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Reachable Set Control of Nonlinear Singular Multi-Agent Systems Realized by State Decomposition Method

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Abstract: This study centers on the consensus problem of singular multi-agent systems featuring nonlinear dynamics, and concurrently derives a more relaxed reachable set estimation (RSE) criterion with a reduced number of decision variables. The characteristics of the Laplacian matrix are utilized for reducing the singular multi-agent system (SMAS), by constructing the Lyapunov function and using singular value decomposition and cross-convex matrix techniques. By employing linear matrix inequalities, sufficient conditions are established for enclosing the reachable set of the singular multi-agent system within a compact ellipsoid. The efficacy of our presented findings is validated through illustrative examples.

Keywords: singular multi-agent systems; state decomposition method; reachable set estimation

1. Introduction

Multi-agent systems (MASs) consist of multiple autonomous and intelligent individual agents, each equipped with independent perception modules, decision-making units, and execution mechanisms, that collaborate to independently perceive external environments, make self-governing decisions, and accomplish complex tasks via information interaction including data sharing, state feedback, and cooperative negotiation as well as collaborative efforts. Boasting the prominent merits of a distributed architecture that avoids single-point failures and enhances system robustness and high flexibility that enables rapid adaptation to dynamic environmental changes, MASs have exhibited substantial application potential in numerous frontier domains such as intelligent transportation, industrial automation, smart grids, unmanned aerial vehicle (UAV) swarm collaboration, and intelligent robot teamwork. Endowed with unique theoretical value and broad practical application prospects, they have garnered widespread attention from researchers across interdisciplinary fields including vehicle formation [1], control engineering [2], artificial intelligence [3], robotics [4], and systems engineering [5–7]. For example, scholars such as Lu proposed a sorting and filtering method based on multi-hop relay for the security assessment of the state of distributed systems against Byzantine agents [8], this method can effectively resist false information transmitted by malicious agents, keeping the states of normal agents within a safe range; The filter-based predefined-time optimal fault-tolerant consensus control for nonlinear MASs via reinforcement learning was proposed in [9]. The fixed-time optimal fault-tolerant control for multi-agent systems under multiple actuator faults/saturations was investigated in [10].

The descriptor system (also termed a generalized state-space system, singular variable system, or semi-state system) refers to a differential system with rigid constraints including algebraic constraints or equality constraints or a coupled system integrating continuous-time evolving differential relations and static functional constraints. Research on descriptor systems originated in the early 1970s, and compared with conventional standard systems, descriptor systems can more naturally and accurately characterize complex dynamic systems in practical scenarios for instance, dynamic input-output models in economic systems that reflect the causal relationship between supply and demand, multi-time-scale problems in power systems involving fast and slow dynamic processes, and chemical production processes comprising material balance and reaction rate constraints can all be precisely described using such systems. Beyond their significant practical significance, their theoretical value has emerged as a focal point of scholarly inquiry in recent years, encompassing key research directions like H_∞ filtering that aims to suppress the impact of external disturbances on system outputs [11], reachability and stabilization that explores whether system states can reach target regions and how to design stable controllers [12], and dissipation analysis that studies energy dissipation characteristics of systems [13]. In the study of descriptor systems, the assessment of impulsiveness and regularity whether the system exhibits impulsive behavior and meets regularity conditions usually complicates the research process, as these two characteristics directly determine the solvability and stability of the system.

In practical engineering scenarios, accounting for nonlinear dynamics is indispensable, as linear approximations are often insufficient to capture the actual characteristics of complex systems. Common nonlinear factors include Lipschitz nonlinearity [14], saturation nonlinearity [15], dead-zone nonlinearity [16], and other forms. For example, formation control for multi-agent networks with Lipschitz-type nonlinear dynamics has been explored in [17], while finite-time containment optimal control for nonlinear MASs that focuses on achieving containment control within a finite time while optimizing performance indices has been investigated in [18]. Nevertheless, these studies failed to fully consider the complex impact of singular terms, namely algebraic constraints in descriptor systems, on the analysis of nonlinear dynamics. In reality, such coupling impacts may lead to inaccurate evaluation of nonlinear dynamic equations, thereby resulting in the design of controllers that do not match the actual dynamic characteristics of the system and ultimately causing severe system malfunctions such as instability or performance degradation. Thus, exploring effective ways to mitigate the adverse influence of nonlinear dynamics in descriptor systems and improve the accuracy of system analysis and control design constitutes a research topic worthy of in-depth attention.

The core objective of RSE for dynamic systems is to quantitatively characterize the full envelope of all feasible state trajectories via constructing an appropriate compact set. Specifically, under all admissible initial states, bounded input signals, external disturbances, and inherent system nonlinearities or parameter uncertainties, RSE aims to find a minimal or conservative compact region that contains all possible states the system can reach within a finite time horizon. This issue holds crucial significance in robust control theory [19,20], as it provides a theoretical basis for system stability analysis, fault diagnosis, and safety verification, and finds widespread applications in diverse engineering fields. For instance, in the safety analysis of automatic aircraft landing, RSE can be used to define the safe range of state variables such as fuselage attitude and speed, ensuring the stability and safety of the landing process [21]. In the power inspection operations of an UAV swarm, it can provide a reference for state monitoring and fault early warning [22]. For a class of differentially flat multi-agent systems affected by uncertainties, Ref. [23] proposed a trajectory optimization control scheme with reachability constraints, combining the multi-agent obstacle-avoidance constraints with the RSE of tracking errors; The real-time security reachable set control of Markov jump cyber-physical systems against actuator attacks based on adaptive neural sliding mode control was investigated in [24]. Consequently, academic interest in addressing the RSE problem for various dynamic systems including linear systems, nonlinear systems, and time-delay systems has grown steadily in recent years. However, it is noteworthy that existing literature features a paucity of RSE studies on SMASs, most existing achievements focus on RSE for conventional linear/nonlinear systems or single descriptor systems, and research on the coupling scenario of multi-agent collaboration and singular characteristics remains relatively scarce. This research gap motivates the present work.

