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The Global Exponential Stability of Neutral-Type Delayed Inertial Neural Networks

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Abstract: This paper presents an analysis of neutral-type delayed inertial neural networks (NDINNs) by using the characteristic method. The study introduces three novel sufficient conditions for the global exponential stability of NDINNs, formulated as a series of linear scalar inequalities. This approach sets them apart from previous results and facilitates straightforward solutions. The theoretical findings are validated through three numerical examples, each accompanied by relevant simulations.

Keywords: inertial neural network; neutral delay; global exponential stability; characteristics method; Lyapunov-Krasovskii functional

1. Introduction

In order to model the behavior of systems exhibiting inertial characteristics, such as mechanical systems defined by mass, electrical circuits containing inductance, or biological systems affected by momentum, Badcock and Westervelt [1,2] introduced the innovative concept of inertial neural networks (INNs). Their foundational work integrated inertial components into neural network architectures, establishing what is now recognized as classical INNs. However, real-world systems often encounter delays due to finite signal transmission times, processing lags, or other constraints. This has driven the development of delayed inertial neural networks (DINNs), which integrate time delays into the inertial framework to better reflect the dynamics of practical systems. Expanding on this, NDINNs have emerged, introducing neutral delays that influence not only the state variables but also their derivatives. This advancement provides a more nuanced representation of systems where the rate of change of the state depends on both historical states and their past rates of change. For a deeper understanding of NDINNs, readers are encouraged to explore studies referenced in [3–16]. The research on NDINNs has primarily focused on two main approaches: the reduced order method and the non-reduced order method. The former approach [3–14] involves converting inertial neural networks into first order differential equations by changing variables. This transformation can lead to complex calculations and an expansion of system dimensions. To address these challenges, the latter method, referred to as the non-reduced order method [15,16], has received considerable attention. Nonetheless, both methodologies heavily rely on the construction of Lyapunov-Krasovskii functionals (LKFs). Designing such LKFs is particularly challenging and complex, frequently requiring the management of complex linear matrix inequalities (LMIs), usually solved using MATLAB toolboxes.

With the aim of surmounting the issues related to the earlier described methods and strategies, we present a new scheme for evaluating the global exponential stability of the following NDINNs:

$$\ddot{w}_i(t) = -\mathcal{A}_i\dot{w}_i(t) - \mathcal{B}_i w_i(t) + \sum_{l=1}^{n}\mathcal{C}_{il}G_l(w_l(t)) + \sum_{l=1}^{n}\mathcal{D}_{il}G_l(w_l(t-\tau_l)) + \sum_{l=1}^{n}\mathcal{E}_{il}\ddot{w}_l(t-\sigma_l) + J_i, \quad (1)$$

where $i \in \mathbb{N} = \{1, 2, \ldots, n\}$, and $w(t) = (w_1(t), w_2(t), \ldots, w_n(t))^T \in \mathbb{R}^n$ symbolizes the state vector, while the second derivative $\ddot{w}_i(t)$ represents the inertial term of the system. The positive constants $\mathcal{A}_i$ and $\mathcal{B}_i$ are fixed, and $\mathcal{C}_{il}$, $\mathcal{D}_{il}$ and $\mathcal{E}_{il}$ describe the connection strength between the ith neuron and lth neuron. Both the transmission delay

τ_l and the neutral delay σ_l are non-negative and do not exceed the upper bound μ, a positive constant, and J_i represents the external input. The activation function $G_l(\cdot)$ is assumed to satisfy the global Lipschitz condition, meaning there exists a positive constant k_l (for each $l \in \mathbb{N}$) such that for any $x, y \in \mathbb{R}$, the following inequality holds:

$$(A_1) \qquad |G_l(x) - G_l(y) \leq k_l|x - y|.$$

The initial conditions for NDINNs (1) are defined as

$$w(s) = \xi(s), \dot{w}(s) = \eta(s), -\mu \leq s \leq 0, \xi \in C([-\mu, 0], \mathbb{R}^n), \eta \in C^1([-\mu, 0], \mathbb{R}^n). \tag{2}$$

In [3], the authors explored the global asymptotic stability of NDINNs (1) with $\tau_l = \sigma_l = \mu$. However, their method depended on the reduced order technique and the development of LKFs, that often results in complex calculations, an expansion of the system's dimensionality and the need to manage LMIs. Since minimizing computational complexity has become a key priority in the neural network dynamics, there is an urgent demand to study the global exponential stability of NDINNs (1) without resorting to reduced order strategy or designing LKFs.

This paper contributes in two key aspects: First, we utilize the characteristic method to investigate NDINNs (1), thereby bypassing the need for either the reduced order method or the creation of LKFs. Second, the suggested conditions can be easily verified through a set of linear scalar inequalities, removing the reliance on MATLAB toolboxes for resolving LMIs.

Below is an outline of the structure of this paper. Section 2 presents essential definitions and preliminary formulations. We derive three sufficient conditions to guarantee the global exponential stability of NDINNs (1) in Section 3. Section 4 includes some practical examples, each backed by numerical simulations, to illustrate the validity of our theoretical outcomes. The paper concludes with Section 5, which offers a summary of the contributions highlighted in this research.

2. Preliminaries

For simplicity, NDINNs (1) can be represented in the following equivalent form:

$$\ddot{w}_i(t) + \mathcal{A}_i\dot{w}_i(t) + \mathcal{B}_i w_i(t) = F_i(t) + \ddot{U}_i(t), \tag{3}$$

where $F_i(t) = \sum_{l=1}^{n} \mathcal{C}_{il}G_l(w_l(t)) + \sum_{l=1}^{n} \mathcal{D}_{il}G_l(w_l(t-\tau_l)) + J_i$ and $U_i(t) = \sum_{l=1}^{n} \mathcal{E}_{il}w_l(t-\sigma_l)$.

Widely recognized is the fact that the homogeneous version of NDINNs, as described in (3), can be formulated as $\ddot{w}_i(t) + \mathcal{A}_i\dot{w}_i(t) + \mathcal{B}_i w_i(t) = 0$ and its can be represented as $\lambda^2 + \mathcal{A}_i\lambda + \mathcal{B}_i = 0$, where the discriminant is given by $\Delta_i = \mathcal{A}_i^2 - 4\mathcal{B}_i$. Following the approach in [17,18], we derive the solution expressions of NDINNs (1) under initial conditions (2) based on the signs of Δ_i, as follows.

$$w_i(t) = \frac{-1}{\sqrt{\Delta_i}}\left[(a_i e^{b_i t} - b_i e^{a_i t})w_i(0) + (e^{a_i t} - e^{b_i t})\dot{w}_i(0) + \int_0^t (e^{a_i(t-s)} - e^{b_i(t-s)})(F_i(s) + \ddot{U}_i(s))ds\right], \tag{4}$$

where $\Delta_i > 0$, $a_i = \frac{-\mathcal{A}_i - \sqrt{\Delta_i}}{2}$ and $b_i = \frac{-\mathcal{A}_i + \sqrt{\Delta_i}}{2}$.

$$w_i(t) = \left[(1 + \frac{\mathcal{A}_i}{2}t)w_i(0) + t\dot{w}_i(0)\right]e^{-\frac{\mathcal{A}_i}{2}t} + \int_0^t (t-s)e^{-\frac{\mathcal{A}_i}{2}(t-s)}(F_i(s) + \ddot{U}_i(s))ds, \tag{5}$$

where $\Delta_i = 0$.

$$w_i(t) = \left[\frac{\sqrt{\mathcal{B}_i}}{\upsilon_i}\sin(\upsilon_i t + \theta_i)w_i(0) + \frac{\sin\upsilon_i t}{\upsilon_i}\dot{w}_i(0)\right]e^{-\frac{\mathcal{A}_i}{2}t} + \int_0^t \frac{\sin\upsilon_i(t-s)}{\upsilon_i}e^{-\frac{\mathcal{A}_i}{2}(t-s)}(F_i(s) + \ddot{U}_i(s))ds, \tag{6}$$

where $\Delta_i < 0$, $\upsilon_i = \frac{1}{2}\sqrt{-\Delta_i}$ and $\theta_i = \arctan\frac{2\upsilon_i}{\mathcal{A}_i}$.

In addition, we employ integration by parts to obtain

$$\begin{cases} \int_0^t (e^{a_i(t-s)} - e^{b_i(t-s)})\ddot{U}_i(s)ds = (b_i e^{b_i t} - a_i e^{a_i t})U_i(0) \\ +(e^{b_i t} - e^{a_i t})\dot{U}_i(0) - \sqrt{\Delta_i}U_i(t) + \int_0^t (a_i^2 e^{a_i(t-s)} - b_i^2 e^{b_i(t-s)})U_i(s)ds \\ \int_0^t (t-s)e^{-\frac{\mathcal{A}_i}{2}(t-s)}\ddot{U}_i(s)ds = -(t\dot{U}_i(0) + (1 - \frac{\mathcal{A}_i}{2}t)U_i(0))e^{-\frac{\mathcal{A}_i}{2}t} + U_i(t) \\ -\int_0^t (\mathcal{A}_i - \frac{\mathcal{A}_i^2}{4}(t-s))e^{-\frac{\mathcal{A}_i}{2}(t-s)}U_i(s)ds \\ \int_0^t \sin\upsilon_i(t-s)e^{-\frac{\mathcal{A}_i}{2}(t-s)}\ddot{U}_i(s)ds = \left[\sqrt{\mathcal{B}_i}\sin(\upsilon_i t - \theta_i)U_i(0)\right. \\ \left.-\sin\upsilon_i t\dot{U}_i(0)\right]e^{-\frac{\mathcal{A}_i}{2}t} + \upsilon_i U_i(t) + \mathcal{B}_i\int_0^t \sin(\upsilon_i(t-s) - 2\theta_i)e^{-\frac{\mathcal{A}_i}{2}(t-s)}U_i(s)ds, \end{cases}$$

formulas (4)–(6) can be alternatively represented as

$$\begin{cases} w_i(t) = \frac{-1}{\sqrt{\Delta_i}} \left[(a_i e^{b_i t} - b_i e^{a_i t}) w_i(0) + (e^{a_i t} - e^{b_i t}) \dot{w}_i(0) + (b_i e^{b_i t} - a_i e^{a_i t}) U_i(0) \right. \\ +(e^{b_i t} - e^{a_i t}) \dot{U}_i(0) - \sqrt{\Delta_i} U_i(t) + \int_0^t (e^{a_i(t-s)} - e^{b_i(t-s)}) F_i(s) ds \\ \left. + \int_0^t (a_i^2 e^{a_i(t-s)} - b_i^2 e^{b_i(t-s)}) U_i(s) ds \right], \end{cases} \tag{7}$$

$$\begin{cases} w_i(t) = \left[(1 + \frac{\mathcal{A}_i}{2} t) w_i(0) + t \dot{w}_i(0) - (1 - \frac{\mathcal{A}_i}{2} t) U_i(0) - t \dot{U}_i(0) \right] e^{-\frac{\mathcal{A}_i}{2} t} + U_i(t) \\ + \int_0^t (t-s) e^{-\frac{\mathcal{A}_i}{2}(t-s)} F_i(s) ds - \int_0^t (\mathcal{A}_i - \frac{\mathcal{A}_i^2}{4}(t-s)) e^{-\frac{\mathcal{A}_i}{2}(t-s)} U_i(s) ds, \end{cases} \tag{8}$$

and

$$\begin{cases} w_i(t) = \frac{1}{\upsilon_i} \left[\sqrt{\mathcal{B}_i} \sin(\upsilon_i t + \theta_i) w_i(0) + \sin \upsilon_i t \dot{w}_i(0) + \sqrt{\mathcal{B}_i} \sin(\upsilon_i t - \theta_i) U_i(0) \right. \\ \left. - \sin \upsilon_i t \dot{U}_i(0) \right] e^{-\frac{\mathcal{A}_i}{2} t} + U_i(t) + \frac{1}{\upsilon_i} \int_0^t \sin \upsilon_i (t-s) e^{-\frac{\mathcal{A}_i}{2}(t-s)} F_i(s) ds \\ + \frac{\mathcal{B}_i}{\upsilon_i} \int_0^t \sin(\upsilon_i (t-s) - 2\theta_i) e^{-\frac{\mathcal{A}_i}{2}(t-s)} U_i(s) ds. \end{cases} \tag{9}$$

For convenience, we introduce the following definition.

Definition 1. *A vector $w^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ is termed an equilibrium of system (1) provided that it satisfies*

$$\mathcal{B}_i w_i^* = \sum_{l=1}^n \mathcal{C}_{il} G_l(w_l^*) + \sum_{l=1}^n \mathcal{D}_{il} G_l(w_l^*) + J_i. \tag{10}$$

Definition 2. *Let w^* denote the equilibrium point of system (1). The NDINNs (1) are said to be globally exponentially stable with decay rate $\kappa > 0$, if there exists a positive constant $\mathcal{L}$ such that the following inequalities hold for all $t \geq 0$ and $i \in \mathbb{N}$*

$$|w_i(t) - w_i^*| < \mathcal{L} e^{-\kappa t}, \quad |\dot{w}_i(t)| < \mathcal{L} e^{-\kappa t}.$$

Lemma 1 ([19])**.** *Suppose $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Then $\mathcal{H}(w)$ is a homeomorphism if it is injective over $\mathbb{R}^n$ and $||\mathcal{H}(w)|| \to \infty$ whenever $||w|| \to \infty$.*

Lemma 2. *Suppose assumption (A_1) is satisfied. System (1) possesses a unique equilibrium point provided that the following inequality holds for all $i \in \mathbb{N}$*

$$\Theta_i > 0 \tag{11}$$

where $\Theta_i = 2\mathcal{B}_i - \sum\limits_{l=1}^n k_l |\mathcal{C}_{il} + \mathcal{D}_{il}| - \sum\limits_{l=1}^n k_i |\mathcal{C}_{li} + \mathcal{D}_{li}|$.

Proof. Equation (10) can be equivalently expressed in matrix form as

$$\mathcal{B} w^* - \mathcal{C} G(w^*) - \mathcal{D} G(w^*) - J = 0 \tag{12}$$

Based on (12), we introduce the mapping

$$\mathcal{H}(w) = \mathcal{B} w - \mathcal{C} G(w) - \mathcal{D} G(w) - J, \tag{13}$$

where $\mathcal{B} = \text{diag}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n)$ is a diagonal matrix, $w = (w_1, w_2, \ldots, w_n)^T$, $\mathcal{C}$ and $\mathcal{D}$ are $n \times n$ matrices with entries $\mathcal{C}_{il}$ and $\mathcal{D}_{il}$, respectively. Furthermore, $G(w) = (G_1(w_1), G_2(w_2), \ldots, G_n(w_n))^T$ denotes a vector-valued function, and $J = (J_1, J_2, \ldots, J_n)^T$.

