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Particle-Wave Critical Velocities and Wave Profiles for Power-Law Energies

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Abstract: The dual particle-wave nature of matter is a guiding principle of modern quantum theory, formally arising from Planck's energy $e = h\nu$, de Broglie's momentum $p = h/\lambda$ and Einstein's energy from special relativity, where h is the Planck constant, ν and λ are the frequency and wavelength respectively of the wave with wave speed $w = \nu\lambda$. Motivated by the symmetry in special relativity between sub-luminal and superluminal motions, these fundamental relations can be generalised to a family that is characterised by a second fundamental constant $h^{'}$ and underpinned by Lorentz invariant power-law particle energy-momentum expressions. By Lorentz invariance, we refer to invariance under the combined special relativistic space-time and energy-momentum transformations. The power-law expressions are important since they share the same relationship with the extension of the Planck-de Broglie energy-momentum relations as does Einstein's energy expression with the Planck-de Broglie relations which are included through the special case $h^{'} = 0$. For $h^{'} \neq 0$, the new relations allow the determination of the stationary wave energy and momentum states and leading to a table for the critical particle and wave velocities in terms of the power law parameter. We further develop these new relations to provide an explicit profile for a Lorentz invariant wave associated with an elementary particle. Overall, we formulate a particle-wave model allowing two distinct energy and momentum states, corresponding to particles and waves, and we propose a mechanism for the particle-wave transition.

Keywords: special relativity; Lorentz invariance; power-law energy momenta; critical particle and wave velocities; Lorentz invariant wave profile

MSC: 70-10; 83A99

1. Introduction

de Broglie held the view that both elementary particles and waves may simultaneously exist. Quantum theory assumes a collapse of normal wave function at the point of measurement and that slowing light waves may progressively become photons travelling at the speed of light and then sub-luminal electrons. Here we formulate a model in which energy and momentum enjoy two distinct formulations corresponding to particle and wave states. Further, for a slowing superluminal wave achieving its minimum or zero-point energy and which can no longer exist in the wave state, we speculate that the entire minimum energy is transferred to the particle state. This assumption is sufficient to guarantee energy and momentum equality from the two formulations.

We emphasise from the outset that the overall approach adopted here and in [1–8], although set within a Lorentz invariant framework, differs from conventional special relativity and conventional quantum mechanics, and that at present there is no experimental evidence to support this approach. However, we suggest that the approach adopted in [6] and here is the more fundamental, since the former is inclusive of the latter. Both special relativistic particle mechanics and quantum mechanics, in the form of Schrödinger's quantum wave theory, are purposely embraced within the theory expounded in [6], in the sense that we may identify the origins of both theories. Within the proposed model, both Newton's applied force as the rate of change of particle momentum and the Schrödinger operator relations for energy and momentum and the wave equation are intentional outcomes of the theory. Normally, Schrödinger's single particle wave equation is derived by first postulating that the wave function satisfies the classical wave equation and then, based on a series of approximations, proceeding to the Schrödinger wave equation and the formulation of an eigenvalue problem, operations which within the proposed theory are systematic.

The major intent of the proposed theory is to start afresh, but with the recognition that both special relativistic particle mechanics and quantum mechanics are fundamental and must be a key component of any new theory. When examined in this manner, it eventuates that special relativistic particle mechanics and Newton's second law address the "space" characteristics while quantum mechanics addresses "time" characteristics. Further, while research within conventional special relativity remains active in areas such as; Lorentz violation, symmetrical relativity, double special relativity and deformed special relativity, there are currently no directly related research developments to the model pursued here.

The motivation for this paper, and in particular for the generalised Planck-de Broglie relations (4) given in [7], rests on the understanding of the symmetry existing in special relativity between sub-luminal and superluminal motion presented in [9]. That is, according to the usual special relativistic rules, if with respect to some fixed inertial frame, we consider the two sets comprising all sub-luminal Lorentz frames and all superluminal Lorentz frames, then as far as the rules of special relativistic velocity addition are concerned, there is no objective means of deciding to which of the two sets do we belong? Of course, superluminal motion is controversial, but within the limitations of a single space dimension, there is a measure of consensus, which predicts a symmetry between the two sets of sub- and superluminal Lorentz reference frames. That is, by velocity manipulation alone, we are not able to detect whether we are all moving at a speed less than or greater than the speed of light? These non-trivial issues, first elaborated in [9], are summarized in Appendix A. Understanding this symmetry gives rise to the notion that many key physical parameters might well originate from both sub- and super contributions.

The purpose of the present paper is to further understand the particle-wave transition through the generalised Planck-de Broglie energy-momentum relations (4) ([7]) which involve a new fundamental constant $h^{'}$ that is additional to the Planck constant h. These new relations are underpinned by power-law energy-momentum expressions (1) ([3]) which involve the non-dimensional power-law parameter $\kappa = h^{'}/h$ and enable an analysis of the stationary wave energy and wave momentum states leading to a new table for the critical particle and wave velocities in terms of κ. In the absence of the power-law relations (1), and in particular the power-law parameter κ, such an analysis is simply not possible. Here, Lorentz invariance refers to the invariance under the combined space-time Lorentz transformations (A1) and the Lorentz energy-momentum transformations (A2).

There is a widespread belief that the phase angle of a plane wave should be Lorentz invariant and the topic has generated much interest (see [10,11] and references therein). Here, using only functions of Lorentz invariants, we employ an argument comparable to "analytical continuation" to determine Lorentz invariant analytical expressions for both the amplitude and phase of the wave (24). This ad hoc argument gives rise to an entirely feasible wave phase, while the "matching" wave amplitude has either exponential growth or decay. In the absence of assuming a mechanical theory, such as quantum mechanics or another dynamical theory, the use of Lorentz invariants provides a very powerful approach to understand the intricacies and the often unexpected behaviour of nonlinear phenomena.

The distinction between waves and particles breaks down at the quantum level and all matter may exhibit both particle-like and wave-like properties. This particle-wave duality means that matter can behave both as a localised particle or as a spreading wave, depending on the circumstance and how it's observed. Particles occupy a specific location in space and are characterised by energy $e = mc^2$ and momentum $p = mu$, where m is the mass and $u \leq c$ is the particle velocity. On the other hand, waves spread out over space and are characterised by frequency ν, wavelength λ, wave velocity $w = \lambda\nu \geq c$ and amplitude A. The numerous particle and wave symbols in play describing particle and wave velocities, wavelength, frequency and amplitude can be found in Table A1 in Appendix B.

Phenomena associated with light may register either from a stream of discrete elementary particles, called photons, or by the presence of diffraction patterns which are characteristic of waves. The Davisson-Germer experiment [12] demonstrates electron diffraction by creating interference patterns for a beam of electrons even when sent through the slits singly. Using a variety of techniques including nano-mechanical gratings, ref. [13] demonstrate the wave-like behaviour and diffraction of fullerenes when encountering obstacles or apertures with comparable dimensions to their wavelength.

