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Field Theory Approach to Eigenstate Thermalization in Random Quantum Circuits

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Abstract: We use a field-theoretic method to explore the statistics of eigenfunctions of the Floquet operator for a large family of Floquet random quantum circuits. The correlation function of the quasienergy eigenstates is calculated and shown to exhibit random matrix circular unitary ensemble statistics, which is consistent with the analogue of Berry's conjecture for quantum circuits. This quantity determines all key metrics of quantum chaos, such as the spectral form factor and thermalizing time-dependence of the expectation value of an arbitrary observable. It also allows us to explicitly show that the off-diagonal matrix elements of local operators possess a variance exponentially small in system size, an essential aspect of the eigenstate thermalization hypothesis (ETH). These results provide analytical evidence for ETH in the family of Floquet random quantum circuits. An outstanding open question for this and most of other sigma-model calculations is a mathematically rigorous proof of the validity of the saddle-point approximation in the large-N limit.

Keywords: eigenstate thermalization hypothesis; quantum chaos; random quantum circuits

1. Introduction

Quantum statistical mechanics is a cornerstone of modern physics. However, understanding of its microscopic origins is still incomplete. How does an isolated quantum many-body system, prepared in a generic initial state, reach thermal equilibrium under its own unitary dynamics? It was proposed by Deutsch [1] and Srednicki [2,3] that the emergence of the statistical mechanical behavior in isolated quantum many-body systems can be understood based on the eigenstate thermalization hypothesis (ETH). It states that the matrix element of a physical observable O in the energy eigenbasis of a thermalizing system has the form [3]

$$O_{\mu\nu} = \bar{O}(\bar{E}_{\mu\nu})\delta_{\mu\nu} + \Omega^{-1/2}(\bar{E}_{\mu\nu}) f(\bar{E}_{\mu\nu}, \Delta E_{\mu\nu}) R_{\mu\nu}. \tag{1}$$

Here $O_{\mu\nu} = \langle\mu| O |\nu\rangle$, where $|\nu\rangle$ represents the energy eigenstate with energy E_ν. $\bar{E}_{\mu\nu} = (E_\mu + E_\nu)/2$ and $\Delta E_{\mu\nu} = E_\mu - E_\nu$. $R_{\mu\nu}$ is a random variable with zero mean and unit variance. $\Omega(\bar{E})$ is the many-body density of states at energy $\bar{E}$. $\bar{O}(\bar{E})$ and $f(\bar{E}, \Delta E)$ are smooth functions of their arguments, and $f(\bar{E}, \Delta E)$ is of order unity. From the ETH ansatz, it follows that for an arbitrary initial state with a small quantum uncertainty in total energy, the expectation value of the observable O reaches its equilibrium value at long time with temporal fluctuations exponentially small in system size.

ETH is closely related to the random matrix theory (RMT) [4–10], and connects thermalization with the notion of quantum chaos [11,12]. For a quantum system whose classical limit is chaotic, it was conjectured that the statistics of its energy levels is the same as that of a random matrix ensemble in a suitable symmetry class [13,14]. The appearance of the universal RMT level statistics in a quantum system has been widely used as a definition of quantum chaos and countless efforts have been undertaken to understand the origin of this universality in a variety of seemingly unrelated systems [15–37].

Apart from the level statistics another metric of quantum chaos is the statistics of energy eigenstates. For quantum systems with classically chaotic counterparts Berry's conjecture [38] is expected to apply, which postulates that an energy eigenstate can be considered as a random superposition of some basis states with Gaussian distributed amplitudes. The ETH follows from Berry's conjecture and connects the statistics of energy eigenstates with the thermalization of physical observables. The ETH has been numerically verified in various systems [11,39–51]. However, its formal proof is still lacking and its range of validity is unknown.

This paper explores the validity of an analogue of Berry's conjecture and the ETH in random quantum circuits (RQCs), which are ideal platforms to study chaos and thermalization (see reviews Refs. [52,53] and references therein). It has been demonstrated that RQCs mimic generic many-body quantum chaotic systems [18,54–66]. In particular, ETH has been carefully investigated in certain Floquet quantum circuits making use of some of their fine-tuned features [21,67,68]. This work puts forward a field-theoretic approach [69] which allows to calculate the statistics of quasienergy eigenstates and investigate ETH in a large family of Floquet RQCs.