It is evident that any equilibrium point of system (1) or (12) coincides with a zero of $\mathcal{H}(w) = 0$. Therefore, a necessary condition for the uniqueness of this equilibrium is that $\mathcal{H}(w)$ constitutes a homeomorphism on $\mathbb{R}^n$. To verify this property, we first demonstrate the injectivity of $\mathcal{H}(w)$. Consider two distinct vectors $w, \widetilde{w} \in \mathbb{R}^n$ where $w \neq \widetilde{w}$ and $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_n)^T$. Substituting into (13) yields

$$\mathcal{H}(w) - \mathcal{H}(\widetilde{w}) = \mathcal{B}(w - \widetilde{w}) - \mathcal{C}[G(w) - G(\widetilde{w})] - \mathcal{D}[G(w) - G(\widetilde{w})] \tag{14}$$

Consequently, invoking assumption (A_1) alongside (11) and (14) yields

$$\begin{array}{rl} & 2(w^T-\widetilde{w}^T)(\mathcal{H}(w)-\mathcal{H}(\widetilde{w})) \\ = & 2(w-\widetilde{w})^T\mathcal{B}(w-\widetilde{w})-2(w-\widetilde{w})^T(\mathcal{C}+\mathcal{D})(G(w)-G(\widetilde{w})) \\ = & \sum\limits_{i=1}^{n}2\mathcal{B}_i(w_i-\widetilde{w}_i)^2-\sum\limits_{i=1}^{n}\sum\limits_{l=1}^{n}2(\mathcal{C}_{il}+\mathcal{D}_{il})(w_i-\widetilde{w}_i)(G_l(w_l)-G_l(\widetilde{w}_l)) \\ \geq & \sum\limits_{i=1}^{n}2\mathcal{B}_i(w_i-\widetilde{w}_i)^2-\sum\limits_{i=1}^{n}\sum\limits_{l=1}^{n}2|\mathcal{C}_{il}+\mathcal{D}_{il}||w_i-\widetilde{w}_i|k_l|w_l-\widetilde{w}_l| \\ \geq & \sum\limits_{i=1}^{n}2\mathcal{B}_i(w_i-\widetilde{w}_i)^2-\sum\limits_{i=1}^{n}\sum\limits_{l=1}^{n}k_l|\mathcal{C}_{il}+\mathcal{D}_{il}|[(w_i-\widetilde{w}_i)^2+(w_l-\widetilde{w}_l)^2] \\ = & \sum\limits_{i=1}^{n}\Theta_i(w_i-\widetilde{w}_i)^2 \\ > & 0. \end{array} \tag{15}$$

This inequality implies $\mathcal{H}(w)\neq\mathcal{H}(\widetilde{w})$ whenever $w\neq\widetilde{w}$. thereby confirming the injectivity of $\mathcal{H}(w)$ on $\mathbb{R}^n$. Furthermore, substituting $\widetilde{w}=0$ into (15) gives

$$\begin{array}{rcl} ||\mathcal{H}(w)-\mathcal{H}(0)|| & = & \frac{2||w||\cdot||\mathcal{H}(w)-\mathcal{H}(0)||}{2||w||} \\ & \geq & \frac{2w^T(\mathcal{H}(w)-\mathcal{H}(0))}{2||w||} \\ & \geq & \frac{\min\limits_{i\in\mathbb{N}}\Theta_i||w||^2}{2||w||} \\ & = & \frac{1}{2}\min\limits_{i\in\mathbb{N}}\Theta_i||w||. \end{array} \tag{16}$$

From (16), we deduce that $||\mathcal{H}(w)||\geq||\mathcal{H}(w)-\mathcal{H}(0)||-||\mathcal{H}(0)||\geq\frac{1}{2}\min\limits_{i\in\mathbb{N}}\Theta_i||w||$, which ensures that $||\mathcal{H}(w)||\to\infty$ as $||w||\to\infty$. According to Lemma 1, this property guarantees that $\mathcal{H}:\mathbb{R}^n\to\mathbb{R}^n$ constitutes a homeomorphism over $\mathbb{R}^n$. Consequently, there exists a unique vector w satisfying $\mathcal{H}(w)=0$, corresponding to the equilibrium of system (1). This concludes the proof of the existence and uniqueness of the equilibrium point for system (1). □

3. Main Results

Within this section, based on the formulations provided in (7)–(9), we investigate three sufficient criteria for achieving the global exponential stability of NDINNs (1).

Theorem 1. *NDINNs (1) achieve global exponential stability with decay rate κ, provided that all hypotheses of Lemma 2 hold and the following inequalities are satisfied:*

$$\begin{cases} \Delta_i>0,\ \frac{\mathcal{M}_i}{\mathcal{B}_i}+\frac{E_i(|a_i|+|b_i|)}{\sqrt{\Delta_i}}+E_i<1, \\ E_i(1+\mathcal{A}_i)+\frac{2\mathcal{M}_i+E_i(a_i^2+b_i^2)}{\sqrt{\Delta_i}}<1, \end{cases} \tag{17}$$

where $E_i=\sum\limits_{l=1}^{n}|\mathcal{E}_{il}|$ and $\mathcal{M}_i=\sum\limits_{l=1}^{n}k_l(|\mathcal{C}_{il}|+|\mathcal{D}_{il}|)$. This means that there exist a $\kappa\in(0,-\frac{1}{2}\max\limits_{i\in\mathbb{N}}b_i)$ and a positive constant $\mathcal{L}$ such that for all $t\geq 0$ and $i\in\mathbb{N}$,

$$|w_i(t)-w_i^*|<\mathcal{L}e^{-\kappa t},\quad |\dot{w}_i(t)|<\mathcal{L}e^{-\kappa t} \tag{18}$$

Proof. According to Lemma 2, there is a unique equilibrium point $w^*=(w_1^*,w_2^*,\ldots,w_n^*)^T$ of system (1). Let $w(t)=(w_1(t),w_2(t),\ldots,w_n(t))^T$ indicate the solution of NDINNs (1) that meets the initial conditions in (2). With $\Delta_i>0$, and by taking the derivative of both sides of (7) with respect to t, we discover that

$$\begin{cases} \dot{w}_i(t) & = & \frac{-1}{\sqrt{\Delta_i}}\Big[\mathcal{B}_i(e^{b_it}-e^{a_it})w_i(0)+(a_ie^{a_it}-b_ie^{b_it})\dot{w}_i(0) \\ & & +(b_i^2e^{b_it}-a_i^2e^{a_it})U_i(0)+(b_ie^{b_it}-a_ie^{a_it})\dot{U}_i(0)-\sqrt{\Delta_i}\dot{U}_i(t) \\ & & +(a_i^2-b_i^2)U_i(t-\mu)+\int_0^t(a_ie^{a_i(t-s)}-b_ie^{b_i(t-s)})F_i(s)ds \\ & & +\int_0^t(a_i^3e^{a_i(t-s)}-b_i^3e^{b_i(t-s)})U_i(s)ds\Big], \end{cases} \tag{19}$$

which, combined with (7) and (10), suggests that

$$\begin{cases} w_i(t)-w_i^* &= \frac{-1}{\sqrt{\Delta_i}}[(a_ie^{b_it}-b_ie^{a_it})(w_i(0)-w_i^*)+(e^{a_it}-e^{b_it})\dot{w}_i(0) \\ &\quad +(b_ie^{b_it}-a_ie^{a_it})(\sum\limits_{l=1}^{n}\mathcal{E}_{il}(w_l(-\sigma_l)-w_l^*))+(e^{b_it}-e^{a_it})\dot{U}_i(0) \\ &\quad -\sqrt{\Delta_i}(\sum\limits_{l=1}^{n}\mathcal{E}_{il}(w_l(t-\sigma_l)-w_l^*))+\int_0^t(e^{a_i(t-s)}-e^{b_i(t-s)})\mathcal{F}_i(s)ds \\ &\quad +\int_0^t(a_i^2e^{a_i(t-s)}-b_i^2e^{b_i(t-s)})(\sum\limits_{l=1}^{n}\mathcal{E}_{il}(w_l(s-\sigma_l)-w_l^*))ds] \\ \dot{w}_i(t) &= \frac{-1}{\sqrt{\Delta_i}}[\mathcal{B}_i(e^{b_it}-e^{a_it})(w_i(0)-w_i^*)+(a_ie^{a_it}-b_ie^{b_it})\dot{w}_i(0) \\ &\quad +(b_i^2e^{b_it}-a_i^2e^{a_it})(\sum\limits_{l=1}^{n}\mathcal{E}_{il}(w_l(-\sigma_l)-w_l^*))+(b_ie^{b_it}-a_ie^{a_it})\dot{U}_i(0) \\ &\quad -\sqrt{\Delta_i}\dot{U}_i(t)+(a_i^2-b_i^2)(\sum\limits_{l=1}^{n}\mathcal{E}_{il}(w_l(t-\sigma_l)-w_l^*)) \\ &\quad +\int_0^t(a_ie^{a_i(t-s)}-b_ie^{b_i(t-s)})\mathcal{F}_i(s)ds \\ &\quad +\int_0^t(a_i^3e^{a_i(t-s)}-b_i^3e^{b_i(t-s)})(\sum\limits_{l=1}^{n}\mathcal{E}_{il}(w_l(s-\sigma_l)-w_l^*))ds], \end{cases} \tag{20}$$

where $\mathcal{F}_i(t)=\sum\limits_{l=1}^{n}\mathcal{C}_{il}(G_l(w_l(t))-G_l(w_l^*))+\sum\limits_{l=1}^{n}\mathcal{D}_{il}(G_l(w_l(t-\tau_l))-G_l(w_l^*))$.

From (17), it follows that there exist κ that is sufficiently small, where $\kappa\in(0,-\frac{1}{2}\max\limits_{i\in\mathbb{N}}b_i)$, and a positive constant $\mathcal{L}$ that is sufficiently large, such that

$$\begin{cases} \frac{(|a_i|+|b_i|+2)(1+E_i)}{\mathcal{L}\sqrt{\Delta_i}}\mathcal{P}_{\xi,\eta}+\frac{\mathcal{M}_i^{\kappa}}{(a_i+\kappa)(b_i+\kappa)}+\frac{E_ie^{\kappa\mu}}{\sqrt{\Delta_i}}(\frac{a_i^2}{|a_i+\kappa|}+\frac{b_i^2}{|b_i+\kappa|})+E_ie^{\kappa\mu}<1, \\ \frac{2\mathcal{B}_i+(|a_i|+|b_i|)(1+E_i)+(a_i^2+b_i^2)E_i}{\mathcal{L}\sqrt{\Delta_i}}\mathcal{P}_{\xi,\eta}+\frac{\mathcal{M}_i^{\kappa}}{\sqrt{\Delta_i}}(\frac{|a_i|}{|a_i+\kappa|}+\frac{|b_i|}{|b_i+\kappa|}) \\ +E_ie^{\kappa\mu}(\mathcal{A}_i+\frac{|a_i|^3}{|a_i+\kappa|\sqrt{\Delta_i}}+\frac{|b_i|^3}{|b_i+\kappa|\sqrt{\Delta_i}})+E_ie^{\kappa\mu}<1, \end{cases} \tag{21}$$

where $\mathcal{M}_i^{\kappa}=\sum\limits_{l=1}^{n}k_l(|\mathcal{C}_{il}|+|\mathcal{D}_{il}|e^{\kappa\mu})$ and $\mathcal{P}_{\xi,\eta}=\max\limits_{i\in\mathbb{N}}\{\max\limits_{s\in[-\mu,0]}|w_i(s)-w_i^*|,\max\limits_{s\in[-\mu,0]}|\dot{w}_i(s)|\}$.

It is evident that inequalities (18) apply for $t\in[-\mu,0]$, $i\in\mathbb{N}$. These inequalities are presumed to be valid for all $t\geq 0$. Should this presumption fail, at least one of the subsequent cases necessarily arises:

Case 1. *There exist $p_1\in\mathbb{N}$ and $q_1>0$ for which it is true that*

$$|w_i(t)-w_i^*|<\mathcal{L}e^{-\kappa t},|\dot{w}_i(t)|<\mathcal{L}e^{-\kappa t},t\in[0,q_1),|w_{p_1}(q_1)-w_{p_1}^*|=\mathcal{L}e^{-\kappa q_1}. \tag{22}$$

Case 2. *There exist $p_2\in\mathbb{N}$ and $q_2>0$ for which it is true that*

$$|w_i(t)-w_i^*|<\mathcal{L}e^{-\kappa t},|\dot{w}_i(t)|<\mathcal{L}e^{-\kappa t},t\in[0,q_2),|\dot{w}_{p_2}(q_2)|=\mathcal{L}e^{-\kappa q_2}. \tag{23}$$

