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# Zonotopic Set-Membership Estimation for Complex Networks Under Measurement Quantization and Weighted Try-Once-Discard Protocol

Yuqing Zhang and Dan Liu *

Department of Mathematics, Yangzhou University, Yangzhou 225002, China
* Correspondence: dliumath@yzu.edu.cn
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**Abstract:** This paper investigates the zonotopic set-membership estimation (ZSME) problem for a class of discrete-time complex networks subject to weighted try-once-discard (WTOD) protocol and logarithmic quantization, where the external disturbances are unknown-but-bounded. To effectively save network resources, signals are first quantized by a quantizer before being transmitted into the network, and the WTOD protocol is introduced to schedule the transmission sequence, thereby alleviating the burden of network communication. This paper aims to develop a ZSME algorithm, which is based on the quantized and protocol-scheduled measurement signals, ensuring that the actual state of each node is enclosed in a zonotope at each moment. Furthermore, by designing a correction matrix, the size of the above zonotope is minimized. Additionally, sufficient conditions for ensuring the boundedness of the developed zonotopes are provided. Finally, the effectiveness of the proposed ZSME algorithm is verified through a simulation example.

**Keywords:** zonotopic set-membership estimation; complex network; weighted try-once-discard protocol; measurement quantization

## 1. Introduction

With the rapid development of information and communication technologies, modern large-scale engineering systems (such as sensor networks, power systems, neural networks, and cyber-physical systems) are often abstracted as complex networks (CNs), which consist of multiple dynamic nodes and their coupling edges [1–3]. Under this framework, each node represents an independent subsystem, while the edges depict the information interaction, collaboration relationships, or physical coupling between nodes. Based on this universal modeling approach, the theory of CNs has developed into an important analytical tool for studying the dynamic behaviors of interconnected systems, and has been widely used to explore synchronization, consensus, stability, and robustness issues [4–7]. Related research not only reveals how local interactions trigger global dynamic behaviors, but also lays a theoretical foundation for ensuring the stability and reliability of large-scale distributed systems.

The state information of each node in a CN serves as the foundation for system monitoring, control, and fault diagnosis. However, in practical engineering, due to factors such as sensor configuration, measurement costs, or physical environment, the system state is often difficult to obtain directly. Therefore, inferring the actual operating state of the system based on measurable output information, that is, the state estimation problem, has become one of the core topics in the theoretical research of CNs. State estimation aims to provide reliable state information support for upper-layer applications such as distributed control, fault diagnosis, and information fusion. Its accuracy directly affects the decision-making quality and operational safety of the entire system [8–11]. For dynamic systems affected by noise and uncertainty, scholars have developed various estimation and filtering methods. Among them, Kalman filtering and its variants exhibit optimal performance under the assumption of Gaussian noise by minimizing the mean square error [12,13]. $H_\infty$ filtering focuses on the worst-case estimation error and is suitable for bounded disturbances with unknown spectral characteristics [14,15]. The moving window estimation improves the robustness and convergence of the estimation by optimizing the batch data within a limited time domain [16]. However, these methods usually rely on prior assumptions about the statistical characteristics of the noise, such as known probability distributions or spectral density functions. When the actual noise characteristics deviate from these idealized assumptions, the estimation performance may deteriorate or even fail.

It is worth noting that in many practical engineering systems, interference and measurement noise are often difficult to be accurately described by precise random models, but instead exhibit unknown-but-bounded (UBB) deterministic characteristics, which means the noise amplitude is bounded and its statistical distribution is unknown. In this context, the set-membership estimation (SME) method has attracted much attention due to its unique theoretical advantages [17, 18]. Unlike traditional methods that provide single-point estimates, SME aims to

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determine a bounded geometric set that contains all system states consistent with the available measurement data and noise boundaries, thereby providing a deterministic guarantee for the robust analysis and control of the system. Under the SME framework, the choice of geometric representation directly affects the computational complexity and estimation accuracy of the algorithm. Common representations include ellipsoids, polyhedrons, and polytopes, etc. Among them, the zonotopic set-membership estimation (ZSME) method, due to its good balance between computational efficiency and estimation accuracy, has become one of the current research hotspots [19–21]. Based on zonotopes (a special form of polytope), the ZSME method can flexibly depict the uncertainty region of the state and has a lower computational burden, making it suitable for online implementation. Although there have been numerous research results on ZSME, the discussion of related issues in the context of CNs has not received sufficient attention. This research gap is precisely the main motivation of this paper.

The above research has laid a solid theoretical foundation for the state estimation of CNs. However, most of these achievements implicitly assume that the measurement data have infinite precision and can be transmitted without loss in an ideal channel. In actual networked systems, due to limitations in communication bandwidth and hardware storage capacity, sensor signals usually need to be quantized before entering the network to reduce the data transmission rate. From a mathematical perspective, quantization is essentially a mapping process of converting continuous amplitude signals into finite discrete amplitude signals. This process inevitably introduces quantization errors, which may lead to a decline in system performance. Therefore, it is crucial to fully consider the impact of quantization in system analysis and design, which has also given rise to a large number of research results on quantization issues [15,22,23]. In the field of quantization estimation, the quantizer is highly regarded due to its unique advantages: its quantization interval increases exponentially with the increase in signal amplitude, which can maintain a constant signal-to-noise ratio when processing wide dynamic range signals. Particularly, the literature [24] proposed a quantization processing method that converts quantization errors into bounded uncertainty of a fan shape. This method has attracted widespread research interest due to its effectiveness and universality, and has demonstrated potential application value in many practical scenarios [25–27].

In actual networked systems, apart from quantization, communication constraints cannot be ignored either. For instance, multiple sensor nodes typically share limited communication bandwidth, inevitably leading to data conflicts and network congestion. Therefore, it is urgent to design a reasonable communication protocol to coordinate the transmission sequence of nodes, thereby ensuring the efficient utilization of network resources [28–31]. Among these protocols, the Weighted Try-Once-Discard (WTOD) protocol, as a typical dynamic scheduling strategy, dynamically grants network access rights to nodes with the highest priority by assigning weights related to the current measurement error to each node. Furthermore, while taking fairness into account, the protocol can effectively reduce the probability of data collision. However, the introduction of the WTOD protocol has broken the assumption in the traditional estimation framework that the measured signals arrive periodically, making the transmission time and sequence of the signals dependent on the real-time evolution of the system state. This brings new challenges to the design of state estimators. In recent years, although some scholars have conducted research on the filtering and control issues under the WTOD protocol (see [32,33]), there are still few reports on how to simultaneously consider quantization effects and develop multicellular set member estimation algorithms within the framework of this protocol. This constitutes the third motivation of this study.

Based on the above discussions, the ZSME problem is studied in this paper for a class of CNs subject to measurement quantization and WTOD protocol scheduling. The addressed problem seems to be nontrivial due primarily to the following substantial difficulties: (1) how to jointly handle logarithmic quantization and WTOD scheduling within a zonotopic estimation framework; (2) how to design an optimal correction matrix that minimizes the zonotope size while accounting for quantization errors and partial measurements; and (3) how to derive sufficient conditions guaranteeing uniform boundedness of the zonotope sizes, explicitly linking quantization density, protocol scheduling, and network coupling. To overcome the above difficulties, the main contributions of this paper can be summarized as follows:

(1) For the first time, the ZSME problem is investigated for CNs in a network environment where both the logarithmic quantizer and the WTOD protocol are simultaneously considered, and thereby we construct an estimation framework that is more consistent with engineering practice.
(2) A ZSME algorithm applicable under the joint effects of quantization and scheduling is proposed by designing a correction matrix, with which it is strictly proven that the real state at each instant is always confined within the constructed zonotope.
(3) A sufficient condition is derived to guarantee the boundedness of the zonotope, and this condition explicitly characterizes the inherent relationship among the quantization parameter, the protocol scheduling mechanism, and the estimation performance.

**Notations:** The symbols $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ represent the $n$-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. The superscript $(\cdot)^T$ denotes matrix transposition and $I_n$ denotes the identity matrix with $n \times n$ dimension. For a matrix $A$, $\mathrm{tr}(A)$, $\|A\|$, $\|A\|_\infty$ and $\|A\|_F$ denote the trace, Euclidean norm, infinity norm and Frobenius norm of $A$, respectively, while $|A|$ denotes the element-wise absolute value of $A$. The symbol $\odot$ denotes the linear image of a set under a matrix transformation. For symmetric matrices $A$ and $B$, $A > B$ ($A \geq B$) means $A - B$ is positive definite (positive semi-definite).