The main contributions of this paper are as follows:

(1) Firstly, most existing fault-tolerant consensus studies only consider single actuator faults such as gain fault or bias fault alone. In contrast, this paper proposes a composite fault-tolerant control strategy for multi-agent systems where both actuator gain and bias faults occur simultaneously. This fills the research gap in handling composite actuator faults under reachable set constraints.

(2) The control framework takes into account the singular effects of non-linear dynamics, and the proposed control strategy can effectively address non-linear effects.

(3) State decomposition is carried out for the singular system, enabling the steady-state control of the system with fewer decision variables.

To verify the effectiveness and conservatism of the proposed results, we design two numerical examples, setting different system parameters, initial conditions, and external disturbance scenarios to comprehensively demonstrate the feasibility and superiority of the proposed method. All mathematical notations adopted in this paper are defined in a unified and standardized manner. Descriptions of some key symbols are shown in Table 1.

Table 1. Nomenclature.

Notations	Physical Meaning
$\mathrm{diag}\{\dots\}$	Block-diagonal matrix (generalized form)
$\mathcal{W}^{-1}$	Multiplicative inverse of matrix $\mathcal{W}$
$\star$	Symmetric placeholder terms in block matrix
$\mathbb{I}_n$	n-order identity matrix (unit matrix)
$\mathcal{S}\{U\}$	$\mathcal{S}\{U\} = U + U^{\mathrm{T}}$ (symmetric summation of matrix U)
$\mathcal{R} > 0$	Matrix $\mathcal{R}$ is positive definite
$\mathbb{E}^n$	n-dimensional Euclidean space (generalized vector space)
$\|\cdot\|_{(2)}$	Euclidean norm (l_2-norm) of a vector

2. Graph Theory and Problem Formulation

In this section, we will first give the relevant concepts of graph theory, and then we will give the mathematical model of the system and some useful lemma together.

2.1. Graph Theory

To depict the communication relations among N agents, the undirected graph $\mathcal{N}(\mathcal{E}, \mathcal{A}, \mathcal{V})$ is utilized as a descriptor for the network topology. Herein, $\mathcal{V}$ comprises the node collection $\{\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_N\}$, and $\mathcal{E}$ is a subset of $\mathcal{V} \times \mathcal{V}$ serving as the edge set an edge $\mathcal{E}_{p,q}$ is included in $\mathcal{E}$ if there exists a communication link between agent p and agent q. Associated with the undirected graph $\mathcal{N}$ is its adjacency matrix $\mathcal{A}$. $\mathcal{N}$ is regarded as a connected graph on the condition that no isolated nodes are contained within it. With respect to graph $\mathcal{N}$, its Laplace matrix $\mathcal{L}$ is formulated as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ denotes the degree matrix corresponding to $\mathcal{N}$.

2.2. Formulation of the Research Problem

Consider the reachable set problem of continuous SMASs composed of N agents, in this context, the model of the i-th agent is given as follows

$$E\dot{x}_i(t){=}Ax_i(t){+}Bu_i(t){+}D\omega_i(t) + f(x_i(t)),\ (i = 1, 2, ..., N), \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ are the state and control input of agent i, $\omega_i(t) \in \mathbb{R}^l$ represents a bounded external perturbation and it satisfies as follow $\omega^T(t)\omega(t) \leqslant \bar{\omega}^2$, where $\bar{\omega}$ is a positive constant. f is non-linear function, and it satisfies the Lipschitz condition: $||f(i) - f(j)||_2 \le \epsilon||i - j||_2$. A, B, D, E are constant matrices with appropriate dimensions, and $rank(E) = r{<}n$. To achieve the desired result, we design a consensus protocol

$$u_i^F(t) = \theta_i K \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) + \phi_i(t). \tag{2}$$

where K is gain to be designed. θ_i denotes the actuator gain fault factor of the i-th agent, which satisfies $\underline{\theta} < \theta_i \le 1$. Specifically, $\theta_i = 1$ means the actuator is healthy, while $\underline{\theta} < \theta_i < 1$ represents partial loss of actuator effectiveness. $\phi_i(t)$ stands for the actuator bias fault of the i-th agent, which is a bounded external perturbation satisfying $||\phi_i(t)||_2 \le \bar{\phi}^2$, where $\bar{\phi}$ is a known constant. Denote $\theta = \mathrm{diag}\{\theta_1, ..., \theta_n\}$ is the global diagonal matrix of actuator gain fault factors. $\phi(t) = [\phi_1^T(t), ..., \phi_n^T(t)]^T$ is the global stacked vector of actuator bias faults.

We denote $x(t) = [x_1^T(t), \dots, x_N^T(t)]^T, \omega(t) = [\omega_1^T(t), \dots, \omega_N^T(t)]^T, f(t) = \left[f^T(x_1(t)), \dots, f^T(x_N(t))\right]^T$ and $u^F(t) = [u_1^T(t), \dots, u_N^T(t)]^T$. Then from (1)–(2), one has

$$\begin{aligned}(I_N \otimes E)x(t) = &(I_N \otimes A)x(t) + (I_N \otimes B)u^F(t) \\ &+ (I_N \otimes D)\omega(t) + f(t).\end{aligned} \tag{3}$$

Let $\bar{x}(t) = \frac{1}{N}\sum\limits_{i=1}^{N} x_i(t), d(t) = [d_1^T(t) \dots, d_N^T(t)]$, where $\bar{x}(t)$ represents the average of $x_i(t)$ and $\nu_i(t) = x_i(t) - \bar{x}(t)$ denotes the deviation from the average, and $\mathcal{W} = I_N - \frac{1}{N}\vec{1}\vec{1}^T$ Then combining (2) to (3) we can obtain the error system:

$$
\begin{aligned}
(I_N \otimes E)\dot{\nu}(t) =&(\mathcal{W} \otimes A + \theta \mathcal{W}\mathcal{L} \otimes B)\nu(t) + (\mathcal{W} \otimes D)\omega(t) \\
&+ (\mathcal{W} \otimes B)\phi(t) + \bar{f}(t).
\end{aligned}
\tag{4}
$$

Obviously, $\bar{f}(t) = [\bar{f}^T(x_1(t)), \ldots, \bar{f}^T(x_N(t))]^T$ where $\bar{f}(x_i(t)) = f(x_i(t)) - \dfrac{1}{N}\sum_{j=1}^N f(x_j(t))$. Then, we introduce some lemmas, which will be helpful for the following work.

