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Letter

A Sensitive Nonclassicality Certification Functional for Continuous-Variable Systems

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Abstract: If the phase space-based Glauber-Sudarshan distribution, P_ϱ, has negative values the quantum state, ϱ, it describes is nonclassical. Due to P's singular behavior, this simple criterion is impractical to use. Recent work has presented a general, sensitive, and noise-tolerant certification functional, $\xi[P]$, for the detection of non-classical behavior of quantum states ϱ. There, it was shown that when this functional takes on negative values somewhere in phase space, $\xi[P](x,p) < 0$, this is sufficient to certify the nonclassicality of a state. Here we give examples where this certification fails. We investigate states which are known to be nonclassical but the certification function is nonnegative, $\xi(x,p) \geq 0$, everywhere in phase space. We generalize ξ, giving it an appealing form, $\mathcal{S}$, which allows for slight improvements in certification, but $\mathcal{S}$ also fails for mixed very weakly nonclassical states. More important than a slight improvement in sensitivity of $\mathcal{S}$ over ξ is that we showed how very sensitive ξ and $\mathcal{S}$ are, and more important still is our simple derivation of $\mathcal{S}$ allowing us to generalize ξ and $\mathcal{S}$ to multiple modes.

Keywords: nonclassical state; certification of nonclassical state; phase space distributions

1. Why Nonclassicality Certification Functionals?

It is desirable to be able to discriminate between nonclassical states on the one hand and classical states on the other. So far no simple approach that works for theoretical studies and experimentally reconstructed states alike has been found to perform this discrimination faithfully. All known approaches to discrimination are either unworkable in the general case, or they do not work unambiguously, or both [1,2].

Here we generalize a recently devised, very sensitive, nonclassicality certification functional, ξ, introduced in Ref. [3]. Our generalization provides insight into the structure of ξ and makes it moderately more sensitive still, yet, the goal of a universally faithful approach to quantum versus classical discrimination remains elusive.

More importantly, our derivation shows how to extend the approach of [3] from the single- to the multi-mode case.

Nonclassicality Based on the P-Distribution

In the 1960s it was established that the Glauber-Sudarshan phase space distribution, P [4–10], can be used for the very definition of nonclassical behavior [3,6,7]. If a state, $\varrho = \frac{1}{2\pi}\iint dxdp P_\varrho(x,p)|\frac{x+ip}{\sqrt{2}}\rangle\langle\frac{x+ip}{\sqrt{2}}|$, has a distribution $P_\varrho(x,p)$ with negative values somewhere at position x and momentum p in (cartesian) phase space (Instead of using complex coherent amplitudes to parameterize phase space we follow the convention for (quantum-mechanical) phase space of Ref. [10] using position x and momentum p.), then that state is nonclassical.

This, 'nonclassicality based on the P-distribution', is the approach we adopt here (Since the P-distribution can be very singular we recommend that a reader might want to use the equivalent definition: a state is nonclassical if any phase space distribution $\mathcal{P}(1-s)$ of Equation (1) has negative values.).

So, the Glauber-Sudarshan distribution P by itself, in principle, successfully discriminates between classical and nonclassical states, but practically speaking, it does not solve our discrimination problem. Both, analytically [6] and numerically [11] P is hard to determine. Typically, it is so singular that it is extremely difficult to analyze or even construct P [12,13].

Useful alternatives to Glauber-Sudarshan's P distribution are Wigner's, W [14], and Husimi's, Q [15], distributions. Both are generated through Gaussian convolutions of the P-distribution (see Equation (1) in Section 2), and therefore more well-behaved functions than P is. But this convolutional smearing washes out details effectively destroying information about the negative values of P such that neither W nor Q can serve as a function to certify that a state is nonclassical. Q has lost the ability to develop negative values and also in the case of W too much Gaussian-convolution-induced smearing has happened: $W(x, p) < 0$ is sufficient to certify that a state is nonclassical but it is not necessary for a state to have regions with negative values of W to be nonclassical [3,12].

This situation has led to many different attempts to devise a reliable strategy for certification of the nonclassical nature of a state (see [3,16–27] and the many references therein). Similarly, a plethora of measures of nonclassicality [16,20,26,28–52] have been devised. We list these here because measures are not always distinguished from certifications [16,20] and also because ξ lends itself to be converted to measures with appealing properties [2,53].