Should Case 1 occur, based on assumption (A_1) as well as (20)–(22), we can deduce that

$$\begin{aligned} & |w_{p_1}(q_1)-w_{p_1}^*| \\ = \;& |\frac{-1}{\sqrt{\Delta_{p_1}}}[(a_{p_1}e^{b_{p_1}q_1}-b_{p_1}e^{a_{p_1}q_1})(w_{p_1}(0)-w_{p_1}^*)+(e^{a_{p_1}q_1}-e^{b_{p_1}q_1})\dot{w}_{p_1}(0) \\ & +(b_{p_1}e^{b_{p_1}q_1}-a_{p_1}e^{a_{p_1}q_1})(\sum_{l=1}^{n}\mathcal{E}_{p_1l}(w_l(-\sigma_l)-w_l^*))+(e^{b_{p_1}q_1}-e^{a_{p_1}q_1})\dot{U}_{p_1}(0) \\ & -\sqrt{\Delta_{p_1}}(\sum_{l=1}^{n}\mathcal{E}_{p_1l}(w_l(q_1-\sigma_l)-w_l^*))+\int_0^{q_1}(e^{a_{p_1}(q_1-s)}-e^{b_{p_1}(q_1-s)})\mathcal{F}_{p_1}(s)ds \\ & +\int_0^{q_1}(a_{p_1}^2e^{a_{p_1}(q_1-s)}-b_{p_1}^2e^{b_{p_1}(q_1-s)})(\sum_{l=1}^{n}\mathcal{E}_{p_1l}(w_l(s-\sigma_l)-w_l^*))ds]| \\ \leq \;& \frac{1}{\sqrt{\Delta_{p_1}}}[(|a_{p_1}|+|b_{p_1}|+2)(1+E_{p_1})\mathcal{P}_{\xi,\eta}e^{b_{p_1}q_1}+\int_0^{q_1}|e^{a_{p_1}(q_1-s)}-e^{b_{p_1}(q_1-s)}| \\ & \times(\sum_{l=1}^{n}|\mathcal{C}_{p_1l}||G_l(w_l(s))-G_l(w_l^*)|+\sum_{l=1}^{n}|\mathcal{D}_{p_1l}||G_l(w_l(s-\tau_l))-G_l(w_l^*)|)ds \\ & +\int_0^{q_1}|a_{p_1}^2e^{a_{p_1}(q_1-s)}-b_{p_1}^2e^{b_{p_1}(q_1-s)}|(\sum_{l=1}^{n}|\mathcal{E}_{p_1l}||w_l(s-\sigma_l)-w_l^*|)ds] \\ & +\sum_{l=1}^{n}|\mathcal{E}_{p_1l}||w_l(q_1-\sigma_l)-w_l^*| \end{aligned}$$

$$\begin{array}{rl}
\leq & \frac{e^{-\kappa q_1}}{\sqrt{\Delta_{p_1}}}[(|a_{p_1}|+|b_{p_1}|+2)(1+E_{p_1})\mathcal{P}_{\xi,\eta}+\int_0^{q_1}(e^{b_{p_1}(q_1-s)}-e^{a_{p_1}(q_1-s)})e^{\kappa q_1} \\
& \times(\sum\limits_{l=1}^{n}|\mathcal{C}_{p_1l}|k_l|w_l(s)-w_l^*|+\sum\limits_{l=1}^{n}|\mathcal{D}_{p_1l}|k_l|w_l(s-\tau_l)-w_l^*|)ds \\
& +\int_0^{q_1}(a_{p_1}^2e^{a_{p_1}(q_1-s)}+b_{p_1}^2e^{b_{p_1}(q_1-s)})e^{\kappa q_1}(\sum\limits_{l=1}^{n}|\mathcal{E}_{p_1l}|\mathcal{L}e^{-\kappa(s-\sigma_l)})ds]+\sum\limits_{l=1}^{n}|\mathcal{E}_{p_1l}|\mathcal{L}e^{-\kappa(q_1-\sigma_l)} \\
\leq & \frac{e^{-\kappa q_1}}{\sqrt{\Delta_{p_1}}}[(|a_{p_1}|+|b_{p_1}|+2)(1+E_{p_1})\mathcal{P}_{\xi,\eta}+\int_0^{q_1}(e^{b_{p_1}(q_1-s)}-e^{a_{p_1}(q_1-s)})e^{\kappa q_1} \\
& \times(\sum\limits_{l=1}^{n}|\mathcal{C}_{p_1l}|k_l\mathcal{L}e^{-\kappa s}+\sum\limits_{l=1}^{n}|\mathcal{D}_{p_1l}|k_l\mathcal{L}e^{-\kappa(s-\tau_l)})ds \\
& +\mathcal{L}E_{p_1}e^{\kappa\mu}\int_0^{q_1}(a_{p_1}^2e^{(a_{p_1}+\kappa)(q_1-s)}+b_{p_1}^2e^{(b_{p_1}+\kappa)(q_1-s)})ds]+E_{p_1}e^{\kappa\mu}\mathcal{L}e^{-\kappa q_1} \\
\leq & \mathcal{L}e^{-\kappa q_1}[\frac{(|a_{p_1}|+|b_{p_1}|+2)(1+E_{p_1})\mathcal{P}_{\xi,\eta}}{\mathcal{L}\sqrt{\Delta_{p_1}}}+\frac{\mathcal{M}_{p_1}^{\kappa}}{\sqrt{\Delta_{p_1}}}\int_0^{q_1}(e^{(b_{p_1}+\kappa)(q_1-s)}-e^{(a_{p_1}+\kappa)(q_1-s)})ds \\
& +\frac{E_{p_1}e^{\kappa\mu}}{\sqrt{\Delta_{p_1}}}\int_0^{q_1}(a_{p_1}^2e^{(a_{p_1}+\kappa)(q_1-s)}+b_{p_1}^2e^{(b_{p_1}+\kappa)(q_1-s)})ds]+E_{p_1}e^{\kappa\mu}\mathcal{L}e^{-\kappa q_1} \\
= & \mathcal{L}e^{-\kappa q_1}[\frac{(|a_{p_1}|+|b_{p_1}|+2)(1+E_{p_1})}{\mathcal{L}\sqrt{\Delta_{p_1}}}\mathcal{P}_{\xi,\eta}+\frac{(1-\delta(q_1))\mathcal{M}_{p_1}^{\kappa}}{(a_{p_1}+\kappa)(b_{p_1}+\kappa)} \\
& +\frac{E_{p_1}e^{\kappa\mu}}{\sqrt{\Delta_{p_1}}}(\frac{a_{p_1}^2(1-e^{(a_{p_1}+\kappa)q_1})}{|a_{p_1}+\kappa|}+\frac{b_{p_1}^2(1-e^{(b_{p_1}+\kappa)q_1})}{|b_{p_1}+\kappa|})+E_{p_1}e^{\kappa\mu}]
\end{array} \tag{24}$$

where $\delta(t)=\frac{1}{\sqrt{\Delta_{p_1}}}[(b_{p_1}+\kappa)e^{(a_{p_1}+\kappa)t}-(a_{p_1}+\kappa)e^{(b_{p_1}+\kappa)t}], t\in[0,+\infty)$.

Given that $\delta'(t)\leq 0$ for all $t\in[0,+\infty)$ is evidently satisfied, it implies that $\delta(t)$ does not increase over $[0,+\infty)$, thus $0=\delta(+\infty)\leq\delta(q_1)<\delta(0)=1$. Combining this observation with (21), (22) and (24), we arrive at the conclusion that

$$\begin{array}{rl}
& \mathcal{L}e^{-\kappa q_1}=|w_{p_1}(q_1)-w_{p_1}^*| \\
< & \mathcal{L}e^{-\kappa q_1}[\frac{(|a_{p_1}|+|b_{p_1}|+2)(1+E_{p_1})}{\mathcal{L}\sqrt{\Delta_{p_1}}}\mathcal{P}_{\xi,\eta}+\frac{\mathcal{M}_{p_1}^{\kappa}}{(a_{p_1}+\kappa)(b_{p_1}+\kappa)} \\
& +\frac{E_{p_1}e^{\kappa\mu}}{\sqrt{\Delta_{p_1}}}(\frac{a_{p_1}^2}{|a_{p_1}+\kappa|}+\frac{b_{p_1}^2}{|b_{p_1}+\kappa|})+E_{p_1}e^{\kappa\mu}] \\
< & \mathcal{L}e^{-\kappa q_1}.
\end{array}$$

Should Case 2 occur, from assumption (A_1) and the conditions stated in (20), (21) and (23), it can be concluded that

$$\begin{array}{rl}
& |\dot{w}_{p_2}(q_2)| \\
= & |\frac{-1}{\sqrt{\Delta_{p_2}}}[\mathcal{B}_{p_2}(e^{b_{p_2}q_2}-e^{a_{p_2}q_2})(w_{p_2}(0)-w_{p_2}^*)+(a_{p_2}e^{a_{p_2}q_2}-b_{p_2}e^{b_{p_2}q_2})\dot{w}_{p_2}(0) \\
& +(b_{p_2}^2e^{b_{p_2}q_2}-a_{p_2}^2e^{a_{p_2}q_2})(\sum\limits_{l=1}^{n}\mathcal{E}_{p_2l}(w_l(-\sigma_l)-w_l^*))+(b_{p_2}e^{b_{p_2}q_2}-a_{p_2}e^{a_{p_2}q_2})\dot{U}_{p_2}(0) \\
& -\sqrt{\Delta_{p_2}}\dot{U}_{p_2}(q_2)+(a_{p_2}^2-b_{p_2}^2)(\sum\limits_{l=1}^{n}\mathcal{E}_{p_2l}(w_l(q_2-\sigma_l)-w_l^*)) \\
& +\int_0^{q_2}(a_{p_2}e^{a_{p_2}(q_2-s)}-b_{p_2}e^{b_{p_2}(q_2-s)})\mathcal{F}_{p_2}(s)ds \\
& +\int_0^{q_2}(a_{p_2}^3e^{a_{p_2}(q_2-s)}-b_{p_2}^3e^{b_{p_2}(q_2-s)})(\sum\limits_{l=1}^{n}\mathcal{E}_{p_2l}(w_l(s-\sigma_l)-w_l^*))ds]| \\
\leq & \frac{1}{\sqrt{\Delta_{p_2}}}[(2\mathcal{B}_{p_2}+(|a_{p_2}|+|b_{p_2}|)(1+E_{p_2})+(a_{p_2}^2+b_{p_2}^2)E_{p_2})\mathcal{P}_{\xi,\eta}e^{b_{p_2}q_2} \\
& +(a_{p_2}^2-b_{p_2}^2)(\sum\limits_{l=1}^{n}|\mathcal{E}_{p_2l}||w_l(q_2-\sigma_l)-w_l^*|)+\int_0^{q_2}|a_{p_2}e^{a_{p_2}(q_2-s)}-b_{p_2}e^{b_{p_2}(q_2-s)}| \\
& \times(\sum\limits_{l=1}^{n}|\mathcal{C}_{p_2l}||G_l(w_l(s))-G_l(w_l^*)|+\sum\limits_{l=1}^{n}|\mathcal{D}_{p_2l}||G_l(w_l(s-\tau_l))-G_l(w_l^*)|)ds \\
& +\int_0^{q_2}|a_{p_2}^3e^{a_{p_2}(q_2-s)}-b_{p_2}^3e^{b_{p_2}(q_2-s)}|(\sum\limits_{l=1}^{n}|\mathcal{E}_{p_2l}||w_l(s-\sigma_l)-w_l^*|)ds] \\
& +\sum\limits_{l=1}^{n}|\mathcal{E}_{p_2l}||\dot{w}_l(q_2-\sigma_l)| \\
\leq & \frac{1}{\sqrt{\Delta_{p_2}}}[(2\mathcal{B}_{p_2}+(|a_{p_2}|+|b_{p_2}|)(1+E_{p_2})+(a_{p_2}^2+b_{p_2}^2)E_{p_2})\mathcal{P}_{\xi,\eta}e^{b_{p_2}q_2} \\
& +(a_{p_2}^2-b_{p_2}^2)(\sum\limits_{l=1}^{n}|\mathcal{E}_{p_2l}|\mathcal{L}e^{-\kappa(q_2-\sigma_l)}+\int_0^{q_2}(|a_{p_2}|e^{a_{p_2}(q_2-s)}+|b_{p_2}|e^{b_{p_2}(q_2-s)}) \\
& \times(\sum\limits_{l=1}^{n}|\mathcal{C}_{p_2l}|k_l|w_l(s)-w_l^*|+\sum\limits_{l=1}^{n}|\mathcal{D}_{p_2l}|k_l|w_l(s-\tau_l)-w_l^*|)ds \\
& +\int_0^{q_2}(|a_{p_2}|^3e^{a_{p_2}(q_2-s)}+|b_{p_2}|^3e^{b_{p_2}(q_2-s)})(\sum\limits_{l=1}^{n}|\mathcal{E}_{p_2l}|\mathcal{L}e^{-\kappa(s-\sigma_l)})ds]+\sum\limits_{l=1}^{n}|\mathcal{E}_{p_2l}|\mathcal{L}e^{-\kappa(q_2-\sigma_l)}
\end{array}$$

$$
\begin{aligned}
&\leq \frac{e^{-\kappa q_2}}{\sqrt{\Delta_{p_2}}}[(2\mathcal{B}_{p_2}+(|a_{p_2}|+|b_{p_2}|)(1+E_{p_2})+(a_{p_2}^2+b_{p_2}^2)E_{p_2})\mathcal{P}_{\xi,\eta}+(a_{p_2}^2-b_{p_2}^2)E_{p_2}\mathcal{L}e^{\kappa\mu} \\
&\quad +\int_0^{q_2}(|a_{p_2}|e^{a_{p_2}(q_2-s)}+|b_{p_2}|e^{b_{p_2}(q_2-s)})e^{\kappa q_2}(\sum_{l=1}^{n}|\mathcal{C}_{p_2 l}|k_l\mathcal{L}e^{-\kappa s}+\sum_{l=1}^{n}|\mathcal{D}_{p_2 l}|k_l\mathcal{L}e^{-\kappa(s-\tau_l)})ds \\
&\quad +\int_0^{q_2}(|a_{p_2}|^3e^{a_{p_2}(q_2-s)}+|b_{p_2}|^3e^{b_{p_2}(q_2-s)})e^{\kappa q_2}E_{p_2}e^{\kappa\mu}\mathcal{L}e^{-\kappa s})ds]+E_{p_2}e^{\kappa\mu}\mathcal{L}e^{-\kappa q_2} \\
&\leq \mathcal{L}e^{-\kappa q_2}[\frac{(2\mathcal{B}_{p_2}+(|a_{p_2}|+|b_{p_2}|)(1+E_{p_2})+(a_{p_2}^2+b_{p_2}^2)E_{p_2})\mathcal{P}_{\xi,\eta}}{\mathcal{L}\sqrt{\Delta_{p_2}}}+\frac{a_{p_2}^2-b_{p_2}^2}{\sqrt{\Delta_{p_2}}}E_{p_2}e^{\kappa\mu} \\
&\quad +\frac{\mathcal{M}_{p_2}^{\kappa}}{\sqrt{\Delta_{p_2}}}\int_0^{q_2}(|a_{p_2}|e^{(a_{p_2}+\kappa)(q_2-s)}+|b_{p_2}|e^{(b_{p_2}+\kappa)(q_2-s)})ds \\
&\quad +\frac{E_{p_2}e^{\kappa\mu}}{\sqrt{\Delta_{p_2}}}\int_0^{q_2}(|a_{p_2}|^3e^{(a_{p_2}+\kappa)(q_2-s)}+|b_{p_2}|^3e^{(b_{p_2}+\mu)(q_2-s)})ds]+E_{p_2}e^{\kappa\mu}\mathcal{L}e^{-\kappa q_2} \\
&= \mathcal{L}e^{-\kappa q_2}[\frac{2\mathcal{B}_{p_2}+(|a_{p_2}|+|b_{p_2}|)(1+E_{p_2})+(a_{p_2}^2+b_{p_2}^2)E_{p_2}}{\mathcal{L}\sqrt{\Delta_{p_2}}}\mathcal{P}_{\xi,\eta}+\mathcal{A}_{p_2}E_{p_2}e^{\kappa\mu} \\
&\quad +\frac{\mathcal{M}_{p_2}^{\kappa}}{\sqrt{\Delta_{p_2}}}(\frac{|a_{p_2}|}{|a_{p_2}+\mu|}(1-e^{(a_{p_2}+\kappa)q_2})+\frac{|b_{p_2}|}{|b_{p_2}+\mu|}(1-e^{(b_{p_2}+\kappa)q_2})) \\
&\quad +\frac{E_{p_2}e^{\kappa\mu}}{\sqrt{\Delta_{p_2}}}(\frac{|a_{p_2}|^3}{|a_{p_2}+\mu|}(1-e^{(a_{p_2}+\kappa)q_2})+\frac{|b_{p_2}|^3}{|b_{p_2}+\mu|}(1-e^{(b_{p_2}+\kappa)q_2}))+E_{p_2}e^{\kappa\mu}] \\
&< \mathcal{L}e^{-\kappa q_2}[\frac{2\mathcal{B}_{p_2}+(|a_{p_2}|+|b_{p_2}|)(1+E_{p_2})+(a_{p_2}^2+b_{p_2}^2)E_{p_2}}{\mathcal{L}\sqrt{\Delta_{p_2}}}\mathcal{P}_{\xi,\eta} \\
&\quad +\frac{\mathcal{M}_{p_2}^{\kappa}}{\sqrt{\Delta_{p_2}}}(\frac{|a_{p_2}|}{|a_{p_2}+\kappa|}+\frac{|b_{p_2}|}{|b_{p_2}+\kappa|}) \\
&\quad +E_{p_2}e^{\kappa\mu}(\mathcal{A}_{p_2}+\frac{|a_{p_2}|^3}{|a_{p_2}+\kappa|\sqrt{\Delta_{p_2}}}+\frac{|b_{p_2}|^3}{|b_{p_2}+\kappa|\sqrt{\Delta_{p_2}}})+E_{p_2}e^{\kappa\mu}] \\
&< \mathcal{L}e^{-\kappa q_2}.
\end{aligned}
$$

