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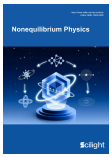

Article

Quantum-to-Classical Transduction in Complex Physical Systems

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Abstract: Quantum-to-classical transduction denotes dynamical processes through which localized microscopic quantum events produce cumulative modifications of mesoscopic and macroscopic classical behavior in hybrid quantum-classical (HQC) systems. Understanding how such cross-scale influence emerges is a central problem in nonequilibrium physics and in the study of complex multiscale systems. Here, we develop a general dynamical perspective in which quantum-to-classical transduction arises naturally in hybrid quantum-classical systems, where quantum degrees of freedom interact continuously with classical environments. From this perspective, we identify physical conditions enabling transduction, including separation of dynamical timescales, metastable classical configurations acting as free-energy reservoirs, environmental fluctuations, and nonlinear feedback between quantum observables and classical motion. These ingredients define distinct dynamical regimes of transduction in hybrid systems. The quasi-Lie bracket formulation of HQC dynamics provides a general description of the coupled evolution, with the quantum-classical Liouville equation representing a minimal realization of the framework. Representative physical contexts in which microscopic quantum processes may influence large-scale dynamics are discussed, including plasmas, driven dissipative media, gravitationally coupled systems, and organized biological matter.

Keywords: quantum-to-classical transduction; hybrid quantum-classical dynamics; nonequilibrium systems; complex systems; multiscale dynamics; open quantum systems

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1. Introduction

Understanding how microscopic dynamics influence macroscopic behavior across space-time scales remains a central problem in nonequilibrium physics [1,2]. In complex open systems operating far from equilibrium,

localized processes may trigger collective dynamical responses that extend over many spatial and temporal scales, particularly in the presence of metastable configurations and distributed free-energy reservoirs.

In this work, we adopt the term *quantum-to-classical transduction* to describe situations in which localized microscopic quantum processes influence mesoscopic and macroscopic classical dynamics in open systems. This idea is conceptually related to early discussions of microscopic triggering and stability in living matter, later reviewed in the context of quantum biology [3], and to Schrödinger's reflections on the physical basis of life [4]. Within this perspective, in appropriate dynamical regimes, quantum fluctuations may bias transitions between metastable classical configurations and contribute to cumulative dynamical amplification across scales. Therefore, amplification represents only one possible dynamical realization of quantum-to-classical transduction, which more generally includes cumulative biasing of classical trajectories, noise-assisted transitions between metastable states, and other forms of cross-scale dynamical influence that do not necessarily involve exponential growth of perturbations.

Hybrid quantum-classical (HQC) theories [5–15] provide a natural framework for describing this type of multiscale physics [16–27], since microscopic quantum variables evolve within classical environments that, in turn, feed back onto the quantum subsystem. From this unified dynamical perspective, the continuous coupling between quantum and classical degrees of freedom modifies transition pathways between classical configurations and enables microscopic fluctuations to influence macroscopic behavior. Within this framework, robustness and adaptability can emerge as intrinsic dynamical properties of coupled quantum-classical systems [16–27]. From this point of view, we develop a Hamiltonian-model-independent framework for studying quantum-to-classical transduction in HQC systems whose time evolution is expressed by a quasi-Lie bracket. Rather than focusing on system-specific mechanisms, we identify minimal physical conditions under which microscopic quantum fluctuations can produce cumulative modifications of dynamics across scales.

The paper is organized as follows. In Section 2, we introduce the HQC framework based on quasi-Lie brackets. In Section 3, we introduce the dynamical indicators and discuss the corresponding regimes of quantum-to-classical transduction. In Section 4, we analyze the corresponding transduction mechanisms in hybrid multiscale dynamics. In Section 5, we discuss the broader multiscale interpretation of quantum-to-classical transduction in complex systems. In Section 6, we consider organized biological systems as a representative realization. Conclusions follow in Section 7.

2. Hybrid Quantum-Classical Dynamics and Quasi-Lie Brackets

The quasi-Lie bracket formulation of hybrid quantum-classical dynamics [12–15] provides a framework for describing quantum subsystems coupled to classical phase-space environments [5–27]. In the main text, we do not reproduce the full derivation of the quantum-classical Liouville equation and of its quasi-Lie generalizations. However, to avoid possible confusion about the physical nature of the formalism used in this paper, we have summarized the most relevant technical points in Appendix A. In particular, we clarify that the quantum-classical Liouville equation is not an ad hoc phenomenological equation and should not be identified with Ehrenfest mean-field dynamics. Rather, it follows from a partial Wigner transform of an underlying quantum Hamiltonian dynamics followed by a controlled expansion in the small mass parameter. We also recall that, in the quasi-Lie formulation, the evolution is generated by the hybrid Hamiltonian together with an antisymmetric bracket structure, so that the resulting dynamics is generally not canonical Hamiltonian dynamics in the ordinary Poisson/Lie sense. Further technical details can be found in Ref. [28] and Appendix B.

Within HQC theories, whose quasi-Lie bracket structure has been discussed at length in Refs. [12–15], the state of a composite system is represented by an operator-valued phase-space density $\tilde{\rho}(X,t)$, where $X=(R,P)$ denotes the classical phase-space coordinates of the environment, conveniently represented in phase space since they describe degrees of freedom with a very short de Broglie wavelength. The remaining degrees of freedom are fully quantum mechanical, having a very long de Broglie wavelength. Please note that we have adopted a notation in which, if we write $\mathcal{O}$ we are considering a pure classical variable; if we write $\hat{\mathcal{O}}$ we are considering a full quantum operator, and if we write $\tilde{\mathcal{O}}$ we are considering a quantum-classical operator, i.e., an operator depending parametrically on the classical phase space coordinates X. In Appendix A, we discuss how the operator-valued phase space density $\tilde{\rho}(X,t)$ is obtained from the full operatorial one by means of a partial Wigner transform.

