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Review

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A Survey of Second-Order Neurodynamic Systems from Optimization Perspective

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Abstract: Over the past two decades, second-order neurodynamic systems have attracted increasing attention due to their strong capability in modeling and solving complex optimization problems. Unlike classical first-order approaches, these continuous neurodynamics exhibit inertial effects, which essentially provide a mechanism for accelerated convergence. We provide an overview of the development of second-order neurodynamic systems from an optimization perspective. We first introduced the basic theoretical framework of second-order neurodynamic systems, including the continuous-time formulations and their connection to optimization models. The review is divided into two main threads: unconstrained and constrained optimization. Based on this categorization, the representative models and solution strategies are analyzed in detail, covering smooth and nonsmooth, as well as convex and nonconvex optimization problems. Finally, we highlight the limitations of the current study and outline some directions for future research.

Keywords: neurodynamic systems; second-order dynamics; unconstrained optimization; constrained optimization; inertial methods

1. Introduction

Optimization theory and the study of dynamical systems have a long history of connection. Many physical phenomena can be presented as differential equations. The paths produced by these equations will move towards a stable state over time, and often, these stable points are also the optimal solutions to energy-minimisation problems. Many optimization algorithms can be regarded as discrete-time versions of continuous-time dynamical systems, and this has now been shown. For a long time, most of the optimization methods in this area have been discrete iterative schemes. Gradient descent [1], Newton's method [2] and proximal algorithms [3] are typical examples. The theoretical foundation of the above methods has been established, particularly for convex problems. However, the optimization problems that arise in practice are gradually becoming more complex. These problems generally have a large volume of data, high-dimensional decision variables, complex constraints and non-convex objectives. In such cases, traditional iterative methods may converge slowly and be highly sensitive to parameter choices. Although many algorithmic improvements have been proposed, these difficulties have also prompted researchers to reexamine optimization from a continuous perspective.

Neurodynamic systems emerged from this context. Rather than constructing an optimization algorithm directly, the central idea is to design a dynamic system whose equilibrium points correspond to solutions of the target problem. Once such a system is established, the optimization task can be transformed into the analysis of trajectory evolution. Over the past several years, this paradigm has found applications in signal processing [4,5], image recovery [6,7], machine learning [8,9], and control [10–12]. Compared with iterative schemes, neurodynamic approaches offer an attractive combination of mathematical interpretability and implementation flexibility, especially in scenarios where parallel computation or real-time processing is desirable [13]. Among various neurodynamic models, the most representative examples are first-order systems and their variants. First-order neurodynamic systems are relatively simple and easy to implement, but they also have certain deficiencies. They have a slow convergence speed and are likely to get stuck in local optima when dealing with ill-conditioned or non-convex problems.

The above deficiencies have promoted the study of accelerated neurodynamic systems, often called second-order neurodynamics. Compared to first-order neurodynamics, second-order systems include an inertia term. As a result, the current motion is affected by a local gradient and also by the previous velocity state. To improve optimization performance, the inertia term can be used to travel in a flat area more quickly and thus accelerate the whole-system convergence. The choice of a suitable damping coefficient can reduce oscillations and improve stability for the system of dynamic equations. Therefore, second-order neurodynamic systems are generally better than first-order systems in both the speed of convergence and stability.

A number of algorithms can be expressed as continuous second-order systems, such as the heavy ball (HB) method [14] and Nesterov's accelerated gradient (NAG) method [15]. The seminal work by Su et al. [16] have established a formula for a second-order dynamical system based on the discrete NAG algorithm, and the convergence speed of this system for convex objectives has been examined. The same neurodynamic system can often be applied to different types of optimization problems and has a unified theoretical framework. Different discretisation strategies can be applied to neurodynamic-based methods to obtain different discrete iterative algorithms. Most of the early studies in this field concentrated on unconstrained settings. In practice, however, the optimization tasks often have constraints or nonsmooth objective functions, or both. Scholars have put forward many ways to solve the above problems. Projection-based neurodynamic methods constrain the motion path to a constraint set and thus ensure that the problem is feasible. Primal-dual (PD) methods introduce auxiliary variables for both equality and inequality constraints. The studies above have increased the application range of second-order neurodynamic methods for optimization problems.

In recent years, second-order neurodynamic methods have shown excellent results in solving optimization problems and are highly generalizable and scalable. This review roughly categorizes the types of optimization problems addressed by existing second-order neurodynamic systems into two classes: unconstrained optimization and constrained optimization. Systematically review the second-order neurodynamic methods related to the two categories of problems, analyze their theoretical foundations and main advantages, and discuss the primary unresolved problems.