None of these approaches faithfully discriminate between classical and nonclassical states: faithful discrimination should always correctly detect and certify that a state is nonclassical, and it should certify only classical states as classical. This ability remains elusive [2].

Nonclassicality certification approaches devised so far, typically, mislabel some states that have weak quantum features as classical [26,43]. In the context of phase space-based approaches, this is often due to the fact that mapping functions from the potentially very singular P-distributions to 'better behaved' ones, such as Wigner and Husimi distributions, washes out the very negative-valued features we intend to detect, see Figure 1.

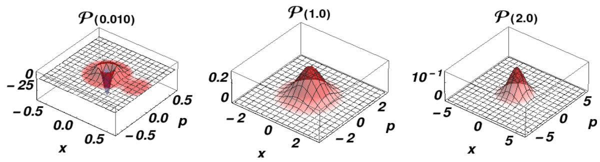

Figure 1. T-parametrized distributions (see Equation (3)) for state (11), with the value $w_1 = 0.0122$, from left to right: $P = \mathcal{P}(T = 0)$ (approximate), $W = \mathcal{P}(T = 1)$ and $Q = \mathcal{P}(T = 2)$.

The phase space-based certification functional, $\tilde{\xi}(S)$ of Ref. [3], or $\varrho\mathcal{L}(\varrho)$ of Ref. [43], use differences of distributions $\tilde{\mathcal{P}}(S)$, these differences are meant to display areas where $\tilde{\xi}(x, p; S) < 0$ (see Section 2 below), if and only if P describes a nonclassical state.

To always work, such certification functionals would effectively have to perform Gaussian deconvolution to retrieve the information held by the P-distribution. But Gaussian de-convolutions are mathematically and computationally hard to achieve and cannot in general be performed by using such difference approaches; this can also be illustrated by the fact that Ref. [54] shows that in the general case an infinite hierarchy of certification functionals is required to faithfully discriminate between classical and nonclassical states. To find a single, general and simple certification functional remains the 'holy grail' in the field of nonclassicality certification and nonclassicality measures. Yet, the approaches of Ref. [3] and its generalization, $\mathcal{S}$, introduced in this work, are surprisingly versatile and powerful.

In Section 4 we show that they can faithfully certify all Gaussian states, mixed and pure. We emphasize that this implies that $\tilde{\xi}(0)$ (based on the Wigner distribution) can faithfully certify all pure states because Hudson's theorem [55] guarantees that the Wigner distribution of non-Gaussian pure states has negative regions [3].

In Section 6 we furthermore demonstrate, that, even for very impure non-Gaussian states, ξ and $\mathcal{S}$ give impressive results and that $\mathcal{S}$ generalizes ξ.

2. Smeared Phase Space Distributions

Smeared phase space distributions, $\tilde{\mathcal{P}}(S)$, expressed in terms of a reference distribution $\tilde{\mathcal{P}}(R)$ via Gaussian convolutions, are parameterized by a smearing convolution parameter S and defined as [10]

$$\tilde{\mathcal{P}}(x, p; S) = \frac{1}{R-S} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dp' \quad \tilde{\mathcal{P}}(x', p'; R) \times \frac{1}{\pi} \exp\left\{ \frac{-1}{R-S} \left[(x-x')^2 + (p-p')^2 \right] \right\}, \quad (1)$$

with allowed parameter values $1 \geq R$ and $R > S$; we see that the Glauber-Sudarshan distribution $P(x,p) = \tilde{\mathcal{P}}(x,p;R=1)$ (whilst taking a suitable limit of $S \uparrow 1$) serves as a starting point of sorts.

Expression (1) connects all S-parameterized phase space distributions $\tilde{\mathcal{P}}(S)$. The $\tilde{\mathcal{P}}(S)$ are real-valued and with dropping values of S increasingly more smeared convolutions of P. They include as specially named cases, the "exceedingly singular" [6] Glauber-Sudarshan P-distribution, as well as Wigner's distribution $W \equiv \tilde{\mathcal{P}}(0)$ [14] and Husimi's distribution $Q \equiv \tilde{\mathcal{P}}(-1)$ [15]. Notably, W can have negative values in phase space but is a well-behaved function [56] and stands out as the only distribution $\tilde{\mathcal{P}}(S)$ whose projections exactly give a state's quadratures. The smearing in Q is such that it constitutes the largest possible value of S (giving rise to the minimal amount of smearing) which is sufficient to guarantee that $Q(x,p) \geq 0$ throughout phase space for any nonclassical state ϱ.