## 2. Preliminaries

**Definition 1.** *(Zonotope [34]) An $m$th-order zonotope $Z = \langle c, G \rangle \subset \mathbb{R}^n$ (with center $c \in \mathbb{R}^n$ and generator matrix $G \in \mathbb{R}^{n \times m}$) is the linear image of the hypercube $H^m = [-1, +1]^m$ defined as*

$$Z = c \oplus GH^m = \{c + Gh \mid h \in H^m\},$$

*where $\oplus$ denotes the Minkowski sum.*

**Definition 2.** *($F$-Radius [35]) For a zonotope $Z = \langle c, G \rangle$, its $F$-radius is given by*

$$\|Z\|_F = \|G\|_F = \sqrt{\mathrm{tr}\{G^T G\}},$$

*where $\|\cdot\|_F$ denotes the Frobenius norm.*

**Property 1.** *(Zonotope Operation Properties [35]) Let $\langle c, G \rangle$ and $\langle c_1, G_1 \rangle$ be two $m$-order zonotopes with $c$, $c_1 \in \mathbb{R}^n$ and $G, G_1 \in \mathbb{R}^{n \times m}$. Then the following properties hold:*

$$\begin{aligned} A \odot \langle c, G \rangle &= \langle Ac,\ AG \rangle, \\ \langle c, G \rangle \oplus \langle c_1, G_1 \rangle &= \langle c + c_1,\ [G \ \ G_1] \rangle, \\ \langle c, G \rangle &\subseteq \langle c,\ \mathrm{rs}(G) \rangle, \end{aligned}$$

*where $A$ is a real matrix of compatible dimensions and $\odot$ denotes the linear image operation. The matrix $\mathrm{rs}(G)$ is diagonal with entries $\mathrm{rs}(G)_{pp} = \sum_{q=1}^{m} |G_{p,q}|$ for $p = 1, \ldots, n$, and $G_{p,q}$ is the element in the $p$th row and $q$th column of $G$. Note that $\langle c, \mathrm{rs}(G) \rangle$ is a box-shaped outer bound of $\langle c, G \rangle$.*

**Property 2.** *(Zonotope Reduction [35]) For a zonotope $Z = \langle c, G \rangle$ with $G = [g_1, \ldots, g_m] \in \mathbb{R}^{n \times m}$, let the columns be sorted in nonincreasing order of their Euclidean norms to form a new matrix $\hat{G} = [\hat{g}_1, \ldots, \hat{g}_m]$ such that $\|\hat{g}_j\| \geq \|\hat{g}_{j+1}\|$ with $\hat{g}_j \in \{g_1, g_2, \ldots, g_m\}$. Given an integer $p$ ($n < p < m$), denoting*

$$\begin{aligned} \hat{G}_x &= \begin{bmatrix} \hat{g}_1 & \hat{g}_2 & \cdots & \hat{g}_{p-n} \end{bmatrix}, \\ \hat{G}_y &= \begin{bmatrix} \hat{g}_{p-n+1} & \hat{g}_{p-n+2} & \cdots & \hat{g}_m \end{bmatrix}, \end{aligned}$$

*then we have*

$$\langle c, G \rangle \subseteq \left\langle c, \begin{bmatrix} \hat{G}_x & \mathrm{rs}(\hat{G}_y) \end{bmatrix} \right\rangle.$$

*The zonotope reduction operator $\downarrow_p$is defined as*

$$\downarrow_p \langle c, G \rangle = \begin{cases} \left\langle c, \begin{bmatrix} \hat{G}_x & \mathrm{rs}(\hat{G}_y) \end{bmatrix} \right\rangle, & \text{if } p < m \\ \langle c, G \rangle, & \text{otherwise} \end{cases}$$

*then it can be inferred that $\langle c, G \rangle \subseteq \downarrow_p \langle c, G \rangle = \langle c, \downarrow_p G \rangle$.*

## 3. Problem Formulation

Consider a class of linear discrete-time CNs with $N$ nodes of the following form:

$$x_i(k+1) = A_i(k)x_i(k) + \sum_{j=1}^{N} m_{ij} \Gamma x_j(k) + B_i(k)w_i(k), \tag{1a}$$

$$y_i(k) = C_i(k)x_i(k) + D_i(k)v_i(k), \tag{1b}$$

where $x_i(k) \in \mathbb{R}^{n_x}$ and $y_i(k) \in \mathbb{R}^{n_y}$ are the state vector and the measurement output of the $i$th $(i = 1, 2, \ldots, N)$ node, respectively; $w_i(k) \in \mathbb{R}^{n_w}$ and $v_i(k) \in \mathbb{R}^{n_v}$ denote the process noise and the measurement noise, respectively. The matrix $M = (m_{ij})_{N\times N}$ is the coupling configuration matrix that characterizes the topological structure of the network, with $m_{ij} > 0$ if node $i$ can receive information from node $j$, and $m_{ij} = 0$ otherwise. The matrix $M$ is assumed to be symmetric, i.e. $M = M^{\mathrm{T}}$, and its diagonal entries satisfy

$$m_{ii} = -\sum_{j=1, j\neq i}^{N} m_{ij}.$$

Moreover, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_{n_x}\} \geq 0$ is a known diagonal inner-coupling matrix, which describes the interconnections between the state components of the nodes; specifically, $\gamma_j \neq 0$ indicates that the $j$th state variable is involved in the coupling. The matrices $A_i(k)$, $B_i(k)$, $C_i(k)$ and $D_i(k)$ are known time-varying matrices of appropriate dimensions.

Note that Assumptions 1 and 2 don't have too many restrictions on the actual systems. Most of the systems will not be restricted by these restrictions. Thus, the results obtained in this article is applicable to most systems.

**Assumption 1.** *The initial state $x_i(0)$, the process noise $w_i(k)$, and the measurement noise $v_i(k)$ are assumed to be bounded by the following zonotopes:*

$$x_i(0) \in \langle \tilde{c}_i(0), \tilde{G}_i(0)\rangle, \quad w_i(k) \in \langle 0, I_{n_w}\rangle, \quad v_i(k) \in \langle 0, I_{n_v}\rangle,$$

*where $\tilde{c}_i(0) \in \mathbb{R}^{n_x}$ is a given vector and $\tilde{G}_i(0) \in \mathbb{R}^{n_x\times n_x}$ is a known matrix.*

**Assumption 2.** *For all time steps $k \in \mathbb{N}$ and each node $i \in \{1, \ldots, N\}$, the measurement output $y_i(k)$ is assumed to be bounded by a known positive scalar $\lambda$. Specifically, it holds that*

$$\|y_i(k)\|_\infty \leq \lambda.$$

**Remark 1.** *In this paper, the process noise $w_i(k)$ and measurement noise $v_i(k)$ are characterized as UBB and are encapsulated within zonotopes, a modeling choice well-suited for practical scenarios where only magnitude bounds are known. Among the typical categories of disturbances in state estimation are stochastic noise, energy-bounded noise, and UBB noise. The UBB model is adopted here to reflect realistic bounded uncertainties. Accordingly, a ZSME approach is developed to guarantee that the system state is confined within a computable geometric region at each time instant. Although Assumptions 1 and 2 may appear somewhat restrictive, they are in fact mild and impose only minimal limitations in practice, as prior knowledge of initial-state bounds, noise supports, and measurement ranges is typically available from physical constraints or sensor specifications. Consequently, the theoretical results derived herein retain broad applicability across a wide range of real-world systems.*

Before being transmitted over the network, the measurement signals are quantized using the a logarithmic quantizer, and the mapping of the quantization process is described by

$$q(y_i(k)) = \begin{bmatrix} q_1(y_i^{(1)}(k)) & q_2(y_i^{(2)}(k)) & \cdots & q_{n_y}(y_i^{(n_y)}(k)) \end{bmatrix}^{\mathrm{T}}.$$

For each $q_j(\cdot)$, the set of quantization levels is given by

$$\mathcal{V}_j = \left\{ \pm v_l^{(j)} \mid v_l^{(j)} = (\rho^{(j)})^l v_0^{(j)},\ l = 0, \pm 1, \pm 2, \ldots \right\} \cup \{0\},$$

where $\rho^{(j)}$ $(j = 1, 2, \ldots, n_y)$ denotes the quantization density satisfying $0 < \rho^{(j)} < 1$. The adopted logarithmic quantizer is described as follows:

$$q_j(y_i^{(j)}(k)) = \begin{cases} v_l^{(j)}, & \dfrac{1}{1+\delta_j} v_l^{(j)} < y_i^{(j)}(k) \leq \dfrac{1}{1-\delta_j} v_l^{(j)}, \\ 0, & y_i^{(j)}(k) = 0, \\ -q_j(-y_i^{(j)}(k)), & y_i^{(j)}(k) < 0, \end{cases}$$

where $\delta_j = \dfrac{1-\rho^{(j)}}{1+\rho^{(j)}}$. From the above definition, we have

$$q_j(y_i^{(j)}(k)) = (1+\Delta(k)^{(j)})y_i^{(j)}(k) \quad \text{with } |\Delta(k)^{(j)}| \leq \delta_j.$$

Setting $\Delta(k) = \mathrm{diag}\{\Delta(k)^{(1)}, \Delta(k)^{(2)}, \ldots, \Delta(k)^{(n_y)}\} \leq \delta I_{n_y}$, the quantized measurements can be rewritten as

$$\tilde{y}_i(k) = q(y_i(k)) = (I+\Delta(k))y_i(k).$$

The quantization error $\bar{y}_i(k)$ is defined as

$$\bar{y}_i(k) = \tilde{y}_i(k) - y_i(k),$$

which yields

$$\bar{y}_i(k) = \Delta(k)y_i(k).$$

Recalling Assumption 2, we have

$$\|\bar{y}_i(k)\|_\infty \leq \|\Delta(k)\|_\infty \|y_i(k)\|_\infty \leq \delta\lambda.$$

Thus, it can be easily obtained that

$$\bar{y}_i(k) \in \langle 0,\ \Lambda_i \rangle, \tag{2}$$

where $\Lambda_i = \delta\lambda I_{n_y}$.