Lemma 1 ([25])**.** *For matrices $\mathcal{W}$ and $\mathcal{L}$, there exists an orthogonal matrix $H \in \mathbb{R}^{N\times N}$ such that*

$$
H^T \mathcal{W} H = \begin{bmatrix} I_{N-1} & \vec{0}_{N-1} \\ * & 0 \end{bmatrix} = \hat{\mathcal{W}}, H^T \mathcal{L} H = \begin{bmatrix} \mathfrak{Q} & \vec{0}_{N-1} \\ * & 0 \end{bmatrix} = \hat{\mathcal{L}}
$$

where $\mathfrak{Q} \in \mathbb{R}^{N-1\times N-1}$ and $\mathcal{L}$ is the Laplacian matrix of a given graph $\mathfrak{G}$.

We denote

$$
\hat{\nu}(t) = (H^T \otimes I_n)\nu(t) = \begin{bmatrix} \hat{\nu}_1(t) \\ \hat{\nu}_2(t) \end{bmatrix},
$$

$$
\hat{\omega}(t) = (H^T \otimes I_l)\omega(t) = \begin{bmatrix} \hat{\omega}_1(t) \\ \hat{\omega}_2(t) \end{bmatrix},
$$

$$
\hat{\phi}(t) = (H^T \otimes I_n)\phi(t) = \begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{bmatrix}.
$$

$$
\hat{f}(t) = (H^T \otimes I_n)\bar{f}(t) = \begin{bmatrix} \hat{f}_1(t) \\ \hat{f}_2(t) \end{bmatrix}.
$$

By mean of Lemma 1, the tree transformation is performed by selecting $H = [H_1 \;\; H_2]$, and $H_2 = \frac{\vec{1}_N}{\sqrt{N}}$. Thus the system (4) can be represented:

$$
\begin{aligned}
(I_N \otimes E)\dot{\hat{\nu}}(t) =& (\hat{\mathcal{W}} \otimes A)\hat{\nu}(t) + (\theta\hat{\mathcal{L}} \otimes BK)\hat{\nu}(t) \\
&+ (\hat{\mathcal{W}} \otimes D)\hat{\omega}(t) + (\hat{\mathcal{W}} \otimes B)\hat{\phi}(t) + \hat{f}(t).
\end{aligned}
\tag{5}
$$

Denote $\delta(t) = \hat{\nu}_1(t),\; w(t) = \hat{\omega}_1(t),\; m = N-1$. Furthermore, Equation (5) admits a decomposition into the two subsequent subsystems:

$$
\begin{aligned}
(I_m \otimes E)\dot{\delta}(t) =& (I_m \otimes A + \tilde{\theta}\mathfrak{Q} \otimes BK)\delta(t) \\
&+ (I_m \otimes B)\hat{\phi}_1(t) \\
&+ (I_m \otimes D)w(t) + \hat{f}_1(t),
\end{aligned}
\tag{6}
$$

$$
\dot{\delta}_2(t) = 0,
\tag{7}
$$

where $\tilde{\theta} = [\theta_1, ..., \theta_{n-1}]$. Clearly, δ_2 fail to exert an influence on $\delta(t)$, thus allowing us to solely take into account $\delta(t)$.

Lemma 2 ([22])**.** *Let $V(x(t))$ be a Lyapunov function and $V(x(t_0)) < \beta\bar{\omega}^2$, where $\beta > 0$, $t_0 \geqslant 0$, $\alpha > 0$. If $\dot{V}(x(t)) + \alpha V(x(t)) - \beta\omega^T(t)\omega(t) < 0$, then $V(x(t)) < \frac{\beta\bar{\omega}^2}{\alpha}$, for $t \geqslant t_0$.*

Lemma 3 ([20])**.** *In the case that the system is without impulsive behavior and regular, it is possible to find two non-singular matrices $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ and $N = [N_1 \quad N_2]$, which satisfy $MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. We denote $MAN = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $\tilde{\delta}(t) = (I_m \otimes N^{-1})\delta(t) = \begin{bmatrix} \tilde{\delta}_1(t) \\ \tilde{\delta}_2(t) \end{bmatrix}$, $\tilde{f}(t) = (I_m \otimes M)\hat{f}(t) = \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \end{bmatrix}$, $MD = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$, $\tilde{K} = KN = [\tilde{K}_1 \quad \tilde{K}_2]$ and then the system (6) can be decomposed into a differential subsystem and a algebraic subsystem:*

$$\begin{aligned}\dot{\tilde{\delta}}_1(t) =&(I_m \otimes A_1)\tilde{\delta}(t) + (\tilde{\theta}\mathfrak{L} \otimes B_1\tilde{K})\tilde{\delta}(t)\\ &+ (I_m \otimes B_1)\hat{\phi}_1(t)\\ &+ (I_m \otimes D_1)w(t) + \tilde{f}_1(t),\end{aligned} \tag{8}$$

$$\begin{aligned}0 =&(I_m \otimes A_2)\tilde{\delta}(t) + (\tilde{\theta}\mathfrak{L} \otimes B_2\tilde{K})\tilde{\delta}(t)\\ &+ (I_m \otimes B_2)\hat{\phi}_1(t)\\ &+ (I_m \otimes D_2)w(t) + \tilde{f}_2(t).\end{aligned} \tag{9}$$

Before carrying out the subsequent research, the relevant control-flow framework is illustrated in Figure 1 to facilitate readers' understanding of our work.