The time evolution of $\tilde{\rho}(X,t)$ is generated by a hybrid Hamiltonian $\tilde{H}(X)$ that contains quantum operators, classical variables, and their couplings. In the minimal canonical case, the equation of motion of $\tilde{\mathcal{O}}$ is directly formulated in terms of the quasi-Lie bracket, see Equations (A4) and (A5):

$$\frac{\partial}{\partial t}\tilde{\rho}(t) = -\left(\tilde{H}, \tilde{\rho}(t)\right)_{\boldsymbol{\Omega}_S} \tag{1}$$

This is one among many possible forms of hybrid quantum-classical dynamics expressed by means of the quasi-Lie bracket formalism. In particular, Equation (1) is the minimal canonical form because it is defined in terms of the symplectic matrix in Equation (A3). It is known as the quantum-classical Liouville equation. Even in this minimal form, the quasi-Lie/quantum-classical bracket in the rhs of Equation (1) fails to satisfy the Jacobi relation [12]. To realize this, we can use the super-operator $\hat{\mathcal{D}}_{\boldsymbol{\Omega}_S}$ defined in (A8), and rewrite the quantum-classical Liouville equation in Equation (1) as

$$\frac{\partial}{\partial t}\tilde{\rho}(t) = -\begin{bmatrix} \tilde{H} \\ \tilde{\rho}(t) \end{bmatrix}^T \hat{\mathcal{D}}_{\boldsymbol{\Omega}_S} \begin{bmatrix} \tilde{H} \\ \tilde{\rho}(t) \end{bmatrix} \tag{2}$$

We note that the dynamics arising from the quasi-Lie/quantum-classical bracket is more general than Hamiltonian dynamics. In Hamiltonian dynamics, H, $\tilde{H}$, or $\hat{H}$, would be the generators of time translations. In quantum-classical/quasi-Lie dynamics, we must specify a pair $(\tilde{H}, \boldsymbol{\Omega}(X))$ or even a triplet $(\tilde{H}, \boldsymbol{\Omega}(X), \boldsymbol{J}(X))$, where $\boldsymbol{\Omega}^T(X) = -\boldsymbol{\Omega}(X)$ and $\boldsymbol{J}^T(X) = -\boldsymbol{J}(X)$ are both antisymmetric tensor fields. The dynamics in these more generalized cases is easier to handle by opportunely modifying Equation (A5). When we have a pair of 'generators', the equation of motion becomes

$$\frac{\partial \tilde{\rho}(t)}{\partial t} = \frac{-i}{\hbar}\begin{bmatrix} \tilde{H} \\ \tilde{\rho}(t) \end{bmatrix}^T \boldsymbol{\Omega}_S \begin{bmatrix} \tilde{H} \\ \tilde{\rho}(t) \end{bmatrix} + \frac{1}{2}\left(\tilde{H}\overleftarrow{\nabla}\right)^T \cdot \boldsymbol{\Omega}(X) \cdot \vec{\nabla}\tilde{\rho}(t) - \frac{1}{2}\left(\tilde{\rho}(t)\overleftarrow{\nabla}\right)^T \cdot \boldsymbol{\Omega}(X) \cdot \vec{\nabla}\tilde{H}, \tag{3}$$

or the more general

$$\frac{\partial \tilde{\rho}(t)}{\partial t} = \frac{-i}{\hbar}\begin{bmatrix} \tilde{H} \\ \tilde{\mathcal{O}}(t) \end{bmatrix}^T \boldsymbol{J}(X) \begin{bmatrix} \tilde{H} \\ \tilde{\mathcal{O}}(t) \end{bmatrix} + \frac{1}{2}\left(\tilde{H}\overleftarrow{\nabla}\right)^T \cdot \boldsymbol{\Omega}(X) \cdot \vec{\nabla}\tilde{\mathcal{O}}(t) + \left(\tilde{\mathcal{O}}(t)\overleftarrow{\nabla}\right)^T \cdot \boldsymbol{\Omega}(X) \cdot \vec{\nabla}\tilde{H}. \tag{4}$$

Equation (3) has been used in order to impose both thermodynamic and holonomic constraints [12–15]. The dynamics defined by the quasi-Lie brackets in Equations (1)–(4) conserve the total Hamiltonian and are time-reversal invariant, unless additional non-Hamiltonian structures, such as thermostatting or barostatting brackets, are explicitly introduced. Moreover, when the quantum-classical coupling term is removed, the quantum and classical sectors decouple and recover their independent limiting dynamics. The transduction mechanisms discussed below should therefore be understood as consequences of the controlled quantum-classical back-reaction encoded in the QCLE and in related quasi-Lie formulations, rather than as artifacts of an arbitrary hybrid prescription.