This case also results in a contradiction. As a consequence, the inequalities (18) are valid for every $t \in [0, +\infty)$ and $i \in \mathbb{N}$. Hence, the proof of Theorem 1 is complete. □

Theorem 2. *NDINNs (1) achieve global exponential stability with decay rate κ, provided that all hypotheses of Lemma 2 hold and the following inequalities are satisfied:*

$$
\begin{cases}
\Delta_i = 0, 4E_i + \frac{4\mathcal{M}_i}{\mathcal{A}_i^2} < 1, \\
E_i(1+\mathcal{A}_i) + \frac{4\mathcal{M}_i + 8\mathcal{B}_i E_i}{\mathcal{A}_i} < 1,
\end{cases}
\tag{25}
$$

where E_i and $\mathcal{M}_i$ are as defined in Theorem 1. This means that there exist a $\kappa \in (0, \frac{1}{4}\min_{i\in\mathbb{N}} \mathcal{A}_i)$ and a positive constant $\mathcal{L}$ such that for all $t \geq 0$ and $i \in \mathbb{N}$,

$$
|w_i(t) - w_i^*| < \mathcal{L}e^{-\kappa t}, \quad |\dot{w}_i(t)| < \mathcal{L}e^{-\kappa t}. \tag{26}
$$

Proof. As stated in Lemma 2, system (1) possesses a unique equilibrium point $w^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$. The notation $w(t) = (w_1(t), w_2(t), \ldots, w_n(t))^T$ represents the solution for NDINNs (1) that meets the initial conditions laid out in (2). When $\Delta_i = 0$, if we differentiate (8) on both sides with respect to t, it follows that

$$
\begin{cases}
\begin{aligned}
\dot{w}_i(t) &= [-\mathcal{B}_i t w_i(0) + (1-\frac{\mathcal{A}_i}{2}t)\dot{w}_i(0) + (\mathcal{A}_i - \mathcal{B}_i t)U_i(0) - (1-\frac{\mathcal{A}_i}{2}t)\dot{U}_i(0)]e^{-\frac{\mathcal{A}_i}{2}t} \\
&\quad + \dot{U}_i(t) - \mathcal{A}_i U_i(t) + \int_0^t (1-\frac{\mathcal{A}_i}{2}(t-s))e^{-\frac{\mathcal{A}_i}{2}(t-s)}F_i(s)ds \\
&\quad + \int_0^t (3-\frac{\mathcal{A}_i}{2}(t-s))\mathcal{B}_i e^{-\frac{\mathcal{A}_i}{2}(t-s)}U_i(s)ds,
\end{aligned}
\end{cases}
\tag{27}
$$

which, together with (8) and (10), implies that

$$
\begin{cases}
\begin{aligned}
w_i(t) - w_i^* &= [(1+\frac{\mathcal{A}_i}{2}t)(w_i(0)-w_i^*) + t\dot{w}_i(0) - (1-\frac{\mathcal{A}_i}{2}t)(\sum_{l=1}^{n}\mathcal{E}_{il}(w_l(-\sigma_l)-w_l^*)) \\
&\quad - t\dot{U}_i(0)]e^{-\frac{\mathcal{A}_i}{2}t} + \sum_{l=1}^{n}\mathcal{E}_{il}(w_l(t-\sigma_l)-w_l^*) + \int_0^t (t-s)e^{-\frac{\mathcal{A}_i}{2}(t-s)}\mathcal{F}_i(s)ds \\
&\quad - \int_0^t (\mathcal{A}_i - \frac{\mathcal{A}_i^2}{4}(t-s))e^{-\frac{\mathcal{A}_i}{2}(t-s)}(\sum_{l=1}^{n}\mathcal{E}_{il}(w_l(s-\sigma_l)-w_l^*))ds \\
\dot{w}_i(t) &= [-\mathcal{B}_i t(w_i(0)-w_i^*) + (1-\frac{\mathcal{A}_i}{2}t)\dot{w}_i(0) + (\mathcal{A}_i - \mathcal{B}_i t)(\sum_{l=1}^{n}\mathcal{E}_{il}(w_l(-\sigma_l)-w_l^*)) \\
&\quad - (1-\frac{\mathcal{A}_i}{2}t)\dot{U}_i(0)]e^{-\frac{\mathcal{A}_i}{2}t} + \dot{U}_i(t) - \mathcal{A}_i(\sum_{l=1}^{n}\mathcal{E}_{il}(w_l(t-\sigma_l)-w_l^*)) \\
&\quad + \int_0^t (1-\frac{\mathcal{A}_i}{2}(t-s))e^{-\frac{\mathcal{A}_i}{2}(t-s)}\mathcal{F}_i(s)ds \\
&\quad + \int_0^t (3-\frac{\mathcal{A}_i}{2}(t-s))\mathcal{B}_i e^{-\frac{\mathcal{A}_i}{2}(t-s)}(\sum_{l=1}^{n}\mathcal{E}_{il}(w_l(s-\sigma_l)-w_l^*))ds.
\end{aligned}
\end{cases}
\tag{28}
$$

Equation (25) indicates the existence of a sufficiently little κ, where $\kappa \in (0, \frac{1}{4}\min_{i\in\mathbb{N}} \mathcal{A}_i)$, and a sufficiently large positive constant $\mathcal{L}$, satisfying that

$$\begin{cases} \frac{1+E_i}{\mathcal{L}}(1+\frac{2+\mathcal{A}_i}{(\mathcal{A}_i-2\kappa)e})\mathcal{P}_{\xi,\eta} + E_i e^{\kappa\mu} + \frac{4\mathcal{M}_i^\kappa}{(\mathcal{A}_i-2\kappa)^2} + \frac{E_i e^{\kappa\mu}\mathcal{A}_i(3\mathcal{A}_i-4\kappa)}{(\mathcal{A}_i-2\kappa)^2} < 1, \\ (1+(1+E_i)(\mathcal{A}_i+\frac{2\mathcal{B}_i+\mathcal{A}_i}{(\mathcal{A}_i-2\kappa)e}))\frac{\mathcal{P}_{\xi,\eta}}{\mathcal{L}} + E_i e^{\kappa\mu}(1+\mathcal{A}_i) + \mathcal{M}_i^\kappa \frac{4(\mathcal{A}_i-\kappa)}{(\mathcal{A}_i-2\kappa)^2} + \mathcal{B}_i E_i e^{\kappa\mu}\frac{4(2\mathcal{A}_i-3\kappa)}{(\mathcal{A}_i-2\kappa)^2} < 1. \end{cases} \tag{29}$$

Evidently, inequalities (26) are satisfied within the interval $t \in [-\mu, 0]$ for all $i \in \mathbb{N}$. We assert that these very inequalities also apply when $t \geq 0$. These inequalities are presumed to be valid for all $t \geq 0$. Should this presumption fail, at least one of the subsequent cases necessarily arises:

Case 3. *There exist $p_3 \in \mathbb{N}$ and $q_3 > 0$ for which it is true that*

$$|w_i(t) - w_i^*| < \mathcal{L}e^{-\kappa t}, |\dot{w}_i(t)| < \mathcal{L}e^{-\kappa t}, t \in [0, q_3), |w_{p_3}(q_3) - w_{p_3}^*| = \mathcal{L}e^{-\kappa q_3}. \tag{30}$$

Case 4. *There exist $p_4 \in \mathbb{N}$ and $q_4 > 0$ for which it is true that*

$$|w_i(t) - w_i^*| < \mathcal{L}e^{-\kappa t}, |\dot{w}_i(t)| < \mathcal{L}e^{-\kappa t}, t \in [0, q_4), |\dot{w}_{p_4}(q_4)| = \mathcal{L}e^{-\kappa q_4}. \tag{31}$$

If Case 3 occurs, using assumption (A_1) along with (28)–(30) and the fact that $\max_{x\geq 0} xe^{-x} = \frac{1}{e}$, we can conclude that

$$\begin{aligned}
& |w_{p_3}(q_3) - w_{p_3}^*| \\
= \; & |[(1+\tfrac{\mathcal{A}_{p_3}}{2}q_3)(w_{p_3}(0)-w_{p_3}^*) + q_3\dot{w}_{p_3}(0) - (1-\tfrac{\mathcal{A}_{p_3}}{2}q_3)(\sum_{l=1}^n \mathcal{E}_{p_3 l}(w_l(-\sigma_l)-w_l^*)) \\
& -q_3\dot{U}_{p_3}(0)]e^{-\frac{\mathcal{A}_{p_3}}{2}q_3} + \sum_{l=1}^n \mathcal{E}_{p_3 l}(w_l(q_3-\sigma_l)-w_l^*) + \int_0^{q_3}(q_3-s)e^{-\frac{\mathcal{A}_{p_3}}{2}(q_3-s)}\mathcal{F}_{p_3}(s)ds \\
& -\int_0^{q_3}(\mathcal{A}_{p_3} - \tfrac{\mathcal{A}_{p_3}^2}{4}(q_3-s))e^{-\frac{\mathcal{A}_{p_3}}{2}(q_3-s)}(\sum_{l=1}^n \mathcal{E}_{p_3 l}(w_l(s-\sigma_l)-w_l^*))ds| \\
\leq \; & (1+E_{p_3})[e^{-(\frac{\mathcal{A}_{p_3}}{2}-\kappa)q_3} + (1+\tfrac{\mathcal{A}_{p_3}}{2})q_3 e^{-(\frac{\mathcal{A}_{p_3}}{2}-\kappa)q_3}]\mathcal{P}_{\xi,\eta}e^{-\kappa q_3} \\
& +\sum_{l=1}^n |\mathcal{E}_{p_3 l}||w_l(q_3-\sigma_l)-w_l^*| + \int_0^{q_3}(q_3-s)e^{-\frac{\mathcal{A}_{p_3}}{2}(q_3-s)} \\
& \times(\sum_{l=1}^n |\mathcal{C}_{p_3 l}||G_l(w_l(s))-G_l(w_l^*)| + \sum_{l=1}^n |\mathcal{D}_{p_3 l}||G_l(w_l(s-\tau_l))-G_l(w_l^*)|)ds \\
& +\int_0^{q_3}|\mathcal{A}_{p_3} - \tfrac{\mathcal{A}_{p_3}^2}{4}(q_3-s)|e^{-\frac{\mathcal{A}_{p_3}}{2}(q_3-s)}(\sum_{l=1}^n |\mathcal{E}_{p_3 l}||w_l(s-\sigma_l)-w_l^*|)ds \\
\leq \; & (1+E_{p_3})(1+\tfrac{2+\mathcal{A}_{p_3}}{(\mathcal{A}_{p_3}-2\kappa)e})\mathcal{P}_{\xi,\eta}e^{-\kappa q_3} + \sum_{l=1}^n |\mathcal{E}_{p_3 l}|\mathcal{L}e^{-\kappa(q_3-\sigma_l)} \\
& +\int_0^{q_3}(q_3-s)e^{-\frac{\mathcal{A}_{p_3}}{2}(q_3-s)}(\sum_{l=1}^n |\mathcal{C}_{p_3 l}|k_l|w_l(s)-w_l^*| + \sum_{l=1}^n |\mathcal{D}_{p_3 l}|k_l|w_l(s-\tau_l)-w_l^*|)ds \\
& +\int_0^{q_3}(\mathcal{A}_{p_3} + \tfrac{\mathcal{A}_{p_3}^2}{4}(q_3-s))e^{-\frac{\mathcal{A}_{p_3}}{2}(q_3-s)}(\sum_{l=1}^n |\mathcal{E}_{p_3 l}||\mathcal{L}e^{-\kappa(s-\sigma_l)})ds \\
\leq \; & e^{-\kappa q_3}[(1+E_{p_3})(1+\tfrac{2+\mathcal{A}_{p_3}}{(\mathcal{A}_{p_3}-2\kappa)e})\mathcal{P}_{\xi,\eta} + E_{p_3}e^{\kappa\mu}\mathcal{L} \\
& +\int_0^{q_3}(q_3-s)e^{-\frac{\mathcal{A}_{p_3}}{2}(q_3-s)}e^{\kappa q_3}(\sum_{l=1}^n |\mathcal{C}_{p_3 l}|k_l\mathcal{L}e^{-\kappa s} + \sum_{l=1}^n |\mathcal{D}_{p_3 l}|k_l\mathcal{L}e^{-\kappa(s-\tau_l)}ds \\
& +\int_0^{q_3}(\mathcal{A}_{p_3} + \tfrac{\mathcal{A}_{p_3}^2}{4}(q_3-s))e^{-\frac{\mathcal{A}_{p_3}}{2}(q_3-s)}e^{\kappa q_3}E_{p_3}e^{\kappa\mu}\mathcal{L}e^{-\kappa s})ds] \\
\leq \; & \mathcal{L}e^{-\kappa q_3}[\tfrac{1+E_{p_3}}{\mathcal{L}}(1+\tfrac{2+\mathcal{A}_{p_3}}{(\mathcal{A}_{p_3}-2\kappa)e})\mathcal{P}_{\xi,\eta} + E_{p_3}e^{\kappa\mu} + \mathcal{M}_{p_3}^\kappa \int_0^{q_3}(q_3-s)e^{-(\frac{\mathcal{A}_{p_3}}{2}-\kappa)(q_3-s)}ds \\
& +E_{p_3}e^{\kappa\mu}\int_0^{q_3}(\mathcal{A}_{p_3} + \tfrac{\mathcal{A}_{p_3}^2}{4}(q_3-s))e^{-(\frac{\mathcal{A}_{p_3}}{2}-\kappa)(q_3-s)}ds] \\
= \; & \mathcal{L}e^{-\kappa q_3}[\tfrac{1+E_{p_3}}{\mathcal{L}}(1+\tfrac{2+\mathcal{A}_{p_3}}{(\mathcal{A}_{p_3}-2\kappa)e})\mathcal{P}_{\xi,\eta} + \mathcal{M}_{p_3}^\kappa\tfrac{4-(4+2q_3(\mathcal{A}_{p_3}-2\kappa))e^{-(\frac{\mathcal{A}_{p_3}}{2}-\kappa)q_3}}{(\mathcal{A}_{p_3}-2\kappa)^2} \\
& +E_{p_3}e^{\kappa\mu}\tfrac{\mathcal{A}_{p_3}(3\mathcal{A}_{p_3}-4\kappa-(3\mathcal{A}_{p_3}-4\kappa+\mathcal{A}_{p_3}q_3(\frac{\mathcal{A}_{p_3}}{2}-\kappa))e^{-(\frac{\mathcal{A}_{p_3}}{2}-\kappa)q_3})}{(\mathcal{A}_{p_3}-2\kappa)^2} + E_{p_3}e^{\kappa\mu}] \\
< \; & \mathcal{L}e^{-\kappa q_3}[\tfrac{1+E_{p_3}}{\mathcal{L}}(1+\tfrac{2+\mathcal{A}_{p_3}}{(\mathcal{A}_{p_3}-2\kappa)e})\mathcal{P}_{\xi,\eta} + E_{p_3}e^{\kappa\mu} + \tfrac{4\mathcal{M}_{p_3}^\kappa}{(\mathcal{A}_{p_3}-2\kappa)^2} \\
& +E_{p_3}e^{\kappa\mu}\tfrac{\mathcal{A}_{p_3}(3\mathcal{A}_{p_3}-4\kappa)}{(\mathcal{A}_{p_3}-2\kappa)^2}] \\
< \; & \mathcal{L}e^{-\kappa q_3}.
\end{aligned}$$