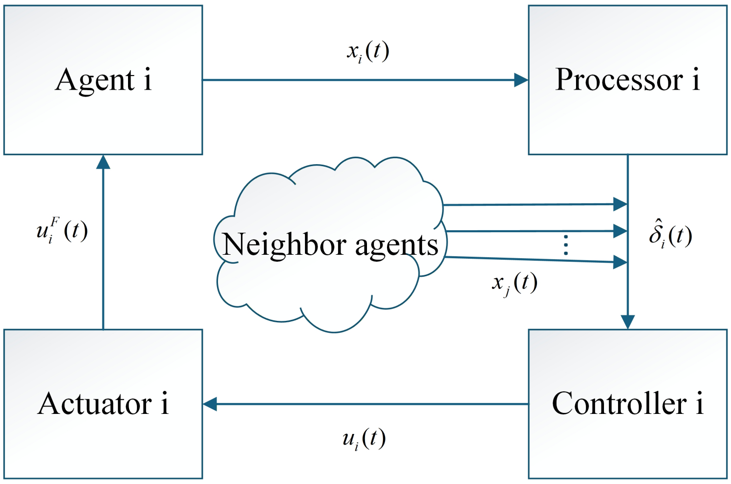

Figure 1. Block diagram.

3. Main Text

Theorem 1. *Given scalar $\alpha > 0$, if there exists an appropriate dimension $X > 0$, such that the following LMIS hold:*

$$\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 \\ * & \Gamma_5 & \Gamma_6 & \Gamma_7 \\ * & * & \Gamma_8 & 0 \\ * & * & * & \Gamma_{10} \end{bmatrix} < 0, \tag{10}$$

where

$$\begin{aligned}&\Gamma_1 = \mathrm{sym}\{(I_m \otimes A_{11}X - \mathfrak{L} \otimes B_1B_1^T)\} + (\alpha + \rho_1 + \frac{\tilde{\epsilon}_1}{\rho_1})(I_m \otimes X),\\ &\Gamma_2 = I_m \otimes (A_{12}X + XA_{21}) - (\mathfrak{L} \otimes B_1B_2^T),\\ &\Gamma_3 = I_m \otimes D_1, \qquad \Gamma_4 = H_1^T \otimes B_1,\\ &\Gamma_5 = \mathcal{S}\{(I_m \otimes A_{22}X - \mathfrak{L} \otimes B_2B_2^T)\} + (\frac{\tilde{\epsilon}_2}{\rho_1} + \frac{\tilde{\epsilon}_4}{\rho_2})(I_m \otimes X), \quad \Gamma_6 = I_m \otimes D_2,\\ &\Gamma_7 = H_2^T \otimes B_2, \quad \Gamma_8 = -\frac{\alpha}{\bar{w}^2}I_m,\\ &\Gamma_{10} = -\frac{\alpha}{\bar{\phi}^2}I_m,\end{aligned}$$

Proof. Construct the following Lyapunov functions

$$V(x(t)) = \tilde{\delta}_1^T(t)(I_{N-1} \otimes P)\tilde{\delta}_1(t), \tag{11}$$

where P is a positive definite matrix. The derivative of Equation (11), in conjunction with Equation (8), gives the following result when considering $\tilde{K} = -\underline{\theta}^{-1}B^TM^TP$.

$$\begin{aligned}\dot{V}(x(t)) =& 2\tilde{\delta}_1^T(t)(I_m \otimes P)\dot{\tilde{\delta}}_1(t)\\ =& 2\tilde{\delta}_1{}^T(t)(I_m \otimes P)[(I_m \otimes A_1 + \tilde{\theta}\mathfrak{Q} \otimes B_1\tilde{K})\tilde{\delta}(t)\\ &+ (I_m \otimes D_1)w(t) + (I_m \otimes B_1)\hat{\phi}_1(t) + \tilde{f}_1(t)]\\ \leq& 2\tilde{\delta}_1{}^T(t)(I_m \otimes P)[(I_m \otimes A_1 - \mathfrak{Q} \otimes B_1B^TM^TP)\tilde{\delta}(t)\\ &+ (I_m \otimes D_1)w(t) + (I_m \otimes B_1)\hat{\phi}_1(t) + \tilde{f}_1(t)]\end{aligned} \tag{12}$$

By using Young's inequality, $2\tilde{\delta}_1^T(t)(I_m \otimes P)\tilde{f}_1(t)$ can be transformed into the following equation.

$$2\tilde{\delta}_1^T(t)(I_m \otimes P)\tilde{f}_1(t) \leq \rho_1\tilde{\delta}_1^T(t)(I_m \otimes P)\tilde{\delta}_1(t) + \frac{1}{\rho_1}\tilde{f}_1^T(t)(I_m \otimes P)\tilde{f}_1(t) \tag{13}$$

where ρ_1 is a given positive constant. It can be obtained from Equations (4), (5) and (8) that $\tilde{f}_1^T(t)(I_m \otimes P)\tilde{f}_1(t)$ can be transformed into

$$\begin{aligned}\tilde{f}_1^T(t)(I_m \otimes P)\tilde{f}_1(t) &= \hat{f}_1^T(t)(I_m \otimes M_1^TPM_1)\hat{f}_1(t)\\ &= \bar{f}^T(t)(H_1^TH_1 \otimes M_1^TPM_1)\bar{f}(t)\end{aligned} \tag{14}$$