The probe we use to study the eigenstate statistics is the following eigenstate correlation function. It is a correlation function of the amplitudes of the (quasi)energy eigenstates $|\nu\rangle$ and $|\mu\rangle$ in a certain fixed basis ($\{|n\rangle\}$) [70]:

$$C_{nn'm'm}(\omega) = \left\langle \sum_{\nu,\mu=1}^{N} \langle n|\nu\rangle \langle \nu|n'\rangle \langle m'|\mu\rangle \langle \mu|m\rangle \, \delta\left(\omega - E_\nu + E_\mu\right) \right\rangle . \tag{2}$$

Here the angular bracket represents the averaging over an ensemble of statistically similar systems and N is the Hilbert space dimension. Note that we sum over all pairs of (quasi)energy eigenstates $|\nu\rangle$ and $|\mu\rangle$ with fixed (quasi)energy separation $\Delta E_{\nu\mu} = \omega$ (For Floquet system, $\delta(\varepsilon)$ in Equation (2) represents the 2π-periodic Dirac delta function $\delta(\varepsilon) = \sum_{k=-\infty}^{\infty} e^{ik\omega}/2\pi$.).

For an ensemble of Floquet (alternatively, Hamiltonian) systems whose time evolution is generated by unitary Floquet operators U from the circular unitary ensemble (CUE) (alternatively, by Hermitian Hamiltonians H from the Gaussian unitary ensemble (GUE)) [9, 10], that is the time evolution operator $U(t) = U^t$ and $U \in$ CUE ($U(t) = e^{-iHt}$ and $H \in$ GUE), one can prove that

$$\begin{aligned} C_{nn'm'm}(\omega) = &\frac{1}{N^2-1} R_2(\omega) \left(\delta_{nn'}\delta_{mm'} - \frac{1}{N}\delta_{mn}\delta_{n'm'} \right) \\ &+ \frac{\delta(\omega)}{N^2-1} \left(N\delta_{mn}\delta_{n'm'} - \delta_{nn'}\delta_{mm'} \right), \end{aligned} \tag{3}$$

where the two-level correlation function [9,10] $R_2(\omega)$ is

$$R_2(\omega) = \sum_{\nu,\mu=1}^{N} \left\langle \delta\left(\omega - E_\nu + E_\mu\right) \right\rangle . \tag{4}$$

The derivation of Equation (3) can be found in the Supplemental Material (see also Refs. [71,72]), and it makes use of the fact that the (quasi)energy eigenvalues and eigenfunctions of the random matrix ensembles are statistically independent. It is expected that the eigenstate correlation function $C_{nn'm'm}(\omega)$ for generic quantum many-body chaotic systems follows the random matrix prediction (Equation (3) for the unitary symmetry class), when the energy separation ω is smaller than the Thouless energy [11,72]. However, to the best of our knowledge, analytical support for this conjecture is only available in a small number of systems, such as the Sachdev-Ye-Kitaev model considered in Ref. [73].

The eigenstate correlation function $C_{nn'm'm}(\omega)$ determines many important physical properties, including the time-evolution of the expectation value of physical observables; the autocorrelation function of physical operators which is directly linked with ETH [3, 67, 68, 74, 75]; the spectral form factor (SFF) [14, 76] which diagnoses the statistics of the (quasi)energy spectra; and the partial spectral form factor (PSFF) [21, 77] which measures the statistical properties of the (quasi)energy eigenstates and eigenvalues and is closely related to the subsystem ETH [78].

In the following, we first examine the connections between the eigenstate correlation function $C_{nn'm'm}(\omega)$ and the aforementioned physical properties, and also explore the behaviors of these properties when the eigenstate correlation function takes the RMT form in Equation (3). Later, we compute the eigenstate correlation function for a large family of Floquet RQCs using a field theoretical approach. We show that it matches the RMT result (to the leading order in large N), indicating that these physical properties are in agreement with the corresponding RMT predictions (when the contribution from the higher order terms in $C_{nn'm'm}(\omega)$ is not important).