The formal relations underpinning the particle-wave duality are the Planck-de Broglie relations for energy and momentum of an elementary particle which apply in conjunction with Einstein's special relativistic mass-energy expression. de Broglie [14–16] was first to supplement Planck's particle energy expression $e = h\nu$ with the equation for particle momentum $p = h/\lambda$ to establish the dual particle-wave nature of matter for all particles, where h is the Planck constant, ν and λ are the frequency and wavelength respectively of the associated wave having a wave speed $w = \nu\lambda$. These three basic laws lie at the heart of quantum theory, particle physics, and calculations involving interacting elementary particles. They have become fundamental pillars of the physics discipline and critical components of modern physics thinking. The particle-wave duality has many practical applications in technologies such as electron microscopy and a proper understanding is crucial for comprehending phenomena at the atomic and subatomic level. A brief history of the background to de Broglie's particle-wave duality is given by [17].

In this paper, we describe the particle motion and the associated wave by means of Lorentz invariant power-law energy-momenta relations first derived in [3]. These relations are subsequently exploited in [7] to deduce the generalised Planck-de Broglie relations (4) for energy and momentum of an elementary particle which involve a new fundamental constant h' that is additional to the Planck constant h. Accordingly, here we merely summarise these developments, and we refer the reader to the original papers. The new elements presented here include Table 1 for $\kappa \neq 0$ and the explicit formulae for the phase and amplitude of the wave function (24), and noting in particular that a table such as Table 1 cannot be constructed in the absence of the power-law energy relations (1) and their connection with the generalised Planck-de Broglie relations (4).

Table 1. Critical particle and wave velocities for $\kappa \neq 0$.

Particle $\frac{u}{c} = \frac{\kappa w/c+1}{w/c+\kappa}$ $0 \leq u/c \leq 1$	**Wave** $\frac{w}{c} = \frac{1-\kappa u/c}{u/c-\kappa}$ $1 \leq w/c \leq \infty$	**Description**
$u/c = 0$	$w/c = -1/\kappa$	Particle velocity zero.
$u/c = 1$	$w/c = 1$	Coincident velocities.
$u/c = \kappa$	$w/c = \infty$	Wave velocity infinite.
$u/c = \frac{1}{\kappa}$	$w/c = 0$	Zero wave velocity.
$u/c = -\kappa$	$w/c = -\frac{1}{2}\left(\kappa + \frac{1}{\kappa}\right)$	Energy stationary.
$u/c = -\frac{1}{\kappa}$	$w/c = -\frac{1}{\frac{1}{2}\left(\kappa+\frac{1}{\kappa}\right)}$	Momentum stationary.

Following de Broglie [18,19], we assume the simultaneous existence of both an elementary particle in motion and a wave which serves to both transfer energy and guide the particle, such that the group velocity of the wave coincides with the particle velocity. Very little else may be said with confidence regarding the particle-wave transition, and in particular, the extent of energy transfer. As previously noted, quantum theory assumes a collapse of normal wave function at the point of measurement and slowing waves may progressively become sub-luminal elementary particles. Looking to the proposed model structure given here for some guidance, we suggest that the energy of the nascent particle is the zero-point energy of the collapsing wave.

On face value, apart from the formal requirement that the group and particle velocities coincide, the particle and wave worlds remain separate independent entities such that the rest energy e_0 is the key parameter fixing the particle motion, while the "zero-point energy" $\mathscr{E}_0$ (lowest attainable energy for a quantum mechanical system) is perhaps the corresponding parameter which fixes the wave. In this paper, we adopt the same exponent κ arising in the power-law energy-momenta relations for both the particle and wave motions. However, in the absence of an assumption connecting the particle and waves, there are no formal requirements that this is necessarily the case, and it is perfectly conceivable that the particle and wave motions may be characterised by distinct exponents κ_1 and κ_2. The assumption $\kappa_1 = \kappa_2$ combined with the requirement that the entire minimum wave energy (zero point wave energy) of the collapsing wave becomes that of the forming elementary particle is sufficient to deduce that the particle and wave energies are coincident.

In the next section we summarise the Lorentz invariant power-law energy-momenta relations (1) from [3] which we suppose apply to an elementary particle. In the following section, we summarise the generalisation of the Planck de Broglie relations (4) from [7]. Assuming that the group velocity of the de Broglie wave coincides with the particle velocity, these relations are underpinned by the power-law energy-momentum relations in the same manner as Einstein's energy underpins the conventional relations.

Making use of the group velocity condition $u = w - \lambda dw/d\lambda$, the two sections thereafter provide an elementary analysis to determine the nature of the stationary energy and stationary momentum states and the determination of the stationary energy $\mathscr{E}_0$ in terms of the particle parameters e_0 and the exponent κ. Using Lorentz invariants, the subsequent two sections provide an ad hoc analysis to determine explicit formulae for the phase and the matching amplitude of the de Broglie wave. The final section of the paper contains a summary and some conclusions. A synopsis of the special relativistic symmetry given in [9] is summarised in Appendix A and Table A1 in Appendix B includes a table of the major particle and wave symbols.

2. Particle Power-Law Energy Relations

In this paper, for a single space dimension x, we suppose (x, t) characterises the location x of an elementary particle at time t which is moving with velocity $u(x, t) = dx/dt$ with energy $e = mc^2$ and momentum $p = mu$, and given by the Lorentz invariant power-law relations

$$\begin{aligned} e(u) &= \frac{e_0}{(1-(u/c)^2)^{1/2}} \left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2}, \\ cp(u) &= \frac{e_0(u/c)}{(1-(u/c)^2)^{1/2}} \left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2}, \end{aligned} \tag{1}$$

where κ denotes an arbitrary non-dimensional constant and e_0 denotes the particle rest energy. These relations, obtained in [3] (see also [6], page 49), extend the Einstein energy-momentum relations which are included by the special case $\kappa = 0$. We note that unlike the Einstein energy relation, the minimum particle energy occurs for $u/c = -\kappa$ in place of the rest energy.

The relations (1) arise from the following Lorentz invariant relations

$$\frac{cp(u)}{e(u)} = \frac{u}{c}, \qquad \frac{de}{dp} = c\left(\frac{\kappa + u/c}{1+\kappa u/c}\right), \tag{2}$$

and also satisfy

$$e(u)^2 - (cp(u))^2 = e_0^2 \left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa}, \tag{3}$$

and we refer the reader to [3] for further details. We comment that Equation (3) forms the basis to generate a new family of nonlinear partial differential equations that are characterised by the constant $\kappa = h^{'}/h$ such that $\kappa = 0$ corresponds to the Klein-Gordon equation ([8]). In this paper we assume that the power-law relations (1) apply to the elementary particle and we derive corresponding expressions for the associated wave and the wave profile.

In the determination of the corresponding energy-momentum relations for the associated wave, we find the same conditions (11) as (2) are satisfied by the wave energy and wave momentum. This means that the corresponding energies and momenta can only differ by at most the same scale factor. Although this result is trivial, it is nevertheless important and most easily appreciated when couched in a simpler notation. If $y/x = Y/X$ and $dy/dx = dY/dX$, then necessarily $x = \delta X$ and $y = \delta Y$ for some constant δ. This simple result follows from the substitution $y = xv$ and $Y = Xv$ and the differential constraint then yields $dx/x = dX/X$ so that on integration $x = \delta X$ for some constant δ and $y = \delta Y$ then follows from the first condition.

3. Wave Power-Law Energy-Momenta Relations

We now suppose that there is a wave associated with the motion of the elementary particle and that both particle and wave may simultaneously exist. The results of this section were first derived in [7], so we merely summarise the essential formulae given in [7] which lead to corresponding Lorentz invariant power-law energy-momentum expressions for the wave and we refer the reader to [7] for the full background.