Based on the Lipschitz condition and Equation (14), the following inequality holds

$$\begin{aligned}\tilde{f}_1^T(t)(I_m \otimes P)\tilde{f}_1(t) \leq& \epsilon\nu^T(t)(H_1^TH_1 \otimes M_1^TPM_1)\nu(t)\\ \leq& \epsilon\hat{\nu}_1^T(t)(I_m \otimes M_1^TPM_1)\hat{\nu}_1(t)\\ \leq& \epsilon\tilde{\delta}_1^T(t)(I_m \otimes N_1^TM_1^TPM_1N_1)\tilde{\delta}_1(t)\\ &+ \epsilon\tilde{\delta}_2^T(t)(I_m \otimes N_2^TM_1^TPM_1N_2)\tilde{\delta}_2(t)\\ \leq& \tilde{\epsilon}_1\tilde{\delta}_1^T(t)(I_m \otimes P)\tilde{\delta}_1(t) + \tilde{\epsilon}_2\tilde{\delta}_2^T(t)(I_m \otimes P)\tilde{\delta}_2(t)\end{aligned} \tag{15}$$

where $\tilde{\epsilon}_1 = ||M_1N_1||_2\epsilon$ and $\tilde{\epsilon}_2 = ||M_1N_2||_2\epsilon$. In order to introduce the $\tilde{\delta}_2$ term, we have

$$\begin{aligned}&\mathcal{S}\{\tilde{\delta}_2^T(t)P^T[(I_m \otimes A_{21} + \tilde{\theta}\mathfrak{Q} \otimes B_2\tilde{K}_1)\tilde{\delta}_1(t)\\ &+ (I_m \otimes A_{22} + \tilde{\theta}\mathfrak{Q} \otimes B_2\tilde{K}_2)\tilde{\delta}_2(t)\\ &+ (I_m \otimes D_2)w(t) + (I_m \otimes B_2)\hat{\phi}_2(t) + \tilde{f}_2(t)]\} = 0.\end{aligned} \tag{16}$$

Similarly, one has

$$2\tilde{\delta}_2(t)(I_m \otimes P)\tilde{f}_2(t) \leq \rho_2\tilde{\delta}_2^T(t)(I_m \otimes P)\tilde{\delta}_2(t) + \frac{1}{\rho_2}\Big[\tilde{\epsilon}_3\tilde{\delta}_1^T(t)(I_m \otimes P)\tilde{\delta}_1(t) + \tilde{\epsilon}_4\tilde{\delta}_2^T(t)(I_m \otimes P)\tilde{\delta}_2(t)\Big],$$

where $\tilde{\epsilon}_3 = ||M_2N_1||_2\epsilon$, $\tilde{\epsilon}_4 = ||M_2N_2||_2\epsilon$ and ρ_2 is a given positive constant. Then, by combining Equations (16) and (12), and introducing the augmented state vector as $\xi(t) = [\tilde{\delta}_1^T(t)\ \tilde{\delta}_2^T(t)\ w^T(t)\ \phi^T(t)]^T$, we can derive

$$\dot{V}(x(t)) + \alpha V(x(t)) - \beta\omega^T(t)\omega(t) - \gamma\phi^T(t)\phi(t) \leq \xi^T(t)\Gamma\xi(t) \leq 0. \tag{17}$$

Denote $X = P^{-1}$, when $\beta = -\frac{\alpha}{\bar{w}^2}$, $\gamma = -\frac{\alpha}{\bar{\phi}^2}$ left and right-multiply $\Gamma < 0$, by $\mathrm{diag}\{X, X, I, I\}$, it is easily get (10), and by using Lemma 2 we can get the following result $\tilde{\delta}_1^T(t)(I_m \otimes P)\tilde{\delta}_1(t) \leq 1$, it yields that

$$\tilde{\delta}_{i,1}^T(t)r_1\tilde{\delta}_{i,1}(t) \leq 1, \tag{18}$$

where $r_1 = \dfrac{1}{\sqrt{\lambda_{min}(P)}}$. Then, according to Equation (9), we can get

$$\begin{aligned}&\Big\|(I_m \otimes A_{22} + \tilde{\theta}\mathfrak{Q} \otimes B_2\tilde{K}_2 + \epsilon(I_m \otimes M_2N_2)\Big\|\Big\|\tilde{\delta}_2(t)\Big\| \leq\\ &\Big\|(I_m \otimes A_{21} + \tilde{\theta}\mathfrak{Q} \otimes B_2\tilde{K}_1) + \epsilon(I_m \otimes M_2N_1)\Big\|r_1 + \|D_2\|\bar{w} + \|B_2\|\bar{\phi},\end{aligned} \tag{19}$$

it is easily to get $\left\|\tilde{\delta}_2\right\| \leq r_2$ that

$$\tilde{\delta}_2^T(t)\frac{1}{r_2^2}\tilde{\delta}_2(t) \leq 1, \tag{20}$$

where $r_2 = \dfrac{\left\|(I_m \otimes A_{21} + \underline{\theta}\mathfrak{Q} \otimes B_2\tilde{K}_1 + \epsilon(I_m \otimes M_2N_1)\right\| r_1 + \|D_2\|\bar{w} + \|B_2\|\bar{\phi}}{\left\|I_m \otimes A_{22} + \mathfrak{Q} \otimes B_2\tilde{K}_2 + \epsilon I_m \otimes M_2N_2\right\|}$. Through combining inequalities in (18) and (20), that

$$\begin{bmatrix}\tilde{\delta}_{i,1}(t)\\ \tilde{\delta}_{i,2}(t)\end{bmatrix}^T \begin{bmatrix}\theta P & \\ & \dfrac{1-\theta}{r_2^2}I\end{bmatrix}\begin{bmatrix}\tilde{\delta}_{i,1}(t)\\ \tilde{\delta}_{i,2}(t)\end{bmatrix} \leq 1, \quad (i = 1, 2, \ldots, m). \tag{21}$$

It yields that $\tilde{\delta}^T(t)(I_m \otimes \hat{P})\tilde{\delta}(t) \leq 1$, which shows $\delta^T(t)(I_m \otimes \check{P})\delta(t) \leq 1$. □

Remark 1. *Comparing the results of [5], we extend the estimation of reachable sets to singular multi-agent systems by constructs a suitable augmented Lyapunov function from the decomposed state vector.*

4. Numerical Example

The examples presented in this section are intended to showcase the effectiveness of the proposed methodology.