The structure of the rest of this survey is as follows. In Section 2, we will introduce the necessary mathematical background and present the optimization problems that will be studied in the paper. The applications of second-order neurodynamics methods in unconstrained and constrained optimization are presented in Sections 3 and 4, respectively. Section 5 presents the main conclusions and lists some possible directions for further study.

2. Preliminaries and Problem Formulation

2.1. Optimization Problems

We consider general optimization problems in both smooth and nonsmooth settings. From an optimization perspective, these problems can be broadly categorized into unconstrained optimization, constrained optimization, and composite optimization.

(1) Unconstrained Optimization:

The unconstrained optimization problem is formulated as

$$\min_{x\in\mathbb{R}^n} F(x), \tag{1}$$

where $F:\mathbb{R}^n\to\mathbb{R}$ is a proper and lower semicontinuous function.

Smooth case: In the classical smooth setting, $F(x)$ is assumed to be continuously differentiable with Lipschitz continuous gradient, i.e., there exists a constant $L>0$ such that

$$|\nabla F(x)-\nabla F(y)|\leq L|x-y|,\ \forall\, x,y\in\mathbb{R}^n. \tag{2}$$

The first-order optimality condition is given by

$$\nabla F(x^*)=0. \tag{3}$$

Nonsmooth case: In many practical applications, the objective function may be nondifferentiable. In this case, the optimality condition is characterized using subdifferential calculus:

$$0 \in \partial F(x^*), \tag{4}$$

where $\partial F(x)$ denotes the (convex) subdifferential of F.

(2) Constrained Optimization:

The constrained optimization problem is given by

$$\min_{x \in \mathbb{R}^n} F(x), \text{ s.t. } x \in \Omega, \tag{5}$$

where $\Omega \subseteq \mathbb{R}^n$ is a closed set.

Smooth case: When $F(x)$ is smooth and Ω is a closed convex set, the optimality condition can be expressed using the normal cone $\mathcal{N}\Omega(x)$:

$$0 \in \nabla F(x^*) + \mathcal{N}\Omega(x^*). \tag{6}$$

This formulation is closely related to variational inequalities and projection-based methods [17,18].

Nonsmooth case: When $F(x)$ is nonsmooth, the optimality condition becomes

$$0 \in \partial F(x^*) + \mathcal{N}_\Omega(x^*), \tag{7}$$

which naturally leads to differential inclusion formulations. Such models are widely used in constrained nonsmooth optimization and provide a foundation for projection and proximal-type neurodynamic systems.

(3) Composite Optimization:

A general type of problem in modern applications can be formulated as a composite problem:

$$\min_{x \in \mathbb{R}^n} F(x) := f(x) + g(x), \tag{8}$$

where $f(x)$ is smooth and $g(x)$ may be nonsmooth.

Unconstrained composite case: The optimality condition is given by

$$0 \in \nabla f(x^*) + \partial g(x^*), \tag{9}$$

which is fundamental in proximal and splitting methods [3,19].

Constrained composite case: Combining composite objectives with constraints result in the general form:

$$\min_{x \in \Omega} f(x) + g(x), \tag{10}$$

with optimality condition

$$0 \in \nabla f(x^*) + \partial g(x^*) + \mathcal{N}_\Omega(x^*). \tag{11}$$

The above is a general form of the current optimization theory. It provides a general form for expressing optimization problems that contain smooth terms, non-smooth regularization or explicit constraints.

2.2. First-Order Neurodynamic Systems

Among all continuous systems used for optimization, first-order neurodynamic systems are likely to be the closest links to dynamic systems theory and first-order optimality conditions. The basic idea is that the system moves along a descent direction derived from local gradient information. A general case is the gradient flow model:

$$\dot{x}(t) = -\nabla F(x(t)), \tag{12}$$

where $x(t)$ is the state variable at time t. Under standard assumptions, every equilibrium point of (12) is a stationary point of problem (1). Solving an optimization problem can be viewed as studying the long-term evolution of the associated dynamic system.

$$\frac{d}{dt} F(x(t)) = \nabla F(x(t))^\top \dot{x}(t) = -|\nabla F(x(t))|^2 \leq 0, \tag{13}$$

which shows that the objective function decreases continuously over time. $F(x(t))$ naturally serves as a Lyapunov function. This property ensures that the system evolves toward an optimal value and provides a guarantee for the convergence analysis.

For constrained optimization problems, one widely used method is projected gradient dynamics

$$\dot{x}(t) = \mathcal{P}_\Omega\big(x(t) - \nabla F(x(t))\big) - x(t), \tag{14}$$

where $\mathcal{P}_\Omega(\cdot)$ denotes the projection operator,which can map the trajectory back within the constraints [20]. Another method relies on Lagrangian equations to transform the constrained optimization problem into an unconstrained optimization problem for solution [21]. These methods have proven particularly useful for equality and inequality constrained optimization and are closely related to variational inequality theory.