2. Eigenstate Statistics and Thermalization

The eigenstate correlation function Equation (2) determines thermalizing time-dependence of the expectation value of observables. Given an initial density matrix $\rho(0)$, its time evolution is given by $\rho(t) = U(t)\rho(0)U^\dagger(t)$. From Equation (3), we find via a simple calculation (see the Supplemental Material)

$$\langle\rho(t)\rangle = \rho_{th}\frac{N^2 - K(t)}{N^2 - 1} + \rho(0)\frac{K(t) - 1}{N^2 - 1}, \tag{5}$$

where $\rho_{th} = \mathbf{1}/N$ is the thermal density matrix. The spectral form factor (SFF) $K(t) = \left\langle |\mathrm{Tr}\, U(t)|^2 \right\rangle$ is the Fourier transform of the two-level correlation function $R_2(\omega)$ (Equation (4)) and can be readily derived from the eigenstate correlation function (see Equation (2)). For generic quantum chaotic systems, the SFF starts off at $K(0) = N^2$, but quickly drops to smaller values via an initial slope and then approaches a plateau at $K(t \to \infty) = N$ via a linear-in-t ramp (for unitary symmetry class in the absence of degeneracies) [9,10]. This result suggests that in chaotic systems the initial thermalization dynamics happens very quickly (compared to the Heisenberg/plateau time) but this behavior is non-universal and is determined by the Fourier transform of the density of states (For generic ergodic systems, Equation (5) is expected to hold for sufficiently large t, and therefore the precise connection between the early time slope and the relaxation process requires further investigation.) [79–81]. The universal features in thermalization are more subtle and appear at long times in temporal fluctuations around the thermal average, the system eventually settles into.

To connect the eigenstate correlation function with ETH [67,68,74], consider the variance of the matrix elements of a Hermitian operator O in the (quasi)energy eigenbasis summed over all pairs of μ, ν with the energy separation $\Delta E_{\nu\mu} = \omega$:

$$C_O(\omega) = \frac{1}{N}\sum_{\nu,\mu=1}^{N} \left\langle |O_{\mu\nu}|^2 \delta(\omega - E_\nu + E_\mu) \right\rangle. \tag{6}$$

It is easy to see that this expression is just the Fourier transform of the autocorrelation function of the operator O, i.e., $C_O(t) = \langle \mathrm{Tr}\,(O(t)O)\rangle / N$, which can be obtained from the eigenstate correlation function $C_{nn'm'm}(\omega)$.

We now focus on Floquet many-body chaotic systems which are expected to thermalize to infinite temperature states [82,83], and consider for simplicity a traceless operator $\mathrm{Tr}(O) = 0$. If $C_{nn'm'm}(\omega)$ coincides with the RMT prediction in Equation (3), we obtain

$$C_O(\omega) \approx R_2(\omega)\, \mathrm{Tr}(O^2)/N^3, \tag{7}$$

in the large $N \to \infty$ limit. This equation suggests that the variance of the matrix element $O_{\mu\nu}$ is given by $\mathrm{Tr}(O^2)/N^2$ [67,68,84]. Specifically, it implies that the spectral function in Equation (1) takes the form $f(\Delta E_{\mu\nu}) = \sqrt{\mathrm{Tr}(O^2)/2\pi N}$, given $\bar{O} = 0$ and $\Omega(\bar{E}_{\mu\nu}) = N/2\pi$ [17]. For physical operator O with eigenvalues of order unity, it follows that $f \sim O(N^0)$, as required by ETH. Therefore, showing the eigenstate correlation function acquires the RMT form in Equation (3) implies the smallness of the variance of the off-diagonal matrix element $O_{\mu\nu}$—a crucial aspect of ETH ensuring the small temporal fluctuation of the expectation value of O around its equilibrium value [2,3].

In fact, $f(\omega) = \sqrt{\mathrm{Tr}(O^2)/2\pi N}$, deduced from the CUE result for the eigenstate correlation function in Equation (3), is in agreement with the RMT prediction for the statistics of the matrix element $O_{\mu\nu}$ (for the unitary symmetry class):

$$O_{\mu\nu} = \frac{\mathrm{Tr}(O)}{N}\delta_{\mu\nu} + \left(\frac{\mathrm{Tr}(O^2)}{N^2} - \frac{\mathrm{Tr}^2(O)}{N^3}\right)^{1/2} R_{\mu\nu}. \tag{8}$$

For a traceless operator O, this expression is consistent with the one in Ref. [11], and leads to $f(\omega) = \sqrt{\mathrm{Tr}(O^2)/2\pi N}$. The fact that Equation (3) (or equivalently Equation (7)) is consistent with Equation (8) is not surprising since they are both derived from the RMT eigenstate statistics. For generic ergodic systems without any antiunitary symmetry and small enough energy separation $\Delta E_{\mu\nu}$, ETH is expected to reduce to RMT [11,85]. For a larger energy separation, ETH, which accounts for more structure beyond RMT, is believed to be applicable (except for the edges of the spectrum).