In [7] we extend the Planck-de Broglie formulae with the following relations for the wave energy $\mathscr{E}$ and wave momentum $\mathscr{P}$, which is assumed to be moving with wavelength λ, frequency ν and wave speed $w = \lambda\nu \geq c$, namely

$$\mathscr{E} = h\nu + \frac{h^{'}c}{\lambda}, \qquad c\mathscr{P} = \frac{hc}{\lambda} + h^{'}\nu, \tag{4}$$

where $h^{'}$ denotes a new fundamental constant with the same dimensions as h and which only comes into play when the wave is not travelling at the speed of light, since in that case $w = c$ and (4) become simply $\mathscr{E} = (h + h')\nu$ and $\mathscr{P} = (h + h')/\lambda$. Physically the relations (4) reflect the parity of space and time in special relativity. Planck's constant is a space characteristic so that $h^{'}$ might be referred as "Planck's constant in time".

These new relations are underpinned by the above power-law formulae (1) with the constant $\kappa = h'/h$ and these expressions bear exactly the same relationship with (4) as does Einstein's energy expression with the Planck-de Broglie formulae. These issues are expounded at length in [7]. Briefly, using $w = \lambda\nu$ and $h' = \kappa h$ the new relations (4) become

$$\mathscr{E} = \frac{h(w + \kappa c)}{\lambda}, \qquad c\mathscr{P} = \frac{h(\kappa w + c)}{\lambda}. \tag{5}$$

If this energy and momentum are either supporting or at least synchronised with the motion of the elementary particle which is moving with velocity u, then the relation $u/c = c\mathscr{P}/\mathscr{E} \leq c$ must continue to hold for the wave as well as for the particle, so that we have

$$\frac{u}{c} = \frac{\kappa w + c}{w + \kappa c} = \frac{\kappa w/c + 1}{w/c + \kappa}, \tag{6}$$

and this is the basic connection between the particle and wave velocities. This equation is a generalisation of the base relation $uw = c^2$ applying for Einstein's energy and which is obtained from (6) when $\kappa = 0$. We comment that we may re-write (6) in a rectangular hyperbola form $(u/c - \kappa)(w/c + \kappa) = 1 - \kappa^2$, and that the $\pm\kappa$ indicate that the velocity relationship is not a simple coordinate transformation from that for $\kappa = 0$. For example, the condition $\kappa^2 < 1$ is sufficient to ensure that a particle velocity $u/c \leq 1$ has an associated wave velocity $w/c \geq 1$.

Using (6), we may verify the following three additional identities, namely

$$\begin{aligned} \frac{w}{c} &= \frac{1 - \kappa u/c}{u/c - \kappa}, \\ ((w/c)^2 - 1)^{1/2} &= \frac{(1-\kappa^2)^{1/2}(1-(u/c)^2)^{1/2}}{((u/c) - \kappa)}, \\ \left(\frac{(w/c)+1}{(w/c)-1}\right)^{\kappa/2} &= \left(\frac{1-\kappa}{1+\kappa}\right)^{\kappa/2}\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2}. \end{aligned} \tag{7}$$

We have in mind that the constant κ is such that $\kappa \neq 0$ and $\kappa \neq \pm 1$ which although physically meaningful would require separate consideration. For instance, if $\kappa = \pm 1$, then (6) gives $u/c = \pm 1$ while the first part of Equation (7) is as an equation for the determination of w/c, would be regarded as indeterminate or undefined. It is also apparent from these relations that zero particle velocity is associated with a non-zero wave velocity and vice-versa.

As fully described in [7], if the proposal (4) is to be physically meaningful and consistent with de Broglie's view of the wave guiding the particle [18,19], then the group velocity of the wave $w_g = w - \lambda dw/d\lambda$ must coincide with the particle velocity u. This condition together with (6) constitutes a differential relation for $w(\lambda)$ which may be readily integrated to yield

$$\frac{1}{\lambda} = \frac{C_1}{((w/c)^2 - 1)^{1/2}}\left(\frac{w/c + 1}{w/c - 1}\right)^{\kappa/2}, \tag{8}$$

where C_1 denotes the constant of integration and the exponent κ is the same as that arising in (6). This equation prescribes the wave velocity $w(\lambda)$ as a function of the wavelength λ, and if satisfied, then the group velocity of the wave $w_g = w - \lambda dw/d\lambda$ coincides with the particle velocity u. From $w = \lambda\nu$ and (8), it is apparent that the wave frequency ν is given by

$$\nu = \frac{C_1 w}{((w/c)^2 - 1)^{1/2}}\left(\frac{w/c + 1}{w/c - 1}\right)^{\kappa/2}. \tag{9}$$

It is convenient to assume that the wavelength λ and frequency ν are given respectively by Equations (8) and (9) and assume that the constant C_1 has been determined from given data such that both expressions are positive.

4. Critical Particle and Wave Velocities in Terms of Power-Law Parameter κ

In this and the following sections we exploit the group velocity condition $u = w - \lambda dw/d\lambda$ to determine the stationary wave energy and momentum states. From (5) together with (8) we obtain

$$\begin{aligned} \mathscr{E} &= \frac{hcC_1(w/c + \kappa)}{((w/c)^2 - 1)^{1/2}}\left(\frac{w/c + 1}{w/c - 1}\right)^{\kappa/2}, \\ c\mathscr{P} &= \frac{hcC_1(\kappa w/c + 1)}{((w/c)^2 - 1)^{1/2}}\left(\frac{w/c + 1}{w/c - 1}\right)^{\kappa/2}, \end{aligned}$$

along with the additional relations

$$\mathscr{E} - \kappa c\mathscr{P} = \frac{hw(1-\kappa^2)}{\lambda}, \qquad c\mathscr{P} - \kappa\mathscr{E} = \frac{hc(1-\kappa^2)}{\lambda}, \tag{10}$$

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and together with the condition $u/c = c\mathscr{P}/\mathscr{E}$, the ratio of these two relations yields the first part of Equation (7). The equation corresponding to (3) becomes

$$\mathscr{E}^2 - (c\mathscr{P})^2 = (hcC_1)^2(1-\kappa^2)\left(\frac{w/c+1}{w/c-1}\right)^{\kappa}. \tag{11}$$

We may show that the wave energy $\mathscr{E}$ and wave momentum $\mathscr{P}$ also satisfy the second condition of (2), so that altogether we have

$$\frac{c\mathscr{P}}{\mathscr{E}} = \frac{u}{c}, \qquad \frac{d\mathscr{E}}{d\mathscr{P}} = c\left(\frac{\kappa + u/c}{1+\kappa u/c}\right). \tag{12}$$