To avoid data collisions and improve communication resource utilization, only one sensor is allowed to transmit its measurement output at each sampling instant. To this end, the WTOD protocol is adopted to schedule the transmission of measurement signals. In this paper, the measurement output of each sensor node $i$ ($i = 1, \ldots, N$) to be transmitted is assumed to consist of $n_y$ components, i.e., $\tilde{y}_i(k) = \begin{bmatrix} \tilde{y}_i^{(1)}(k) & \tilde{y}_i^{(2)}(k) & \cdots & \tilde{y}_i^{(n_y)}(k) \end{bmatrix}^{\mathrm{T}}$.

Let $\sigma_i(k) \in \{1, \ldots, n_y\}$ denote the selected component of node $i$ for transmitting its measurement data at time instant $k$, governed by the following selection principle:

$$\sigma_i(k) \triangleq \min\left\{ \arg \max_{1 \leq l \leq n_y} e_i^{(l)}(k)^T P_i^{(l)} e_i^{(l)}(k) \right\} \tag{3}$$

with

$$e_i^{(l)}(k) \triangleq \tilde{y}_i^{(l)}(k) - \vec{y}_i^{(l)}(k-1),$$

where $\vec{y}_i^{(l)}(k-1)$ is the last transmitted component from node $i$ before time instant $k$, and $P_i^{(l)} > 0$ is a scalar weighting coefficient assigned to the $l$-th component of node $i$.

Under the WTOD protocol, the measurement signal transmitted from sensor $i$ via a zero-order holder is denoted by

$$\vec{y}_i^{(l)}(k) = \begin{cases} \tilde{y}_i^{(l)}(k), & \text{if} \quad l = \sigma_i(k) \\ \vec{y}_i^{(l)}(k-1), & \text{otherwise.} \end{cases}$$

It is worth noting that the signal is also affected by noise in the transmission channel during transmission. Thus, the actual measurement signal received by estimator $i$ is given by

$$\vec{y}_i(k) = \Phi_{\sigma_i(k)}\tilde{y}_i(k) + (I - \Phi_{\sigma_i(k)})\vec{y}_i(k-1) + o_i(k), \tag{4}$$

where $\vec{y}_i(k) = \begin{bmatrix} \vec{y}_i^{(1)}(k) & \cdots & \vec{y}_i^{(n_y)}(k) \end{bmatrix}^{\mathrm{T}}$ denotes the vector of received signals of estimator $i$, the matrix $\Phi_{\sigma_i(k)} = \mathrm{diag}\{\delta(\sigma_i(k)-1), \delta(\sigma_i(k)-2), \ldots, \delta(\sigma_i(k)-n_y)\}$ with $\delta(\cdot)$ being the Kronecker delta function, and $o_i(k) \in \mathbb{R}^{n_y}$ is the transmission noise satisfying $o_i(k) \in \langle 0, I_{n_y} \rangle$.

Let $\zeta_i(k) \triangleq \mathrm{col}\{x_i(k), \vec{y}_i(k-1)\} \in \mathbb{R}^{\bar{n}}$ with $\bar{n} = n_x + n_y$, and define the augmented disturbance vector as

$$\eta_i(k) \triangleq \mathrm{col}\{w_i(k), v_i(k), \bar{y}_i(k), o_i(k)\}.$$

Then, combining (1a), (1b) and (4), the augmented system can be represented as

$$\zeta_i(k+1) = \mathcal{A}_i(k)\zeta_i(k) + \sum_{j=1}^{N} m_{ij}\bar{\Gamma}\,\zeta_j(k) + \mathcal{B}_i(k)\eta_i(k), \tag{5a}$$

$$\vec{y}_i(k) = \mathcal{C}_i(k)\zeta_i(k) + \mathcal{D}_i(k)\eta_i(k), \tag{5b}$$

where

$$\begin{aligned}
\mathcal{A}_i(k) &= \begin{bmatrix} A_i(k) & 0 \\ \Phi_{\sigma_i(k)}C_i(k) & I - \Phi_{\sigma_i(k)} \end{bmatrix}, \quad \bar{\Gamma} = \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix}, \\
\mathcal{B}_i(k) &= \begin{bmatrix} B_i(k) & 0 & 0 & 0 \\ 0 & \Phi_{\sigma_i(k)}D_i(k) & \Phi_{\sigma_i(k)} & I_{n_y} \end{bmatrix}, \\
\mathcal{C}_i(k) &= \begin{bmatrix} \Phi_{\sigma_i(k)}C_i(k) & I - \Phi_{\sigma_i(k)} \end{bmatrix}, \\
\mathcal{D}_i(k) &= \begin{bmatrix} 0 & \Phi_{\sigma_i(k)}D_i(k) & \Phi_{\sigma_i(k)} & I_{n_y} \end{bmatrix}.
\end{aligned}$$

In addition, for $t \le 0$ and $i \in \{1, \ldots, N\}$, the received measurement satisfies $\vec{y}_i(t) = y_i(0)$, where $y_i(0)$ is the initial measurement. Consequently, $\vec{y}_i(t)$ is bounded by a known zonotope, i.e., $\vec{y}_i(t) \in \langle y_i(0), I_{n_y}\rangle$. Together with Assumption 1, the augmented initial state $\zeta_i(0)$ is then enclosed by the following zonotope:

$$\zeta_i(0) \in \left\langle \begin{bmatrix} \tilde{c}_i(0) \\ y_i(0) \end{bmatrix}, \begin{bmatrix} \tilde{G}_i(0) & 0 \\ 0 & I_{n_y} \end{bmatrix} \right\rangle \triangleq \langle c_i(0),\, G_i(0)\rangle. \tag{6}$$

Moreover, under Assumption 1, the augmented disturbance vector $\eta_i(k)$ can also be enclosed by the following zonotope:

$$\eta_i(k) \in \langle 0, H_i\rangle, \tag{7}$$

where $H_i = \mathrm{blkdiag}\{I_{n_w},\, I_{n_v},\, \Lambda_i,\, I_{n_y}\}$.

In order to obtain a zonotope $\hat{\mathcal{Z}}_i(k)$ that encloses the state of system (5a), the following ZSME method is proposed:

(S1) Prediction: Given system (5a), obtain a predicted zonotope $\bar{\mathcal{Z}}_i(k)$ such that $\zeta_i(k) \in \bar{\mathcal{Z}}_i(k)$ at every moment $k$.

(S2) Measurement: Based on the quantized and protocol-based measurement $\vec{y}_i(k)$, obtain a set $\mathcal{Z}_{\vec{y}_i(k)}$ such that $\zeta_i(k) \in \mathcal{Z}_{\vec{y}_i(k)}$.

(S3) Correction: Obtain a zonotope $\mathcal{Z}_i(k)$ as the outer approximation of the intersection between $\bar{\mathcal{Z}}_i(k)$ and $\mathcal{Z}_{\vec{y}_i(k)}$, i.e., $\bar{\mathcal{Z}}_i(k) \cap \mathcal{Z}_{\vec{y}_i(k)} \subseteq \mathcal{Z}_i(k)$.

This paper aims to achieve two main objectives: (i) to obtain the zonotope $\mathcal{Z}_i(k)$; and (ii) to investigate the stability performance of the proposed ZSME method by examining the boundedness of $\|\mathcal{Z}_i(k)\|_F^2$.