Example 1. *Consider the subsequent parameters associated with SMASs (1).*

$$E = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \; A = \begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}, M = \begin{bmatrix}0.5 & 0.5\\ -0.5 & 0.5\end{bmatrix}, D = \begin{bmatrix}0.5\\ 0\end{bmatrix},$$
$$B = \begin{bmatrix}1\\ 2\end{bmatrix}, N = \begin{bmatrix}0.5 & -0.5\\ 0.5 & 0.5\end{bmatrix}, \mathcal{L} = \begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}.$$

Choosing $\alpha = 0.42, f(x_i) = \begin{bmatrix}\xi_{i,1}\sin x_{i,1}\cos x_{i,1}\\ \xi_{i,2}\sin x_{i,2}\cos x_{i,2}\end{bmatrix}, \theta = \begin{bmatrix}0.5 & 0.6 & 0.8 & 1\end{bmatrix}^T, \bar{\phi} = \bar{w} = 1$ *where* $\xi_{1,1} = 0.3, \xi_{1,2} = 0.2, \xi_{2,1} = 0.1, \xi_{2,2} = 0.2, \xi_{3,1} = \xi_{4,1} = 0.2, \xi_{3,2} = \xi_{4,2} = 0.3$, *and solving the LMIs in Theorem 1, we have*

$$\tilde{K} = \begin{bmatrix}-2.1970 & -0.7323\end{bmatrix}.$$

By taking various values of the parameter θ, we derive the relevant matrices, and the specific results are summarized in Table 2. Figure 2 depicts the reachable set and the bounding ellipsoids. Subsequently, Figure 3 shows the variation of the consensus error of the MASs during the system operation, while Figure 4 indicates that the state of the error system fails to reach the consensus protocol in a short time under open-loop operation. This demonstrates that the control scheme proposed in this paper has good control performance for this type of system.

Example 2. *This section takes a DC motor driving system as a semi-physical simulation example, and its overall structure is shown in [26]. Let $\omega(t)$, $i(t)$ and $v_{in}(t)$ denote the rotor angular velocity, armature current and input control voltage of the DC motor, respectively. Based on Kirchhoff's voltage law and mechanical rotation dynamics, the practical dynamic model of the DC motor is established as*

$$\begin{cases}\dot{\omega}(t) = -\dfrac{d}{J}\omega(t) + \dfrac{K_m}{J}i(t),\\ v_{in}(t) = K_e\omega(t) + R_a i(t).\end{cases}$$

Define the time-varying comprehensive inertia and damping parameters as $J = J_0 + \dfrac{J_L}{k^2}$ and $d = d_0 + \dfrac{d_L}{k^2}$, where J_0 and d_0 are the inherent inertia and damping of the motor, J_L and d_L represent time-varying load inertia and load damping under different operating conditions, and $k = 10$ is the gear ratio. Following the singular *modeling method, we choose state variables $x_1(t) = \omega(t), x_2(t) = i(t)$. The original dynamic equations are rearranged into a standard singular system form:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -\dfrac{d}{J} & \dfrac{K_m}{J} \\ K_e & R_a \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} v_{in}(t).$$

In this example, the physical parameters are set as: $K_e = 0.3$, $K_m = 0.15$, $R_a = 0.3$, $J_0 = 1$, $J_L = 50$, $d_0 = 2.8$, $d_L = 20$. The corresponding system matrices are calculated as

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 0.1 \\ 0.3 & 0.3 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

We select $\alpha = 0.3$. It may be advisable to define other parameters to be consistent with those in Example 1. Then, through the solution of Theorem 1, we can obtain $\tilde{K} = \begin{bmatrix} 0 & -2.1099 \end{bmatrix}$. This practical DC motor case further verifies the rationality and engineering applicability of the proposed singular multi-agent cooperative control strategy.

As shown in Figure 5, the consensus error of the multi-agent like model composed of four DC motors converges to 0 within a certain period of time. This indicates that the control gain given by Theorem 1 can effectively resist the impacts of external disturbances, nonlinear dynamics, actuator failures, etc. Figure 6 shows that the system state operates within the given reachable set.

Table 2. Parameter tolerance test of Example 1.

ε	**0.3**	**0.5**	**0.7**
$\hat{P}$	$\begin{bmatrix} 0.2197 & 0 \\ 0 & 0.0841 \end{bmatrix}$	$\begin{bmatrix} 0.3662 & 0 \\ 0 & 0.0601 \end{bmatrix}$	$\begin{bmatrix} 0.5126 & 0 \\ 0 & 0.0360 \end{bmatrix}$

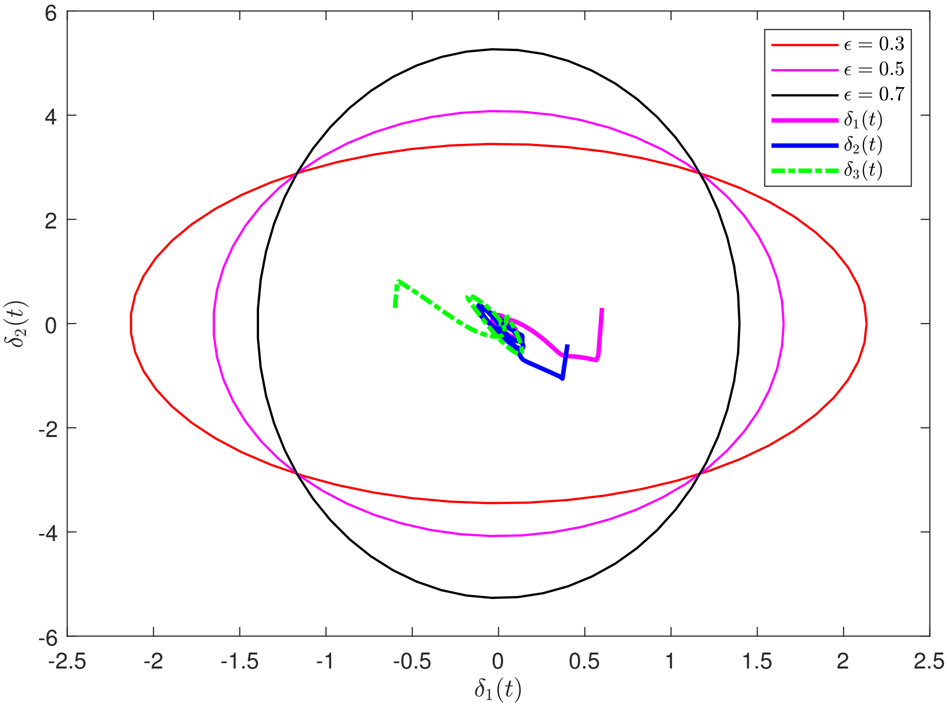

Figure 2. Comparison of reachable sets and boundary ellipsoids for Example 1 with different ε.