This case evidently leads to a contradiction.

If Case 4 occurs, utilizing assumption (A_1) in combination with (28), (29), (31) and taking into account the fact $\max_{x\geq 0} xe^{-x} = \frac{1}{e}$, it follows that

$$\begin{aligned}
& |\dot{w}_{p_4}(q_4)| \\
= \;& |[-\mathcal{B}_{p_4} q_4 (w_{p_4}(0) - w^*_{p_4}) + (1 - \tfrac{\mathcal{A}_{p_4}}{2} q_4)\dot{w}_{p_4}(0) + (\mathcal{A}_{p_4} - \mathcal{B}_{p_4} q_4)(\sum_{l=1}^{n} \mathcal{E}_{p_4 l}(w_l(-\sigma_l) - w^*_l)) \\
& -(1 - \tfrac{\mathcal{A}_{p_4}}{2} q_4)\dot{U}_{p_4}(0)]e^{-\frac{\mathcal{A}_{p_4}}{2} q_4} + \dot{U}_{p_4}(q_4) - \mathcal{A}_{p_4}(\sum_{l=1}^{n} \mathcal{E}_{p_4 l}(w_l(q_4 - \sigma_l) - w^*_l)) \\
& + \int_0^{q_4} (1 - \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s))e^{-\frac{\mathcal{A}_{p_4}}{2}(q_4 - s)} \mathcal{F}_{p_4}(s) ds \\
& + \int_0^{q_4} (3 - \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s))\mathcal{B}_{p_4} e^{-\frac{\mathcal{A}_{p_4}}{2}(q_4 - s)}(\sum_{l=1}^{n} \mathcal{E}_{p_4 l}(w_l(s - \sigma_l) - w^*_l))ds| \\
\leq \;& [(1 + (1 + E_{p_4})\mathcal{A}_{p_4})e^{-(\frac{\mathcal{A}_{p_4}}{2} - \kappa)q_4} + (1 + E_{p_4})(\mathcal{B}_{p_4} + \tfrac{\mathcal{A}_{p_4}}{2})q_4 e^{-(\frac{\mathcal{A}_{p_4}}{2} - \kappa)q_4}]\mathcal{P}_{\xi,\eta} e^{-\kappa q_4} \\
& + \sum_{l=1}^{n} |\mathcal{E}_{p_4 l}||\dot{w}_l(q_4 - \sigma_l)| + \mathcal{A}_{p_4}(\sum_{l=1}^{n} |\mathcal{E}_{p_4 l}||w_l(q_4 - \sigma_l) - w^*_l|) \\
& + \int_0^{q_4} |1 - \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s)|e^{-\frac{\mathcal{A}_{p_4}}{2}(q_4 - s)} \\
& \times (\sum_{l=1}^{n} |\mathcal{C}_{p_4 l}||G_l(w_l(s)) - G_l(w^*_l)| + \sum_{l=1}^{n} |\mathcal{D}_{p_4 l}||G_l(w_l(s - \tau_l)) - G_l(w^*_l)|)ds \\
& + \int_0^{q_4} |3 - \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s)|\mathcal{B}_{p_4} e^{-\frac{\mathcal{A}_{p_4}}{2}(q_4 - s)}(\sum_{l=1}^{n} |\mathcal{E}_{p_4 l}||w_l(s - \sigma_l) - w^*_l|)ds \\
\leq \;& [1 + (1 + E_{p_4})\mathcal{A}_{p_4} + (1 + E_{p_4})\tfrac{2\mathcal{B}_{p_4} + \mathcal{A}_{p_4}}{(\mathcal{A}_{p_4} - 2\kappa)e}]\mathcal{P}_{\xi,\eta} e^{-\kappa q_4} + \sum_{l=1}^{n} |\mathcal{E}_{p_4 l}|\mathcal{L}e^{-\kappa(q_4 - \sigma_l)} \\
& + \mathcal{A}_{p_4}(\sum_{l=1}^{n} |\mathcal{E}_{p_4 l}|\mathcal{L}e^{-\kappa(q_4 - \sigma_l)}) + \int_0^{q_4} (1 + \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s))e^{-\frac{\mathcal{A}_{p_4}}{2}(q_4 - s)} \\
& \times (\sum_{l=1}^{n} |\mathcal{C}_{p_4 l}|k_l|w_l(s) - w^*_l| + \sum_{l=1}^{n} |\mathcal{D}_{p_4 l}|k_l|w_l(s - \tau_l) - w^*_l|)ds \\
& + \int_0^{q_4} (3 + \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s))\mathcal{B}_{p_4} e^{-\frac{\mathcal{A}_{p_4}}{2}(q_4 - s)}(\sum_{l=1}^{n} |\mathcal{E}_{p_4 l}|\mathcal{L}e^{-\kappa(s - \sigma_l)})ds \\
\leq \;& e^{-\kappa q_4}[(1 + (1 + E_{p_4})\mathcal{A}_{p_4} + (1 + E_{p_4})\tfrac{2\mathcal{B}_{p_4} + \mathcal{A}_{p_4}}{(\mathcal{A}_{p_4} - 2\kappa)e})\mathcal{P}_{\xi,\eta} + E_{p_4} e^{\kappa\mu}\mathcal{L} + \mathcal{A}_{p_4} E_{p_4} e^{\kappa\mu}\mathcal{L} \\
& + \int_0^{q_4} (1 + \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s))e^{-\frac{\mathcal{A}_{p_4}}{2}(q_4 - s)} e^{\kappa q_4}(\sum_{l=1}^{n} |\mathcal{C}_{p_4 l}|k_l \mathcal{L}e^{-\kappa s} + \sum_{l=1}^{n} |\mathcal{D}_{p_4 l}|k_l|\mathcal{L}e^{-\kappa(s - \tau_l)})ds \\
& + \int_0^{q_4} (3 + \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s))\mathcal{B}_{p_4} e^{-\frac{\mathcal{A}_{p_4}}{2}(q_4 - s)} e^{\kappa q_4} E_{p_4} e^{\kappa\mu}\mathcal{L}e^{-\kappa s} ds] \\
\leq \;& \mathcal{L}e^{-\kappa q_4}[(1 + (1 + E_{p_4})(\mathcal{A}_{p_4} + \tfrac{2\mathcal{B}_{p_4} + \mathcal{A}_{p_4}}{(\mathcal{A}_{p_4} - 2\kappa)e}))\tfrac{\mathcal{P}_{\xi,\eta}}{\mathcal{L}} + E_{p_4} e^{\kappa\mu}(1 + \mathcal{A}_{p_4}) \\
& + \mathcal{M}^{\kappa}_{p_4} \int_0^{q_4} (1 + \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s))e^{-(\frac{\mathcal{A}_{p_4}}{2} - \kappa)(q_4 - s)} ds \\
& + \mathcal{B}_{p_4} E_{p_4} e^{\kappa\mu} \int_0^{q_4} (3 + \tfrac{\mathcal{A}_{p_4}}{2}(q_4 - s))e^{-(\frac{\mathcal{A}_{p_4}}{2} - \kappa)(q_4 - s)} ds] \\
= \;& \mathcal{L}e^{-\kappa q_4}[(1 + (1 + E_{p_4})(\mathcal{A}_{p_4} + \tfrac{2\mathcal{B}_{p_4} + \mathcal{A}_{p_4}}{(\mathcal{A}_{p_4} - 2\kappa)e}))\tfrac{\mathcal{P}_{\xi,\eta}}{\mathcal{L}} + E_{p_4} e^{\kappa\mu}(1 + \mathcal{A}_{p_4}) \\
& + \mathcal{M}^{\kappa}_{p_4}(\tfrac{4(\mathcal{A}_{p_4} - \kappa)}{(\mathcal{A}_{p_4} - 2\kappa)^2} - \tfrac{4(\mathcal{A}_{p_4} - \kappa) + \mathcal{A}_{p_4}(\mathcal{A}_{p_4} - 2\kappa)q_4}{(\mathcal{A}_{p_4} - 2\kappa)^2} e^{-(\frac{\mathcal{A}_{p_4}}{2} - \kappa)q_4}) \\
& + \mathcal{B}_{p_4} E_{p_4} e^{\kappa\mu}(\tfrac{4(2\mathcal{A}_{p_4} - 3\kappa)}{(\mathcal{A}_{p_4} - 2\kappa)^2} - \tfrac{4(2\mathcal{A}_{p_4} - 3\kappa) + \mathcal{A}_{p_4}(\mathcal{A}_{p_4} - 2\kappa)q_4}{(\mathcal{A}_{p_4} - 2\kappa)^2} e^{-(\frac{\mathcal{A}_{p_4}}{2} - \kappa)q_4})] \\
< \;& \mathcal{L}e^{-\kappa q_4}[(1 + (1 + E_{p_4})(\mathcal{A}_{p_4} + \tfrac{2\mathcal{B}_{p_4} + \mathcal{A}_{p_4}}{(\mathcal{A}_{p_4} - 2\kappa)e}))\tfrac{\mathcal{P}_{\xi,\eta}}{\mathcal{L}} + E_{p_4} e^{\kappa\mu}(1 + \mathcal{A}_{p_4}) + \mathcal{M}^{\kappa}_{p_4}\tfrac{4(\mathcal{A}_{p_4} - \kappa)}{(\mathcal{A}_{p_4} - 2\kappa)^2} \\
& + \mathcal{B}_{p_4} E_{p_4} e^{\kappa\mu}\tfrac{4(2\mathcal{A}_{p_4} - 3\kappa)}{(\mathcal{A}_{p_4} - 2\kappa)^2}] \\
< \;& \mathcal{L}e^{-\kappa q_4}.
\end{aligned}$$

Similarly, this case leads to a contradiction. Therefore, inequalities (26) hold true for all $t \in [0, +\infty)$ and $i \in \mathbb{N}$. Consequently, the proof of Theorem 2 is completed. □

Theorem 3. *NDINNs (1) achieve global exponential stability with decay rate κ, provided that all hypotheses of Lemma 2 hold and the following inequalities are satisfied:*

$$\begin{cases} \Delta_i < 0, E_i + \frac{4\mathcal{M}_i + 4\mathcal{B}_i E_i}{\sqrt{-\Delta_i}\mathcal{A}_i} < 1, \\ E_i(1 + \mathcal{A}_i) + \frac{4\sqrt{\mathcal{B}_i}\mathcal{M}_i + 4\mathcal{B}_i\sqrt{\mathcal{B}_i}E_i}{\sqrt{-\Delta_i}\mathcal{A}_i} < 1, \end{cases} \tag{32}$$

where E_i and $\mathcal{M}_i$ are as defined in Theorem 1. This means that there exist a $\kappa \in (0, \frac{1}{4}\min_{i\in\mathbb{N}} \mathcal{A}_i)$ and a positive constant $\mathcal{L}$ such that for all $t \geq 0$ and $i \in \mathbb{N}$,

$$|w_i(t) - w^*_i| < \mathcal{L}e^{-\kappa t}, \quad |\dot{w}_i(t)| < \mathcal{L}e^{-\kappa t}. \tag{33}$$

Proof. As mentioned in Lemma 2, system (1) has a unique equilibrium point $w^* = (w^*_1, w^*_2, \ldots, w^*_n)^T$. The solution for NDINNs (1), denoted by $w(t)=(w_1(t),w_2(t),\ldots,w_n(t))^T$, satisfies the initial conditions specified in (2). If $\Delta_i<0$, differentiating both sides of (9) with respect to t yields the following result.