Symmetries of P-Distribution Nonclassicality

Since phase space displacements and rotations do not change the quantum character of a state, we follow the quantum optics' community convention that nonclassicality certification functionals should obey the following symmetries [3]:

Displacement symmetry in phase space allows us to always shift states (e.g., to the phase space origin).

Rotation symmetry allows us to rotate states (such that they align with coordinate axes).

3. Bohmann and Agudelo's Phase Space-Based Certification Functional

The explicit form of the family of S-parameterized certification functionals devised in Ref. [3] is

$$\tilde{\xi}[\tilde{\mathcal{P}}](S) = \tilde{\mathcal{P}}(S) - 4\pi(1-S)\,(\tilde{\mathcal{P}}(2S-1))^2, \tag{2}$$

for some details see Appendix A.

For transparency, in subsequent calculations we prefer to employ the *positive T-parametrization*: $T = 1 - S$, $T \geq 0$ in Equation (1) whilst dropping the tilde symbols, used above. Hence, $\tilde{\mathcal{P}}(S) \mapsto \mathcal{P}(1-S) = \mathcal{P}(1-(1-T)) = \mathcal{P}(T)$. With this convention, the Glauber-Sudarshan distribution is $P \equiv \mathcal{P}(0)$, the Wigner distribution $W \equiv \mathcal{P}(1)$ and Husimi's distribution $Q \equiv \mathcal{P}(2)$, see Figure 1.

In expression (2) $\tilde{\mathcal{P}}(2S-1) \mapsto \mathcal{P}(1-[2(1-T)-1]) = \mathcal{P}(2T)$ which, in *positive T-parametrization*, becomes

$$\xi[\mathcal{P}](T) = \mathcal{P}(T) - 4\pi\,T\,(\mathcal{P}(2T))^2. \tag{3}$$

Bohmann and Agudelo showed in Ref. [3] that all classical single-mode states obey $\xi[\mathcal{P}](x,p;T) \geq 0$. This is to be understood as the statement that the occurrence of negative values $\xi[\mathcal{P}](x,p;T) < 0$ somewhere in phase space (x,p) is *sufficient* to show that a state is nonclassical [3]. What is missing for ξ of [3], however, is *necessity*: to be *faithfully discriminating* it would be *necessary* that whenever a state is nonclassical we find that $\xi(x,p;T) < 0$ in phase space somewhere. In Section 6 below, we will show that there are states $\mathcal{P}$ which are (weakly) nonclassical and yet for these states $\xi[\mathcal{P}] \geq 0$, across phase space.

4. Generalizing from ξ to $\mathcal{S}$

$\xi[\mathcal{P}](x,p;T)$ is a functional of fixed form independent of context such as the type of system or the type of state the system is in. $\xi[\mathcal{P}]$ applies to all states of one-dimensional continuous-variable systems, irrespective of whether the state is pure or mixed and whether it has small, large, or even macroscopic [57] extent, irrespective of context, hamiltonian, environment or measurement procedures [1], in this sense ξ is *universally* applicable for all cases and to all states of single-mode systems.

We now generalize ξ by constructing the certification functional, $\mathcal{S}$, requiring that it displays negative values certifying a state's nonclassicality, somewhere in phase space. Whereas the derivation for ξ presented in Ref. [3] relies on a Chebyshev integral inequality which only applies to one dimension, our approach is guided by the idea that coherent states form the dividing line between classical and nonclassical states. For ξ applied to coherent states, α, this is represented by the *zero-identity* $\xi_\alpha(x,p) \equiv 0$. We thus seek to find a simple way to map coherent states onto the vanishing function $\mathcal{S}_\alpha(x,p) \equiv 0$.