2.3. Second-Order Neurodynamic Systems

Second-order neurodynamic systems are extended first-order systems by adding an inertia term, and their general form can be expressed as

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla F(x(t)) = 0, \tag{15}$$

where $\gamma(t) : [t_0, +\infty) \to \mathbb{R}_+$ is a damping coefficient. Damping term $\gamma(t)\dot{x}(t)$ is used for stability. The inertia term facilitates faster convergence. Research shows that with a suitable choice of $\gamma(t)$, second-order systems generally converge much faster than first-order systems. A constant $\gamma(t) = \gamma$ is equivalent to the continuous form of the HB method. Setting $\gamma(t) = \gamma/t$ with $\gamma \geq 3$ recovers the continuous system of the NAG method [16].

For constrained optimization problems, operators that account for the constraints can be added to extend second-order neurodynamic systems. The two are combined as follows:

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla F(x(t)) + \mathcal{N}\Omega(x(t)) \ni 0, \tag{16}$$

where $\mathcal{N}\Omega(x)$ is the normal cone to the set Ω. This kind of formula is naturally suitable for a differential inclusion framework because it can deal with both non-smoothness and constraints at the same time [22].

3. Second-Order Neurodynamics for Unconstrained Optimization

To speed up the optimization of neurodynamic systems, inertia can be introduced in continuous systems, resulting in second-order neurodynamic systems. The system has a more rich dynamics and better convergence. Most of the existing second-order systems can be regarded as continuous analogues of accelerated iterative algorithms. It thus connects dynamic systems to optimization theory directly. We will present a general introduction to unconstrained optimization of second-order neurodynamic systems in this part. Based on the different structures and basic design ideas of dynamic systems, several representative categories of the existing methods will be introduced.

3.1. Classical Inertial Neurodynamic Systems

First, research on second-order neurodynamics focused on inertial mechanisms. By adding an acceleration term to the first-order gradient flow, the system will be dependent on its current position but also influenced by its past motion states. The most representative system is

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla F(x(t)) = 0, \tag{17}$$

where $\gamma > 0$ is the damping coefficient. Research has shown that the system can achieve faster convergence under the assumption of strongly convex functions [23]. Theoretical studies of these systems can be traced back to the first work by Alvarez [24]. He has studied the asymptotic behavior of inertial dynamics in Hilbert spaces and shown that trajectories converge to the optimum solution. The above studies provided a foundation for a number of subsequent investigations into second-order neurodynamics. Bégout and others [25] conducted an all-encompassing study of damped gradient systems and obtained refined asymptotic convergence results for the non-convex case. The above results also show that the damping factor will affect how quickly and whether a system converges to an equilibrium point. A fixed damping coefficient is relatively easy to analyze, but it is not possible to achieve both fast convergence and sustained stable.

This class of systems does not have constant damping, and the form of time-varying damping is usually given by:

$$\ddot{x}(t) + \frac{\gamma}{t}\dot{x}(t) + \nabla F(x(t)) = 0, \tag{18}$$

where $\gamma > 0$ is damping decay rate. Su and others [16] have conducted an all-encompassing study of this system. They were the first to provide a continuous interpretation of the NAG method and showed that the system converges to $O(1/t^2)$ for convex optimization problems. Compared with the system (17), the most prominent difference of the NAG system is that the damping is not constant. With the reduction in friction over time, the trajectory will have accumulated more momentum by this time. The following studies have expanded this system. Attouch and others [22] have proposed a more general mechanism of asymptotically vanishing damping and achieved rapid convergence without assuming strong convexity. Attouch and Cabot [26] have studied the effect of time-varying viscous terms on trajectory convergence and found sufficient conditions for convergence to a minimizer. Later work [27] has studied inertial systems with general damping coefficients and derived corresponding convergence rate bounds. At the same time, many scholars have proposed various theoretical reasons for the speed-up of NAG neurodynamics. Wibisono et al. [28] proposed the Bregman Lagrangian framework and extended acceleration methods to the variational principle. Wilson and others [29] subsequently used Lyapunov analysis to build a unified stability theory and provided a new view on continuous acceleration neurodynamics.