The second condition is most easily established using the group velocity condition $\lambda dw/d\lambda = w - u$, so that from the two expressions (5)

$$\begin{aligned} \frac{d\mathscr{E}}{d\lambda} &= \frac{h}{\lambda}\frac{dw}{d\lambda} - \frac{h(w+\kappa c)}{\lambda^2} = -\frac{h(u+\kappa c)}{\lambda^2}, \\ c\frac{d\mathscr{P}}{d\lambda} &= \frac{\kappa h}{\lambda}\frac{dw}{d\lambda} - \frac{h(\kappa w + c)}{\lambda^2} = -\frac{h(\kappa u + c)}{\lambda^2}, \end{aligned} \tag{13}$$

and therefore by division, we obtain the desired result. Since the particle and wave energies and momenta each satisfy (2), we may conclude that the respective quantities differ by at most the same scale factor. Formally, we may use the various identities (7) to verify this result and show that

$$\begin{aligned} \mathscr{E}(u) &= \frac{hcC_1(1-\kappa^2)^{1/2}}{(1-(u/c)^2)^{1/2}}\left(\frac{1-\kappa}{1+\kappa}\right)^{\kappa/2}\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2}, \\ c\mathscr{P}(u) &= \frac{hcC_1(1-\kappa^2)^{1/2}}{(1-(u/c)^2)^{1/2}}\left(\frac{1-\kappa}{1+\kappa}\right)^{\kappa/2}\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2}\frac{u}{c}. \end{aligned} \tag{14}$$

The group velocity condition $\lambda dw/d\lambda = w - u$, is also important to identify the stationary values of the wave speed w, since these necessarily occur when $u = w$, and from (6), this only occurs for $u/c = \pm 1$.

A table of critical particle and wave velocities is presented below. The values $u/c = -\kappa$ and $w/c = -\frac{1}{2}\left(\kappa + \frac{1}{\kappa}\right)$ arise immediately from the numerator of the second part of Equation (2) and gives rise to the possibility of that energy state which in the literature is commonly referred to as the "zero-point energy", although it does not necessarily refer to a "zero" energy state. In the literature the "zero-point energy" is the lowest possible energy that a quantum mechanical system may have. It arises throughout physics and an improved understanding would be of benefit to many branches of the discipline. In this context, notice that the table given below has little meaning for the conventional Planck de Broglie relations coupled with Einstein's energy relation, since in that case $\kappa = 0$. The particular factors $\left(\frac{u}{c} + \kappa\right)$ or $\left(\frac{w}{c} + \frac{1}{2}\left(\kappa + \frac{1}{\kappa}\right)\right)$ arise throughout the subsequent analysis and it is important to connect their appearance with the "zero-point energy" state.

From the energy-momenta symmetry in the above analysis, there is evidently a related "stationary momentum" concept which arises from the velocity values $\frac{u}{c} = -\frac{1}{\kappa}$ and $\frac{w}{c} = -\frac{1}{\frac{1}{2}\left(\kappa + \frac{1}{\kappa}\right)}$ and with the corresponding factors $\left(\frac{u}{c} + \frac{1}{\kappa}\right)$ and $\left(\frac{w}{c} + \frac{1}{\frac{1}{2}\left(\kappa + \frac{1}{\kappa}\right)}\right)$. These values arise immediately from the denominator of the second part of Equation (2) and give rise to the possibility that the momentum is stationary. Notice at this point we have indicated only that these critical velocity values correspond to stationary energy or stationary momentum states. In order to establish the nature of these states, either as maximum or minimum, further analysis is given in the following section, and we show that the nature of the energy state depends on the sign of κ.

5. Nature of Stationary Energy and Momentum

In this section we examine the nature of the stationary energy and momentum and the determination of the arbitrary constant C_1 in terms of the particle parameters e_0 and the exponent κ. By differentiation with respect to λ of the group velocity condition $\lambda dw/d\lambda = w - u$ and (6), we may deduce

$$\frac{1}{c}\frac{dw}{d\lambda} = \frac{1}{\lambda}\left(\frac{\left(\frac{w}{c}\right)^2 - 1}{\left(\frac{w}{c}\right) + \kappa}\right), \qquad \frac{1}{c}\frac{d^2w}{d\lambda^2} = \frac{(1-\kappa^2)}{\lambda^2}\left(\frac{\left(\frac{w}{c}\right)^2 - 1}{\left(\left(\frac{w}{c}\right) + \kappa\right)^3}\right). \tag{15}$$

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On further differentiating (13) and using $\lambda \frac{d^2 w}{d\lambda^2} = -\frac{du}{d\lambda}$, we obtain

$$\frac{d^2\mathscr{E}}{d\lambda^2} = -\frac{h}{\lambda^2}\frac{du}{d\lambda} + \frac{2h(u+\kappa c)}{\lambda^3} = \frac{h}{\lambda}\frac{d^2 w}{d\lambda^2} + \frac{2h(u+\kappa c)}{\lambda^3},$$
$$c\frac{d^2\mathscr{P}}{d\lambda^2} = -\frac{h\kappa}{\lambda^2}\frac{du}{d\lambda} + \frac{2h(\kappa u + c)}{\lambda^3} = \frac{h\kappa}{\lambda}\frac{d^2 w}{d\lambda^2} + \frac{2h(\kappa u + c)}{\lambda^3}.$$

On noting that the final terms in each of these expressions vanishes at the respective stationary points, below we include only those terms that are relevant to the determination of the sign of the second derivative at the stationary points. Thus, on making use of the second part of the relation (15), we obtain

$$\begin{aligned} \frac{d^2\mathscr{E}}{d\lambda^2} &= \frac{hc(1-\kappa^2)}{\lambda^3}\left(\frac{\left(\frac{w_0}{c}\right)^2 - 1}{\left(\left(\frac{w_0}{c}\right)+\kappa\right)^3}\right) = -\frac{2hc\kappa}{\lambda^3}, \\ c\frac{d^2\mathscr{P}}{d\lambda^2} &= \frac{hc\kappa(1-\kappa^2)}{\lambda^3}\left(\frac{\left(\frac{c}{w_0}\right)^2 - 1}{\left(\left(\frac{c}{w_0}\right)+\kappa\right)^3}\right) = \frac{hc(1+\kappa^2)}{\lambda^3\kappa^2}, \end{aligned} \tag{16}$$

where in each case we are assuming that the wavelength $\lambda > 0$ refers to the expression (8) evaluated at the appropriate wave speed, namely $w = w_0$ in the case of stationary energy and $w = c^2/w_0$ in the case of stationary momentum, and where $\frac{w_0}{c} = -\frac{1}{2}\left(\kappa + \frac{1}{\kappa}\right)$ denotes the wave speed at the stationary energy state and we have used the identity $\left(\frac{w_0}{c}\right)^2 - 1 = \frac{1}{4}\left(\kappa - \frac{1}{\kappa}\right)^2$. Thus from (16), the energy $\mathscr{E}$ is a minimum at $w = w_0$ only for $\kappa < 0$, while the momentum $\mathscr{P}$ is a minimum at $w = c^2/w_0$ for all values of κ.

