Astron. Nachr.* **1994**, *315*, 399–408. <https://doi.org/10.1002/asna.2103150602>.

21. Poletti, S.J.; Twamley, J.; Wiltshire, D.L. Dyonic Dilaton Black Holes. *Class. Quantum Gravity* **1995**, *12*, 1753–1769. <https://doi.org/10.1088/0264-9381/12/7/017>; Erratum in *Class. Quantum Gravity* **1995**, *12*, 2335.
22. Bronnikov, K.A. On Spherically Symmetric Solutions in D-Dimensional Dilaton Gravity. *Gravit. Cosmol.* **1995**, *1*, 67–78.
23. Lü, H.; Pope, C.N. P-Brane Solitons in Maximal Supergravities. *Nucl. Phys. B* **1996**, *465*, 127–156. [https://doi.org/10.1016/0550-3213\(96\)00048-x](https://doi.org/10.1016/0550-3213(96)00048-x).
24. Duff, M.J.; Lü, H.; Pope, C.N. The Black Branes of M-Theory. *Phys. Lett. B* **1996**, *382*, 73–80. [https://doi.org/10.1016/0370-2693\(96\)00521-7](https://doi.org/10.1016/0370-2693(96)00521-7).
25. Lü, H.; Pope, C.N.; Xu, K.W. Liouville and Toda Solitons in M-Theory. *Mod. Phys. Lett. A* **1996**, *11*, 1785–1795. <https://doi.org/10.1142/s0217732396001776>.
26. Bronnikov, K.A. Block-Orthogonal Brane Systems, Black Holes and Wormholes. *Gravit. Cosmol.* **1998**, *4*, 49.
27. Ivashchuk, V.D.; Melnikov, V.N. Toda P-Brane Black Holes and Polynomials Related to Lie Algebras. *Class. Quantum Gravity* **2000**, *17*, 2073–2092. <https://doi.org/10.1088/0264-9381/17/10/303>.
28. Lü, H.; Yang, W. SL(N,R)-Toda Black Holes. *Class. Quantum Gravity* **2013**, *30*, 235021. <https://doi.org/10.1088/0264-9381/30/23/235021>.
29. Ivashchuk, V.D. Black Brane Solutions Governed by Fluxbrane Polynomials. *J. Geom. Phys.* **2014**, *86*, 101–111.
30. Davydov, E.A. Discreteness of Dyonic Dilaton Black Holes. *Theor. Math. Phys.* **2018**, *197*, 1663–1676. <https://doi.org/10.1134/s0040577918110107>.
31. Zadora, A.; Gal'tsov, D.V.; Chen, C.M. Higher-N Triangular Dilatonic Black Holes. *Phys. Lett. B* **2018**, *779*, 249–256. <https://doi.org/10.1016/j.physletb.2018.02.017>.
32. Abishev, M.E.; Ivashchuk, V.D.; Malybayev, A.N.; et al. Dyon-like Black Hole Solutions in the Model with Two Abelian Gauge Fields. *Gravit. Cosmol.* **2019**, *25*, 374–382. <https://doi.org/10.1134/s0202289319040029>.
33. Ivashchuk, V.D.; Malybayev, A.N.; Nurbakova, G.S.; et al. Photon Spheres near Dilatonic Dyon-like Black Holes in a Model with Two Abelian Gauge Fields and Two Scalar Fields. *Gravit. Cosmol.* **2023**, *29*, 411–418. <https://doi.org/10.1134/s0202289323040114>.
34. Ivashchuk, V.D.; Kayumov, U.S.; Malybayev, A.N.; et al. Photon Sphere for a Dilatonic Dyonic Black Hole in a Model with an Abelian Gauge Field and a Scalar Field. *Gravit. Cosmol.* **2025**, *31*, 591–599. <https://doi.org/10.1134/s0202289325700471>.
35. Lü, H.; Pang, Y.; Pope, C.N. AdS Dyonic Black Hole and Its Thermodynamics. *J. High Energy Phys.* **2013**, *2013*, 33. [https://doi.org/10.1007/jhep11\(2013\)033](https://doi.org/10.1007/jhep11(2013)033).
36. Vertogradov, V.; Övgün, A. General Approach on Shadow Radius and Photon Spheres in Asymptotically Flat Spacetimes and the Impact of Mass-Dependent Variations. *Phys. Lett. B* **2024**, *854*, 138758. <https://doi.org/10.1016/j.physletb.2024.138758>.