Consider the T-parameterized phase space distribution of a general Gaussian state

$$\mathcal{P}_g(A,B;T) = e^{-\frac{(x-x_0)^2}{A} - \frac{(p-p_0)^2}{B}} / (\pi\sqrt{AB})\,. \tag{4}$$

where, using rotation symmetry we assumed alignment with the coordinate axes and using displacement symmetry we move its center to the origin ($x_0 = p_0 = 0$).

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$\mathcal{P}_g(A,B;T)$ has respective widths A and B; in this representation Heisenberg's uncertainty principle reads $AB \geq T^2$. If one of these widths, say $A < T$, is below that of the vacuum state, $\mathcal{P}_0(T) = e^{-\frac{x^2}{T}-\frac{p^2}{T}}/(\pi T)$, then that state with $A < T$ is squeezed and thus nonclassical [12].

Inspired by Ref. [3], we form $\mathcal{P}_g(x,p;T+\Delta T) = e^{-\frac{x^2}{A+\Delta T}-\frac{p^2}{B+\Delta T}}/(\pi\sqrt{A+\Delta T}\sqrt{B+\Delta T})$ from the convolution of $\mathcal{P}_g(T)$ with $e^{-\frac{x^2}{\Delta T}-\frac{p^2}{\Delta T}}/(\pi\Delta T)$.

$\mathcal{P}_g(T+\Delta T)$ is wider and its maximum lower than that of $\mathcal{P}_g(T)$.

To compensate for this increase in width we exponentiate $\mathcal{P}_g(T+\Delta T)$ by $(1+\Delta T/T)$ which maps the width of coherent states back to T. Subsequent multiplication by the factor $(T+\Delta T)^{\frac{T+\Delta T}{T}}((A+\Delta T)(B+\Delta T))^{-\frac{T+\Delta T}{2T}}$ increases the height of a coherent state represented by $\mathcal{P}_g(T+\Delta T)^{(1+\Delta T/T)}$ back to that of a coherent state represented by $\mathcal{P}_g(T)$. Forming the difference gives us

$$\begin{aligned}
&\mathcal{S}[\mathcal{P}_g(A,B;T)](x,p;T,\Delta T) \\
&= \frac{e^{-\frac{x^2}{A}-\frac{p^2}{B}}}{\pi\sqrt{AB}} - \frac{e^{\left(-\frac{x^2(T+\Delta T)}{(A+\Delta T)T}-\frac{p^2(T+\Delta T)}{(B+\Delta T)T}\right)}}{\pi T} \times (T+\Delta T)^{\frac{T+\Delta T}{T}}((A+\Delta T)(B+\Delta T))^{-\frac{T+\Delta T}{2T}}.
\end{aligned} \tag{5}$$

Inserting a coherent state ($A = B = T$), yields $\mathcal{S}[\mathcal{P}_g(T,T;T)](T,\Delta T) \equiv 0$, as desired.
Re-expressed in terms of T-parameterized general states $\mathcal{P}$, expression (5) gives us the certification functional

$$\mathcal{S}[\mathcal{P}](x,p;T,\Delta T) = \mathcal{P}(x,p;T) - \frac{\pi^{\frac{\Delta T}{T}}}{T}\left[(T+\Delta T)\,\mathcal{P}(x,p;T+\Delta T)\right]^{\frac{T+\Delta T}{T}} \tag{6}$$

$$= \mathcal{P}(x,p;T) - N_0(T)\left(\frac{\mathcal{P}(x,p;T+\Delta T)}{N_0(T+\Delta T)}\right)^{\frac{T+\Delta T}{T}}, \quad \text{where } N_0(T) = \tfrac{1}{\pi T}. \tag{7}$$

We emphasize that $\mathcal{S}[\mathcal{P}]$ should only be applied to *nonnegative states*, namely, states which obey $\mathcal{P}(x,p;T+\Delta T) \geq 0$, since non-integer exponents $(T+\Delta T)/T$ can generate complex values if a state has negative values somewhere in phase space.

This restriction to nonnegative states is an acceptable price to pay for two reasons: Firstly, *negative states*, namely those for which $\mathcal{P}(x,p;T+\Delta T) < 0$ somewhere in phase space, are already certified to be non-classical by displaying their negative values. Secondly, for the special case of $T = \Delta T$ we have that $\mathcal{S}[\mathcal{P}](T,T) = \xi(T)$ with the associated exponent $(T+\Delta T)/T = 2$ which allows us to form the difference in Expression (6) as desired, also for negative states, and, as we show in Section 6, $\xi(T)$ tends to work nearly as well as $\mathcal{S}[\mathcal{P}](T,\Delta T)$.