Equations (1)–(4) provide a systematic framework for simulating open-system dynamics [29] while retaining selected environmental degrees of freedom explicitly [12–28]. They describe a system belonging to the quantum sector, e.g., Q with its Hamiltonian operator $\hat{H}_Q$, a system belonging to the classical sector, e.g., C with its Hamiltonian H_C, and the coupling Hamiltonian $\tilde{H}_{QC}$. The total Hamiltonian of the system, i.e.,

$$\tilde{H}(X) = \hat{H}_Q + \tilde{H}_{QC} + H_C \tag{5}$$

by its very nature describes multilevel phenomena. If the quantum degrees of freedom of *S* are many, the numerical simulation of the physical problem faces the same difficulty of the many-body problem. However, if there are only a few quantum degrees of freedom and many classical degrees of freedom, standard numerical simulations of classical systems can be combined with numerical diagonalizations and stochastic dynamics for treating quantum transitions to provide effective algorithms.

The dynamical properties of the quantum-classical system are given in terms of operator-valued phase space functions, which we generically denote by $\tilde{\mathcal{O}}(X)$. Expectation values of $\tilde{\mathcal{O}}(X)$ are computed as phase-space integrals over the classical variables X, combined with a partial trace, Tr', over the quantum degrees of freedom:

$$\overline{\tilde{\mathcal{O}}(t)} = Tr' \int_\Gamma dX \tilde{\rho}(X,t)\tilde{\mathcal{O}}(X) \tag{6}$$

where Γ is the phase space volume.

Although the quantum-classical Liouville equation provides a canonical realization of HQC dynamics, the quantum-to-classical transduction mechanisms discussed here do not depend on this specific representation and apply more generally to hybrid dynamical theories in which quantum observables influence classical evolution. When the classical variables are frozen, the first term in the rhs of Equation (4) reduces to the quantum commutator, whereas in the limit of commuting quantum operators, or purely classical variables, the second term in the rhs of Equation (4) reduces to the classical Poisson bracket [30].

HQC dynamics [5–11] formulated via quasi-Lie brackets [12–15] has proven effective in modeling nonadiabatic processes [16–19]. Importantly, HQC dynamics defined via quasi-Lie brackets [12–15] is not restricted to near-equilibrium regimes. The formalism can describe driven nonequilibrium conditions and dissipative environments through the inclusion of thermostatting schemes [14]. This hybrid dynamical structure provides a natural theoretical setting for analyzing quantum-to-classical transduction mechanisms in multiscale complex open systems.

3. Dynamical Regimes of Quantum-to-Classical Transduction in Hybrid Systems

By quantum-to-classical transduction, we mean a multiscale dynamical process through which localized microscopic quantum events produce cumulative modifications of mesoscopic and macroscopic classical behavior in HQC systems. In this section, we want to discuss the dynamical conditions under which the phenomenon of quantum-to-classical transduction can take place in physical systems and how it can be identified in quantum-classical dynamics [5–11] as formulated by means of quasi-Lie brackets [12–15].

We consider the Hamiltonian $\tilde{H}(X)$ defined in Equation (5) and note that the interaction Hamiltonian $\tilde{H}_{QC}$ is in general a nontrivial coupling term. If this term is not present, the possibility of microscopic quantum events influencing classical dynamics disappears. The Hamiltonian structure given by Equation (5) is standard in the formulation of HQC theory through quasi-Lie brackets [12–15].

There are at least two dimensionless parameters by which we can identify different dynamical regimes of the HQC system. The first one is

$$\chi = \frac{\tau_Q}{\tau_C} \tag{7}$$

where τ_Q is the typical period of motion of the quantum degrees of freedom and τ_C is the typical period of the classical coordinates. In concrete applications, τ_Q and τ_C should be understood as characteristic dynamical times extracted from the dominant frequencies, autocorrelation decay times, or transition times of the corresponding quantum and classical observables. Such quantities are naturally emerging from the numerical simulations [16–21]. In Appendix B, we sketch how one particular numerical implementation of HQC dynamics works [28].

Within these numerical schemes, χ can be calculated easily. When $\chi << 1$, the classical subsystem evolves slowly relative to the quantum dynamics (adiabatic HQC regime). When $\chi \sim 1$, quantum and classical degrees of freedom evolve on comparable timescales, leading to nonadiabatic hybrid dynamics.

The physical discussion is clarified by introducing a second dimensionless parameter that measures the relative fluctuation scale of the quantum-classical force with respect to the intrinsic classical force scale. The following discussion considers that the dynamics is most naturally described within the adiabatic basis. In this case, the Hamiltonian in Equation (5) is written as

$$\tilde{H}(X) = \frac{P^2}{2M} + V_C + \hat{h}(R) \tag{8}$$

and the adiabatic eigenvalues and eigenvectors are defined by

$$\hat{h}(R)|\alpha; R\rangle = \epsilon_\alpha |\alpha; R\rangle \tag{9}$$

The adiabatic Hamiltonian $\hat{h}(R)$ is defined as

$$\hat{h}(R) = \hat{H}_Q + \tilde{H}_{QC} \tag{10}$$

From Equations (8) and (10), it follows that the classical Hamiltonian H_C has its standard form

$$H_C = \frac{P^2}{2M} + V_C(R) \tag{11}$$

As it is better explained in Appendix B, the numerical simulation of HQC dynamics from quasi-Lie brackets is based on the generation of a swarm of trajectories. Each trajectory ‘feels’ a classical force

$$F_C(R) = -\nabla_R V_C(R) \tag{12}$$

and a quantum-classical force

$$\tilde{F}_{QC}(R) = -\nabla_R \tilde{H}_{QC}(R) \tag{13}$$

Arising from the coupling Hamiltonian $\tilde{H}_{QC}(R)$, in Equation (13) we have assumed that the coupling Hamiltonian depends only on R. Each trajectory is generated by the sum of the forces in Equations (12) and (13). Since Equation (13) is an operator-dependent function, its representation in any basis will be a matrix with indices corresponding to pair of quantum states. This shows that the quantum-classical force in Equation (13) is not a mean-field force, and so are the trajectories of the swarm, see Ref. [28]. The discussion in Appendix B also makes clear that HQC dynamics from quasi-Lie brackets is fundamentally distinct from a mean-field dynamics.