In order to determine the arbitrary constant C_1, we need to speculate on the extent of energy transfer in the transition from a collapsing wave to a particle. Here, for definiteness, we suppose that the entire minimum energy of the wave $\mathscr{E}_0$ (zero-point energy state for $\kappa < 0$) becomes particle energy, and this turns out to be equivalent to assuming that $e(u) = \mathscr{E}(w)$ for all particle and wave velocities u and w. Firstly, from the first part of Equation (14) together with $u/c = -\kappa$, for the stationary energy state, we have

$$\mathscr{E}_0 = hcC_1\left(\frac{1-\kappa}{1+\kappa}\right)^{\kappa}. \tag{17}$$

Secondly, from the particle energy $e(u)$ given by the first part of Equation (1) evaluated at $u/c = -\kappa$ and the assumption that the entire minimum energy of the wave $\mathscr{E}_0$ is transferred to the particle we have

$$\mathscr{E}_0 = \frac{e_0}{(1-\kappa^2)^{1/2}}\left(\frac{1-\kappa}{1+\kappa}\right)^{\kappa/2}. \tag{18}$$

Together (17) and (18) imply that the arbitrary constant C_1 is determined from the relation

$$e_0 = hcC_1(1-\kappa^2)^{1/2}\left(\frac{1-\kappa}{1+\kappa}\right)^{\kappa/2},$$

and from the first parts of equations (1) and (14), this condition is sufficient to ensure that $e(u) = \mathscr{E}(w)$ for all particle and wave velocities u and w connected by the velocity relations (7).

6. de Broglie Wave Profile

A simple harmonic wave of the form $e^{i(kx-\omega(k)t)} = e^{2\pi i(x/\lambda - \nu t)}$ has wave velocity $w = \omega(k)/k$ and group velocity $u = d\omega/dk$. For a particle moving with velocity u and with an associated wave having a wave velocity $w = c^2/u$, de Broglie in his PhD thesis ([15]) (see also [17]) makes use of the relations $\mathscr{E} = h\nu$ and $\mathscr{P} = h/\lambda$ to formulate the wave function

$$\psi(x,t) = \exp\left\{\frac{2\pi i}{\lambda}(x - wt)\right\}, \tag{19}$$

and identifies the wave velocity $w = \lambda\nu$. This expression can be seen to arise most simply using the Lorentz invariants $\xi = \mathscr{E}x - c^2\mathscr{P}t$ and $\eta = \mathscr{P}x - \mathscr{E}t$. These are discussed at length in [6] (see pages 20 and 102) and remain unchanged under the simultaneous Lorentz coordinate transformations (A1) and the Lorentz energy-momenta transformations (A2).

Assuming the conventional Planck-de Broglie formulae, waves of the form $e^{2\pi i\eta/h} = e^{2\pi i(\mathscr{P}x-\mathscr{E}t)/h}$ and $e^{2\pi i\xi/hc} = e^{2\pi i(\mathscr{E}x-c^2\mathscr{P}t)/hc}$ are examined in [6] and a table of corresponding wave and group velocities is given in [6] (see page 286), where h is Planck's constant. de Broglie's wave function (19) emerges from the Lorentz invariant $\eta = \mathscr{P}x - \mathscr{E}t$, thus

$$\psi(x,t) = \exp\left\{\frac{2\pi i\eta}{h}\right\} = \exp\left\{\frac{2\pi i(\mathscr{P}x-\mathscr{E}t)}{h}\right\} = \exp\left\{\frac{2\pi i}{\lambda}(x-\lambda\nu t)\right\},$$

on using $\mathscr{E} = h\nu$ and $\mathscr{P} = h/\lambda$ and which identifies a wave speed $w = \lambda\nu$. On the other hand, the Lorentz invariant $\xi = \mathscr{E}x - c^2\mathscr{P}t$ gives

$$\psi(x,t) = \exp\left\{\frac{2\pi i\xi}{hc}\right\} = \exp\left\{\frac{2\pi i(\mathscr{E}x-c^2\mathscr{P}t)}{hc}\right\} = \exp\left\{\frac{2\pi i\nu}{c}\left(x-\frac{c^2}{\lambda\nu}t\right)\right\},$$

which identifies a wave speed $w = c^2/\lambda\nu$ for a particle moving with velocity c^2/u and with an associated wave speed u. Accordingly, from these observations, it is natural to pose the question as to whether a wave function might be meaningfully determined from a suitable linear combination of ξ and η.

There is a general belief within the physics community that the phase angle of a plane wave should be Lorentz invariant (see for example [10,11]). For Galilean phase invariance and the historical context we refer the reader to [20] (pages 6–8). We expect any fundamental wave profile to have a general wave profile of the form $y = A(\xi,\eta)e^{iB(\xi,\eta)}$, where the amplitude $A(\xi,\eta)$ and phase $B(\xi,\eta)$ denote certain real functions of the Lorentz invariants $\xi = \mathscr{E}x - c^2\mathscr{P}t$ and $\eta = \mathscr{P}x - \mathscr{E}t$ to be determined.

Now since for the phase $B(\xi,\eta)$ we have in mind an expression which is linear in x and t, namely $B(\xi,\eta) = kx - \omega(k)t$, it is natural to ask first whether there exists a constant α such that an expression of the form $e^{i(\xi+\alpha c\eta)/c\hbar}$ admits the desired wave and group velocity requirements.

In order to examine this question, we need to exploit the relations (6) and (12), and $\xi + \alpha c\eta$ becomes

$$\begin{aligned} \xi + \alpha c\eta &= (\mathscr{E}x - c^2\mathscr{P}t) + \alpha c(\mathscr{P}x - \mathscr{E}t) \\ &= (\mathscr{E} + \alpha c\mathscr{P})x - (c\mathscr{P} + \alpha\mathscr{E})ct, \end{aligned}$$

so that the wave velocity condition becomes

$$\frac{w}{c} = \frac{c\mathscr{P} + \alpha\mathscr{E}}{\mathscr{E} + \alpha c\mathscr{P}} = \frac{\alpha + \frac{c\mathscr{P}}{\mathscr{E}}}{1 + \alpha\frac{c\mathscr{P}}{\mathscr{E}}} = \frac{\alpha + \left(\frac{\kappa w/c+1}{w/c+\kappa}\right)}{1 + \alpha\left(\frac{\kappa w/c+1}{w/c+\kappa}\right)}, \tag{20}$$

while the group velocity condition becomes

$$\frac{u}{c} = \frac{cd\mathscr{P} + \alpha d\mathscr{E}}{d\mathscr{E} + \alpha cd\mathscr{P}} = \frac{1 + \frac{\alpha}{c}\frac{d\mathscr{E}}{d\mathscr{P}}}{\alpha + \frac{1}{c}\frac{d\mathscr{E}}{d\mathscr{P}}} = \frac{1 + \alpha\left(\frac{\kappa+u/c}{1+\kappa u/c}\right)}{\alpha + \left(\frac{\kappa+u/c}{1+\kappa u/c}\right)}. \tag{21}$$

With re-arrangement, Equations (20) and (21) yield the identical constraint, so that with either $z = w/c$ or $z = u/c$, the same relation is obtained, namely

$$z = \frac{(\alpha+\kappa)z + (1+\alpha\kappa)}{(1+\alpha\kappa)z + (\alpha+\kappa)},$$

and on simplification, this equation yields $(1+\alpha\kappa)(z^2-1) = 0$, so that either $u = w = \pm c$ or $\alpha = -1/\kappa$. A commonly occurring characteristic of such relations is the redundancy of the constraint in the event that the particle or wave is travelling at the speed of light. Thus, having identified $\alpha = -1/\kappa$, we are tacitly assuming that $\kappa \neq 0$ and therefore, unlike the previous formulae, there is no smooth passage to the conventional situation in the following analysis, since in this context $\kappa = 0$ corresponds to $\alpha = \infty$.