Using the RMT prediction for $C_{nn'm'm}(\omega)$ in Equation (3), one can also show that, the PSFF [21,77] defined as $K_{\mathcal{A}}(t) = \left\langle \mathrm{Tr}_{\bar{\mathcal{A}}} \left[\mathrm{Tr}_{\mathcal{A}}\, U(t)\, \mathrm{Tr}_{\mathcal{A}}\, U^\dagger(t) \right] \right\rangle$ acquires a form

$$K_{\mathcal{A}}(t) = \frac{N N_{\mathcal{A}} - N/N_{\mathcal{A}}}{N^2 - 1} K(t) + \frac{N^3/N_{\mathcal{A}} - N N_{\mathcal{A}}}{N^2 - 1}. \tag{9}$$

Here $\mathrm{Tr}_{\mathcal{A}}$ and $\mathrm{Tr}_{\bar{\mathcal{A}}}$ indicate the trace operators over the Hilbert spaces of the subsystem $\mathcal{A}$ and its complement $\bar{\mathcal{A}}$, respectively. $N_{\mathcal{A}}$ denotes the dimension of the Hilbert space of the subsystem $\mathcal{A}$. The fact that the PSFF follows the RMT form is in agreement with the subsystem ETH [77,78], which implies ETH for all observables in sufficiently small subsystems.

3. Supersymmetric Sigma Model

We will now prove that for a large family of Floquet random quantum circuits, the eigenstate correlation function is given by the CUE result in Equation (3). As explained earlier, this in turn implies that the variance of the off-diagonal matrix element $O_{\mu\nu}$ for these RQCs is of order $O(N^{-1})$, consistent with ETH. We consider RQCs [69], which consist of an arbitrary dimensional simple square lattice of qudits, evolving under periodic applications of local quantum gates. Specifically, random unitary matrices, drawn independently from the CUE of dimension q^2, are applied to different pairs of neighboring qudits at various substeps within one period. Each qudit is coupled to all its neighbors at different and arbitrary substeps by independent CUE matrices. We focus on the limit where the single-particle Hilbert space dimension of each qudit $q \to \infty$. It is straightforward to see that the second moment of the Floquet operator for any of these RQCs is given by [69]

$$\left\langle U_{ij} U^\dagger_{j'i'} \right\rangle = \frac{1}{N} \delta_{ii'} \delta_{jj'}. \tag{10}$$

To the leading order in large q, the fourth moment of the Floquet operator is also identical to that of CUE [69].

In Ref. [18], Chan et al. considered a special case of the one-dimensional brickwork model, and studied the SFF, the bipartite entanglement, as well as the two-point and out-of-time-order correlators of the local observables (see also Ref. [21] for finite q case). Later in Ref. [69], we considered more general models and employed a sigma model approach [86] to prove that the statistics of the quasienergy spectrum of any model in this family of Floquet RQCs agrees with the CUE result, irrespective of the dimensionality of the qudit lattice, the choice of the boundary condition (periodic or open) and the ordering of the local quantum gates. We will employ a similar sigma model approach to show that the eigenstate correlation function of these models also follows the CUE result Equation (3).

For an arbitrary ensemble of Floquet systems, the correlation function $C_{nn'm'm}(\omega)$ (Equation (2)) can be obtained from the generating function

$$\mathcal{Z}[J] = \left\langle \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{\det\left(1 - \alpha_{\mathrm{F}} e^{i\phi} U + J^+\right)}{\det\left(1 - \alpha_{\mathrm{B}} e^{i\phi} U\right)} \frac{\det\left(1 - \beta_{\mathrm{F}} e^{-i\phi} U^\dagger + J^-\right)}{\det\left(1 - \beta_{\mathrm{B}} e^{-i\phi} U^\dagger\right)} \right\rangle, \tag{11}$$

where the angular bracket stands for averaging over the ensemble of Floquet operators U. $J^\pm$ is a $N \times N$ source matrix, while $\alpha_F, \alpha_B, \beta_F$ and β_B are complex variables related to ω. Taking derivatives of the generating function $\mathcal{Z}[J]$ with respect to $J^\pm$, we find