Once these concepts are clarified, we can proceed and define our second parameter. We denote by $\overline{F_C}(t)$ the average value of F_C over all phase space and all quantum states at a certain time t. During the numerical simulation, one can calculate the positive quantity

$$\Delta_{F_C} = \sqrt{\overline{F_C^2(t)} - \overline{F_C(t)}^2} \tag{14}$$

Similarly, we denote by $\overline{\tilde{F}_{QC}(t)}$ the average value of the quantum-classical force over all phase space trajectories and all quantum states at each instant of time. The average quantum-classical force is calculated in standard quantum-classical numerical simulations [28]. We can now introduce the quantity $\Delta_{F_{QC}}$:

$$\Delta_{F_{QC}} = \sqrt{\overline{F_{QC}^2(t)} - \overline{F_{QC}(t)}^2} \tag{15}$$

For multidimensional classical coordinates, F_C^2 and F_{QC}^2 denote the corresponding scalar products, $F_C \cdot F_C$ and $F_{QC} \cdot F_{QC}$, respectively. Finally, we can define the second parameter ξ as

$$\xi = \frac{\Delta_{F_{QC}}}{\Delta_{F_{QC}} + \Delta_{F_C}} \tag{16}$$

By construction, the dimensionless parameter ξ satisfies $0 \leq \xi \leq 1$. The separate use of timescale and coupling-strength indicators, such as (χ, ξ), is common in nonadiabatic quantum dynamics, although the literature often adopts model-specific forms rather than the compact two-parameter scheme introduced here. For instance, in trajectory-based nonadiabatic methods, the transition between adiabatic and nonadiabatic behavior is commonly discussed in terms of dimensionless adiabaticity measures (e.g., Landau-Zener/Massey-type criteria) and related decoherence thresholds [22–24]. Likewise, in open-system and driven dissipative settings, the qualitative crossover between weak and strong coupling dynamics is routinely organized in terms of dimensionless coupling parameters, controlling noise-assisted relaxation versus interaction-dominated behavior [25]. From a more general viewpoint, HQC admits a decomposition in quantum-classical and classical-quantum channels, which naturally motivates a regime-based discussion [26]. In the present work, we adopt the minimal pair (χ, ξ) as a Hamiltonian-model-independent parametrization of the widely used adiabatic and coupling notions within HQC [5–11] formulated by means of quasi-Lie brackets [12–15], and we explicitly complement it with the location of free energy in metastable classical configurations as a necessary ingredient for transduction.

Quantum-to-classical transduction depends on the combined values of χ and ξ. In the regime

$$\chi << 1;\ \xi << 1 \tag{17}$$

the quantum and classical subsystems evolve almost independently, and transduction effects are negligible. When

$$\chi << 1 \text{ and } \xi \sim \mathcal{O}(1) \tag{18}$$

microscopic quantum fluctuations can appreciably bias transitions between metastable classical configurations, leading to cumulative modifications of classical dynamics.

When

$$\chi \sim 1 \text{ and } \xi \to 1 \tag{19}$$

the hybrid dynamics becomes interaction-dominated and strong quantum-to-classical feedback may occur. These parameters provide a minimal dynamical characterization of quantum-to-classical transduction in hybrid multiscale systems. Besides timescale separation and coupling strength, the possibility of transduction also

depends on where free energy is stored in the hybrid system. In many amplification scenarios, the classical subsystem acts as a metastable energy reservoir, while the quantum subsystem provides a microscopic trigger or bias. When most of the available free energy resides in the classical sector, small quantum-induced perturbations can initiate the release of macroscopic energy through nonlinear classical dynamics. Conversely, when no classical free-energy reservoir is present, hybrid coupling typically leads to relaxation or damping rather than amplification. In the interaction-dominated regime $(\chi \sim 1, \xi \to 1)$, quantum-to-classical transduction may become explosive. In this case, microscopic quantum events act as triggers that destabilize classical configurations storing free energy. Amplification arises from nonlinear classical dynamics rather than from the quantum subsystem itself. Several well-known physical processes illustrate this interaction-dominated limit. In the following, we list some of these.

Population-inverted optical media near laser threshold [31,32] provide a clear example, where spontaneous emission seeds stimulated emission [31,32] and rapid macroscopic field growth. The stored energy of the inverted medium is released once the instability threshold is crossed [31,32]. In strong electric fields, microscopic tunneling or impact-ionization events can trigger carrier multiplication and macroscopic current formation [33]. The electric field acts as a metastable energy reservoir, while nonlinear transport dynamics produce amplification [33–35]. In superheated liquids, localized ionization events can initiate nucleation and macroscopic bubble growth [35,36]. The metastable thermodynamic state of the liquid provides the energy reservoir driving the phase transition [35,36], while microscopic fluctuations select the transition pathway and control nucleation dynamics [35,36]. In Josephson devices and SQUID systems, macroscopic quantum tunneling can trigger transitions between metastable macroscopic states [37–39], while phase-slip events provide another mechanism for switching between superconducting configurations [37–39]. The electromagnetic energy stored in the circuit is released through classical dissipative dynamics in superconducting circuits and SQUID devices [37–39]. These examples highlight the interaction-dominated limit of HQC dynamics, in which metastability, nonlinear classical dynamics, and hybrid coupling allow microscopic quantum events to trigger the rapid macroscopic release of stored classical free energy. In the next section, we turn to the more generic regime in which quantum fluctuations bias noise-assisted transitions between metastable classical configurations, leading to cumulative quantum-to-classical transduction across scales.