Thus, in the absence of further information on the amplitude, we may say that a possible de Broglie wave admits the profile

$$\begin{aligned}\psi(x,t) &= \exp\left\{i\frac{(\kappa\xi - c\eta)}{c\kappa\hbar}\right\} = \exp\left\{i\frac{((\kappa\mathscr{E} - c\mathscr{P})x - (\kappa c\mathscr{P} - \mathscr{E})ct)}{c\kappa\hbar}\right\}, \\ &= \exp\left\{-i\frac{2\pi(1-\kappa^2)}{\kappa\lambda}(x - wt)\right\},\end{aligned} \tag{22}$$

on using the relations (10) where the wavelength λ is assumed to be given by (8), and with the arbitrary constant C_1 as yet undetermined. The major contribution of the expression (22) is the explicit dependence upon the parameter κ, which has the effect of modulating the wavelength λ. The dependence on wavelength and the (x,t) dependence is evidently correct and might be deduced immediately from $e^{i(kx-\omega(k)t)} = e^{2\pi i(x/\lambda - \nu t)} = e^{\frac{2\pi i}{\lambda}(x-wt)}$. We have purposely followed the Lorentz invariant formalism since the determination of the amplitude in the next section follows a related approach.

7. Amplitude of de Broglie Wave

In the previous section, with a general wave profile in mind, $y = A(\xi,\eta)e^{iB(\xi,\eta)}$, where $A(\xi,\eta)$ and $B(\xi,\eta)$ denote certain real functions of the Lorentz invariants ξ and η, we have purposely structured the wave profile in terms of the invariants, thus $y = e^{i(\kappa\xi - c\eta)/c\kappa\hbar}$. The question arises as to whether there might exist a natural extension of this expression and thereby provide a means to determine the amplitude. Any such generalisation is not unique and can at best be only an educated guess. Our ad hoc approach is to seek the product of two complex numbers such that the imaginary part coincides with $(\kappa\xi - c\eta)/c\kappa\hbar$, and one possibility is $(\xi - ic\eta)(1 + i\kappa)/c\kappa\hbar$, thus

$$\frac{(\xi - ic\eta)(1+i\kappa)}{c\kappa\hbar} = \frac{(\xi + c\kappa\eta)}{c\kappa\hbar} + i\frac{(\kappa\xi - c\eta)}{c\kappa\hbar}. \tag{23}$$

While the Lorentz invariants ξ and η have a known symmetry with respect to the particle and wave worlds, so that the complex number $\xi - ic\eta$ has a meaningful structure, the equivalent symmetry in the constant $(1+i\kappa)/c\kappa\hbar$ only becomes apparent when expressed in the original variables, namely

$$\frac{(1+i\kappa)}{c\kappa\hbar} = \frac{2\pi}{c}\left(\frac{1}{h'} + \frac{i}{h}\right),$$

on using the basic relations $\kappa = h'/h$ and $\hbar = h/2\pi$.

From the real part of (23) we might deduce the following expression for the amplitude $A(w)$, thus

$$\begin{aligned}A(w) &= \exp\left\{\frac{(\xi + c\kappa\eta)}{c\kappa\hbar}\right\} = \exp\left\{\frac{((\mathscr{E} + c\kappa\mathscr{P})x - (c\mathscr{P} + \kappa\mathscr{E})ct)}{c\kappa\hbar}\right\}, \\ &= \exp\left\{\frac{2\pi}{\lambda\kappa}\left(\left((1+\kappa^2)\frac{w}{c} + 2\kappa\right)x - \left(2\kappa\frac{w}{c} + (1+\kappa^2)\right)ct\right)\right\}, \\ &= \exp\left\{-\frac{4\pi}{\lambda}\left(\left(\frac{ww_0}{c^2} - 1\right)x - (w_0 - w)t\right)\right\},\end{aligned}$$

where w_0 denotes the wave speed corresponding to the "zero point energy" state which is given by $\frac{w_0}{c} = -\frac{1}{2}\left(\kappa + \frac{1}{\kappa}\right)$. This wave speed is superluminal since $\left(\frac{w_0}{c}\right)^2 - 1 = \frac{1}{4}\left(\kappa - \frac{1}{\kappa}\right)^2 > 0$, and $w_0 > c$, which means that the stationary momentum state with wave speed c^2/w_0 is necessarily sub-luminal.

The amplitude values at the stationary energy wave speed $w = w_0$ and at the stationary momentum wave speed $w = c^2/w_0$ are given respectively by

$$\begin{aligned}A(w_0) &= \exp\left\{-\frac{4\pi}{\lambda}\left(\left(\frac{w_0}{c}\right)^2 - 1\right)x\right\}, \\ A\left(\frac{c^2}{w_0}\right) &= \exp\left\{4\pi\left(\left(\frac{w_0}{c}\right)^2 - 1\right)\nu t\right\},\end{aligned}$$

where the wavelength λ and frequency ν are understood to be those values determined from (8) and (9) at the appropriate wave speed $w = w_0$ or $w = c^2/w_0$. We also observe that, on using $w = \lambda\nu$, the given expression for the amplitude may be alternatively written as

$$A(w) = \exp\left\{4\pi\left(\frac{(x + w_0 t)}{\lambda} - \left(t + \frac{xw_0}{c^2}\right)\nu\right)\right\},$$

and the above amplitude values $A(w_0)$ and $A\left(c^2/w_0\right)$ may be obtained from this expression along the lines $ct = -xw_0/c$ and $x = -w_0 t$ respectively.

On making use of $\left(\left(\frac{w_0}{c}\right)^2 - 1\right)^{1/2} = \pm\frac{1}{2}\left(\kappa - \frac{1}{\kappa}\right)$ and (22), the wave profile in terms of w_0 becomes

$$y = \exp\left\{\pm i\frac{4\pi}{\lambda}\left(\left(\frac{w_0}{c}\right)^2 - 1\right)^{1/2}(x - wt)\right\}.$$

Thus, altogether the de Broglie wave is given by

$$\begin{aligned} y = \exp&\left\{-\frac{4\pi}{\lambda}\left(\left(\frac{ww_0}{c^2} - 1\right)x - (w_0 - w)t\right)\right\} \\ \times \exp&\left\{\pm i\frac{4\pi}{\lambda}\left(\left(\frac{w_0}{c}\right)^2 - 1\right)^{1/2}(x - wt)\right\}, \end{aligned} \tag{24}$$

where $w_0 = -\frac{c}{2}\left(\kappa + \frac{1}{\kappa}\right)$ is the "zero-point energy" state wave speed.

8. Summary and Conclusions

Motivated by the symmetry between sub-luminal and superluminal motions in special relativity, we have considered the duality problem of the simultaneous existence of both an elementary particle and a wave which serves to both guide and transfer energy to the particle. The notion of Lorentz invariance throughout this paper refers to the invariance under the combined space-time Lorentz transformations and the Lorentz energy-momentum transformations given respectively by (A1) and (A2). We have briefly summarised the Lorentz invariant power-law particle energy-momentum expressions (1) from [3] and their connection with the generalised Planck de Broglie relations (4) from [7]. In [3] the power-law particle energy-momentum expressions are based on Lorentz invariance, while in the wave context [7], these expressions emerge naturally from (4) and the hypothesis that the group velocity of the wave coincides with the particle velocity.