$$\begin{aligned} \bar{C}_{nn'm'm}(\alpha_{\mathrm{F}}, \beta_{\mathrm{F}}) &\equiv \left. \frac{\partial^2 \mathcal{Z}[J]}{\partial J^+_{n'n} \partial J^-_{mm'}} \right|_{J=0, \alpha_{\mathrm{B}}=\alpha_{\mathrm{F}}, \beta_{\mathrm{B}}=\beta_{\mathrm{F}}} \\ &= \sum_{k=0}^{\infty} \left\langle (U^k)_{nn'} (U^{-k})_{m'm} \right\rangle (\alpha_{\mathrm{F}} \beta_{\mathrm{F}})^k, \end{aligned} \tag{12}$$

where we have set $J = 0$, $\alpha_{\mathrm{B}} = \alpha_{\mathrm{F}}$ and $\beta_{\mathrm{B}} = \beta_{\mathrm{F}}$ (As will be apparent later, we find it more convenient for the calculation to set $\alpha_{\mathrm{B}} = \alpha_{\mathrm{F}}$ and $\beta_{\mathrm{B}} = \beta_{\mathrm{F}}$ after taking the derivatives. $|\alpha_{\mathrm{B}}|, |\beta_{\mathrm{B}}|$ are chosen to be infinitesimally smaller than 1 to avoid singularity and to ensure that the bosonic integral for the denominator is well defined.). Here $\left\langle (U^k)_{nn'} (U^{-k})_{m'm} \right\rangle$ is just the Fourier transform of $C_{nn'm'm}(\omega)$ at discrete time k in units of the period. We therefore set $\alpha_{\mathrm{F}} \beta_{\mathrm{F}} = e^{i\omega}$, which leads to

$$C_{nn'm'm}(\omega) = \frac{1}{2\pi} \left(\bar{C}_{nn'm'm} + \bar{C}^*_{n'nmm'} - \delta_{nn'} \delta_{mm'} \right). \tag{13}$$

Following a standard derivation of the supersymmetric (SUSY) sigma model [87] for the Floquet systems with the help of the color-flavor transformation [88–91], we find that the generating function $\mathcal{Z}[J]$ can be rewritten as an integral over $2N \times 2N$ supermatrices Z and $\tilde{Z}$ (see the Supplementary Material, the review [86] and references therein):

$$\begin{aligned}\mathcal{Z}[J] &= \det(1+J^+)\det(1+J^-)\left\langle \int \mathcal{D}(\tilde{Z},Z)e^{-S[\tilde{Z},Z]}\right\rangle, \\ S[\tilde{Z},Z] &= -\mathrm{STr}\ln(1-\tilde{Z}Z) + \mathrm{STr}\ln\left(1-\tilde{Z}\alpha_J\mathbf{U}Z\beta_J\mathbf{U}^\dagger\right).\end{aligned} \tag{14}$$

Here Z^{ab}_{ij} and $\tilde{Z}^{ab}_{ij}$ carry indices $a,b=\mathsf{B},\mathsf{F}$ labeling the boson-fermion (BF) space and indices $i,j=1,2,...,N$ labeling the Hilbert ($\mathcal{H}$) space wherein the Floquet operator acts. They are subject to the constraints $\tilde{Z}^{\mathsf{FF}}=-Z^{\mathsf{FF}\,\dagger}$, $\tilde{Z}^{\mathsf{BB}}=Z^{\mathsf{BB}\,\dagger}$, $|\tilde{Z}^{\mathsf{BB}}Z^{\mathsf{BB}}|<1$. STr denotes the supertrace over the $\mathsf{BF}\otimes\mathcal{H}$ space. α_J and β_J are $2N\times 2N$ supermatrices defined as

$$\begin{aligned}\alpha_J &= \begin{bmatrix}\alpha_{\mathrm{B}}\mathbf{1}_{\mathcal{H}} & 0 \\ 0 & \alpha_{\mathrm{F}}(\mathbf{1}_{\mathcal{H}}+J^+)^{-1}\end{bmatrix}_{\mathsf{BF}}, \\ \beta_J &= \begin{bmatrix}\beta_{\mathrm{B}}\mathbf{1}_{\mathcal{H}} & 0 \\ 0 & \beta_{\mathrm{F}}(\mathbf{1}_{\mathcal{H}}+J^-)^{-1}\end{bmatrix}_{\mathsf{BF}},\end{aligned} \tag{15}$$

and $\mathbf{U}=U\otimes\mathbf{1}_{\mathsf{BF}}$. We use $\mathbf{1}_{\mathcal{H}}$ ($\mathbf{1}_{\mathsf{BF}}$) to denote the identity matrix in the $\mathcal{H}$ (BF) space.