When $\Delta T = T$, $\mathcal{S}(T,T) = \xi(T)$, see Equation (3). Thus, $\mathcal{S}(T,\Delta T)$ generalizes $\xi(T)$, see Section 6.

Gaussian States

A particular strength of the certification function ξ of Ref. [3] is its ability, without fail, to detect the nonclassical [12] squeezing of fluctuations below the vacuum level in *all Gaussian states*; irrespective of whether the states are pure or impure states. This follows from a slight modification of the arguments given when deriving Equation (6):

When deriving Equation (6), we were guided by the principle that we aim to map pure coherent states, the states that define the boundary between classical and quantum states, onto the identically vanishing function $\mathcal{S} \equiv 0$. It is plausible then that $\mathcal{S}$ maps nonclassical Gaussian states to functions which in parts of phase space fall 'below' this zero-function, forming negative regions, whereas $\mathcal{S}$ maps classical states to functions greater than zero everywhere in phase space, see Section 5.1.

Let us first consider nonclassical Gaussian states:

For Gaussians which (in their squeezed coordinate) decay faster than vacuum states, the spreading when mapping $\mathcal{P}(T)$ to $\mathcal{P}(T+\Delta T)$ is not fully compensated for by the narrowing due to the exponentiation by $E = T + \Delta T/T$: therefore, in the squeezed direction, the $\mathcal{P}(T)$-term in Equation (4) decays faster than the $\mathcal{P}(T+\Delta T)$-term and, upon forming their difference, $\mathcal{S}$ forms regions with negative values on the squeezed 'narrow flanks' of the phase space distribution (for illustration see Figure 2 and also Figure 1 of [3]). $\mathcal{S}[\mathcal{P}_g(A,B;T)](x,p;T,\Delta T)$ faithfully certifies suppression of fluctuations to nonclassical levels (here, below the level of vacuum fluctuations [12]) for all Gaussian states.

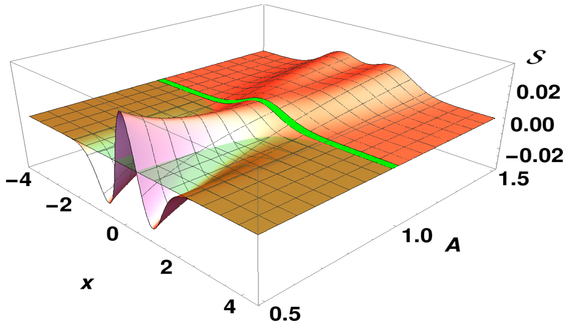

Figure 2. $\mathcal{S}[\mathcal{P}_g(A, B=2; T=1)](x, p=0; T=1, \Delta T=\frac{3}{4})$ based on the Wigner distribution ($T=1$) of the impure squeezed states $\mathcal{P}_g(A, B=2; T=1)$ as a function of the squeezed coordinate x with varied x-squeezing width A and fixed p-coordinate anti-squeezing width $B=2$. The green line at $T=1$ divides values of $A<T$ where the nonclassical character of the state is always certified by the occurrence of negative values of $\mathcal{S}$, whilst for $A \geq T$, $\mathcal{S} \geq 0$, just as desired.

5. $\mathcal{S}(T, \Delta T)$ Generalizes $\xi(T)$

Our derivation also allows us to generalize to the multimode case of K cartesian phase space modes:

$$\mathcal{S}(T, \Delta T) = \mathcal{P}\,(T) - N_0^K\,(T)\left(\frac{\mathcal{P}(T+\Delta T)}{N_0^K(T+\Delta T)}\right)^{\frac{T+\Delta T}{T}}. \tag{8}$$

This can be confirmed by insertion of the K-mode vacuum state, $(\mathcal{P}_{|0\rangle\langle 0|}(T))^{\otimes K}$ into (8) yielding $\mathcal{S}(\boldsymbol{x}, \boldsymbol{p}) \equiv 0$ (where bold fonts, $\boldsymbol{x}$, describe K-dimensional vectors).