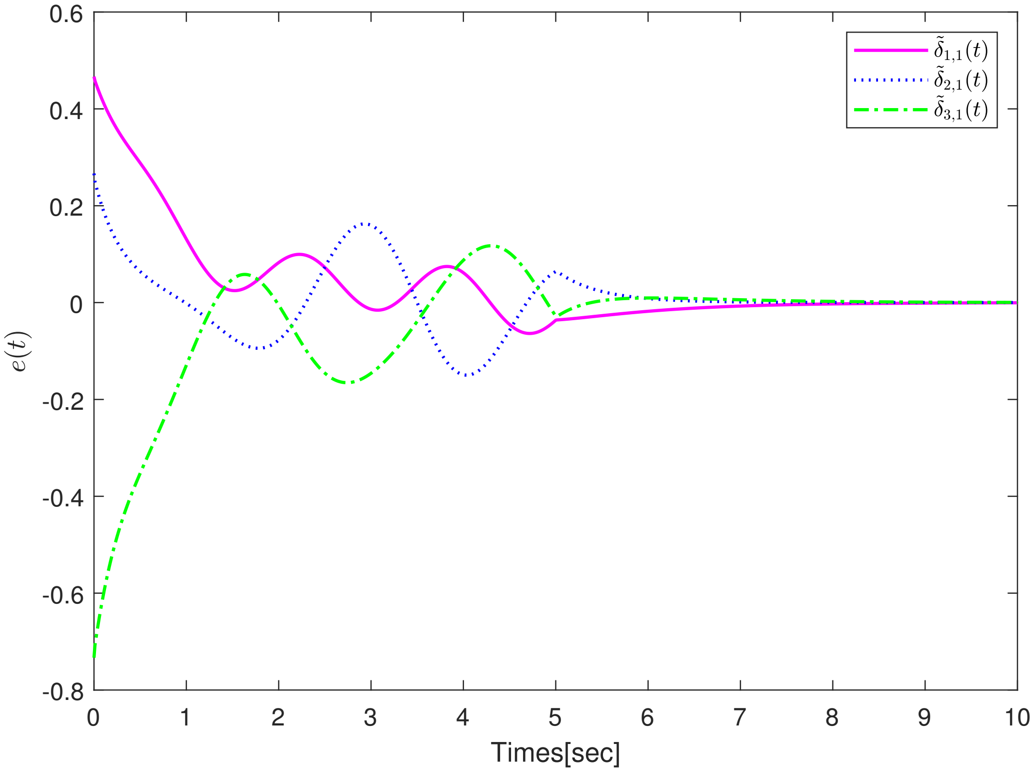

Figure 3. State responses of the error system for Example 1 under closed-loop condition.

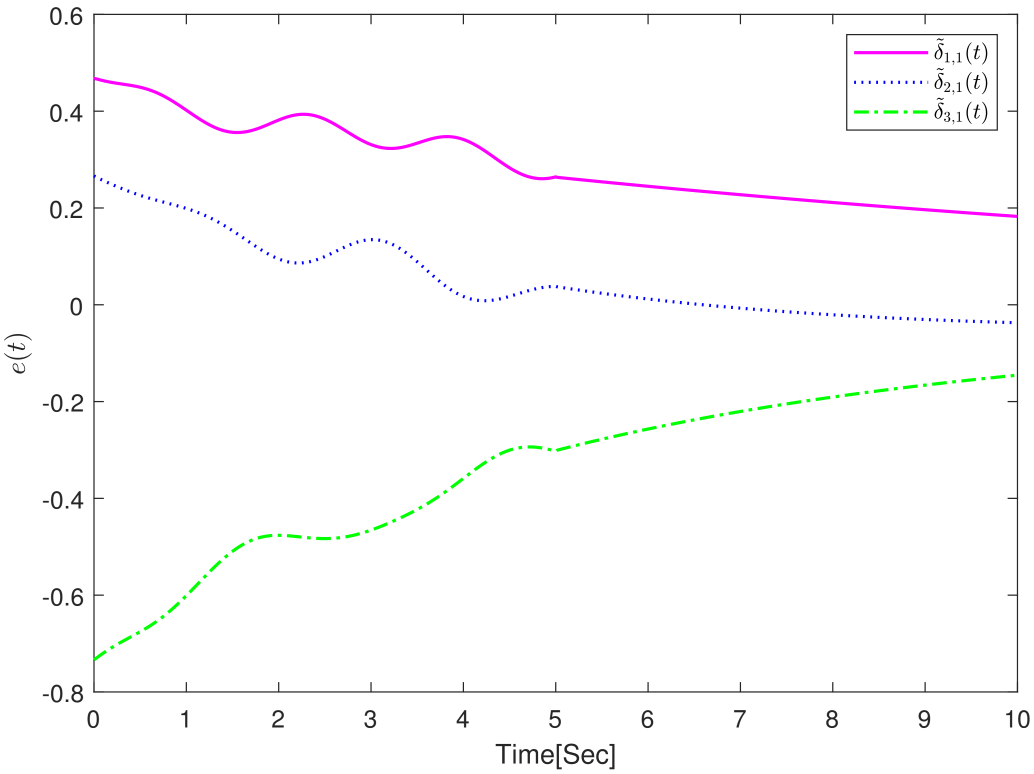

Figure 4. State responses of the error system for Example 1 under open-loop condition.

Table 3. Parameter tolerance test of Example 2.