Note that the sigma model Equation (14) is exact and applicable to any ensemble of Floquet systems. We will now turn to the specific model under consideration here—the family of Floquet RQCs with local Haar-distributed unitary gates, and use Equation (14) to evaluate the correlation function $C_{nn'm'm}(\omega)$ in the limit where the local Hilbert space dimension is large. Specifically, $1/q$ serves as the small parameter in our perturbative sigma model calculation. In Ref. [69], we have proven that, for the current model, the higher order fluctuations of the supermatrix Z give rise to a contribution of higher order in large q and therefore can be neglected. As a result, we will focus on the saddle points and the quadratic fluctuations around them.

Variation of the action $S[\tilde{Z},Z]$ (Equation (14)) with respect to $\tilde{Z}$ yields the saddle point equation:

$$Z(1-\tilde{Z}Z)^{-1} = \alpha_J\mathbf{U}Z\beta_J\mathbf{U}^\dagger\left(1-\tilde{Z}\alpha_J\mathbf{U}Z\beta_J\mathbf{U}^\dagger\right)^{-1}, \tag{16}$$

for an arbitrary configuration of Floquet operator U in the ensemble. It is easy to see that the standard saddle point $Z^{(s)}=0$ solves this equation, and the corresponding action vanishes at this saddle point $S^{(s)}=0$. Another solution is the nonstandard saddle point of the form [86,92–95]

$$Z^{(ns)} = \begin{bmatrix}0 & 0 \\ 0 & z\end{bmatrix}_{\mathsf{BF}}, \qquad \tilde{Z}^{(ns)} = \begin{bmatrix}0 & 0 \\ 0 & -z^\dagger\end{bmatrix}_{\mathsf{BF}}. \tag{17}$$

Here z is a $N\times N$ matrix acting in the Hilbert space, and can be expressed as $z=c_0z'$, with z' being an arbitrary invertible matrix of order unity and the coefficient $c_0\to\infty$. The action at the nonstandard saddle point is

$$S^{(ns)} = -\,\mathrm{Tr}\ln\left[\alpha_F\beta_F(1+J^+)^{-1}(1+J^-)^{-1}\right]. \tag{18}$$

We note that the standard and nonstandard saddle points are solutions to Equation (16) for every configuration of the Floquet operator U in the ensemble. There also exist other saddle points (given by Equation (17) with singular matrix z') which solve the saddle point equation for one or a few specific realizations of the Floquet operator U in the ensemble. We believe that these saddle points are not important to the ensemble averaged theory for the current model in the large q limit, and therefore ignore them in the following calculation.

If one introduces a positive infinitesimal η to $\ln(\alpha_{\mathrm{F}}\beta_{\mathrm{F}})=i\omega-\eta$, the correlation function $C_{nn'm'm}(\omega)$ obtained from Equations (12) and (13) becomes its smoothed version where the 2π-periodic Dirac delta function $\delta(\varepsilon)$ in the definition Equation (2) is replaced with the 2π-periodic "Lorentzian" with the width η: $\delta(\varepsilon)\to\sum_{k=-\infty}^{\infty}e^{ik\varepsilon-|k|\eta}/2\pi$ [10,86,96]. In the case where $\eta N\gtrsim 1$, we can focus on the dominant standard saddle point as its real part of the action at zero source field ($\mathrm{Re}\,S^{(s)}=0$) is small compared with that of the nonstandard saddle point ($\mathrm{Re}\,S^{(ns)}|_{J=0}=N\eta$) [94]. In Ref. [69], the smoothed two-point level correlation function $R_2^{(s)}(\omega)$ was obtained from the consideration of the standard saddle point only. We will show that the inclusion of the nonstandard saddle point allows us to extract the fine structure of nearby quasienergy eigenstates.

For the standard saddle point, the quadratic fluctuations around it ($\delta Z=Z-Z^{(s)}=Z$) are governed by

$$\delta S_2^{(s)}[\delta\tilde{Z},\delta Z] = \mathrm{STr}\left(\delta\tilde{Z}\delta Z\right) - \mathrm{STr}\left(\delta\tilde{Z}\alpha_J\mathbf{U}\delta Z\beta_J\mathbf{U}^\dagger\right). \tag{19}$$

We approximate the corresponding contribution to the generating function by [69]

$$\mathcal{Z}^{(s)} \approx \det\left((1+J^+)(1+J^-)\right)\int \mathcal{D}(\tilde{Z},Z)e^{-\left\langle \delta S_2^{(s)}\right\rangle}. \tag{20}$$