More generally, hybrid quantum-classical dynamics provides a natural description of physical situations in which quantum subsystems evolve within slowly varying classical environments, even outside explosive transduction regimes. Representative contexts include semiclassical electromagnetic fields [31,32], quantum particles in plasmas [5,40,41], and quantum subsystems propagating in classical gravitational backgrounds [6,42–45].

The parameters introduced above should be understood as operational indicators rather than conserved quantities. Similarly, metastability is understood here in a dynamical sense, as the existence of regions of classical phase space in which the system displays residence times longer than local relaxation times, while still allowing finite transition probabilities under environmental noise or quantum-classical perturbations. The associated free-energy reservoir need not be represented by a global thermodynamic potential; it may correspond more generally to stored mechanical, electromagnetic, chemical, configurational, or field energy that can be released through nonlinear classical dynamics. Within this interpretation, the quantum origin of a transduction effect is not inferred merely from the occurrence of amplification, but from the modification of classical trajectory statistics induced by the quantum-classical force F_{QC}, for example, by comparing the dynamics with and without the coupling term $\tilde{H}_{QC}$, or with a classical stochastic source chosen to reproduce only selected fluctuation statistics.

4. Quantum-to-Classical Transduction across Scales

The discussion in Section 3 identified the dynamical conditions under which quantum-to-classical transduction can occur in hybrid systems, in terms of timescale separation, coupling strength, and the presence of free energy stored in metastable classical configurations. These conditions determine whether hybrid coupling leads to damping, cumulative transduction, or explosive amplification.

For quantum-to-classical transduction to propagate beyond the directly coupled classical sector, the classical environment must contain multiple interacting spatial and temporal scales. In addition to the quantum subsystem Q and the directly coupled classical subsystem C, we consider a larger classical environment, or bath, B, interacting with C [18,19]. A schematic hybrid Hamiltonian can then be written as

$$\tilde{H}_T = \hat{H}_Q + H_C + \tilde{H}_{QC} + H_B + H_{CB} \tag{20}$$

where B represents slower or larger-scale classical degrees of freedom that are not directly coupled to the quantum subsystem. Thus, H_{CB} is a purely classical term describing the coupling between B and C. In this formulation, quantum-to-classical transduction is not identified with the persistence of quantum coherence at macroscopic

scales, but with the process by which microscopic quantum fluctuations modify the statistical structure of classical trajectories or selected classical observables. Operationally, this modification can be expressed as a change in the probability distribution of classical trajectories of C, induced by the quantum-classical force F_{QC} acting at the Q-C level. In the hierarchical structure represented by the Hamiltonian $\tilde{H}_T$ in Equation (13), quantum-to-classical transduction first occurs at the Q-C level. At this level, microscopic quantum fluctuations bias the dynamics of metastable classical configurations. This mechanism relies on the ingredients identified in Section 3. A separation of time scales allows quantum degrees of freedom to evolve rapidly relative to the classical subsystem C, while metastable classical configurations provide a landscape in which small perturbations can accumulate. Nonlinear feedback between quantum observables and classical motion enables microscopic fluctuations to modify classical trajectories, and environmental noise facilitates transitions between classical states, allowing these quantum-induced biases to become dynamically relevant. The resulting classical response can then propagate to larger scales through the C-B coupling, provided that this coupling transmits the modified classical statistics on timescales shorter than, or comparable to, the relevant relaxation times of the intermediate classical levels.

In the adiabatic regime, this propagation occurs through slow cumulative modifications of classical dynamics. When the characteristic time scales of the coupled classical levels become comparable, nonadiabatic effects can dominate the evolution, and transduction may occur through direct dynamical mixing between classical levels rather than through gradual accumulation in metastable configurations. From a dynamical-systems perspective, the coupling between quantum and classical degrees of freedom modifies the stability landscape explored by the classical subsystem. Quantum variables act as fast dynamical components that reshape transition pathways between metastable classical states, while the coupling between classical levels enables these modifications to propagate across scales. This mechanism is not specific to any particular physical realization. It may arise in plasmas with quantum particles, in quantum subsystems evolving in gravitational or electromagnetic classical backgrounds, in driven dissipative media, and in organized biological matter. In all these cases, the HQC description provides a natural framework for linking microscopic quantum processes to emergent classical dynamics.

5. Hybrid Quantum-Classical Dynamics in Complex Systems

The multiscale transduction mechanism discussed in Section 4 naturally extends to complex physical systems, which can be viewed as hierarchies of coupled dynamical levels [46,47]. In such systems, localized quantum processes interact with mesoscopic and macroscopic classical degrees of freedom across wide ranges of spatial and temporal scales. Elementary quantum processes typically occur on fast microscopic timescales, whereas collective classical dynamics may extend over many orders of magnitude in space and time.