$$\begin{cases} \dot{w}_i(t) &= -\frac{1}{\upsilon_i}[\mathcal{B}_i\sin\upsilon_i t w_i(0)+\sqrt{\mathcal{B}_i}\sin(\upsilon_i t-\theta_i)\dot{w}_i(0)+\mathcal{B}_i\sin(\upsilon_i t-2\theta_i)U_i(0) \\ & +\sqrt{\mathcal{B}_i}\sin(\upsilon_i t-\theta_i)\dot{U}_i(0)]e^{-\frac{\mathcal{A}_i}{2}t}+\dot{U}_i(t)-\mathcal{A}_iU_i(t) \\ & -\frac{\sqrt{\mathcal{B}_i}}{\upsilon_i}\int_0^t\sin(\upsilon_i(t-s)-\theta_i)e^{-\frac{\mathcal{A}_i}{2}(t-s)}F_i(s)ds \\ & -\frac{\mathcal{B}_i\sqrt{\mathcal{B}_i}}{\upsilon_i}\int_0^t\sin(\upsilon_i(t-s)-3\theta_i)e^{-\frac{\mathcal{A}_i}{2}(t-s)}U_i(s)ds, \end{cases} \tag{34}$$

which, in conjunction with (9) and (10), indicates that

$$\begin{cases} w_i(t)-w_i^* &= \frac{1}{\upsilon_i}[\sqrt{\mathcal{B}_i}\sin(\upsilon_i t+\theta_i)(w_i(0)-w_i^*)+\sin\upsilon_i t\dot{w}_i(0) \\ & +\sqrt{\mathcal{B}_i}\sin(\upsilon_i t-\theta_i)(\sum\limits_{l=1}^n\mathcal{E}_{il}(w_l(-\sigma_l)-w_l^*))-\sin\upsilon_i t\dot{U}_i(0)]e^{-\frac{\mathcal{A}_i}{2}t} \\ & +\sum\limits_{l=1}^n\mathcal{E}_{il}(w_l(t-\sigma_l)-w_l^*)+\frac{1}{\upsilon_i}\int_0^t\sin\upsilon_i(t-s)e^{-\frac{\mathcal{A}_i}{2}(t-s)}\mathcal{F}_i(s)ds \\ & +\frac{\mathcal{B}_i}{\upsilon_i}\int_0^t\sin(\upsilon_i(t-s)-2\theta_i)e^{-\frac{\mathcal{A}_i}{2}(t-s)}(\sum\limits_{l=1}^n\mathcal{E}_{il}(w_l(s-\sigma_l)-w_l^*))ds \\ \dot{w}_i(t) &= -\frac{1}{\upsilon_i}[\mathcal{B}_i\sin\upsilon_i t(w_i(0)-w_i^*)+\sqrt{\mathcal{B}_i}\sin(\upsilon_i t-\theta_i)\dot{w}_i(0) \\ & +\mathcal{B}_i\sin(\upsilon_i t-2\theta_i)(\sum\limits_{l=1}^n\mathcal{E}_{il}(w_l(-\sigma_l)-w_l^*))+\sqrt{\mathcal{B}_i}\sin(\upsilon_i t-\theta_i)\dot{U}_i(0)]e^{-\frac{\mathcal{A}_i}{2}t} \\ & +\dot{U}_i(t)-\mathcal{A}_i(\sum\limits_{l=1}^n\mathcal{E}_{il}(w_l(t-\sigma_l)-w_l^*)) \\ & -\frac{\sqrt{\mathcal{B}_i}}{\upsilon_i}\int_0^t\sin(\upsilon_i(t-s)-\theta_i)e^{-\frac{\mathcal{A}_i}{2}(t-s)}\mathcal{F}_i(s)ds \\ & -\frac{\mathcal{B}_i\sqrt{\mathcal{B}_i}}{\upsilon_i}\int_0^t\sin(\upsilon_i(t-s)-3\theta_i)e^{-\frac{\mathcal{A}_i}{2}(t-s)}(\sum\limits_{l=1}^n\mathcal{E}_{il}(w_l(s-\sigma_l)-w_l^*))ds. \end{cases} \tag{35}$$

Equation (32) suggests the presence of a sufficiently little κ, with κ belonging to $(0,\frac{1}{4}\min\limits_{i\in\mathbb{N}}\mathcal{A}_i)$, along with a sufficiently large positive constant $\mathcal{L}$, such that they fulfill the condition

$$\begin{cases} \frac{2(1+E_i)}{\sqrt{-\Delta_i}\mathcal{L}}(\sqrt{\mathcal{B}_i}+1)\mathcal{P}_{\xi,\eta}+E_ie^{\kappa\mu}+\frac{4\mathcal{M}_i^{\kappa}+4\mathcal{B}_iE_ie^{\kappa\mu}}{\sqrt{-\Delta_i}(\mathcal{A}_i-2\kappa)}<1, \\ \frac{2(1+E_i)}{\sqrt{-\Delta_i}\mathcal{L}}(\mathcal{B}_i+\sqrt{\mathcal{B}_i})\mathcal{P}_{\xi,\eta}+E_ie^{\kappa\mu}(1+\mathcal{A}_i)+\frac{4\sqrt{\mathcal{B}_i}\mathcal{M}_i^{\kappa}+4\mathcal{B}_i\sqrt{\mathcal{B}_i}E_ie^{\kappa\mu}}{\sqrt{-\Delta_i}(\mathcal{A}_i-2\kappa)}<1. \end{cases} \tag{36}$$

Clearly, inequalities (33) are satisfied for $t\in[-\mu,0]$, $i\in\mathbb{N}$. We assert that these inequalities also hold true for all $t\geq 0$. Should this assertion be incorrect, then one of the subsequent cases must arise:

Case 5. *There exist* $p_5\in\mathbb{N}$ *and* $q_5>0$ *for which it is true that*

$$|w_i(t)-w_i^*|<\mathcal{L}e^{-\kappa t},|\dot{w}_i(t)|<\mathcal{L}e^{-\kappa t},t\in[0,q_5),|w_{p_5}(q_5)-w_{p_5}^*|=\mathcal{L}e^{-\kappa q_5}. \tag{37}$$

Case 6. *There exist* $p_6\in\mathbb{N}$ *and* $q_6>0$ *for which it is true that*

$$|w_i(t)-w_i^*|<\mathcal{L}e^{-\kappa t},|\dot{w}_i(t)|<\mathcal{L}e^{-\kappa t},t\in[0,q_6),|\dot{w}_{p_6}(q_6)|=\mathcal{L}e^{-\kappa q_6}. \tag{38}$$

In the case where Case 5 happens, by applying assumption (A_1) in conjunction with (35)–(37), it follows that

$$\begin{aligned} & |w_{p_5}(q_5)-w_{p_5}^*| \\ = & \frac{1}{\upsilon_{p_5}}|[\sqrt{\mathcal{B}_{p_5}}\sin(\upsilon_{p_5}q_5+\theta_{p_5})(w_{p_5}(0)-w_{p_5}^*)+\sin\upsilon_{p_5}q_5\dot{w}_{p_5}(0) \\ & +\sqrt{\mathcal{B}_{p_5}}\sin(\upsilon_{p_5}q_5-\theta_{p_5})(\sum_{l=1}^n\mathcal{E}_{p_5l}(w_l(-\sigma_l)-w_l^*))-\sin\upsilon_{p_5}q_5\dot{U}_{p_5}(0)]e^{-\frac{\mathcal{A}_{p_5}}{2}q_5} \\ & +\sum_{l=1}^n\mathcal{E}_{p_5l}(w_l(q_5-\sigma_l)-w_l^*)+\frac{1}{\upsilon_{p_5}}\int_0^t\sin\upsilon_{p_5}(q_5-s)e^{-\frac{\mathcal{A}_{p_5}}{2}(q_5-s)}\mathcal{F}_{p_5}(s)ds \\ & +\frac{\mathcal{B}_{p_5}}{\upsilon_{p_5}}\int_0^{q_5}\sin(\upsilon_{p_5}(q_5-s)-2\theta_{p_5})e^{-\frac{\mathcal{A}_{p_5}}{2}(q_5-s)}(\sum_{l=1}^n\mathcal{E}_{p_5l}(w_l(s-\sigma_l)-w_l^*))ds| \\ \leq & \frac{1}{\upsilon_{p_5}}(1+E_{p_5})(\sqrt{\mathcal{B}_{p_5}}+1)e^{-(\frac{\mathcal{A}_{p_5}}{2}-\kappa)q_5}\mathcal{P}_{\xi,\eta}e^{-\kappa q_5}+\sum_{l=1}^n|\mathcal{E}_{p_5l}||w_l(q_5-\sigma_l)-w_l^*| \\ & +\frac{1}{\upsilon_{p_5}}\int_0^{q_5}e^{-\frac{\mathcal{A}_{p_5}}{2}(q_5-s)}(\sum_{l=1}^n|\mathcal{C}_{p_5l}||G_l(w_l(s))-G_l(w_l^*)| \\ & +\sum_{l=1}^n|\mathcal{D}_{p_5l}||G_l(w_l(s-\tau_l))-G_l(w_l^*)|)ds \\ & +\frac{\mathcal{B}_{p_5}}{\upsilon_{p_5}}\int_0^{q_5}e^{-\frac{\mathcal{A}_{p_5}}{2}(q_5-s)}(\sum_{l=1}^n|\mathcal{E}_{p_5l}||w_l(s-\sigma_l)-w_l^*|)ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\upsilon_{p5}}(1+E_{p5})(\sqrt{\mathcal{B}_{p5}}+1)\mathcal{P}_{\xi,\eta}e^{-\kappa q_5}+\sum_{l=1}^{n}|\mathcal{E}_{p5l}|\mathcal{L}e^{-\kappa(q_5-\sigma_l)}\\
&\quad+\frac{1}{\upsilon_{p5}}\int_0^{q_5}e^{-\frac{\mathcal{A}_{p5}}{2}(q_5-s)}(\sum_{l=1}^{n}|\mathcal{C}_{p5l}|k_l|w_l(s)-w_l^*|+\sum_{l=1}^{n}|\mathcal{D}_{p5l}|k_l|w_l(s-\tau_l)-w_l^*|)ds\\
&\quad+\frac{\mathcal{B}_{p5}}{\upsilon_{p5}}\int_0^{q_5}e^{-\frac{\mathcal{A}_{p5}}{2}(q_5-s)}(\sum_{l=1}^{n}|\mathcal{E}_{p5l}|\mathcal{L}e^{-\kappa(s-\sigma_l)})ds\\
&\leq e^{-\kappa q_5}[\frac{1+E_{p5}}{\upsilon_{p5}}(\sqrt{\mathcal{B}_{p5}}+1)\mathcal{P}_{\xi,\eta}+E_{p5}\mathcal{L}e^{\kappa\mu}+\frac{1}{\upsilon_{p5}}\int_0^{q_5}e^{-\frac{\mathcal{A}_{p5}}{2}(q_5-s)}e^{\kappa q_5}\\
&\quad\times(\sum_{l=1}^{n}|\mathcal{C}_{p5l}|k_l\mathcal{L}e^{-\kappa s}+\sum_{l=1}^{n}|\mathcal{D}_{p5l}|k_l\mathcal{L}e^{-\kappa(s-\tau_l)})ds\\
&\quad+\frac{\mathcal{B}_{p5}}{\upsilon_{p5}}\int_0^{q_5}e^{-\frac{\mathcal{A}_{p5}}{2}(q_5-s)}e^{\kappa q_5}E_{p5}e^{\kappa\mu}\mathcal{L}e^{-\kappa s}ds]\\
&\leq \mathcal{L}e^{-\kappa q_5}[\frac{1+E_{p5}}{\upsilon_{p5}\mathcal{L}}(\sqrt{\mathcal{B}_{p5}}+1)\mathcal{P}_{\xi,\eta}+E_{p5}e^{\kappa\mu}+\frac{\mathcal{M}_{p5}^{\kappa}}{\upsilon_{p5}}\int_0^{q_5}e^{-(\frac{\mathcal{A}_{p5}}{2}-\kappa)(q_5-s)}ds\\
&\quad+\frac{\mathcal{B}_{p5}E_{p5}e^{\kappa\mu}}{\upsilon_{p5}}\int_0^{q_5}e^{-(\frac{\mathcal{A}_{p5}}{2}-\kappa)(q_5-s)}ds]\\
&= \mathcal{L}e^{-\kappa q_5}[\frac{2(1+E_{p5})}{\sqrt{-\Delta_{p5}}\mathcal{L}}(\sqrt{\mathcal{B}_{p5}}+1)\mathcal{P}_{\xi,\eta}+E_{p5}e^{\kappa\mu}+\frac{4\mathcal{M}_{p5}^{\kappa}+4\mathcal{B}_{p5}E_{p5}e^{\kappa\mu}}{\sqrt{-\Delta_{p5}}(\mathcal{A}_{p5}-2\kappa)}(1-e^{-(\frac{\mathcal{A}_{p5}}{2}-\kappa)q_5})]\\
&< \mathcal{L}e^{-\kappa q_5}[\frac{2(1+E_{p5})}{\sqrt{-\Delta_{p5}}\mathcal{L}}(\sqrt{\mathcal{B}_{p5}}+1)\mathcal{P}_{\xi,\eta}+E_{p5}e^{\kappa\mu}+\frac{4\mathcal{M}_{p5}^{\kappa}+4\mathcal{B}_{p5}E_{p5}e^{\kappa\mu}}{\sqrt{-\Delta_{p5}}(\mathcal{A}_{p5}-2\kappa)}]\\
&< \mathcal{L}e^{-\kappa q_5}.
\end{aligned}
$$