As shown in Ref. [69], the correction to this approximation involves higher order fluctuations and is negligible in the large N limit. The ensemble averaged effective action $\left\langle \delta S_2^{(s)}\right\rangle$ can be further simplified to,

$$\left\langle \delta S_2^{(s)}\right\rangle = \sum s_a \delta\tilde{Z}_{ij}^{ab}\delta Z_{j'i'}^{ba}\left(\delta_{ii'}\delta_{jj'} - (\alpha_J)_{ji}^{bb}(\beta_J)_{i'j'}^{aa}/N\right), \tag{21}$$

where $s_{\mathsf{B}/\mathsf{F}} = \pm 1$. We have used the fact that the second moment of the Floquet operator for the Floquet RQCs under consideration is given by Equation (10).

Performing the Gaussian integration in Equation (20) and using Equation (12), we obtain

$$\begin{aligned}\bar{C}^{(s)}_{nn'm'm}(\alpha_F,\beta_F) = \delta_{nn'}\delta_{mm'}&\left[1+\frac{1}{N^2}\left(\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}\right)^2\right]\\ &+\delta_{nm}\delta_{n'm'}\frac{1}{N}\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}{1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}}},\end{aligned} \tag{22}$$

where we have set $\alpha_{\mathrm{B}} = \alpha_{\mathrm{F}}$ and $\beta_{\mathrm{B}} = \beta_{\mathrm{F}}$. When substituted into Equation (13), this leads to the smoothed correlation function $C^{(s)}_{nn'm'm}(\omega)$,

$$C^{(s)}_{nn'm'm}(\omega) = \delta_{nn'}\delta_{mm'}\left(\frac{R_2^{(s)}(\omega)}{N^2}+\frac{1}{2\pi N^2}\right) - \frac{\delta_{nm}\delta_{n'm'}}{2\pi N}, \tag{23a}$$

$$R_2^{(s)}(\omega) = \frac{N^2}{2\pi} - \frac{1}{4\pi}\frac{1}{\sin^2(\omega/2)}. \tag{23b}$$

The action for the quadratic fluctuations around the nonstandard saddle point ($\delta Z = Z - Z^{(ns)}$) can be reduced to an expression almost identical to that of the standard saddle point in Equation (19), but with α_J^{FF} and β_J^{FF} replaced by $(\beta_J^{\mathsf{FF}})^{-1} = \beta_{\mathrm{F}}^{-1}(1+J^-)$ and $(\alpha_J^{\mathsf{FF}})^{-1} = \alpha_{\mathrm{F}}^{-1}(1+J^+)$, respectively. As a result, the contribution from the nonstandard saddle point to $\bar{C}_{nn'm'm}$ can be deduced from that of the standard saddle point. It assumes the form

$$\bar{C}^{(ns)}_{nn'm'm}(\alpha_{\mathrm{F}},\beta_{\mathrm{F}}) = -\delta_{nn'}\delta_{mm'}\frac{1}{N^2}(\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^N\frac{\alpha_{\mathrm{F}}\beta_{\mathrm{F}}}{(1-\alpha_{\mathrm{F}}\beta_{\mathrm{F}})^2}. \tag{24}$$

Combining the contributions of the standard and nonstandard saddle points (Equations (22) and (24)), we find, with the help of Equation (13), the explicit expression for $C_{nn'm'm}(\omega)$:

$$\begin{aligned}C_{nn'm'm}(\omega) &= \delta_{nn'}\delta_{mm'}\left(\frac{R_2(\omega)}{N^2}+\frac{1}{2\pi N^2}\right) - \frac{\delta_{nm}\delta_{n'm'}}{2\pi N}\\ &+\left(\frac{1}{N}\delta_{nm}\delta_{n'm'} - \frac{1}{N^2}\delta_{nn'}\delta_{mm'}\right)\delta(\omega),\end{aligned} \tag{25a}$$

$$R_2(\omega) = \frac{N^2}{2\pi} - \frac{1}{2\pi}\frac{\sin^2(N\omega/2)}{\sin^2(\omega/2)} + N\delta(\omega). \tag{25b}$$

Here $R_2(\omega)$ is identical to the two-level correlation function for the CUE [9,10]. This result is equivalent to the random matrix prediction Equation (3) to the leading order in $1/N$. Moreover, the smoothed correlation function $C^{(s)}_{nn'm'm}(\omega)$ in Equation (23a), arising from the contribution of the standard saddle point only, is also consistent with Equation (3) after replacing $R_2(\omega)$ with its smoothed version $R_2^{(s)}(\omega)$. This is not surprising as the expression for the second moment of the Floquet operator (Equation (10)) also holds for the CUE. At the quadratic level, the Floquet RQCs are therefore described by the same effective field theory as the CUE [69]. Moreover, the higher order fluctuations for both the current model and the CUE give rise to negligible contributions in the large q limit [69].