**Remark 2.** *Equation* (4) *models the measurement received by the estimator under the WTOD protocol and transmission noise. The diagonal matrix* $\Phi_{\sigma_i(k)} \in \mathbb{R}^{n_y \times n_y}$ *is defined as* $\Phi_{\sigma_i(k)} = \mathrm{diag}\{\delta(\sigma_i(k)-1), \delta(\sigma_i(k)-2), \ldots, \delta(\sigma_i(k)-n_y)\}$, *where* $\delta(\cdot)$ *is the Kronecker delta. It selects the component* $\sigma_i(k)$ *(determined by the WTOD rule in* (3)*) to be updated with the current quantized measurement* $\bar{y}_i(k)$*, while the remaining components keep their previous transmitted values via a zero-order holder, i.e.,* $(I - \Phi_{\sigma_i(k)})\bar{y}_i(k-1)$*. The term* $o_i(k) \in \mathbb{R}^{n_y}$ *denotes the transmission noise, which is assumed to be UBB and satisfies* $o_i(k) \in \langle 0, I_{n_y}\rangle$*. This formulation captures the essential features of communication constraints (WTOD scheduling) and channel imperfections within a unified set-membership estimation framework.*

**Remark 3.** *Due to the limited bandwidth in communication networks, it is essential to adopt appropriate protocols for coordinating data transmission and alleviating communication congestion. The WTOD protocol defined in* (4) *serves this purpose by scheduling the transmission order among sensors based on assigned weights, where only the sensor with the highest priority is permitted to broadcast its measurement at each sampling instant. In this work, we investigate the ZSME problem in the presence of both the quantization effect and the WTOD protocol, thereby extending the applicability of the SME methodology to more complex networked environments.*

## 4. Main Results

### *4.1. Zonotopic Set-Membership Estimation*

With the proposed ZSME approach, the subsequent theorem constructs a zonotope that contains the system state of each node.

**Theorem 1.** *Given a zonotope $\mathcal{Z}_i(k)$ satisfying*

$$\begin{aligned} \mathcal{Z}_i(k) &=\downarrow_p \hat{\mathcal{Z}}_i(k) = \langle c_i(k), G_i(k) \rangle, \\ \hat{\mathcal{Z}}_i(k) &= \langle c_i(k), \hat{G}_i(k) \rangle, \quad G_i(k) =\downarrow_p \hat{G}_i(k), \end{aligned} \tag{8}$$

*where*

$$\begin{aligned} c_i(k) &= (I - K_i(k)\mathcal{C}_i(k))\bar{c}_i(k) + K_i(k)\vec{y}_i(k), \\ \hat{G}_i(k) &= \big[(I - K_i(k)\mathcal{C}_i(k))\bar{G}_i(k) \quad -K_i(k)\mathcal{D}_i(k)H_i\big], \\ \bar{c}_i(k) &= \mathcal{A}_i(k-1)c_i(k-1) + \sum_{j=1}^{N} m_{ij}\bar{\Gamma}c_j(k-1), \\ \bar{G}_i(k) &= [\mathcal{A}_i(k-1)G_i(k-1) \ \widehat{T}_i \ \mathcal{B}_i(k-1)H_i], \\ \widehat{T}_i &\triangleq [\tilde{T}_{i,1} \ \cdots \ \tilde{T}_{i,N}], \\ \tilde{T}_{i,j} &\triangleq m_{ij}\bar{\Gamma}G_j(k-1), \quad j = 1, 2, \ldots, N, \end{aligned} \tag{9}$$

*and $K_i(k)$ is a correction matrix, then one obtains*

$$\zeta_i(k) \in \mathcal{Z}_i(k).$$

**Proof.** The proof of the theorem is carried out by mathematical induction. From (6), it follows that $\zeta_i(0) \in \mathcal{Z}_i(0)$. Supposing that $\zeta_i(k-1) \in \mathcal{Z}_i(k-1)$, we aim to verify $\zeta_i(k) \in \mathcal{Z}_i(k)$.

To this end, we first construct the prediction zonotope $\bar{\mathcal{Z}}_i(k)$ as described in (S1) and the measurement set $\mathcal{Z}_{\vec{y}_i(k)}$ as in (S2) for the ZSME method.

With (5a), (7) and Property 1, we obtain

$$\zeta_i(k) \in \ \mathcal{A}_i(k-1) \odot \mathcal{Z}_i(k-1) \oplus \sum_{j=1}^{N} m_{ij}\bar{\Gamma} \odot \mathcal{Z}_j(k-1) \oplus \mathcal{B}_i(k-1) \odot \langle 0, H_i \rangle,$$

where

$$\begin{aligned} \sum_{j=1}^{N} m_{ij}\bar{\Gamma} \odot \mathcal{Z}_j(k-1) &= m_{i1}\bar{\Gamma} \odot \mathcal{Z}_1(k-1) \oplus m_{i2}\bar{\Gamma} \odot \mathcal{Z}_2(k-1) \oplus \cdots \oplus m_{iN}\bar{\Gamma} \odot \mathcal{Z}_N(k-1) \\ &= \left\langle \sum_{j=1}^{N} m_{ij}\bar{\Gamma}c_j(k-1), \widehat{T}_i \right\rangle. \end{aligned}$$

Thus, it follows that

$$\zeta_i(k) \in \langle \bar{c}_i(k), \bar{G}_i(k) \rangle = \bar{\mathcal{Z}}_i(k) \tag{10}$$

with $\bar{c}_i(k)$ and $\bar{G}_i(k)$ defined in (9).

Combining (5b) and (7) yields

$$\vec{y}_i(k) - \mathcal{C}_i(k)\zeta_i(k) \in \ \mathcal{D}_i(k) \odot \langle 0, H_i \rangle = \langle 0, \mathcal{D}_i(k)H_i \rangle.$$

Accordingly, the measurement set $\mathcal{Z}_{\vec{y}_i(k)}$ can be obtained as

$$\zeta_i(k) \in \mathcal{Z}_{\vec{y}_i(k)} = \{\zeta_i(k) | \vec{y}_i(k) - \mathcal{C}_i(k)\zeta_i(k) \in \langle 0, \mathcal{D}_i(k)H_i \rangle\}. \tag{11}$$

Next, we proceed to verify $\zeta_i(k) \in \hat{\mathcal{Z}}_i(k)$. According to Definition 1 and $\zeta_i(k) \in \bar{\mathcal{Z}}_i(k)$ in (10), there is a vector $p_1 \in [-1,1]^{m_1}$ such that

$$\zeta_i(k) = \bar{c}_i(k) + \bar{G}_i(k)p_1. \tag{12}$$

Similarly, from $\zeta_i(k) \in \mathcal{Z}_{\vec{y}_i(k)}$ in (11), there is another vector $p_2 \in [-1,1]^{m_2}$ to guarantee

$$\mathcal{C}_i(k)\zeta_i(k) = \vec{y}_i(k) - \mathcal{D}_i(k)H_i p_2. \tag{13}$$

In addition, from (10) and (11), it follows that

$$\zeta_i(k) \in \bar{\mathcal{Z}}_i(k) \cap \mathcal{Z}_{\vec{y}_i(k)}. \tag{14}$$

With the introduction of a correction matrix $K_i(k)$ and the incorporation of zero term

$$K_i(k)\mathcal{C}_i(k)\bar{G}_i(k)p_1 - K_i(k)\mathcal{C}_i(k)\bar{G}_i(k)p_1$$

into (12), we obtain

$$\zeta_i(k) = \bar{c}_i(k) + (I - K_i(k)\mathcal{C}_i(k))\bar{G}_i(k)p_1 + K_i(k)\mathcal{C}_i(k)\bar{G}_i(k)p_1. \tag{15}$$

Multiplying (12) on both sides by $K_i(k)\mathcal{C}_i(k)$ gives

$$K_i(k)\mathcal{C}_i(k)\zeta_i(k) = K_i(k)\mathcal{C}_i(k)\bar{c}_i(k) + K_i(k)\mathcal{C}_i(k)\bar{G}_i(k)p_1,$$

which, combined with (13), leads to

$$K_i(k)(\vec{y}_i(k) - \mathcal{D}_i(k)H_i p_2) = K_i(k)\mathcal{C}_i(k)\bar{c}_i(k) + K_i(k)\mathcal{C}_i(k)\bar{G}_i(k)p_1. \tag{16}$$

Substituting (16) into (15), it is clear that

$$\begin{aligned}\zeta_i(k) &= (I - K_i(k)\mathcal{C}_i(k))\bar{G}_i(k)p_1 + K_i(k)\vec{y}_i(k) + \bar{c}_i(k) - K_i(k)\mathcal{C}_i(k)\bar{c}_i(k) - K_i(k)\mathcal{D}_i(k)H_i p_2 \\ &= c_i(k) + \hat{G}_i(k)p \\ &\in \langle c_i(k), \hat{G}_i(k)\rangle \triangleq \hat{\mathcal{Z}}_i(k)\end{aligned}$$

with $p = [p_1^T \quad p_2^T]^T$.