ε	**0.3**	**0.5**	**0.7**
$\hat{P}$	$\begin{bmatrix} 0.6330 & 0 \\ 0 & 3.6863 \end{bmatrix}$	$\begin{bmatrix} 1.0549 & 0 \\ 0 & 2.6331 \end{bmatrix}$	$\begin{bmatrix} 1.4769 & 0 \\ 0 & 1.5799 \end{bmatrix}$

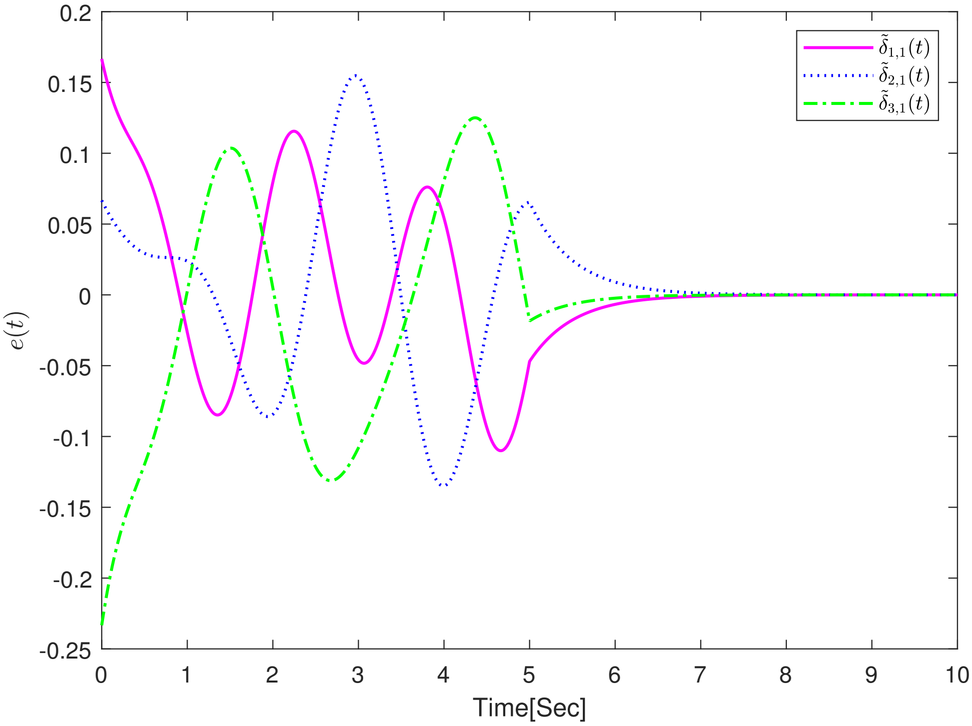

Figure 5. State responses of the error system for Example 2 under closed-loop condition.

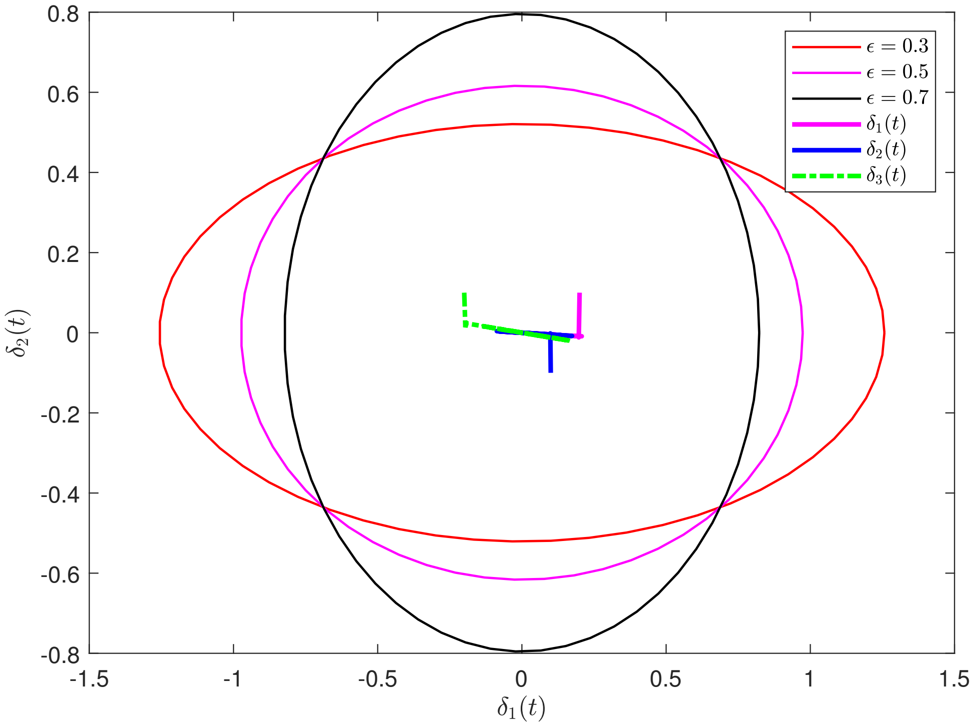

Figure 6. Comparison of reachable sets and boundary ellipsoids for Example 2 with different ε.

5. Conclusions

This study addresses the reachability issue for delay-free continuous-time singular nonlinear multi-agent systems. Leveraging Linear Matrix Inequalities (LMI), a sufficient condition is derived to restrict the RSE of singular systems within an ellipsoid. The proposed findings are anchored in strict LMI, which significantly improves the condition's feasibility and ease of application. Numerical simulations and a DC motor model are conducted to validate the efficacy of the developed approach. For future research directions, we intend to explore more practical methodologies and extend the study to broader scenarios—including Wirtinger-based inequalities, time-delay effects, and switching communication topologies.

Author Contributions

M.L.: conceptualization, methodology, software; L.Z.: data curation, writing—original draft preparation; N.Z.: visualization, investigation; X.W. (Xiaomei Wang): supervision, software, validation; X.W. (Xinjun Wang): writing—reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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