Recently, it has been found that the time scale of the system is also related to its performance. Attouch et al. [30] considered an inertial system of the form

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \beta(t)\nabla F(x(t)) = 0, \tag{19}$$

where $\gamma(t) = \gamma/t$ and a time-scaling function $\beta(t)$ is added to the gradient term. The system is not a standard NAG, but it is also relatively flexible. Assuming convexity, the convergence rate is of the order $O(1/(t^2\beta(t)))$. Thus, it can be seen that the time scale and convergence performance are related. In addition, the research in [31] also examined the stability of system (19) under external disturbance. The authors have derived a convergence estimate of the form $O(1/(\beta(t)\Gamma(t)^2))$, and $\Gamma(t)$ depends on the accumulated damping. Based on the above data, the speed of convergence is not only determined by inertia but is also affected by other time-varying factors. Consequently, the design of dynamic parameters has become a subject of extensive research aimed at achieving rapid convergence. Balhag et al. [32] analyzed inertial systems with constant damping and time-scale mechanisms. They proved that linear (exponential) convergence can be achieved under appropriate parameter conditions.

In recent years, researchers have begun to interpret these continuous systems from different perspectives. Aujol et al. [33] proved that system (19) achieves a better rate of convergence under some additional geometric conditions (such as the Łojasiewicz property). The high-resolution analysis proposed by Shi et al. [34] revealed higher-order information that is overlooked in classical systems and explains why the continuous acceleration mechanism remains effective over the long term. Luo and Chen [35] established a connection between neurodynamics and discrete optimization algorithms from the perspective of numerical ODE solvers. They have collectively advanced the understanding of the nature of inertial acceleration.

3.2. Hessian-Driven Damping Inertial Neurodynamic Systems

In many ill-conditioned problems, classical inertial neurodynamics often produce oscillations. Suppression of these oscillations and maintaining the acceleration have gradually become objects of study in neurodynamics. A typical way is to add a damping term that depends on the local geometry of the objective function. This is called Hessian-driven (HD) damping. The above methods change the energy dissipation dynamically according to the curvature information of the objective function and adjust the damping direction based on local geometry. A common formulation is

$$\ddot{x}(t) + \gamma\dot{x}(t) + \beta\nabla^2 F(x(t))\dot{x}(t) + \nabla F(x(t)) = 0. \tag{20}$$

Alvarez et al. proposed the model (20) in [36]. It adds the Hessian matrix to the damping term explicitly and adjusts the dissipation effect automatically based on changes in curvature. This design has reduced transverse oscillation and improved the stability of the system in ill-conditioned optimization problems compared with the classical HB dynamic. It can be seen from the above that the inertial effect and local curvature are not independent of each other.

Attouch et al. [37] extended the neurodynamic system (20) to nonsmooth convex optimization problems. Their research has shown that the system is still asymptotically stable when the objective function is not smooth. At the same time, problems of well-posedness, energy decay and trajectory convergence have also been studied systematically to provide strong theoretical support for the development of HD damping. With the development of NAG neurodynamics, researchers have begun to investigate the combination of time-varying damping mechanisms and HD concepts [38]. According to the above research, for strongly convex, convex and non-smooth convex problems, the combination of curvature information and time-varying damping can enhance the convergence characteristics of the system further.

Adly and others [39] have introduced an inertial system that combines dry friction, viscous friction and HD damping to improve the energy dissipation capability of the system. The specific expression is as follows:

$$\ddot{x}(t) + \gamma\dot{x}(t) + \partial\psi(\dot{x}(t)) + \beta\nabla^2 F(x(t))\dot{x}(t) + \nabla F(x(t)) \ni 0, \tag{21}$$

where the dry friction damping function ψ is convex and has a sharp minimum at the origin. They showed that the trajectories generated by the system are stable in finite time. A discrete algorithm for the system (21) was proposed in [40]. It has been shown that this algorithm is finite-convergent and behaves similarly to system (21). Based on (20), the study in [41] proposed a more general second-order neurodynamic system with HD damping, described as

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \beta(t)\nabla^2 F(x(t))\dot{x}(t) + \nabla F(x(t)) = 0, \tag{22}$$

where $\gamma(t)$ and $\beta(t)$ are time-varying damping parameters. This form offers greater design flexibility than systems with fixed parameters. The authors proved the convergence rates for the system with $\gamma(t) = \gamma/t$ for general convex objectives and with $\gamma(t) = 2\sqrt{\mu}$ for μ-strongly convex objectives. Meanwhile, a more precise convergence rate estimate was obtained by adopting a new Lyapunov construction method [42]. The influence of geometric conditions on the convergence rate was further elucidated in [43]. Beyond convergence speed, the issue of robustness has also garnered increasing attention. Attouch et al. [44] discussed the stability of HD inertial neurodynamic systems under external perturbations and showed that their acceleration properties can be maintained within a certain range. Similar ideas were subsequently extended to nonconvex optimization settings [45]. Under appropriate assumptions, it is guaranteed that the trajectory converges to a critical point. Further related results can be found in [46,47].