At last, noting (14), it can be deduced that

$$\bar{\mathcal{Z}}_i(k) \cap \mathcal{Z}_{\vec{y}_i(k)} \subseteq \hat{\mathcal{Z}}_i(k).$$

From Property 2, we have $\hat{\mathcal{Z}}_i(k) \subseteq \mathcal{Z}_i(k) =\downarrow_p \hat{\mathcal{Z}}_i(k)$, which means $\zeta_i(k) \in \mathcal{Z}_i(k)$. Thus, the original state $x_i(k)$ in (1a) satisfies

$$x_i(k) \in \begin{bmatrix} I_{n_x} & 0 \end{bmatrix} \odot \mathcal{Z}_i(k).$$

□

Clearly, the size of the zonotope $\hat{\mathcal{Z}}_i(k)$ depends on the correction matrix $K_i(k)$. Hence, the next goal is to choose an appropriate $K_i(k)$ that minimizes the size of $\hat{\mathcal{Z}}_i(k)$ as measured by its $F$-radius.

**Theorem 2.** *The size of zonotope $\hat{\mathcal{Z}}_i(k)$ is minimized by the following correction matrix*

$$K_i(k) = S_i(k)L_i(k)^{-1},$$

*where*

$$\begin{aligned}S_i(k) &= \bar{G}_i(k)\bar{G}_i(k)^T\mathcal{C}_i(k)^T, \\ L_i(k) &= \mathcal{C}_i(k)\bar{G}_i(k)\bar{G}_i(k)^T\mathcal{C}_i(k)^T + \mathcal{D}_i(k)H_iH_i^T\mathcal{D}_i(k)^T.\end{aligned}$$

**Proof.** From Definition 2 and (8)–(9), one can infer that

$$\begin{aligned}\|\hat{\mathcal{Z}}_i(k)\|_F^2 &= \|\hat{G}_i(k)\|_F^2 \\ &= \mathrm{tr}\{\hat{G}_i(k)^T\hat{G}_i(k)\} \\ &= \mathrm{tr}\{\bar{G}_i(k)\bar{G}_i(k)^T\} + \mathrm{tr}\{K_i(k)\mathcal{C}_i(k)\bar{G}_i(k)\bar{G}_i(k)^T\mathcal{C}_i(k)^TK_i(k)^T\} \\ &\quad - \mathrm{tr}\{K_i(k)\mathcal{C}_i(k)\bar{G}_i(k)\bar{G}_i(k)^T\} - \mathrm{tr}\{\bar{G}_i(k)\bar{G}_i(k)^T\mathcal{C}_i(k)^TK_i(k)^T\} \\ &\quad + \mathrm{tr}\{K_i(k)\mathcal{D}_i(k)H_iH_i^T\mathcal{D}_i(k)^TK_i(k)^T\}.\end{aligned} \tag{17}$$

Taking the partial derivative of $\|\hat{\mathcal{Z}}_i(k)\|_F^2$ with respect to $K_i(k)$, we have

$$\frac{\partial\|\hat{\mathcal{Z}}_i(k)\|_F^2}{\partial K_i(k)} = 2K_i(k)\mathcal{C}_i(k)\bar{G}_i(k)\bar{G}_i(k)^T\mathcal{C}_i(k)^T + 2K_i(k)\mathcal{D}_i(k)H_iH_i^T\mathcal{D}_i(k)^T - 2\bar{G}_i(k)\bar{G}_i(k)^T\mathcal{C}_i(k)^T.$$

To minimize $\|\hat{\mathcal{Z}}_i(k)\|_F^2$, we set

$$\frac{\partial\|\hat{\mathcal{Z}}_i(k)\|_F^2}{\partial K_i(k)} = 0,$$

which yields

$$K_i(k) = S_i(k)L_i(k)^{-1}. \tag{18}$$

□

For ease of later implementation, the estimation strategy proposed above is summarized in the algorithm below.

**Remark 4.** *Algorithm 1 seamlessly integrates the results of Theorem 1 and Theorem 2. In Step 2, the predicted center $\bar{c}_i(k)$ and generator matrix $\bar{G}_i(k)$ are constructed according to Theorem 1, which guarantees that the true augmented state $\zeta_i(k)$ lies in the predicted zonotope $\bar{\mathcal{Z}}_i(k)$. Step 3 adopts the optimal correction matrix $K_i(k)$ derived in Theorem 2, thereby minimizing the $F$-radius of the intermediate zonotope $\hat{\mathcal{Z}}_i(k)$ at each time instant. Step 4 then applies the reduction operator $\downarrow_p$ to obtain the final zonotope $\mathcal{Z}_i(k)$; this step preserves the inclusion property while controlling the growth of the generator matrix order. Consequently, Algorithm 1 produces, at every step, the smallest possible zonotopic enclosure (in the $F$-radius sense) that is guaranteed to contain the true state, striking a balance between estimation accuracy and computational efficiency.*

---

**Algorithm 1:** ZSME algorithm

---

**Step 1.** Set $k = 0$ and let $\zeta_i(0) \in \langle c_i(0),\, G_i(0)\rangle$.

**Step 2.** Compute the predicted center $\bar{c}_i(k)$ and generator matrix $\bar{G}_i(k)$ according to (9).

**Step 3.** Design the correction matrix $K_i(k)$ according to (18).

**Step 4.** Compute the corrected zonotope $\hat{\mathcal{Z}}_i(k) = \langle c_i(k), \hat{G}_i(k)\rangle$ via (9), and then apply the reduction operator $\downarrow_p$ to obtain $\mathcal{Z}_i(k) =\downarrow_p \hat{\mathcal{Z}}_i(k) = \langle c_i(k), G_i(k)\rangle$.

**Step 5.** Extract the original state estimate and its bounds from the projection of $\mathcal{Z}_i(k)$ onto the first $n_x$ coordinates:
$$\begin{cases}\underline{x}_i(k) = [I_{n_x}\ 0]c_i(k) - \big|[I_{n_x}\ 0]G_i(k)\big|\mathbf{1}, \\ \overline{x}_i(k) = [I_{n_x}\ 0]c_i(k) + \big|[I_{n_x}\ 0]G_i(k)\big|\mathbf{1}.\end{cases}$$
**Step 6.** Set $k = k+1$, and return to Step 2.

---

*4.2. Boundedness Analysis*

**Lemma 1.** *Let $\langle c, \hat{G}\rangle \subseteq \langle c, G\rangle$ with $G =\downarrow_p \hat{G}$, where $c \in \mathbb{R}^{\bar{n}}$ and $\hat{G} \in \mathbb{R}^{\bar{n}\times q}$. Then there exists a positive constant $\tau > 0$ such that*

$$\|G\|_F^2 - \|\hat{G}\|_F^2 \le \tau\|\hat{G}\|_F^2,$$

*where*

$$\tau = \frac{(q-p+\bar{n}-1)(q-p+\bar{n})}{q}$$

*and $\bar{n} \le p \le q$.*

**Proof.** This proof follows straightforwardly from the approach presented in [35]; therefore, the detailed steps are omitted. □

**Assumption 3.** *There exist positive scalars $\bar{a}$, $\bar{b}$, $\bar{r}$, $\bar{h}$ and $\xi(0)$ such that for any $i = 1, 2, \ldots, N$ and $k \geq 0$,*

$$\mathcal{A}_i(k)\mathcal{A}_i^T(k) \leq \bar{a}^2 I_{\bar{n}}, \quad \mathcal{B}_i(k)\mathcal{B}_i^T(k) \leq \bar{b}^2 I_{\bar{n}}, \quad \bar{\Gamma}\bar{\Gamma}^T \leq \bar{r}^2 I_{\bar{n}},$$
$$H_i H_i^T \leq \bar{h}^2 I_{n_w+n_v+2n_y}, \quad G_i(0)G_i(0)^T \leq \frac{\xi(0)}{\bar{n}} I_{\bar{n}}.$$

**Theorem 3.** *Consider the systems (5a)–(5b) under Algorithm 1. Based on Assumption 3, there exists an upper bound $\xi(k)$ such that*

$$\|\mathcal{Z}_i(k)\|_F^2 \leq \xi(k), \quad \forall k \geq 0,$$

*where*

$$\begin{aligned}
\xi(k) &= (1+\tau(k))\bar{n}[(\bar{a}^2 + N\bar{r}^2)\xi(k-1) + \bar{h}^2\bar{b}^2],\\
\tau(k) &= \frac{(q(k)-p+\bar{n}-1)(q(k)-p+\bar{n})}{q(k)},\\
q(k) &= \bar{q}(k) + n_w + 2n_y + n_v,\\
\bar{q}(k) &= (N+1)p + n_w + 2n_y + n_v.
\end{aligned} \tag{19}$$

*Furthermore, if*

$$\kappa = (Np + 2n_w + 4n_y + 2n_v + \bar{n})\bar{n}(\bar{a}^2 + N\bar{r}^2) < 1, \tag{20}$$

*then there exists a positive constant $\pi$ such that*

$$\|\mathcal{Z}_i(k)\|_F^2 \leq \xi(k) \leq \pi \quad \forall k \geq 0. \tag{21}$$

**Proof.** To begin, we use mathematical induction to demonstrate $\|\mathcal{Z}_i(k)\|_F^2 = \|G_i(k)\|_F^2 \leq \xi(k)$. By Assumption 3, it is true for $k = 0$: $\|\mathcal{Z}_i(0)\|_F^2 = \|G_i(0)\|_F^2 \leq \xi(0)$. Then, as the induction hypothesis, we take $\|\mathcal{Z}_i(k-1)\|_F^2 = \|G_i(k-1)\|_F^2 \leq \xi(k-1)$.