The negative values of Gaussian states form just like in the single-mode case (4), see Figure 2, such that the results of the single-mode case carry over to the multi-mode case. This generalization of the single-mode expression (7) to the multi-mode expression (8) is the main result of this work.

5.1. $\xi(T)$ and $\mathcal{S}$ Can always Identify Classical States

$\mathcal{S}$, just like ξ, without fail, certifies classical states (incoherent mixtures of Glauber coherent states) as classical, since the nonlinear terms in Equation (6) for a mixture between states $\mathcal{P}_1$ and $\mathcal{P}_2$, parameterized by $(0 < w < 1)$, form the positive concave difference function $\mathcal{S}[w\mathcal{P}_1 + (1-w)\mathcal{P}_2] - (w\mathcal{S}[\mathcal{P}_1] + (1-w)\mathcal{S}[\mathcal{P}_2]) \geq 0$. This renders mixtures $\mathcal{S}$ more positive than the mixture's individual contributions, as desired.

We now show this in two steps.

By construction $\mathcal{S} \equiv 0$ for pure coherent states, see Equation (5). Since classical states are incoherent mixtures of Glauber coherent states, we first establish that a coherent state contributing to a mixture with below-unit weight w $(0 < w < 1)$ adds a positive contribution to $\mathcal{S}[w\mathcal{P}_0] > 0$: recall that Equation (6) implies that $\mathcal{S}[w\mathcal{P}_0](T, \Delta T) = w \times \left(\mathcal{P}_0\,(T) - w^{\frac{\Delta T}{T}} N_0\,(T)\left(\frac{\mathcal{P}_0(T+\Delta T)}{N_0(T+\Delta T)}\right)^{\frac{T+\Delta T}{T}}\right) > w\,\mathcal{S}[\mathcal{P}_0] \equiv 0$, since $w^{\frac{\Delta T}{T}} < 1$.

$\xi(T)$ and $\mathcal{S}$ Are Concave Functionals of the Mixing of States

In this second step, we now generalize this result by showing, for all states, mixtures render $\mathcal{S}$ more positive than the mixture's individual contributions. This reduces the telltale negative values of nonclassical, nonnegative states for mixtures.

For nonnegative states $\mathcal{S}$ is concave in the mixing weight w of the mixed state $w\mathcal{P}_1 + (1-w)\mathcal{P}_2$ (with $0 < w < 1$):

$$\Delta\mathcal{S}(T, \Delta T) = \mathcal{S}[w\mathcal{P}_1 + (1-w)\mathcal{P}_2] - \Big(w\mathcal{S}[\mathcal{P}_1] + (1-w)\mathcal{S}[\mathcal{P}_2]\Big) \tag{9}$$

$$\propto -\left(w\mathcal{P}_1' + (1-w)\mathcal{P}_2'\right)^E + \left(w(\mathcal{P}_1')^E + (1-w)(\mathcal{P}_2')^E\right) > 0; \tag{10}$$

here we have used the abbreviations $\mathcal{P}' = \mathcal{P}(T+\Delta T)$ and $E = (T+\Delta T)/T > 1$ and we assumed that $(\mathcal{P}_1(x,p;T) \geq 0, \mathcal{P}_2(x,p;T) \geq 0)$, and $\mathcal{P}_1 \neq \mathcal{P}_2$, and that inequality (10) obeys a Jensen's inequality for concave functions, making $\Delta\mathcal{S}$ concave under mixing.

These results imply that $\mathcal{S} < 0$ (just like $\xi < 0$) certifies that a state is nonclassical.

Because multimode classical states can be described by sums of (unentangled) product states these result carry over to the multimode case of Equation (8).

5.2. $\xi(T)$ *Can Be Faithfully Discriminating for Some Classes of States*

Additionally to Gaussian states, there are a number of cases when the certification functional $\xi(T)$ of Ref. [3] is faithfully discriminating.

For example the mixed state $\varrho = \cos(w)^2|0\rangle\langle 0| + \sin(w)^2|1\rangle\langle 1|$, to first order in w, yields $\xi(T) = w\, e^{-\frac{p^2+x^2}{T}}\left(p^2+x^2-T^2\right)/(\pi T^3)$: even for vanishing nonclassical contributions ($w \downarrow 0$), ξ assumes negative values near the origin, faithfully detecting the presence of the single photon Fock state $|1\rangle\langle 1|$ in this mixture. ξ similarly detects the presence of such mixtures if Fock states of order higher than one are used, and also if those are displaced with respect to the dominant (classical) vacuum state $\varrho = |0\rangle\langle 0|$.