In the case where Case 6 arises, by employing assumption (A_1) together with (35), (36) and (38), we deduce that

$$
\begin{aligned}
&|\dot{w}_{p6}(q_6)|\\
=\;&|-\frac{1}{\upsilon_{p6}}[\mathcal{B}_{p6}\sin\upsilon_{p6}q_6(w_{p6}(0)-w_{p6}^*)+\sqrt{\mathcal{B}_{p6}}\sin(\upsilon_{p6}q_6-\theta_{p6})\dot{w}_{p6}(0)\\
&+\mathcal{B}_{p6}\sin(\upsilon_{p6}q_6-2\theta_{p6})(\sum_{l=1}^{n}\mathcal{E}_{p6l}(w_l(-\sigma_l)-w_l^*))+\sqrt{\mathcal{B}_{p6}}\sin(\upsilon_{p6}q_6-\theta_{p6})\dot{U}_{p6}(0)]e^{-\frac{\mathcal{A}_{p6}}{2}q_6}\\
&+\dot{U}_{p6}(q_6)-\mathcal{A}_{p6}(\sum_{l=1}^{n}\mathcal{E}_{p6l}(w_l(q_6-\sigma_l)-w_l^*))\\
&-\frac{\sqrt{\mathcal{B}_{p6}}}{\upsilon_{p6}}\int_0^{q_6}\sin(\upsilon_{p6}(q_6-s)-\theta_{p6})e^{-\frac{\mathcal{A}_{p6}}{2}(q_6-s)}\mathcal{F}_{p6}(s)ds\\
&-\frac{\mathcal{B}_{p6}\sqrt{\mathcal{B}_{p6}}}{\upsilon_{p6}}\int_0^{q_6}\sin(\upsilon_{p6}(q_6-s)-3\theta_{p6})e^{-\frac{\mathcal{A}_{p6}}{2}(q_6-s)}(\sum_{l=1}^{n}\mathcal{E}_{p6}(w_l(s-\sigma_l)-w_l^*))ds|\\
\leq\;&\frac{1}{\upsilon_{p6}}(1+E_{p6})(\mathcal{B}_{p6}+\sqrt{\mathcal{B}_{p6}})e^{-(\frac{\mathcal{A}_{p6}}{2}-\kappa)q_6}\mathcal{P}_{\xi,\eta}e^{-\kappa q_6}+\sum_{l=1}^{n}|\mathcal{E}_{p6l}||\dot{w}_l(q_6-\sigma_l)|\\
&+\mathcal{A}_{p6}(\sum_{l=1}^{n}|\mathcal{E}_{p6l}||w_l(q_6-\sigma_l)-w_l^*|)+\frac{\sqrt{\mathcal{B}_{p6}}}{\upsilon_{p6}}\int_0^{q_6}e^{-\frac{\mathcal{A}_{p6}}{2}(q_6-s)}\\
&\times(\sum_{l=1}^{n}|\mathcal{C}_{p6l}||G_l(w_l(s))-G_l(w_l^*)|+\sum_{l=1}^{n}|\mathcal{D}_{p6l}||G_l(w_l(s-\tau_l))-G_l(w_l^*)|)ds\\
&+\frac{\mathcal{B}_{p6}\sqrt{\mathcal{B}_{p6}}}{\upsilon_{p6}}\int_0^{q_6}e^{-\frac{\mathcal{A}_{p6}}{2}(q_6-s)}(\sum_{l=1}^{n}|\mathcal{E}_{p6}||w_l(s-\sigma_l)-w_l^*|)ds\\
\leq\;&\frac{1+E_{p6}}{\upsilon_{p6}}(\mathcal{B}_{p4}+\sqrt{\mathcal{B}_{p6}})\mathcal{P}_{\xi,\eta}e^{-\kappa q_6}+\sum_{l=1}^{n}|\mathcal{E}_{p6l}|\mathcal{L}e^{-\kappa(q_6-\sigma_l)}+\mathcal{A}_{p6}(\sum_{l=1}^{n}|\mathcal{E}_{p6l}|\mathcal{L}e^{-\kappa(q_6-\sigma_l)})\\
&+\frac{\sqrt{\mathcal{B}_{p6}}}{\upsilon_{p6}}\int_0^{q_6}e^{-\frac{\mathcal{A}_{p6}}{2}(q_6-s)}(\sum_{l=1}^{n}|\mathcal{C}_{p6l}|k_l|w_l(s)-w_l^*|+\sum_{l=1}^{n}|\mathcal{D}_{p6l}|k_l|w_l(s-\tau_l)-w_l^*|)ds\\
&+\frac{\mathcal{B}_{p6}\sqrt{\mathcal{B}_{p6}}}{\upsilon_{p6}}\int_0^{q_6}e^{-\frac{\mathcal{A}_{p6}}{2}(q_6-s)}(\sum_{l=1}^{n}|\mathcal{E}_{p6}|\mathcal{L}e^{-\kappa(s-\sigma_l)})ds\\
\leq\;&e^{-\kappa q_6}[\frac{1+E_{p6}}{\upsilon_{p6}}(\mathcal{B}_{p4}+\sqrt{\mathcal{B}_{p6}})\mathcal{P}_{\xi,\eta}+E_{p6}e^{\kappa\mu}\mathcal{L}+\mathcal{A}_{p6}E_{p6}e^{\kappa\mu}\mathcal{L}\\
&+\frac{\sqrt{\mathcal{B}_{p6}}}{\upsilon_{p6}}\int_0^{q_6}e^{-\frac{\mathcal{A}_{p6}}{2}(q_6-s)}e^{\kappa q_6}(\sum_{l=1}^{n}|\mathcal{C}_{p6l}|k_l\mathcal{L}e^{-\kappa s}+\sum_{l=1}^{n}|\mathcal{D}_{p6l}|k_l\mathcal{L}e^{-\kappa(s-\tau_l)})ds\\
&+\frac{\mathcal{B}_{p6}\sqrt{\mathcal{B}_{p6}}}{\upsilon_{p6}}\int_0^{q_6}e^{-\frac{\mathcal{A}_{p6}}{2}(q_6-s)}e^{\kappa q_6}E_{p6}e^{\kappa\mu}\mathcal{L}e^{-\kappa s}ds]\\
\leq\;&\mathcal{L}e^{-\kappa q_6}[\frac{1+E_{p6}}{\upsilon_{p6}\mathcal{L}}(\mathcal{B}_{p4}+\sqrt{\mathcal{B}_{p6}})\mathcal{P}_{\xi,\eta}+E_{p6}e^{\kappa\mu}(1+\mathcal{A}_{p6})\\
&+\frac{\sqrt{\mathcal{B}_{p6}}\mathcal{M}_{p6}^{\kappa}}{\upsilon_{p6}}\int_0^{q_6}e^{-(\frac{\mathcal{A}_{p6}}{2}-\kappa)(q_6-s)}ds+\frac{\mathcal{B}_{p6}\sqrt{\mathcal{B}_{p6}}E_{p6}e^{\kappa\mu}}{\upsilon_{p6}}\int_0^{q_6}e^{-(\frac{\mathcal{A}_{p6}}{2}-\kappa)(q_6-s)}ds]\\
=\;&\mathcal{L}e^{-\kappa q_6}[\frac{2(1+E_{p6})}{\sqrt{-\Delta_{p6}}\mathcal{L}}(\mathcal{B}_{p4}+\sqrt{\mathcal{B}_{p6}})\mathcal{P}_{\xi,\eta}+E_{p6}e^{\kappa\mu}(1+\mathcal{A}_{p6})\\
&+\frac{4\sqrt{\mathcal{B}_{p6}}\mathcal{M}_{p6}^{\kappa}+4\mathcal{B}_{p6}\sqrt{\mathcal{B}_{p6}}E_{p6}e^{\kappa\mu}}{\sqrt{-\Delta_{p6}}(\mathcal{A}_{p6}-2\kappa)}(1-e^{-(\frac{\mathcal{A}_{p6}}{2}-\kappa)q_6})]\\
<\;&\mathcal{L}e^{-\kappa q_6}[\frac{2(1+E_{p6})}{\sqrt{-\Delta_{p6}}\mathcal{L}}(\mathcal{B}_{p4}+\sqrt{\mathcal{B}_{p6}})\mathcal{P}_{\xi,\eta}+E_{p6}e^{\kappa\mu}(1+\mathcal{A}_{p6})\\
&+\frac{4\sqrt{\mathcal{B}_{p6}}\mathcal{M}_{p6}^{\kappa}+4\mathcal{B}_{p6}\sqrt{\mathcal{B}_{p6}}E_{p6}e^{\kappa\mu}}{\sqrt{-\Delta_{p6}}(\mathcal{A}_{p6}-2\kappa)}]\\
<\;&\mathcal{L}e^{-\kappa q_6}.
\end{aligned}
$$

In the same manner, this case results in a contradiction. Hence, inequalities (33) are valid for every $t \in [0, +\infty)$ and $i \in \mathbb{N}$. As a result, the proof of Theorem 3 is concluded. □

Remark 1. *As far as we know, only a handful of scholars have studied the stability analysis of NDINNs (1), with the exception of the authors in [3], who examined a simplified version of NDINNs (1) where $\tau_l = \sigma_l = \mu$. They derived delay-dependent criteria for ensuring global asymptotic stability through solving LMIs, utilizing* *both the reduced order method and advancements in LKFs. Despite this, applying the reduced order technique frequently leads to intricate computations and heightened system intricacy. Additionally, constructing LKFs poses significant difficulties and demands considerable effort, typically involving the handling of sophisticated LMIs. These LMIs are generally addressed with the aid of MATLAB software (MATLAB R2022b) suites. On the other hand, our approach avoids employing both the reduced order technique and traditional methods for constructing LKFs. By utilizing the characteristic method, we propose a set of delay-independent criteria that are sufficient to ensure the global exponential stability of system (1). One key benefit of our proposed method is its ability to bypass the complexities associated with the reduced order method along with traditional LKFs formulation, thereby sidestepping convoluted analyses and computationally intensive tasks. Several recent works [17,18,20] have utilized the characteristic method for inertial neural networks. Importantly, unlike [18], which applies the characteristic method to neutral-type inertial neural networks involving only the first order derivative $\dot{w}_l(t-\sigma_l)$, this study targets the second order derivative $\ddot{w}_l(t-\sigma_l)$. We compare these existing results with both [3] and our current findings; the details are presented in Table 1.*

Table 1. Comparison of the results and methods between this paper and related references

Comparison	[3]	[17]	[18]	[20]	**This Paper**
Type of neural network	Neutral	Delayed	Neutral	Delayed	Neutral
Whether inertial terms are included	Yes	Yes	Yes	Yes	Yes
Type of method	Reduced-order, LKFs, LMIs	Characteristic method (CM)	CM	CM	CM
Delay-dependent (DD) or delay-independent (DI) criteria	DD	DI	DI	DI	DI
Type of stability	Global asymptotic	Global exponential (GE)	GE	GE	GE
Improvement of this paper	Avoiding the reduced order method, eliminating the need to construct LKFs, and bypassing the solution of LMIs; Delay-independent criteria that can be readily verified as a set of linear scalar inequalities.				

Remark 2. *The three cases $\Delta_i > 0$, $\Delta_i = 0$ and $\Delta_i < 0$ correspond respectively to over-damping, critical-damping and under-damping regimes. Theorems 1–3 enumerate conditions (17), (25) and (32) under which system (1) achieves exponential stability. Crucially, E_i and $\mathcal{M}_i$ must remain bounded; this necessitates that the coupling coefficients $\mathcal{C}_{il}$, $\mathcal{D}_{il}$, the Lipschitz constant k_l, and the neutral delay coefficients $\mathcal{E}_{il}$ be sufficiently small.*

4. Three Numerical Examples

The validity of our theoretical results is confirmed via three numerical simulations, corresponding to cases where the discriminant $\Delta_i = \mathcal{A}_i^2 - 4\mathcal{B}_i$ assumes positive, zero, and negative values respectively.

Example 1. *Consider the NDINNs (1) with dimension $n = 2$ and the parameters specified below: $\mathcal{A}_1 = 5$, $\mathcal{A}_2 = 6$, $\mathcal{B}_1 = 4$, $\mathcal{B}_2 = 5$, $\mathcal{C}_{11} = 0.1$, $\mathcal{C}_{12} = -0.1$, $\mathcal{C}_{21} = 0.2$, $\mathcal{C}_{22} = -0.2$, $\mathcal{D}_{11} = -0.2$, $\mathcal{D}_{12} = 0.2$, $\mathcal{D}_{21} = 0.3$, $\mathcal{D}_{22} = -0.3$, $\mathcal{E}_{11} = 0.01$, $\mathcal{E}_{12} = -0.02$, $\mathcal{E}_{21} = 0.02$, $\mathcal{E}_{22} = -0.01$, $J_1 = 12$, $J_2 = -13$, $\tau_1 = 1$, $\tau_2 = 2$, $\sigma_1 = 2$, $\sigma_2 = 1$, $G_1(x) = 0.3\sin x + 0.7x$, $G_2(x) = 0.7\cos x + 0.3x$.*

It is straightforward to confirm that both G_1 and G_2 satisfy the global Lipschitz condition (A_1) with Lipschitz constants $k_1 = k_2 = 1$. Moreover, the following inequalities hold:

$$\begin{cases} 0.8 = \sum_{l=1}^{2} k_l|\mathcal{C}_{1l} + \mathcal{D}_{1l}| + \sum_{l=1}^{n} k_1|\mathcal{C}_{l1} + \mathcal{D}_{l1}| < 2\mathcal{B}_1 = 8, \\ 1.6 = \sum_{l=1}^{2} k_l|\mathcal{C}_{2l} + \mathcal{D}_{2l}| + \sum_{l=1}^{n} k_1|\mathcal{C}_{l2} + \mathcal{D}_{l2}| < 2\mathcal{B}_2 = 10, \end{cases}$$

$$\begin{cases} \Delta_1 = 9 > 0, \Delta_2 = 16 > 0, \\ 0.23 = \frac{\mathcal{M}_1}{\mathcal{B}_1} + \frac{E_1(|a_1|+|b_1|)}{\sqrt{\Delta_1}} + E_1 < 1, \\ 0.325 = \frac{\mathcal{M}_2}{\mathcal{B}_2} + \frac{E_1(|a_2|+|b_2|)}{\sqrt{\Delta_2}} + E_2 < 1, \\ 0.75 = E_1(1+\mathcal{A}_1) + \frac{2\mathcal{M}_1+E_1(a_1^2+b_1^2)}{\sqrt{\Delta_1}} < 1, \\ 0.905 = E_2(1+\mathcal{A}_2) + \frac{2\mathcal{M}_2+E_2(a_2^2+b_2^2)}{\sqrt{\Delta_2}} < 1. \end{cases}$$

Consequently, in accordance with Theorem 1, the NDINNs presented in Example 1 are proven to possess a unique equilibrium point $(2.9188, -2.2755)$ and demonstrate globally exponential stability with the decay rate 0.015. This theoretical conclusion is further validated by the computational results depicted in Figures 1–3.

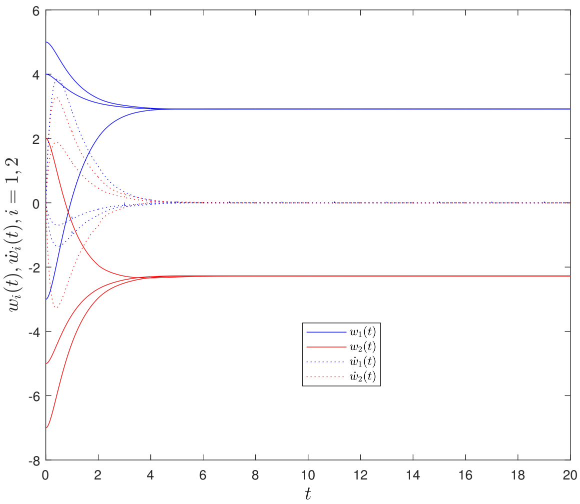

Figure 1. Trajectories of state variables and their derivatives in Example 1 with initial values: $(\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s)) = (5, -7, 0, 0)$, $(4, -5, 0, 0)$, $(-3, 2, 0, 0)$, $s \in [-2, 0]$.