Within this hierarchical structure, the classical subsystem directly coupled to the quantum degrees of freedom plays the role of the intermediate level C, while larger-scale collective variables form a multiscale classical environment B. The classical sector in complex systems is typically characterized by a broad spectrum of dynamical timescales and by high-dimensional metastable landscapes. The coexistence of multiple metastable configurations supports noise-assisted transitions and rare events, providing a dynamical setting in which small perturbations can accumulate and propagate across scales.

From a dynamical viewpoint, this multiscale classical structure acts as a distributed reservoir of free energy stored in metastable configurations. When the conditions identified in Section 3 are satisfied, in particular, a separation of timescales between quantum and classical motion and a finite hybrid coupling strength, microscopic quantum fluctuations can bias transitions between classical configurations. Through the coupling between classical levels, these modifications can propagate to larger spatial and temporal scales, producing cumulative quantum-to-classical transduction.

Classical nonequilibrium thermodynamics provides powerful tools for describing instabilities, feedback, and self-organization [1,2]. In the present framework, these mechanisms operate primarily within the classical sector, while quantum degrees of freedom act as microscopic sources of fluctuations capable of selecting transition pathways between metastable states. The coexistence of fast quantum dynamics and slower classical evolution thus generates multiscale feedback loops consistent with the transduction mechanisms identified in Section 4.

From a broader physical perspective, the emergence of dynamical complexity appears to be closely connected to the formation of effective quasi-classical sectors in open systems. Purely unitary quantum dynamics does not naturally produce the extended metastability, dissipation, and hierarchical timescale structure characteristic of complex nonequilibrium behavior. These features emerge when coarse-grained classical variables arise from many-body dynamics and interact with localized quantum degrees of freedom.

Within this perspective, hybrid quantum-classical dynamics provides a consistent theoretical framework for describing complex systems in which microscopic quantum processes and multiscale classical dynamics are

continuously coupled. Quantum-to-classical transduction then appears as a general dynamical mechanism linking microscopic quantum fluctuations to emergent classical organization across scales.

6. Organized Biological Systems as Hybrid Quantum-Classical Multiscale Systems

The model-independent HQC framework for multiscale quantum-to-classical transduction developed in the preceding sections can be illustrated by considering organized biological systems as illustrative complex systems. Biological matter operates across a hierarchy of spatial and temporal scales, ranging from subnanometer electronic dynamics in ion channels, enzymes, and cytoskeletal proteins to mesoscopic electromagnetic fields, mechanical stresses, and collective dynamical modes evolving over seconds or longer. At different levels, distinct physical degrees of freedom dominate: quantum processes govern molecular recognition, charge transfer, and polarization, whereas classical variables describe ionic diffusion, fluid flow, and mechanical deformation.

Standard biological modeling captures parts of this hierarchy through compartmental equations, conductance-based models, reaction-diffusion systems, and network dynamics. These approaches have been highly successful in describing excitability, transport, and large-scale signal propagation. However, they typically treat ionic currents, effective potentials, and rate equations as fundamental, while the molecular and electronic degrees of freedom generating them are incorporated only implicitly through phenomenological parameters.

From a physical perspective, this separation reflects a modeling approximation rather than a fundamental divide. Ion selectivity, gating kinetics, molecular transport, and cytoskeletal regulation depend on quantum processes such as tunneling, polarization, and electronic rearrangement in structured protein environments [48,49]. These microscopic processes occur within warm, driven, and strongly dissipative environments formed by membranes, water, ionic gradients, and electromagnetic fields. Organized biological matter thus provides representative realizations of complex physical systems in which quantum and classical degrees of freedom are dynamically intertwined.

This hybrid structure is also manifested at the level of collective biological organization. Many living systems exhibit oscillatory behavior, synchronization, and long-range correlations across broad frequency ranges and spatial scales [50–53]. Such patterns are mediated by classical electromagnetic, mechanical, and chemical fields, yet they emerge from molecular assemblies whose properties ultimately depend on quantum-level interactions. Collective temporal structure and large-scale coordination, therefore, arise from the interplay between microscopic excitability and macroscopic coupling.

Within the HQC framework, biological dynamics can be viewed as a hierarchy of interacting subsystems in which localized quantum degrees of freedom are continuously coupled to classical collective variables evolving across multiple spatial and temporal scales. Rather than requiring long-lived quantum coherence at the macroscopic level, quantum-to-classical transduction can occur through the biasing of noise-assisted transitions between metastable classical configurations, followed by propagation of the resulting dynamical modifications across levels of organization.

In this perspective, the classical sector of organized biological matter, characterized by dissipation, continuous energy throughput, and multiscale metastability, provides the dynamical substrate for amplification, while microscopic quantum processes contribute fluctuations and pathway selection mechanisms. Organized biological systems may therefore provide physically motivated candidate settings for multiscale quantum-to-classical transduction in complex open systems described by hybrid quantum-classical dynamics.

7. Conclusions

In this work, we have introduced a general dynamical perspective on quantum-to-classical transduction in complex physical systems based on hybrid quantum-classical dynamics. Within this framework, localized quantum degrees of freedom interact continuously with structured classical environments, allowing microscopic fluctuations to influence mesoscopic and macroscopic dynamics.

We have identified the dynamical ingredients that enable quantum-to-classical transduction in multiscale systems, including timescale separation between quantum and classical motion, metastable classical configurations acting as free-energy reservoirs, environmental fluctuations, and nonlinear feedback between quantum observables and classical dynamics. Under these conditions, microscopic quantum events can bias transitions between classical states and produce cumulative dynamical effects across scales.