In Equation (25a) (and Equation (23a)), the second and third terms are of higher order in $1/N$ compared with the leading order first term there. They can be rewritten as $\delta_{nn'}\delta_{mm'}R_2^{(\mathsf{dis})}(\omega)/N^4$ and $-\delta_{mn}\delta_{n'm'}R_2^{(\mathsf{dis})}(\omega)/N^3$. Here $R_2^{(\mathsf{dis})}(\omega) = N^2/2\pi$ denotes the disconnected part of the two-level correlation function, and is the dominating term in $R_2(\omega)$ for ω much larger than mean level spacing $2\pi/N$. We note that they do not agree with the corresponding higher order terms in the RMT prediction Equation (3), i.e., $\delta_{nn'}\delta_{mm'}R_2(\omega)/N^4$ and $-\delta_{mn}\delta_{n'm'}R_2(\omega)/N^3$. In fact, the higher order fluctuations ignored in the current calculation can give rise to contributions of the same orders as these terms for $\omega \lesssim 2\pi/N$. In the Supplementary Material, we investigate the role played by the higher order fluctuations of the supermatrix Z for the CUE, as a simple example. We recover the next leading higher order term $-\delta_{mn}\delta_{n'm'}R_2^{(s)}(\omega)/N^3$ in the smoothed correlation function $C^{(s)}_{nn'm'm}(\omega)$ by inclusion of higher order fluctuations around the standard saddle point.

Using our result for $C_{nn'm'm}(\omega)$ in Equation (25a), one can show that, for the current model, the SFF $K(t)$, the PSFF $K_{\mathcal{A}}(t)$ with subsystem Hilbert space dimension $N_{\mathcal{A}} \gg 1$, and the auto-correlation functions of local traceless operators agree with the corresponding RMT predictions. However, the higher order terms in $C_{nn'm'm}(\omega)$ are non-negligible for the PSFF with $N_{\mathcal{A}} \sim O(1)$ or the auto-correlation function of the traceful operator $\mathrm{Tr}\, O \neq 0$. In these cases, consideration of higher order fluctuations is required.

4. Conclusions

In summary, this paper investigates the statistics of quasienergy eigenstates for a family of Floquet random quantum circuits with Haar-distributed two-qudit random unitary gates. We compute the eigenstate correlation function $C_{nn'm'm}(\omega)$ (Equation (2)) via a sigma-model calculation. $C_{nn'm'm}(\omega)$ follows the RMT prediction Equation (3) to the leading order in $1/N$, which shows that the SFF, the PSFF with large subsystem size, and the correlation function of local traceless operators are in agreement with the corresponding RMT predictions. Moreover, this result is consistent with ETH and shows that $f(\omega)$ entering the ETH ansatz also matches the RMT prediction $f(\omega) = \sqrt{\mathrm{Tr}(O^2)/2\pi N}$. We note that the eigenstate correlation function studied in this paper can only provide information about the ensemble averaged result of the temporal relaxation towards thermal equilibrium. To examine sample-to-sample fluctuations, higher-order correlation functions are needed. The current sigma model calculation can be generalized to compute these higher order correlation function of quasienergy eigenstates from the higher order derivatives of the generating function $\mathcal{Z}[J]$ with respect to the source J. These higher-order eigenstate correlation functions are related to the higher-order correlation functions of physical operators (such as the out-of-time-order correlators) and contain additional information about the statistics of the matrix elements of operators required for a full analysis of the generalized ETH proposed in Ref. [74] (see also Refs. [68,75]). We leave this for a future work.

Supplementary Materials

The following supporting information can be downloaded at: https://media.sciltp.com/articles/others/2608041135402348/NP-25120150-SI.pdf. Refs. [97–104] have been cited in the Supplementary Materials.

Author Contributions

Both authors contributed to conceptualization, investigation, writing-original draft preparation, writing-reviewing and editing. Both authors have read and agreed to the published version of the manuscript.

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The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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