It follows from (9) that

$$\begin{aligned}
&\hat{G}_i(k)\hat{G}_i(k)^T\\
&= \left[(I - K_i(k)\mathcal{C}_i(k))\bar{G}_i(k) \quad -K_i(k)\mathcal{D}_i(k)H_i\right]\left[(I - K_i(k)\mathcal{C}_i(k))\bar{G}_i(k) \quad -K_i(k)\mathcal{D}_i(k)H_i\right]^T\\
&= (I - K_i(k)\mathcal{C}_i(k))\bar{G}_i(k)\bar{G}_i(k)^T(I - K_i(k)\mathcal{C}_i(k))^T + K_i(k)\mathcal{D}_i(k)H_iH_i^T\mathcal{D}_i(k)^T K_i(k)^T.
\end{aligned} \tag{22}$$

Substituting (18) into (22), we obtain

$$\hat{G}_i(k)\hat{G}_i(k)^T = \bar{G}_i(k)\bar{G}_i(k)^T - S_i(k)L_i(k)^{-1}S_i(k)^T.$$

Since the last term is positive semi-definite, we have

$$\hat{G}_i(k)\hat{G}_i(k)^T \leq \bar{G}_i(k)\bar{G}_i(k)^T, \quad k \geq 0.$$

According to (9) and Assumption 3, we have

$$\begin{aligned}
\bar{G}_i(k)\bar{G}_i(k)^T &= \left[\mathcal{A}_i(k-1)G_i(k-1) \quad \widehat{T}_i \quad \mathcal{B}_i(k-1)H_i\right]\left[\mathcal{A}_i(k-1)G_i(k-1) \quad \widehat{T}_i \quad \mathcal{B}_i(k-1)H_i\right]^T\\
&= \mathcal{A}_i(k-1)G_i(k-1)G_i(k-1)^T\mathcal{A}_i(k-1)^T + \widehat{T}_i\widehat{T}_i^T + \mathcal{B}_i(k-1)H_iH_i^T\mathcal{B}_i(k-1)^T\\
&\leq \bar{a}^2\xi(k-1)I_{\bar{n}} + N\bar{r}^2\xi(k-1)I_{\bar{n}} + \bar{h}^2\bar{b}^2 I_{\bar{n}}.
\end{aligned}$$

Hence,

$$\mathrm{tr}\{\hat{G}_i(k)\hat{G}_i(k)^T\} \le \mathrm{tr}\{\bar{G}_i(k)\bar{G}_i(k)^T\} \le \bar{n}[(\bar{a}^2 + N\bar{r}^2)\xi(k-1) + \bar{h}^2\bar{b}^2].$$

From Lemma 1 , it follows that

$$\begin{aligned}\|\mathcal{Z}_i(k)\|_F^2 &= \|G_i(k)\|_F^2 \\ &\le (1+\tau(k))\|\hat{G}_i(k)\|_F^2 \\ &\le (1+\tau(k))\bar{n}[(\bar{a}^2 + N\bar{r}^2)\xi(k-1) + \bar{h}^2\bar{b}^2] \\ &= \xi(k)\end{aligned}$$

with $\xi(k)$ and $\tau(k)$ defined in (19).

Secondly, we demonstrate that $\xi(k)$ has an upper bound. From the expression of $\bar{G}_i(k)$ in (9), one can observe that there exists some time step $t$ for which $\bar{q}(k) = (N+1)p + n_w + 2n_y + n_v$ holds whenever $k \ge t$.

According to Lemma 1, one has

$$\begin{aligned}1 + \tau(k) &< q(k) - p + \bar{n} \\ &= \bar{q}(k) + n_w + 2n_y + n_v - p + \bar{n} \\ &= Np + 2(n_w + 2n_y + n_v) + \bar{n}, \quad \forall k \ge t.\end{aligned} \tag{23}$$

Together (19) and (23), we have

$$\xi(k) < \kappa\xi(k-1) + \kappa\beta$$

with $\varsigma$ defined in (20) and $\beta = \frac{\bar{h}^2\bar{b}^2}{\bar{a}^2 + N\bar{r}^2}$.

We introduce an auxiliary sequence $\Xi(k)$ defined by

$$\Xi(k) = \kappa\Xi(k-1) + \kappa\beta, \quad k \ge t,$$

with the initial condition $\Xi(t) = \xi(t)$. Suppose that $\xi(k-1) \le \Xi(k-1)$ holds. Then, by mathematical induction, we obtain $\xi(k) \le \Xi(k)$. Given the condition $\kappa < 1$ from (20), it follows that

$$\limsup_{k\to\infty} \xi(k) \le \frac{\kappa\beta}{1-\kappa},$$

which implies the existence of a constant $\varepsilon$ such that $\xi(k) \le \Xi(k) \le \varepsilon$ for all $k - 1 \ge t$.

Moreover, for the finite time horizon $k \in [0, t)$, the boundedness of $\xi(k)$ is automatically assured. Consequently, from the above arguments, we conclude that $\xi(k)$ remains bounded for every $k \ge 0$. In other words, when $\kappa < 1$, the result (21) follows directly. □

**Remark 5.** *This paper provides a thorough investigation of the ZSME problem for CNs under the effects of quantization and WTOD protocol. First, in Theorem 1, we construct the desired zonotopes that are guaranteed to cover all possible system states. Then, Theorem 2 shows how the sizes of these zonotopes can be minimized by appropriately selecting the correction matrices. Finally, Theorem 3 examines the boundedness of the obtained zonotopes and offers a set of sufficient conditions to ensure it.*

**Remark 6.** *Up to now, a ZSME algorithm has been developed for time-varying systems in the presence of quantization and protocol effects. Relative to existing state estimation results, the key innovations of this work can be summarized as follows: (i) the ZSME problem for CNs is investigated for the first time, taking into account both logarithmic quantization and the WTOD protocol; (ii) the proposed ZSME algorithm is capable of achieving the minimal zonotope sizes at each time step; and (iii) the boundedness of the resulting zonotopes is rigorously guaranteed through the derivation of sufficient conditions.*

## 5. Numerical Example

In this section, the performance of the proposed ZSME algorithm is validated through a numerical example. Consider the discrete-time system defined by (1a) and (1b), with the parameters given as follows:

$$A_1(k) = \begin{bmatrix} 0.90 & 0.84\cos(k) \\ 0.84 & 0.16 \end{bmatrix}, \qquad A_2(k) = \begin{bmatrix} 0.90 & 0.87\cos(k) \\ 0.88 & 0.17 \end{bmatrix}, \qquad A_3(k) = \begin{bmatrix} 0.84 & -0.83\cos(k) \\ 0.86 & -0.17 \end{bmatrix},$$

$$B_1(k) = \begin{bmatrix} 0.85 \\ -0.2 + 0.15\sin(k) \end{bmatrix}, \qquad B_2(k) = \begin{bmatrix} 0.90 \\ -0.22 + 0.15\cos(k) \end{bmatrix}, \qquad B_3(k) = \begin{bmatrix} 0.92 \\ -0.2 + 0.15\sin(k) \end{bmatrix},$$

$$C_1(k) = \begin{bmatrix} 0.70\sin(k) & 0.58 \\ 0.45 & 0.75 \end{bmatrix}, \qquad C_2(k) = \begin{bmatrix} 0.75\sin(k) & 0.73 \\ 0.60 & -0.94 \end{bmatrix}, \qquad C_3(k) = \begin{bmatrix} 0.80\sin(k) & 0.72 \\ 0.48 & 0.82 \end{bmatrix},$$

$$D_1(k) = \begin{bmatrix} 0.6\cos(k) & 0.5 \\ -0.2\sin(k) & 0.7 \end{bmatrix}, \qquad D_2(k) = \begin{bmatrix} -0.7\cos(k) & 0.4 \\ -0.1\sin(k) & -0.7 \end{bmatrix}, \qquad D_3(k) = \begin{bmatrix} -0.8\cos(k) & 0.4 \\ 0.1\sin(k) & 0.7 \end{bmatrix},$$

$$M = \begin{bmatrix} -0.4 & 0.25 & 0.15 \\ 0.4 & -0.55 & 0.15 \\ 0.1 & 0.25 & -0.35 \end{bmatrix} \qquad \Gamma = \mathrm{diag}(0.1, 0.3).$$