Within this perspective, long-lived quantum coherence is not required for quantum mechanics to influence macroscopic behavior. Instead, transduction emerges as a dynamical property of coupled quantum and classical degrees of freedom in open nonequilibrium systems. In regimes characterized by metastability and environment-assisted transitions, these mechanisms may lead to effective dynamical amplification across spatial and temporal levels.

Hybrid quantum-classical dynamics thus provides a consistent physical language for describing complex open systems in which microscopic quantum processes and multiscale classical organization are continuously coupled. Organized biological systems may provide physically motivated candidate settings for this dynamical architecture, but the framework applies more generally to plasmas, driven dissipative media, and quantum subsystems evolving in classical gravitational or electromagnetic backgrounds.

Author Contributions

A.S.: Conceptualization, methodology, writing—original draft preparation; A.M.: Writing—reviewing & editing, supervision; G.M.: Writing—reviewing & editing; M.A.N.: Writing—reviewing & editing, validation; M.T.C.: Writing—reviewing & editing; S.M.: Writing—reviewing & editing, project administration. All authors have read and agreed to the published version of the manuscript.

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Not applicable.

Data Availability Statement.

No new data were created or analysed in this study. Data sharing is not applicable to this article.

Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

Appendix A

The QCLE should not be regarded as an ad hoc hybrid postulate. It can be obtained from a fully quantum Liouville equation by separating light quantum degrees of freedom, $\hat{x} = (\hat{r}, \hat{p})$, from massive degrees of freedom, $\hat{X} = (\hat{R}, \hat{P})$, and by performing a partial Wigner transform over the latter:

$$\tilde{\rho}(X,t) = \frac{1}{(2\pi\hbar)^{3N}} \int_{-\infty}^{\infty} d\sigma e^{iP\cdot\sigma} \langle R - \frac{\sigma}{2} | \hat{\rho}(t) | R + \frac{\sigma}{2} \rangle \tag{A1}$$

Quantum-classical operators are obtained through a similar equation:

$$\tilde{\mathcal{O}}(X) = \int_{-\infty}^{\infty} d\sigma e^{iP\cdot\sigma} \langle R - \frac{\sigma}{2} | \hat{\mathcal{O}} | R + \frac{\sigma}{2} \rangle \tag{A2}$$

Up to this stage, the transformation is exact, and the massive variables are represented in phase space while retaining their quantum origin. The QCLE is obtained by expanding the exact partially Wigner-transformed evolution equation to first order in the small mass parameter $\mu = (m/M)^{1/2}$ [7]. In this sense, the QCLE is a controlled semiclassical approximation to an underlying quantum dynamics, valid when a clear mass separation exists.

If we denote by $\boldsymbol{\Omega}_s$ the symplectic matrix

$$\boldsymbol{\Omega}_s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{A3}$$

the commutator of two operators $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ (or quantum-classical operators $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$) can be written as

$$\left[\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2\right] = \begin{bmatrix} \tilde{\mathcal{O}}_1 \\ \tilde{\mathcal{O}}_2 \end{bmatrix}^T \boldsymbol{\Omega}_s \begin{bmatrix} \tilde{\mathcal{O}}_1 \\ \tilde{\mathcal{O}}_2 \end{bmatrix}, \tag{A4}$$

where T stands for transpose. The Poisson bracket can also be written in terms of the symplectic matrix as

$$\left\{\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2\right\} = \left(\tilde{\mathcal{O}}_1 \overleftarrow{\nabla}\right)^T \cdot \boldsymbol{\Omega}_s \cdot \vec{\nabla} \tilde{\mathcal{O}}_2 \tag{A5}$$

where ∇ is the phase-space gradient operator

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial R} \\ \frac{\partial}{\partial P} \end{bmatrix} \tag{A6}$$

with the above quantities, one can finally define the canonical form of the quantum classical bracket

$$\left(\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2\right)_{\boldsymbol{\Omega}_s} = \frac{i}{\hbar} \begin{bmatrix} \tilde{\mathcal{O}}_1 \\ \tilde{\mathcal{O}}_2 \end{bmatrix}^T \boldsymbol{\Omega}_s \begin{bmatrix} \tilde{\mathcal{O}}_1 \\ \tilde{\mathcal{O}}_2 \end{bmatrix} - \frac{1}{2}\left(\tilde{\mathcal{O}}_1 \overleftarrow{\nabla}\right)^T \cdot \boldsymbol{\Omega}_s \cdot \vec{\nabla} \tilde{\mathcal{O}}_2 + \frac{1}{2}\left(\tilde{\mathcal{O}}_2 \overleftarrow{\nabla}\right)^T \cdot \boldsymbol{\Omega}_s \cdot \vec{\nabla} \tilde{\mathcal{O}}_1 \tag{A7}$$

In this form, it is easy to devise mathematical methods to develop generalizations. As a matter of fact, we can introduce the superoperator $\hat{\hat{\mathcal{D}}}_{\boldsymbol{\Omega}_s}$

$$\hat{\hat{\mathcal{D}}}_{\boldsymbol{\Omega}_s} = \begin{bmatrix} 0 & \frac{i}{\hbar} - \frac{1}{2} \overleftarrow{\nabla} \cdot \boldsymbol{\Omega}_s \cdot \vec{\nabla} \\ -\frac{i}{\hbar} + \frac{1}{2} \overleftarrow{\nabla} \cdot \boldsymbol{\Omega}_s \cdot \vec{\nabla} & 0 \end{bmatrix} \tag{A8}$$