On face value, the particle and wave worlds are independent such that the rest energy e_0 fixes the particle motion, while the "zero-point energy" $\mathscr{E}_0$ (lowest attainable quantum mechanical energy) fixes the wave. Up to this point, there are no restrictions on the power-law exponents of the particle and the wave which might differ. However, here for definiteness, we make the assumption that the collapsing wave transfers its entire minimum or zero-point energy $\mathscr{E}_0$ to the particle with rest energy e_0 and if the power-law exponents of both particle and wave coincide, then this assumption is sufficient to guarantee particle and wave energies coincide for all velocities. The final result (18) constitutes the major contribution of the paper and for which there is no comparable formula within conventional theory, namely

$$\mathscr{E}_0 = \frac{e_0}{(1-\kappa^2)^{1/2}}\left(\frac{1-\kappa}{1+\kappa}\right)^{\kappa/2}.$$

We comment that equally well, we might have assumed that a certain fraction of the minimum wave energy is transferred. Assuming the same power-law exponents for the particle and wave, we have given a table for the critical velocities, including those values giving rise to stationary energy and stationary momentum. We have shown that the wave energy $\mathscr{E}$ is a minimum at $w = w_0$ only for $\kappa < 0$, while the wave momentum $\mathscr{P}$ is a minimum at $w = c^2/w_0$ for all values of κ where $w_0 = -\frac{c}{2}\left(\kappa + \frac{1}{\kappa}\right)$.

Further, we have exploited the Lorentz invariants $\xi = \mathscr{E}x - c^2\mathscr{P}t$ and $\eta = \mathscr{P}x - \mathscr{E}t$ to first determine the simplest and most likely wave profile and then to use this structure to propose a possible matching amplitude. Thus, any wave of the form $y = A(\xi,\eta)e^{iB(\xi,\eta)}$ is possible, where $A(\xi,\eta)$ and $B(\xi,\eta)$ denote real functions of the Lorentz invariants ξ and η. This ad hoc approach gives rise to an entirely feasible wave phase, except that in the derivation, we have introduced the linear combination $\xi + \alpha c\eta$ where $\alpha = -1/\kappa$. We are tacitly assuming that $\kappa \neq 0$ and therefore in this particular instance there is no passage to the conventional Planck de-Broglie limit. Further, the "matching" wave amplitude has either exponential growth or decay which at first sight might be unexpected. However, in the absence of adopting another approach, such as quantum mechanics or a variational theory, the use of Lorentz invariants provides the most powerful approach. Both the absence of a smooth limit $\kappa \longrightarrow 0$ and the possible unexpected exponential growth or decay are typical characteristics of nonlinear theory which are "picked up" by the invariance approach. However, this ad hoc approach is by no means unique and similar methods might be used to generate other possible wave forms.

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Original contributions in this study are included in the article and inquiries may be directed to the author.

Conflicts of Interest

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Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

Appendix A. Symmetry of Special Relativity beyond the Speed of Light

In special relativity, sub-luminal and superluminal motions enjoy a certain symmetry, which gives rise to the idea that key physical parameters might involve characteristics having both particle and wave based origins. Accordingly, in this appendix we present a synopsis of certain aspects of special relativity beyond the speed of light, and in particular as described in [9,21]. For the continuation of Einstein's special relativity theory beyond the speed of light there is a symmetry between the sub-luminal (particle) and superluminal (wave) worlds such that each is the mirror image of the other. Here we provide a synopsis of this symmetry and we refer the reader to [9] for the full discussion.

The problem of extending special relativity beyond the speed of light and the possible existence of superluminal particles (tachyons) has a long and colourful history with numerous contributors over a long period. There are many individual perspectives pursuing particular points of view and there are many competing claims for precedence. For an indication of some of the history and the development of superluminal physics we refer the reader to [9,21–28] and the numerous references contained therein. It is a developing subject that remains controversial. However, the main controversial issues revolve around the correct numbers of superluminal space and time dimensions and the apparent difficulties associated with reconciling three spatial dimensions with only one time dimension. By restricting attention to one space dimension x and one time dimension t, some controversial aspects are reduced and there is measure of agreement on the essential structure of the sub-luminal and superluminal Lorentz transformations.

With reference to Figure A1, capital letters refer to the fixed (X, T) reference frame and lower case letters refer to the moving (x, t) reference frame. In the notation of Figure 1, Einstein's addition of velocity law (A3) or (A4) arises from both the following sub-luminal Lorentz transformations and superluminal Lorentz transformations connecting two inertial frames as shown in the figure. If v denotes the constant relative frame velocity, then the sub-luminal Lorentz transformations $(0 \leqslant v < c)$ are given by

$$x = \frac{X - vT}{[1-(v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1-(v/c)^2]^{1/2}}, \tag{A1}$$

with the identity transformation $x = X$ and $t = T$ given by $v = 0$. For $0 \leqslant v < c$, the Lorentz invariant energy-momentum relations are given by

$$p = \frac{P - Ev/c^2}{[1-(v/c)^2]^{1/2}}, \quad e = \frac{E - Pv}{[1-(v/c)^2]^{1/2}}, \tag{A2}$$

from which the Lorentz invariant $e^2 - (pc)^2 = E^2 - (Pc)^2$ is apparent.

For $c < v < \infty$, the superluminal Lorentz transformations are given by

$$x = \frac{\epsilon\,(X - vT)}{[(v/c)^2 - 1]^{1/2}}, \quad t = \frac{\epsilon\left(T - vX/c^2\right)}{[(v/c)^2 - 1]^{1/2}},$$

where $\epsilon^2 = 1$. The two cases $\epsilon = \pm 1$ arise from the two distinct alternatives according to the assumed behaviour at $v = \infty$, namely,

$$x = -\epsilon cT, \quad t = -\epsilon X/c, \quad v = \infty.$$

Further, for $c < v < \infty$, the Lorentz invariant energy-momentum relations become

$$p = \frac{\epsilon(P - Ev/c^2)}{((v/c)^2 - 1)^{1/2}}, \quad e = \frac{\epsilon(E - Pv)}{((v/c)^2 - 1)^{1/2}},$$

and we refer the reader to [21] for details of these Lorentz transformations.

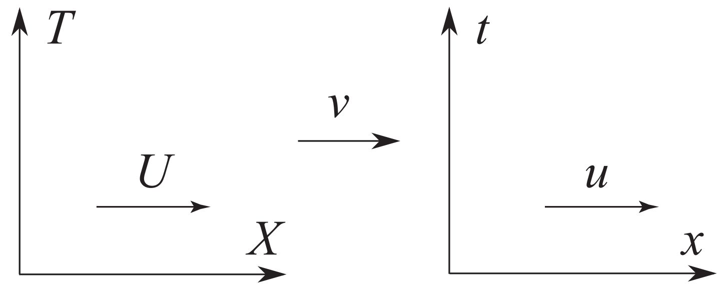

Figure A1. Two inertial frames moving along x-axis with relative velocity v.