For the coherent superposition state $|\psi\rangle = \cos(w)|0\rangle + \sin(w)|1\rangle$, between vacuum and single-photon Fock state, we get, to second order in w, that $\xi = w^2\left(p^2-x^2\right) \times \exp[-\frac{p^2+x^2}{T}]/(\pi T^3)$. Again, if Fock states of higher order are combined with the dominant (classical) vacuum state and also if they are displaced with respect to each other, in all these cases, $\xi < 0$ in phase space somewhere, certifying the presence of (vanishingly small) nonclassical contributions to the state.

5.3. $\xi(1)$ *Always Detects Non-Gaussian Pure States*

Wigner distributions of non-Gaussian pure states have negative regions [55]. Because $Q \geq 0$ this implies that, for $T = 1$, $\xi(1) = W - 4\pi Q^2 < 0$ when $W < 0$.

We believe that pure states might exist for which more smeared versions of $\xi(T)$ and $\mathcal{S}(T, \Delta T)$ with $T > 1$ fail to certify nonclassicality.

6. A Case Where First $\xi(T)$ and then $\mathcal{S}(T, \Delta T)$ Fails to Be Discriminating

In order to investigate a scenario where $\xi(T)$ fails to be discriminating, we consider a state with a mixed positive background distribution consisting of a vacuum state, $W_0(x,p)$, overlapping with a coherent state, $W_0(x-\frac{19}{40},p)$, and a (nonclassical) single-photon Fock state, W_1, which carries a small weight $w_1 \approx 1\%$, namely:

$$W(w_1, s) = w_1 W_1(x,p) + (1-w_1)\Big(W_0(x,p)\cos^2(s) + \sin^2(s) W_0(x-\tfrac{19}{40},p)\Big), \tag{11}$$

with $s = \frac{\pi}{20}$. This state is nonclassical, see Figure 1. Figure 3 shows that for small values of ΔT the certification functional $\mathcal{S}(T, \Delta T)$ (6) can be slightly more discriminating than $\mathcal{S}(T,T) = \xi(T)$ of Ref. [3], but both fail when the single-photon state's weight w_1 drops further, below 0.0122. This is due to inequality (10).

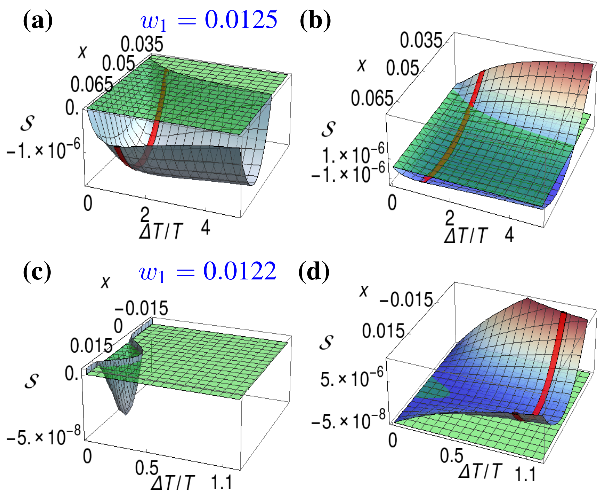

Figure 3. Display of $\xi(T=1)$ (red bands) and $\mathcal{S}(T=1,\Delta T)$ (color sheets) for Wigner's distribution, $\mathcal{P}(T=1)$. We consider the weakly nonclassical state (11): for values of $w_1 = 0.0125$ (top panels (**a**,**b**)) both certification functionals $\xi(x, p=0; T=1)$ of Ref. [3] and $\mathcal{S}(x, p=0; T=1, \Delta T)$ of Equation (6) drop to negative values (below green transparent zero-value sheets), certifying the nonclassicality of state (11), see Figure 1. But for the value $w_1 = 0.0122$ (bottom panels (**c**,**d**)), whereas the certification functional $\mathcal{S}(x, 0; T=1, \Delta T)$ still shows negative values, ξ fails to certify that this state, depicted in Figure 1, is nonclassical (red band does not dip below green zero-sheet). For slightly smaller values of $w_1 < 0.0122$ (not shown) $\mathcal{S}$ fails as well. In short, $\mathcal{S}$ can be more discriminating than ξ but only very moderately so.