The initial conditions are chosen as follows: $\tilde{c}_1(0) = x_1(0) = \begin{bmatrix} 1.2 & 0.5 \end{bmatrix}^T$, $\tilde{c}_2(0) = x_2(0) = \begin{bmatrix} -0.9 & 1.0 \end{bmatrix}^T$, $\tilde{c}_3(0) = x_3(0) = \begin{bmatrix} 1.0 & -0.6 \end{bmatrix}^T$, $\tilde{G}_1(0) = \tilde{G}_2(0) = \tilde{G}_3(0) = 0.02I_2$. For the logarithmic quantizer, the parameters are given by $\rho = 0.87$, $\delta = 0.0695$, $\lambda = 12$, and the WTOD weights are uniformly selected as $P_i^{(l)} = 1$ $(i = 1, 2, 3; l = 1, 2)$. Moreover, the zonotope reduction order is specified as $p = 35$, and the simulation horizon is $T = 150$.

With the above parameters, the proposed ZSME algorithm yields, for each of the three nodes, the desired zonotopes $\mathcal{Z}_i(k)$, the point estimates, as well as the upper and lower bounds of the system states. Following Assumption 1, the corresponding simulation outcomes are presented in Figures 1–3. Specifically, Figure 1 presents the estimate, the upper bound, and the lower bound of two components of the actual state for the 1st node. Similarly, the corresponding outcomes for nodes 2 and 3 are provided in Figures 2 and 3. By examining the curves of the true states and their estimates for each node, it can be seen that all actual states are accurately tracked. Moreover, it is evident that the real state always lies between its upper and lower bounds for every node, confirming that the true states are indeed enclosed within the obtained zonotopes.