For these superluminal Lorentz transformations, a key feature is that the various versions of the Einstein addition of velocity law remain unchanged, and there is the implication that at least one of the three velocities $u = dx/dt$, v and $U = dX/dT$ must be less than the speed of light. Thus, Einstein's formula for the addition of velocities still applies and both sets of Lorentz transformations give rise to Einstein's relative velocity formula which can be expressed in either of the forms,

$$U = \frac{u+v}{(1+uv/c^2)}, \qquad c^2/U = \frac{c^2/u+v}{(1+v/u)}, \tag{A3}$$

so that both the particle velocity u and the wave velocity c^2/u transform in exactly the same manner. Alternatively (A3) can be expressed in the form

$$\left(\frac{1+U/c}{1-U/c}\right) = \left(\frac{1+u/c}{1-u/c}\right)\left(\frac{1+v/c}{1-v/c}\right), \tag{A4}$$

and both (A3) and (A4) apply for both sub-luminal and superluminal motion. It is apparent from the Formula (A4) that at least one of the velocities u, v or U must not exceed the speed of light so that if the frame velocity v exceeds the speed of light, then one and only one of u or U will be superluminal and the other sub-luminal.

We now present a synopsis from [9] of the symmetry of the sub-luminal and superluminal worlds existing within the limited confines of special relativistic velocity manipulation. We note that beyond this limited perspective there may be other means of distinguishing between the two sets such as the different Poincare group representations believed to exist for sub-luminal and superluminal elementary particles (see [29]).

If with respect to some fixed inertial frame we consider the two sets comprising all sub-luminal and all superluminal Lorentz frames then in order to illustrate the symmetrical nature of the two sets of reference frames, a table of relative velocities is constructed in [9] for three inertial frames. These three frames comprise an initial rest frame A and two additional frames B and C, which are moving with non-dimensionalized relative velocities v and w, respectively, relative to A and both v and w have been non-dimensionalized with respect to the speed of light c ($|v| < 1$, and $|w| < 1$). Subsequently, three additional frames are introduced which are the reciprocal frames for A, B and C, denoted by A', B' and C'. These comprise the non-dimensionalized reciprocals in the unprimed frames, which have velocities given by $\pm\infty$, $1/v$ and $1/w$, respectively, as measured by an observer at rest in A. Both cases $\pm\infty$ arise in consequence of the two superluminal sets of Lorentz transformations.

If we change the rest frame to be either of the unprimed frames, then the relative velocities of A, B and C change in the usual way and the absolute value of each of the relative velocities of the unprimed frames remain less than c. Further, the primed frames maintain their reciprocal velocity relationship regardless from which of the unprimed frames the measurements are made. The notion of a reciprocal frame is independent of the choice of the rest frame, and all frames that are superluminal when measured by an observer in A, remain superluminal when measured by observers in B or C, or any observer at rest in a frame which is measured as sub-luminal by an observer in the initial rest frame A. For an observer at rest in the primed frame A', where the world view is symmetrically inverted, meaning that all the primed frames now have velocities which are less than c and all the unprimed frames have reciprocal, superluminal velocities and precisely the same velocities are measured by observers in any of the primed frames as originally measured by observers in the unprimed frames.

The data presented in the table of relative velocities in [9] may be viewed as a 6×6 matrix $\mathbf{Q}$ relating each of the six frames A, B, C, A', B' and C'. In [9] the matrix $\mathbf{Q}$ is shown to be made up of four 3×3 matrices

$$\mathbf{Q} = \begin{pmatrix} \mathbf{R} & \mathbf{R}' \\ \mathbf{R}' & \mathbf{R} \end{pmatrix},$$

where as given below, the matrices $\mathbf{R}$ on the diagonal are the standard special relativity addition formulae and the matrices $\mathbf{R}'$ on the anti-diagonal are the reciprocals of the corresponding elements in the diagonal matrices, thus

$$\mathbf{R} = \begin{pmatrix} 0 & -v & -w \\ v & 0 & \dfrac{v-w}{1-vw} \\ w & \dfrac{w-v}{1-vw} & 0 \end{pmatrix}, \qquad \mathbf{R}' = \begin{pmatrix} \pm\infty & -1/v & -1/w \\ 1/v & \pm\infty & \dfrac{1-vw}{v-w} \\ 1/w & \dfrac{1-vw}{w-v} & \pm\infty \end{pmatrix}.$$

Each of the 3×3 matrices $\mathbf{R}$ and $\mathbf{R}'$ and the full 6×6 matrix $\mathbf{Q}$ are skew-symmetric, with the unconventional inclusion of the $\pm\infty$ terms that are essentially without sign in this system.

An important outcome of this symmetry is that within the limited confines of velocity manipulation within special relativity, there is no objective way for any observer to determine whether or not they are in the primed or the unprimed frames of reference. This leads to the idea that not only that the speed of light is relative to all observers in frames which are sub-luminal to some reference frame, but also there is a separate and symmetric reciprocal set of frames of reference which have an identical world view, in the sense that observers in the reciprocal frames consider their velocities to be sub-luminal and those from the original set of frames to be superluminal. The only velocity which lacks a distinct reciprocal is the speed of light itself, which is its own reciprocal and is not properly a member of either set. The implications to causality are that either set of frames can interact with all others in their set without violating the conventional notion of causality, since all velocities are less than c for an observer at rest in any of the frames in a single set. Interactions between an observer at rest in a frame and observers at rest in a frame which is not a member of the set have the potential to violate causality. If causality paradoxes are to be avoided then there must be restrictions on the manner in which the two sets of frames can interact.

Appendix B. Table of Particle and Wave Symbols

Table A1. Particle and wave symbols.

Quantity	Equation
Wave form $\exp[i(kx-\omega t)]$	$\exp[i(kx-\omega(k)t)]$
Wave number k	$k=\frac{2\pi}{\lambda}$
Wavelength of wave (peak to peak) λ	$\lambda=\frac{2\pi}{k}$
Frequency of wave ν	$\nu=\frac{\omega}{2\pi}=\frac{1}{T}$
Angular frequency ω	$\omega(k)=2\pi\nu=\frac{2\pi}{T}$
Period of wave T	$T=\frac{1}{\nu}=\frac{2\pi}{\omega}$
Wave velocity (velocity of peak) w	$w=\frac{\omega(k)}{k}=\lambda\nu$
Particle and group velocity u	$u=\frac{d\omega(k)}{dk}=w+k\frac{dw}{dk}=w-\lambda\frac{dw}{d\lambda}$
Mass of elementary particle m	$m=e/c^2$
Elementary particle energy e	$e=mc^2$
Elementary particle momentum p	$p=mu$
Wave energy $\mathscr{E}$	$\mathscr{E}=h\nu+\frac{h^{'}c}{\lambda}$
Wave momentum $c\mathscr{P}$	$c\mathscr{P}=\frac{hc}{\lambda}+h^{'}\nu$
Planck's constant h	$h=2\pi\hbar=6.626\times10^{-34}$ (J-Secs)
Planck's constant divided by 2π $\hbar$	$\hbar=\frac{h}{2\pi}$
Fundamental constant $h^{'}$	$h^{'}=\kappa h$
Power-law exponent κ	$\kappa=h^{'}/h$
Power-law particle energy	$e(u)=\frac{e_0}{(1-(u/c)^2)^{1/2}}\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2}$
Power-law particle momentum	$cp(u)=\frac{e_0(u/c)}{(1-(u/c)^2)^{1/2}}\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2}$
First Lorentz invariant ξ	$\xi=ex-c^2pt$
Second Lorentz invariant η	$\eta=px-et$

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