It is worth noting that the aforementioned HD neurodynamics generally rely on the explicit computation of the Hessian matrix. Thereby typically requiring the objective function to be at least twice continuously differentiable. However, this assumption is often difficult to satisfy in practical problems. To address this, researchers have proposed the concept of implicit HD damping. This concept indirectly characterizes curvature effects through gradient information. A representative system is

$$\ddot{x}(t) + \frac{\gamma}{t}\dot{x}(t) + \nabla F(x(t) + \beta(t)\dot{x}(t)) = 0, \tag{23}$$

where $\gamma \geq 3$, $\beta(t) = \alpha + \beta/t$, and $\alpha, \beta > 0$. Since $\nabla F(x(t) + \beta(t)\dot{x}(t)) \approx \nabla F(x(t)) + \beta(t)\nabla^2 F(x(t))\dot{x}(t)$, this gradient correction term implicitly incorporates the HD damping effect [48]. This approach avoids the explicit computation of the Hessian matrix while effectively suppressing oscillations. Subsequently, the analysis of its asymptotic behavior was extended to settings with different time-scale parameters $\beta(t) = \beta_0(\alpha + 1)/t$ [49]. Recent studies have demonstrated the robust stability and acceleration capabilities of system (20) within complex objective landscapes by applying it to nonconvex optimization problems [50].

3.3. Tikhonov Regularization-Based Inertial Neurodynamic Systems

Another general direction of development for second-order neurodynamic systems is the addition of Tikhonov regularization terms. This way was used to study the problems of ill-posedness. It has been found that it is necessary to ensure a strong convergence of the trajectory in cases with multiple local minima for the optimization problem. Tikhonov regularization offers a particular way to choose a solution. Generally speaking, this will result in the solution with the minimum norm and still converge well.

The beginning of this idea can be traced back to the study of evolution equations for maximal monotone operators. Early results showed that a vanishing regularization term might change the limit behavior of the trajectories. As shown in [51], under certain decay conditions for the regularization parameter, the analysis conducted here also achieves strong convergence to minimum-norm solutions. These results were later applied in the optimization work. Attouch et al. [52] introduced Tikhonov regularization into Nesterov-type inertial dynamics, leading to the system

$$\ddot{x}(t) + \frac{\gamma}{t}\dot{x}(t) + \nabla F(x(t)) + \epsilon(t)x(t) = 0, \tag{24}$$

where $\epsilon(t) \to 0$ as $t \to +\infty$. If the regularization is small, then the accelerated property of NAG neurodynamics will essentially be retained. A longer decay promotes a strong convergence to the minimum-norm solution. Therefore, a wider range of studies has been conducted on the effects of Tikhonov regularization and other inertial structures. In addition to the initial NAG, researchers have added time-scaling coefficients and modified damping strategies and curvature information in the dynamics. For example, Ref. [53] showed that a suitable time-scaling mechanism can improve the convergence speed while maintaining strong convergence. At the same time, Tikhonov regularization was combined with HD damping and perturbed HB systems that had vanishing damping to achieve good convergence results and a better balance among acceleration, stabilisation and solution selection [54,55].

With the progress of time, the focus of research on this theory has moved from convergence to the effects of regularization on accelerated neurodynamics. Under suitable decay conditions, strong convergence to the minimum-norm solution can be achieved simultaneously with accelerated convergence speed [56]. Therefore, other research has explored regularised systems with HD damping and asymptotic vanishing parameters. It has clarified the respective roles of inertia, curvature information, and regularization in shaping the long-term behavior of trajectories [57,58].

These methods have been incorporated into the settings of discrete optimization algorithms, leading to fast inertial convergence schemes for convex optimization problems and Newton-inspired inertial systems for monotone inclusions [59,60]. More recent work has further combined Tikhonov regularization with implicit or explicit HD damping and inertial proximal methods for nonsmooth optimization [61–63]. These developments indicate that the Tikhonov regularization term ensures the strong convergence of a trajectory generated by neurodynamic system.

4. Second-Order Neurodynamics for Constrained Optimization

In recent years, second-order neurodynamic systems have become an increasingly attractive approaches for addressing constrained optimization problems. We provide a systematic review of second-order neurodynamic systems for constrained optimization in this section. We focus on two main categories: PD neurodynamic systems and projection neurodynamics.

4.1. Primal-Dual Second-Order Neurodynamic Systems

Among the various approaches for constrained optimization, PD neurodynamics occupy a particularly important position. Instead of being external corrections, these methods incorporate the primal and dual variables into a single dynamic framework. They turn the optimization problem into a saddle-point system. Their respective paths have shown faster convergence speeds and can also accommodate more complex constraint structures.