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The example state (11) appears contrived and Figure 3 reports only a very modest increase in sensitivity of our certification functional $\mathcal{S}$ over ξ. Instead, of taking this point of view we advise to adopt the following perspective:

Figure 3 firstly confirms that $\mathcal{S}(T, \Delta T)$ really differs from $\xi(T)$, hence, $\mathcal{S}$ truly generalizes ξ of Ref. [3].

Secondly, it illustrates how very discriminating between classical and nonclassical states $\mathcal{S}$ and ξ really are, namely, a roughly 1.2% nonclassical contribution to state (11) is detected; also see Section 5.2.

And, thirdly, when we apply $\mathcal{S}$, typically little is gained by using $\Delta T \neq T$ in $\mathcal{S}(T, \Delta T)$ instead of sticking with $\Delta T = T$ such that $\mathcal{S}(T, T) = \xi(T)$. Therefore, sticking with ξ and its multimode generalization $\mathcal{S}(T, T)$ of Equation (8) is typically good enough.

7. Conclusions

We have generalized the nonclassicality certification functional of Ref. [3]. We have proved that the nonclassicality functionals (7) always identify classical states correctly, but for weakly nonclassical states they can fail, incorrectly mislabeling these as classical. We proved that these nonclassicality certification functionals always identify mixtures of positive-valued states as the functionals are concave in the mixing probabilities of the states' constituents, see Equation (10). Our generalized certification functionals' transparent form hopefully opens up the door for further improvements eventually resulting in a general and useful discriminating approach to nonclassicality certification. Its transparent form has allowed us to generalize to multimode settings, see Equation (8).

Author Contributions

O.S. and R.-K.L.: Conceptualization, methodology, writing—reviewing and editing; O.S.: Software, writing—original draft preparation, visualization. Both authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

Appendix A. Smeared Unbalanced Certification Functionals $\tilde{\xi}(S, k)$

In Ref. [3] the two-parameter families of nonclassicality certification functions

$$\tilde{\xi}(S,k) = \tilde{\mathcal{P}}(S) - \frac{\pi(1-S)}{k(1-k)}\, \tilde{\mathcal{P}}(S_k)\, \tilde{\mathcal{P}}(S_{1-k}), \tag{A1}$$

with $S_k = 1 - \frac{1-S}{k}$, were introduced. Here the parameter $S \leq 1$ selects the reference distribution $\tilde{\mathcal{P}}(x, p; S)$, see Equation (1). Expression (A1) can be useful for some field correlation measurements [3].

The parameter $k \in (0, 1)$ linearly interpolates between different exponents in the distributions $\tilde{\mathcal{P}}(S_k)\, \tilde{\mathcal{P}}(S_{1-k}) = \tilde{\mathcal{P}}(S_{1-k})\tilde{\mathcal{P}}(S_k)$ which are smeared with respect to $\tilde{\mathcal{P}}(S)$. Note that, because of the symmetry in k with respect to the midpoint $k = \frac{1}{2}$, effectively $k \in (0, 0.5]$.

Also, for vanishing k the limit is $\lim_{k\downarrow 0} S_k = -\infty$ and $\lim_{k\downarrow 0} S_{1-k} = S$ and we get, according to Equation (1), an infinitely spread-out distribution $\tilde{\mathcal{P}}(-\infty)$ (giving us an essentially constant background, which vanishes with increasing smearing values $|S|$) multiplied with $\tilde{\mathcal{P}}(S)$: in short, for vanishing k, in phase space regions where it assumes negative values, expression (A1) rises towards zero.

As a function of k, negative minimum values $\tilde{\xi}_-(S_k)$ with the greatest magnitude, occur at the balanced mid-point, $k = \frac{1}{2}$, providing the greatest contrast when applying $\tilde{\xi}$ [2]. This is why here, without loss of generality, we confine our discussion of ξ to the form of Equation (2) only.

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