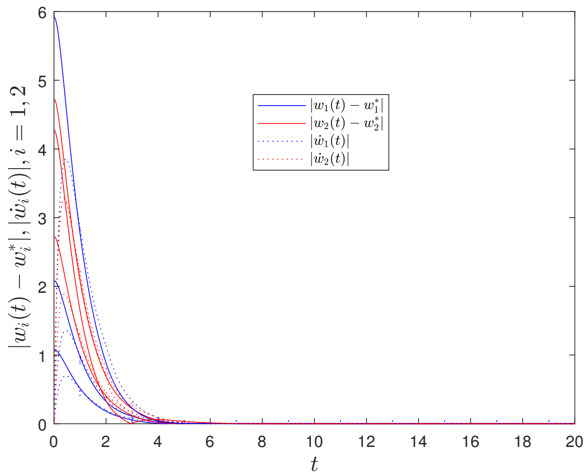

Figure 2. Errors of $|w_i(t) - w_i^*|$ and $|\dot{w}_i(t)|$ in Example 1.

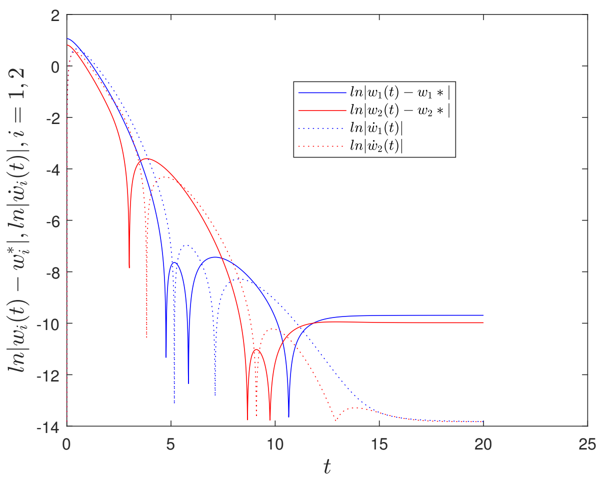

Figure 3. Semilogarithmic plots of $ln|w_i(t) - w_i^*|$ and $ln|\dot{w}_i(t)|$ in Example 1.

Example 2. *The system parameters here are identical to those in Example 1, with the following modifications:* $\mathcal{A}_1 = 4$, $\mathcal{A}_2 = 2\sqrt{5}$, $\mathcal{C}_{21} = 0.1$, $\mathcal{C}_{22} = -0.1$, $\mathcal{D}_{21} = 0.2$, $\mathcal{D}_{22} = -0.2$.

We can readily check that

$$\begin{cases} 0.6 = \sum\limits_{l=1}^{2} k_l|\mathcal{C}_{1l} + \mathcal{D}_{1l}| + \sum\limits_{l=1}^{n} k_1|\mathcal{C}_{l1} + \mathcal{D}_{l1}| < 2\mathcal{B}_1 = 8, \\ 1 = \sum\limits_{l=1}^{2} k_l|\mathcal{C}_{2l} + \mathcal{D}_{2l}| + \sum\limits_{l=1}^{n} k_1|\mathcal{C}_{l2} + \mathcal{D}_{l2}| < 2\mathcal{B}_2 = 10, \end{cases}$$

$$\begin{cases} \Delta_1 = \Delta_2 = 0, \\ 0.3 = 4E_1 + \frac{4\mathcal{M}_1}{\mathcal{A}_1^2} < 1, \ \ 0.3642 = 4E_2 + \frac{4\mathcal{M}_2}{\mathcal{A}_2^2} < 1 \\ 0.99 = E_1(1+\mathcal{A}_1) + \frac{4\mathcal{M}_1 + 8\mathcal{B}_1 E_1}{\mathcal{A}_1} < 1, \\ 0.9691 = E_2(1+\mathcal{A}_2) + \frac{4\mathcal{M}_2 + 8\mathcal{B}_2 E_2}{\mathcal{A}_2} < 1. \end{cases}$$

By Theorem 2, the NDINNs in Example 2 are shown to admit a unique equilibrium point $(2.9163, -2.3994)$ with global exponential stability with the decay rate 0.013, a result validated numerically in Figures 4–6.

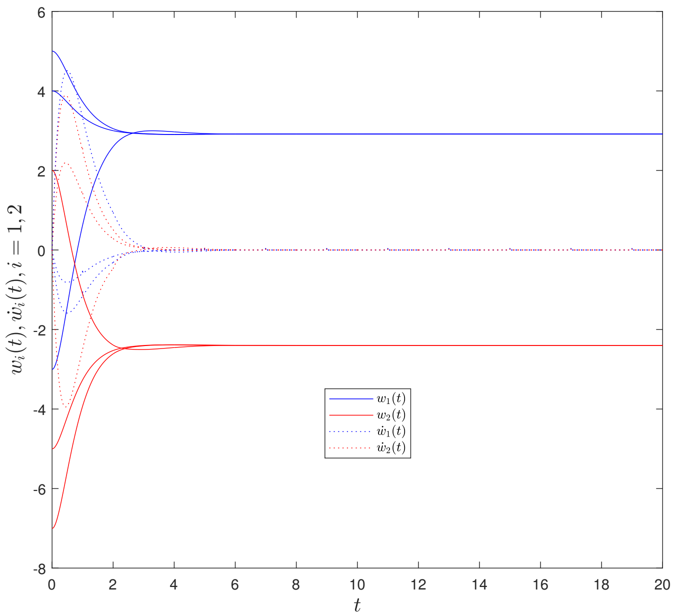

Figure 4. Trajectories of state variables and their derivatives in Example 2 with initial values: $(\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s)) = (5, -7, 0, 0), (4, -5, 0, 0), (-3, 2, 0, 0), s \in [-2, 0]$.

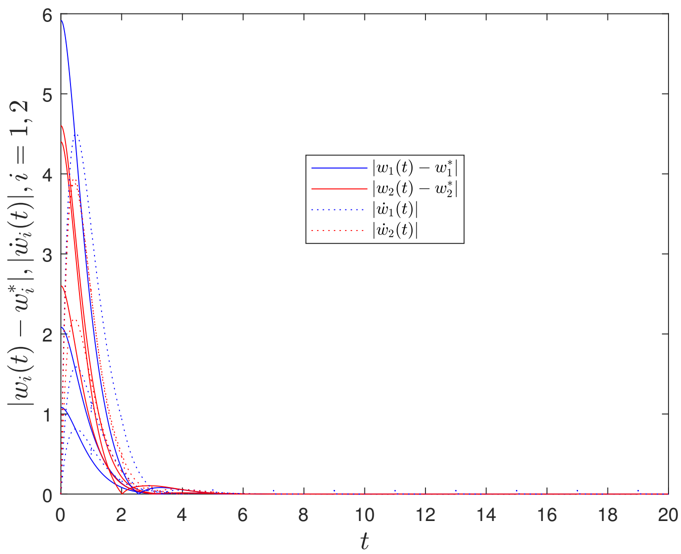

Figure 5. Errors of $|w_i(t) - w_i^*|$ and $|\dot{w}_i(t)|$ in Example 2.

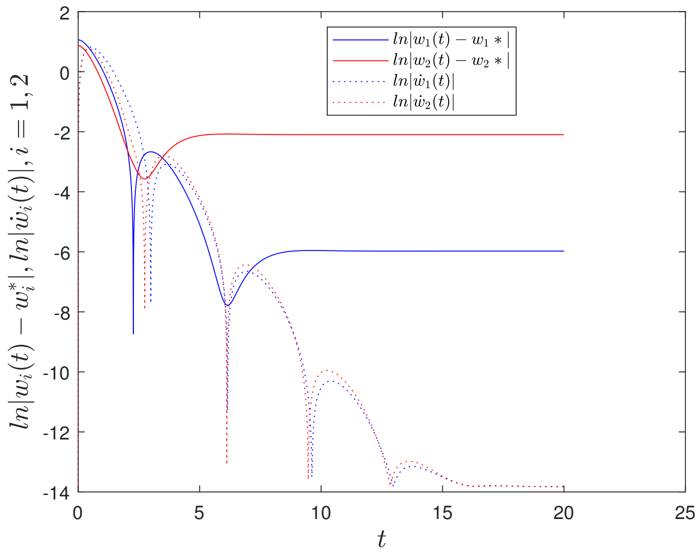

Figure 6. Semilogarithmic plots of $ln|w_i(t) - w_i^*|$ and $ln|\dot{w}_i(t)|$ in Example 2.

Example 3. *With the exception of $\mathcal{B}_1 = 5$ and $\mathcal{B}_2 = 6$, all remaining parameters in this case are identical to those specified in Example 2.*

It is straightforward to confirm that

$$\begin{cases} 0.6 = \sum_{l=1}^{2} k_l|\mathcal{C}_{1l} + \mathcal{D}_{1l}| + \sum_{l=1}^{n} k_1|\mathcal{C}_{l1} + \mathcal{D}_{l1}| < 2\mathcal{B}_1 = 10, \\ 1 = \sum_{l=1}^{2} k_l|\mathcal{C}_{2l} + \mathcal{D}_{2l}| + \sum_{l=1}^{n} k_1|\mathcal{C}_{l2} + \mathcal{D}_{l2}| < 2\mathcal{B}_2 = 12, \end{cases}$$

$$\begin{cases} -4 = \Delta_1 = \Delta_2 < 0, \\ 0.405 = E_1 + \frac{4\mathcal{M}_1 + 4\mathcal{B}_1 E_1}{\sqrt{-\Delta_1}\mathcal{A}_1} < 1, \\ 0.4783 = E_2 + \frac{4\mathcal{M}_2 + 4\mathcal{B}_2 E_2}{\sqrt{-\Delta_2}\mathcal{A}_2} < 1, \\ 0.8958 = E_1(1 + \mathcal{A}_1) + \frac{4\sqrt{\mathcal{B}_1}\mathcal{M}_1 + 4\mathcal{B}_1\sqrt{\mathcal{B}_1}E_1}{\sqrt{-\Delta_1}\mathcal{A}_1} < 1, \\ 0.9019 = E_2(1 + \mathcal{A}_2) + \frac{4\sqrt{\mathcal{B}_2}\mathcal{M}_2 + 4\mathcal{B}_2\sqrt{\mathcal{B}_2}E_2}{\sqrt{-\Delta_2}\mathcal{A}_2} < 1. \end{cases}$$

Consequently, as established by Theorem 3, the NDINNs in Example 3 are shown to admit a unique equilibrium point $(2.3445, -2.0280)$ with globally exponential stability with the decay rate 0.011. This theoretical assertion is further validated through computational evidence provided in Figures 7–9.

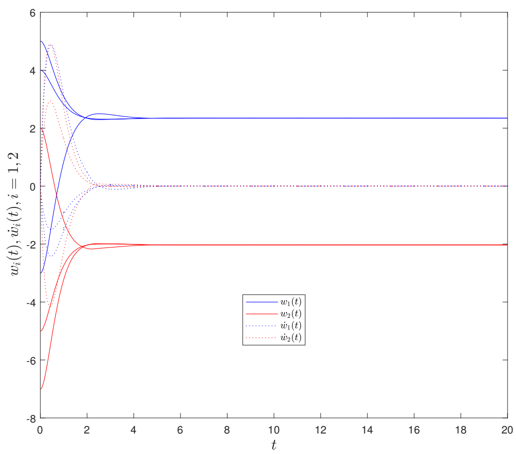

Figure 7. Trajectories of state variables and their derivatives in Example 3 with initial values: $(\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s)) = (5, -7, 0, 0)$, $(4, -5, 0, 0)$, $(-3, 2, 0, 0)$, $s \in [-2, 0]$.

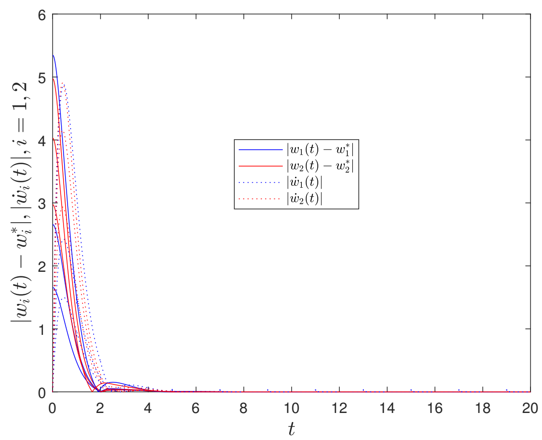

Figure 8. Errors of $|w_i(t) - w_i^*|$ and $|\dot{w}_i(t)|$ in Example 3.

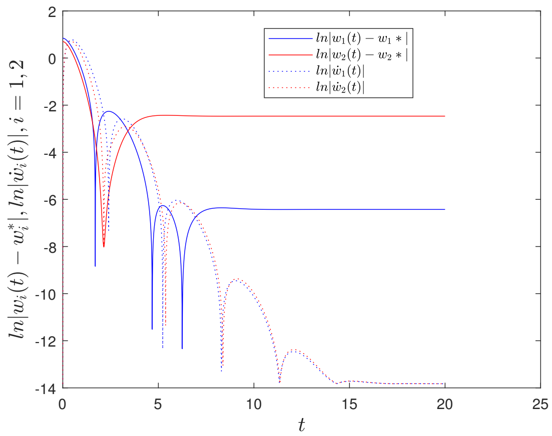

Figure 9. Semilogarithmic plots of $ln|w_i(t) - w_i^*|$ and $ln|\dot{w}_i(t)|$ in Example 3.

Remark 3. *Existing literatures lack any application of the characteristic method to analyze NDINNs (1). While [3] addresses only the restricted NDINNs (1) with $\tau_l = \sigma_l = \mu$, our methodology bypasses traditional order reduced method and LKFs construction. This implies that approaches in [3–16] fail to generalize to Examples 1–3.*

5. Concluding Remarks

By employing the characteristic method, we investigated the global exponential stability of a particular class of NDINNs. Through a comprehensive analysis centered on the discriminant Δ_i, we derived explicit solution representations for system (1). Furthermore, we established several delay-independent criteria sufficient to ensure globally exponentially stable dynamics. Three numerical examples accompanied by computational simulations were provided to validate these theoretical results. Notably, this approach reduces the analytical complexity to solving a series of elementary scalar inequalities, thus obviating the need for MATLAB LMI solver-based techniques. Motivated by the methods in [20–24], future work will address the global exponential stability of NDINNs under less restrictive activation function assumptions.

Author Contributions

W.W. drafted the main text of the manuscript, and W.C. carried out the numerical simulations for the three examples, as well as reviewed and edited the manuscript. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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