So that the canonical quasi-Lie/quantum-classical bracket can be written as

$$\left(\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2\right)_{\boldsymbol{\Omega}_s} = \begin{bmatrix} \tilde{\mathcal{O}}_1 \\ \tilde{\mathcal{O}}_2 \end{bmatrix}^T \hat{\hat{\mathcal{D}}}_{\boldsymbol{\Omega}_s} \begin{bmatrix} \tilde{\mathcal{O}}_1 \\ \tilde{\mathcal{O}}_2 \end{bmatrix} \tag{A9}$$

Appendix B

In this Appendix, while restricting ourselves to the equation of motion (1), we sketch how the quantities needed to calculate parameters χ and ξ, defined in Equations (7) and (16), respectively, naturally arise from the numerical algorithm of the computer simulation.

The representation of Equation (1) in the adiabatic basis, which is defined by Equation (9), is

$$\frac{\partial}{\partial t}\rho_{\alpha\alpha'} = -i \sum_{\beta\beta'} \mathcal{L}_{\alpha\alpha',\beta\beta'}\rho_{\beta\beta'} \tag{A10}$$

The quantum-classical super Liouville operator $i\mathcal{L}_{\alpha\alpha',\beta\beta'}$ is

$$i\mathcal{L}_{\alpha\alpha',\beta\beta'} = (i\omega_{\alpha\alpha'} + iL_{\alpha\alpha'})\delta_{\alpha\beta}\delta_{\alpha'\beta'} + \mathcal{T}_{\alpha\alpha',\beta\beta'} \tag{A11}$$

The term $i\omega_{\alpha\alpha'} + iL_{\alpha\alpha'}$ is clearly diagonal. The quantities $\omega_{\alpha\alpha'} = (\epsilon_\alpha(R) - \epsilon_{\alpha'}(R))/\hbar$ are analogous to the Bohr frequencies, while the Liouville operator $iL_{\alpha\alpha'}$ is

$$iL_{\alpha\alpha'} = \frac{P}{M} \cdot \frac{\partial}{\partial R} + \left[F_C + \frac{1}{2}F_Q^\alpha + \frac{1}{2}F_Q^{\alpha'}\right] \cdot \frac{\partial}{\partial R} \tag{A12}$$

The transition operator $\mathcal{T}_{\alpha\alpha',\beta\beta'}$ is

$$\mathcal{T}_{\alpha\alpha',\beta\beta'} = \left(\frac{P}{M} \cdot d_{\alpha\beta} + \frac{\hbar\omega_{\alpha\beta}d_{\alpha\beta}}{2} \cdot \frac{\partial}{\partial P}\right)\delta_{\alpha'\beta'} + \left(\frac{P}{M} \cdot d^*_{\alpha'\beta'} + \frac{\hbar\omega^*_{\alpha'\beta'}d^*_{\alpha'\beta'}}{2} \cdot \frac{\partial}{\partial P}\right)\delta_{\alpha\beta} \tag{A13}$$

The transition operator is purely off-diagonal and governs the transitions among the swarms of trajectories used to unfold Equation (A10).

To discuss the definitions of the parameters χ and ξ, defined in Equations (7) and (16), respectively, it is important to consider the Liouville operators in Equation (A11). The forces used in ξ appear naturally in $iL_{\alpha\alpha'}$

and are therefore not arbitrary. On the contrary, these arise from the representation of the equation of motion (A10) in its most physical representation (away from avoided or conical intersections). They are computed routinely by the programs that simulate HQC dynamics [28]. Upon representing Equation (6) in the adiabatic basis, one can realize how the swarms of trajectories are involved in the calculation of quantum-classical averages

$$\overline{\tilde{\mathcal{O}}(t)} = \sum_{\alpha\alpha',\beta\beta'} \int_\Gamma dX \mathcal{O}_{\alpha\alpha'}(X)(e^{-i\mathcal{L}t})_{\alpha\alpha',\beta\beta'}\rho_{\beta\beta'}(X) \tag{A14}$$

Equation (A14) can be implemented through a piecewise deterministic algorithm, according to which deterministic continuous trajectories over a specific adiabatic surface, specified by the pair of indices (μ, μ'), for example, are interspersed by stochastic jumps to a different adiabatic surface. The sampling involves the action of either the first term or the second term in the r.h.s of Equation (A13). This whole discussion clarifies that HQC as formulated by quasi-Lie brackets is fundamentally distinct from a mean-field dynamics, since it treats explicitly the motion over all available adiabatic surfaces. The characteristic times τ_Q of the quantum coordinates and τ_C of the classical degrees of freedom, respectively, can be calculated along the piecewise adiabatic trajectories. The Wiener–Khinchin Theorem allows one to directly calculate the power spectrum of the classical motion and from this estimate the relevant frequencies. At the zeroth order, the characteristic times of the quantum motion can be calculated by Fourier transforming since, during adiabatic motion, the quantum coordinates see the classical ones as frozen. Hence, the characteristic time of a single trajectory is

$$\tau_Q^{\alpha\alpha'} = \frac{2\pi\hbar}{\epsilon_\alpha(R) - \epsilon_{\alpha'}(R)} \tag{A15}$$

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