A relatively large number of recent studies have been spurred by continuous explorations of operator splitting algorithms. Therefore, inertial PD neurodynamics can be regarded as dynamical counterparts of the alternating direction method of multipliers (ADMM) method. The system takes the form

$$\begin{cases} \ddot{x}(t)+\gamma(t)\dot{x}(t)+b(t)\nabla_x\mathcal{L}_\alpha\left(x(t),\lambda(t)+\beta(t)\dot{\lambda}(t)\right)=0, \\ \ddot{\lambda}(t)+\gamma(t)\dot{\lambda}(t)-b(t)\nabla_\lambda\mathcal{L}_\alpha\left(x(t)+\beta(t)\dot{x}(t),\lambda(t)\right)=0, \end{cases} \tag{25}$$

where $\gamma(t)$, $b(t)$, and $\beta(t)$ are time-dependent coefficients. This form shows that accelerated PD neurodynamics are directly connected to ADMM iterative schemes, and it also provides a natural platform for convergence analysis of separable convex optimization with linear constraints [64]. Different choices of $\gamma(t)$, $b(t)$ and $\beta(t)$ can lead to various kinds of asymptotic behavior for the trajectories. According to the investigations in [65,66], vanishing damping and appropriate time-scaling mechanisms can significantly enhance the convergence speed and potentially alter the chosen limiting solution. Based on the above findings, a large number of PD neurodynamic systems that incorporate inertial effects and constraint coupling have been introduced in recent years [67,68].

A distinct modeling strategy was introduced in [69], where a mixed-order system combining second-order primal dynamics and first-order dual dynamics was analyzed:

$$\begin{cases} \ddot{x}(t)+\gamma\dot{x}(t)+b(t)\nabla_x\mathcal{L}_\alpha\left(x(t),\lambda(t)\right)=0, \\ \dot{\lambda}(t)-b(t)\nabla_\lambda\mathcal{L}_\alpha\left(x(t)+\beta\dot{x}(t),\lambda(t)\right)=0, \end{cases} \tag{26}$$

which has been relatively good for equality constrained optimization. At about the same time, ideas that came from NAG and mirror descent began to affect PD systems. Zeng et al. [70] have proposed a NAG PD neurodynamic system for network optimization and distributed computation that faces communication constraints in its performance. Mirror-based formulations have further expanded the scope of Euclidean geometry to apply PD neurodynamics in a more general setting [71].

Over the years, with the development of this theory, the focus gradually moved from acceleration to the interaction between multiple dynamic structures. In [72], a highly structured second-order PD system was proposed by integrating time-scaling, HD damping, and Tikhonov regularization:

$$\begin{cases} \ddot{x}(t) + \dfrac{\gamma}{t^p}\dot{x}(t) + \mu\dfrac{d}{dt}\left(\nabla_x \mathcal{L}_\alpha\left(x(t), \lambda(t) + \beta(t)\dot{\lambda}(t)\right)\right) + t^q \nabla_x \mathcal{L}_\alpha\left(x(t), \lambda(t) + \beta(t)\dot{\lambda}(t)\right) = 0, \\ \ddot{\lambda}(t) + \dfrac{\gamma}{t^p}\dot{\lambda}(t) - \mu\dfrac{d}{dt}\left(\nabla_\lambda \mathcal{L}_\alpha\left(x(t) + \beta(t)\dot{x}(t), \lambda(t)\right)\right) - t^q \nabla_\lambda \mathcal{L}_\alpha\left(x(t) + \beta(t)\dot{x}(t), \lambda(t)\right) = 0, \end{cases} \tag{27}$$

where $\gamma > 1$, $0 < p < 1$, $\mu > 0$, and $q > 0$. This system is not a single-acceleration type. Instead, it uses several combined components at the same time to achieve a stronger convergence guarantee for the convex-concave saddle point problem. One of the problems recently studied is strong convergence and selection of the solution. Tikhonov regularization has been added to the PD system to guide the trajectory toward a minimum-norm solution, similar to developments in unconstrained optimization [73]. Research on inertial acceleration in nonsmooth constrained optimization has been conducted in [74]. More recently, mixed-order neurodynamics, HD damping mechanisms and time-scaling strategies have all been added to regularised PD frameworks to achieve improved convergence under weaker assumptions [75–77].