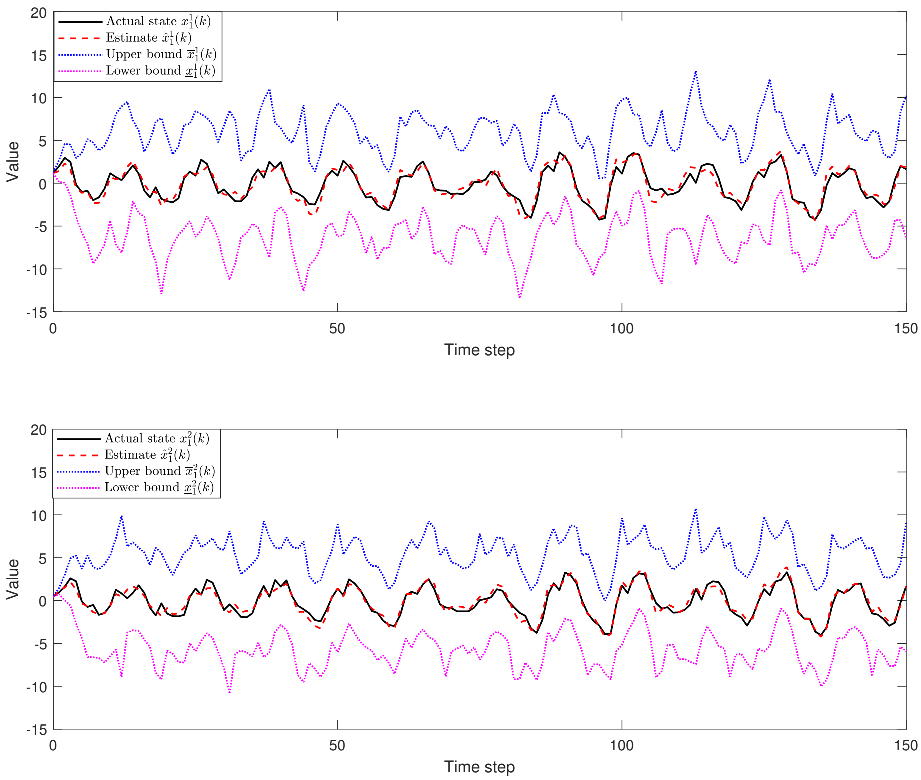


**Figure 1.** State $x_1^1$, $x_1^2$, their estimates and bounds.

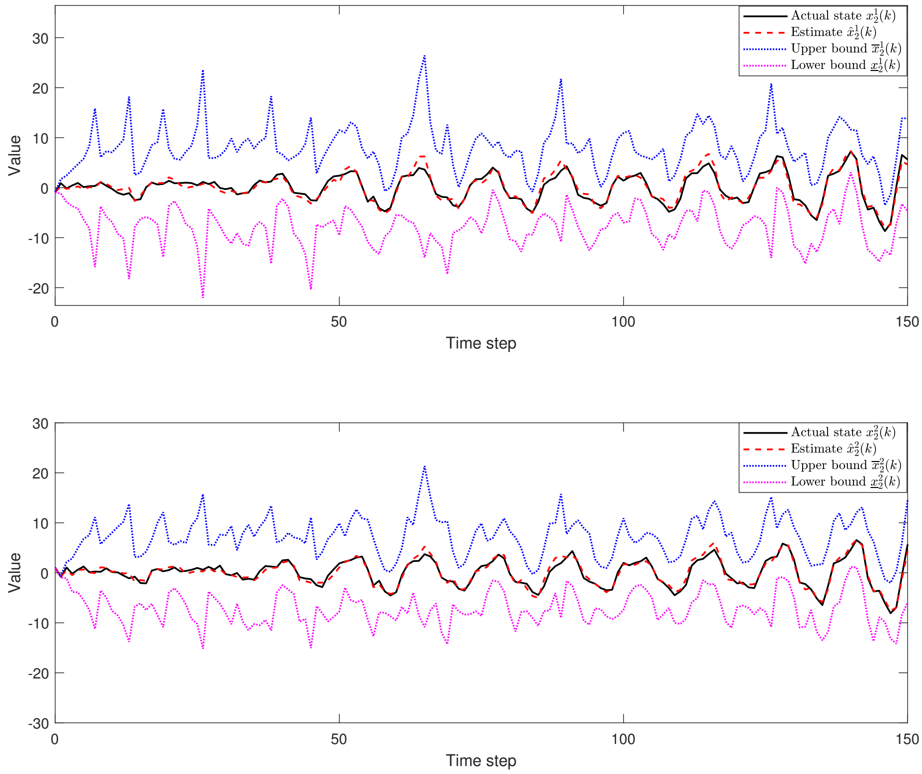


**Figure 2.** State $x_2^1$, $x_2^2$, their estimates and bounds.

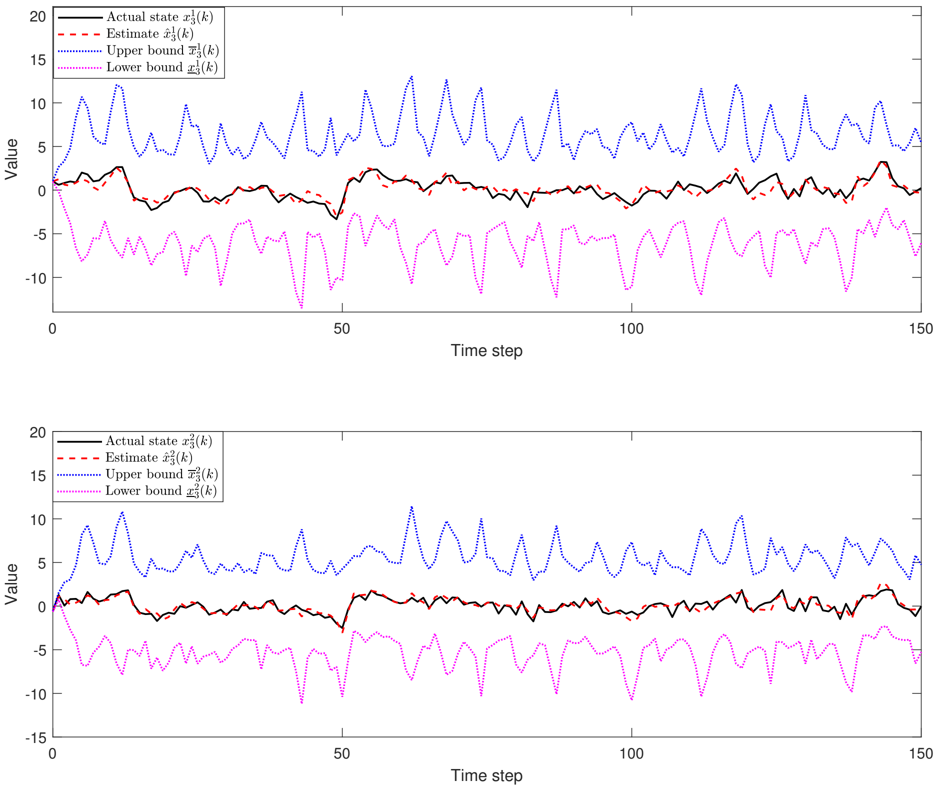


**Figure 3.** State $x_3^1$, $x_3^2$, their estimates and bounds.

Figure 4 illustrates the evolution of the zonotope scale $\|G_i(k)\|_F^2$ for all nodes. The generator matrix norms experience a brief transient increase and then stabilize within a bounded region. The dashed curve represents the maximum scale among all nodes and closely tracks the individual trajectories, indicating consistent boundedness across the network. Although Theorem 3 offers a sufficient (albeit possibly conservative) condition for boundedness, the numerical results confirm that the obtained zonotopes remain bounded under the considered setting.

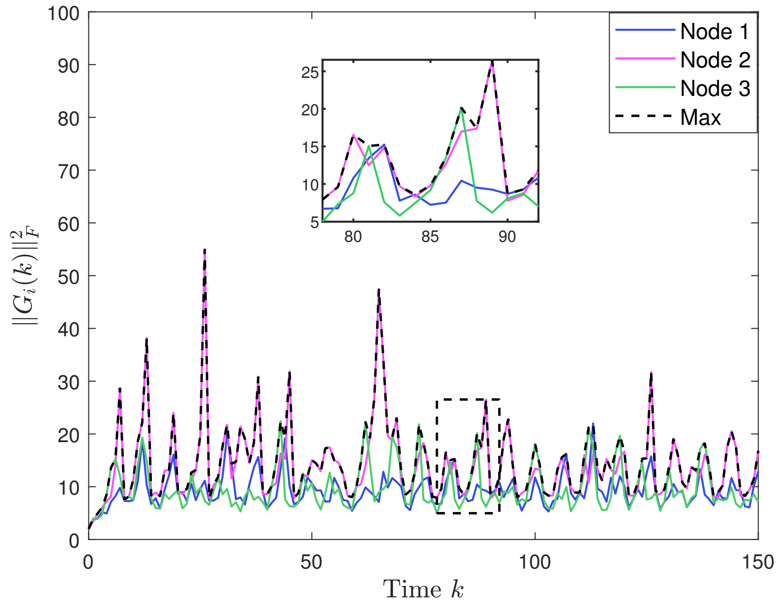


**Figure 4.** Evolution of the zonotope scale $\|G_i(k)\|_F^2$ for all nodes.

To evaluate the overall performance of the proposed scheme, we compare it with the ideal case (without quantization and protocol), the quantization-only case, and the WTOD-only case, from the perspectives of communication cost and estimation accuracy. Figure 5 shows the evolution of the cumulative transmitted bits over time steps for each scheme. From Figure 5 and Table 1, it can be seen that the total bits required by the ideal, quantization-only, WTOD-only, and proposed schemes are 28,800, 7200, 14,400, and 3600 bits, respectively, which means the proposed scheme reduces the communication cost by 87.5%, 50%, and 75% compared to the ideal, quantization-only, and WTOD-only cases, respectively. The cumulative bit curve of the proposed scheme has the smallest slope in Figure 5, indicating optimal communication efficiency. Figure 6 presents the corresponding RMSE curves. Together with Table 1, it can be seen that the average RMSE values achieved by the ideal, quantization-only, WTOD-only, and proposed schemes are 0.4023, 0.4089, 0.5810, and 0.6048, respectively. Although the accuracy of the proposed scheme is slightly lower than the first two, it achieves comparable performance to the quantization-only scheme with the lowest communication cost, whereas the quantization-only scheme requires twice as many bits.

In summary, the proposed ZSME scheme achieves a significant reduction in communication load at a small cost in estimation accuracy under bandwidth-constrained environments, demonstrating a good trade-off and practicality.

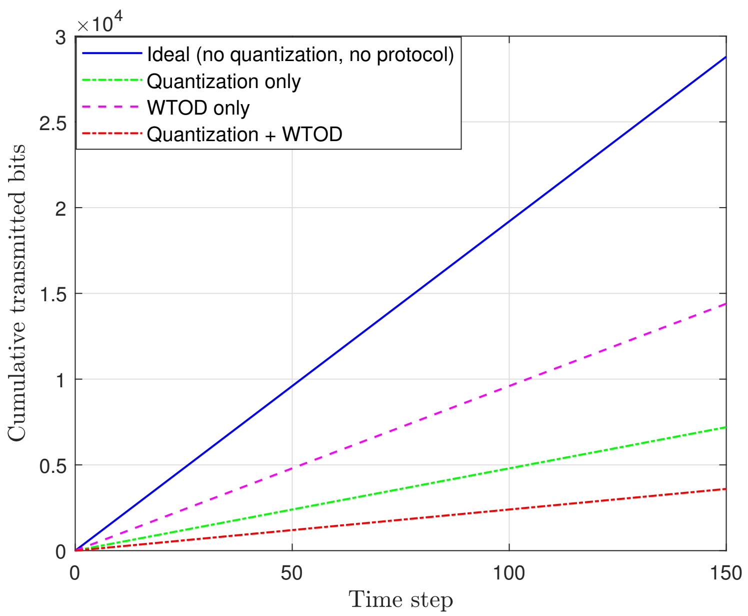


**Figure 5.** Cumulative transmitted bits under different schemes.

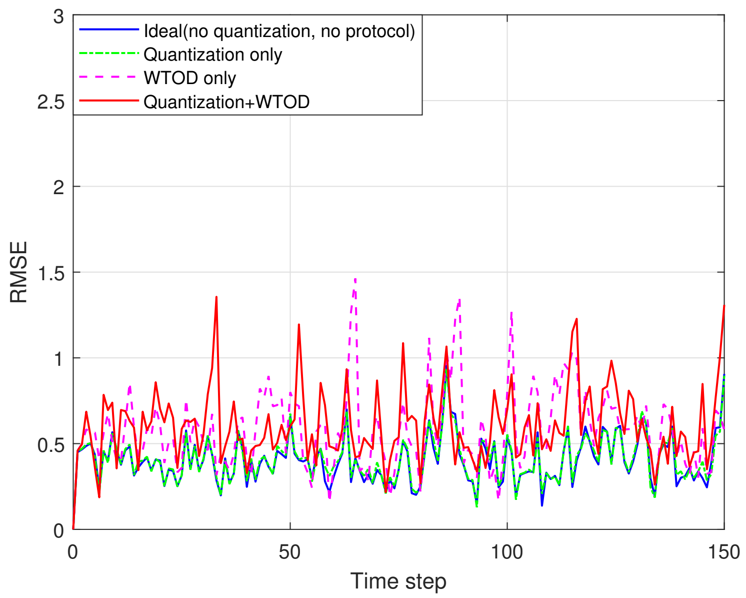


**Figure 6.** RMSE comparison among different schemes.

**Table 1.** Comparison of communication cost and estimation accuracy under different schemes.

| Scheme | Total Transmitted Bits | Average RMSE |
|---|---|---|
| Ideal | 28,800 | 0.4023 |
| Quantization only | 7200 | 0.4089 |
| WTOD only | 14,400 | 0.5810 |
| Quantization + WTOD | 3600 | 0.6048 |

## 6. Conclusions

In this paper, the ZSME problem has been addressed for a class of CNs subject to logarithmic quantization and WTOD protocol. To reduce the consumption of limited communication resources, a logarithmic quantizer has been first employed to handle the measurements, after which the WTOD protocol has been adopted to coordinate the transmission sequence of the quantized data. By applying a prediction-measurement-correction framework, the proposed ZSME algorithm constructs, for each node, a zonotope that is guaranteed to enclose the true system state, and its size is further minimized through an appropriate choice of the correction matrix. Moreover, the boundedness of the resulting zonotope has been rigorously examined. Finally, an illustrative numerical example has been provided to confirm the effectiveness of the developed ZSME algorithm. Future work may involve extending the key findings of this paper to estimation problems in a broader family of multi-sensor systems affected by various communication disturbances.

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