Presently, research is also moving towards richer dynamical architectures. Sun et al. [78] proposed PD neurodynamics with variable mass and curvature-aware damping, allowing adaptive control of inertia along trajectories:

$$\begin{cases} m(t)\ddot{x}(t) + \dfrac{\gamma}{t^p}\dot{x}(t) + \mu\dfrac{d}{dt}\left(\nabla_x \mathcal{L}_\alpha\left(x(t), \lambda(t)\right)\right) + t^q \nabla_x \mathcal{L}_\alpha\left(x(t), \lambda(t)\right) = 0, \\ \dot{\lambda}(t) - (\gamma - 1)(t^{p+q} - \mu p t^{p-1})\nabla_\lambda \mathcal{L}_\alpha\left(x(t) + \beta(t)\dot{x}(t), \lambda(t)\right) = 0, \end{cases} \tag{28}$$

where $m : [t_0, +\infty) \to (0, +\infty)$ is a decreasing positive function. By making the effective mass change over time, more degrees of freedom for momentum are added along the trajectory. Related studies have introduced implicit HD damping in PD frameworks to improve numerical stability [79]. Another method is a new type of closed-loop controlled PD neurodynamics. The feedback control law continuously adjusts system parameters according to the current state to increase the robustness of uncertainty and model error [80]. Based on the above research, PD neurodynamic systems are gradually evolving from simple inertial extensions into highly adaptable dynamical frameworks that can solve increasingly difficult constrained optimization problems.

4.2. Projection Second-Order Neurodynamics Systems

Projection neurodynamic systems are a general class of methods for constrained optimization. Feasibility is explicitly enforced by projection operators onto the constraint set. However, this way also has some problems. Inertia will push the path away from the feasible region, but projection will continuously guide the dynamics to remain within the admissible region. Together, the two mechanisms provide the foundation for projection second-order neurodynamics.

One of the earliest attempts to combine acceleration and feasibility can be traced to the projection HB system introduced in [81],

$$\ddot{x}(t) + \gamma\dot{x}(t) + x(t) - \mathcal{P}_\Omega\big(x(t) - \alpha\nabla F(x(t))\big) = 0. \tag{29}$$

The additional projection term can be viewed as a geometric correction to the HB neurodynamics. At that time, the same idea was also used for non-standard optimization problems. According to the analysis in [82], the projection system is still stable and effective when using a more general monotonic operator instead of an objective function.

Another development has emerged from the intersection of projection dynamics and neural network implementation. Projection has been applied to recurrent neural network architecture to solve the problem of non-smooth optimization directly in several studies, rather than being used merely as a mathematical operator. A typical case is the two-layer neurodynamic system in [83]. Embed the projection operator in the evolution of the network to maintain feasibility and handle non-differentiable objective functions. The above architectures have strong support for parallelism and are therefore suitable for large-scale environments. Sparse optimization and signal recovery generally have many non-smooth penalties, thus increasing the difficulty of both theory and numerical implementation. Therefore, the two are often combined into a projection and smoothing method. This idea turned out to be effective for sparse reconstruction based on L_{p-q} minimization [84], and was later extended to L_p-regularised formulations [85]. Simplify the analysis and reduce numerical oscillations to improve the stability of the whole system of inertial dynamics through smoothing.

Recently, most of them have changed away from autonomous projection systems and been replaced by flexible non-autonomous frameworks. To enhance the convergence properties, the following changes have been made: time-varying parameters, adaptive gains and a variable-projection method. The accelerated projection neurodynamics in [86] are a typical example of this. The system does not use fixed inertial parameters. It adapts its development based on the present stage of the optimization process to provide stronger theoretical support and broader application. Traditional inertial systems are generally sublinearly convergent and may be unsuitable for large-scale applications. Given the above deficiency, the latest studies have proposed a new type of projection-based model with rapid convergence. The second-order projection neurodynamics proposed in [87] for sparse signal reconstruction have been shown to converge exponentially under suitable assumptions. Therefore, the carefully designed damping and projection mechanism of the neurodynamic system will have a significantly improved asympotic property.

The practical optimization problem often has a more complex constraint structure than a single feasible set. Linear equalities, affine inequalities and coupled constraints frequently occur together. To meet the above circumstances, projection inertial systems have been generalised in many ways. The system proposed in [88] adds projection neurodynamics and set constraints and equality constraints. The models in [89] have also extended the method to address affine equality and inequality constraints simultaneously in a unified system. The above improvements will significantly increase the model-building power of projection neurodynamics. Recently, some scholars have combined projection mechanisms and PD systems. Together, they can form a hybrid neurodynamic system to address the more complex optimization problem. A representative example is the time-scaled PD projection system proposed in [90],

$$\begin{cases} \dot{x}(t) = \frac{\gamma}{t}\big(\mathcal{P}_\Omega(z(t)) - x(t)\big), \\ \dot{z}(t) = -\frac{\gamma}{t}\omega(t)\Big(\nabla F(x(t)) + \alpha A^T(Ax(t) - b) + A^T\lambda(t) + z(t) - \mathcal{P}_\Omega(z(t))\Big) - \dot{x}(t), \\ \dot{\lambda}(t) = t\omega(t)\big(A\mathcal{P}_\Omega(z(t)) - b\big), \end{cases} \tag{30}$$

where $\gamma \geq 2$ and $\omega(t)$ satisfies $\dot{\omega}(t) \leq (\gamma - 2)/t\omega(t)$. The projection operator ensures the feasibility of the set, and the dual variable drives the satisfaction of the equality constraint. Thus, the interaction of these can improve the stability and convergence of problems with multiple constraint structures. Based on the neurodynamic system (30), Zhao et al. [91] proposed a new PD projection neurodynamic method for solving constrained convex optimization problems. Numerical experiments show that this method can effectively reconstruct the sparse signal. Recently, new achievements have continued to expand the scope of application for second-order projection neurodynamics. Liu and others [92] have proposed a new second-order projection neurodynamics method for solving nonconvex non-smooth $\tau L_1 - \upsilon L_2$ optimization problems and applied it to signal recovery and image reconstruction.

5. Conclusions

We have presented a review of the development of second-order neurodynamic systems from an optimization perspective in this paper. Second-order neurodynamic systems are used to realise acceleration, stability and constraint handling flexibly. Based on the existing body of work, the second-order neurodynamic system has been developed far from the original HB and NAG dynamics. With the addition of time-varying damping, HD damping, Tikhonov regularization, projection operators and PD systems, a large number of other types of optimization problems have also been solved. The above results have strengthened the connections among neurodynamics and optimization algorithms and provided new ideas for their theoretical exploration and algorithm design.

However, many of the current theories still have assumptions that are difficult to meet in practice at a large scale. The performance of many second-order systems depends on the selection of damping and scaling factors. The second problem is how to build a practical numerical algorithm for the continuous-time model that does not violate the good dynamic characteristics of this model. The three closely related directions of future research are as follows. The development of self-adjusting mechanisms for inertia and damping parameters at present is the focus of research. There is also a pressing need for convergence theories that extend to broader classes of nonconvex and nonsmooth optimization problems. Finally, the application of second-order neurodynamic methodologies to large-scale tasks in machine learning, signal processing and image processing.

Figure 1 shows a summary of the second-order neurodynamic methods for unconstrained and constrained optimization, and lists representative methods and key research directions in this paper.

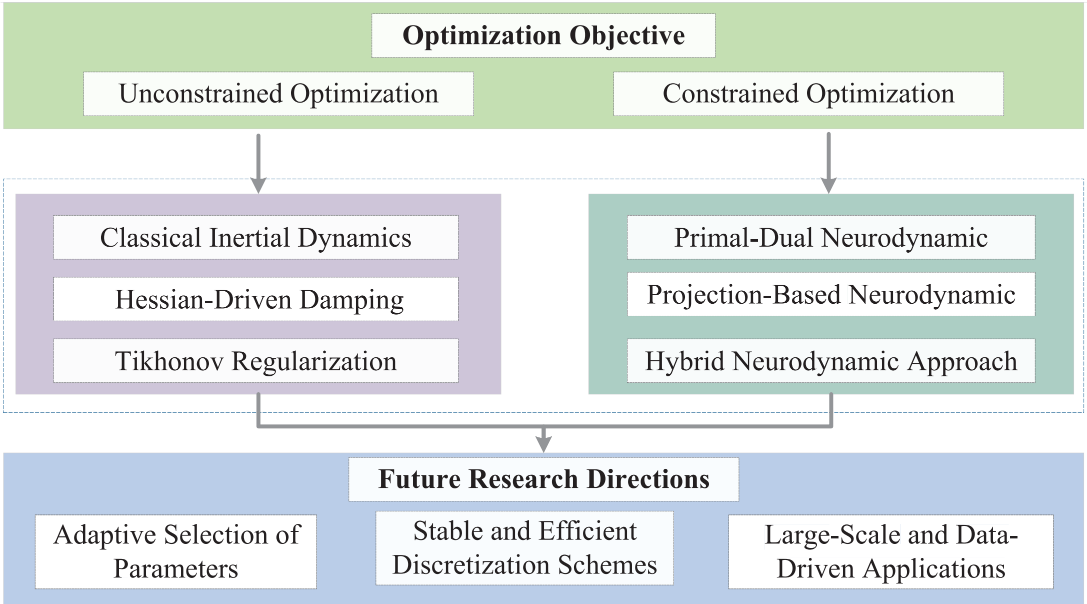

Figure 1. Second-order neurodynamic approaches for unconstrained and constrained optimization.

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During the preparation of this review, the authors used OpenAI's ChatGPT to refine the language. After employing this tool, the authors carefully reviewed and revised the content as necessary and take full responsibility for the final published version of the article.

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