

Numerical Assessment of Linear and Nonlinear Models for Compressible Flows

Mutaz Mohammad ^{1,*}, Nicola Sottocornola ², Abdelwahab Kharab ¹ and Ronald B. Guenther ³

¹ Department of Mathematics and Statistics, Zayed University, Abu Dhabi P.O. Box 144534, United Arab Emirates

² Department of Mathematics, New York University Abu Dhabi, Abu Dhabi PO Box 129188, United Arab Emirates

³ Department of Mathematics, Oregon State University, Corvallis, OR 97333, USA

* Correspondence: Mutaz.Mohammad@zu.ac.ae

How To Cite: Mohammad, M.; Sottocornola, N.; Kharab, A.; et al. Numerical Assessment of Linear and Nonlinear Models for Compressible Flows. *Nonlinear Analysis and Computer Simulations* **2026**, *1*(3), 13. <https://doi.org/10.53941/nacs.2026.100013>

Received: 20 March 2026

Revised: 1 May 2026

Accepted: 22 June 2026

Published: 4 August 2026

Abstract: This paper presents a numerical comparison between linearized and nonlinear models for compressible fluid flow in several configurations: a Cauchy-type setting, a half-space model with prescribed free-surface velocity, a bounded-domain model, and a pressure-dependent compressibility model. The purpose is to assess, under identical initial and boundary data, when the linear approximation reproduces the pressure response of the nonlinear formulation and when nonlinear effects become significant. The governing equations are written in terms of the logarithmic density variable $\sigma = \ln(\rho/\rho_0)$ and the velocity field. A three-dimensional Lax–Wendroff finite-difference discretization is used for the computations. The revision clarifies the computational setup, including the grid $31 \times 31 \times 21$, the dimensional time step $\Delta t = 600$ s, the representative grid spacing $\Delta x = 6667$ m, and the corresponding Courant number 0.045 based on a maximum velocity magnitude of 0.5 m/s. The initial temperature and pressure ranges used in the numerical setup are also stated explicitly. The results indicate that the linear and nonlinear models are generally close in the interior of the computational domain for the reported parameter regimes, while larger discrepancies occur near boundaries and grow in time. The pressure-dependent compressibility case exhibits the strongest mismatch, which is consistent with the stronger nonlinear pressure-gradient scaling in that model. The paper also states the limitations of the present computations: density is used locally as a diagnostic variable in the pressure update, total mass is not tracked as a separate diagnostic, and no grid-refinement study was performed in the reported simulations.

Keywords: compressible flow; linear and nonlinear models; pressure-dependent compressibility; numerical simulation; Lax–Wendroff method

2020 MSC: 76N15; 76M20

1. Introduction

The study of compressible fluid dynamics remains a central topic in continuum mechanics and scientific computing. In contrast with incompressible models, compressible models couple the velocity field, density, pressure, and thermodynamic response of the fluid. This coupling produces nonlinear effects that may be small in some parameter regimes but decisive in others, particularly near boundaries or when the equation of state produces a strong pressure-density feedback [1,2].

A common practical question is whether a linearized compressible model is sufficiently accurate for a given computational regime. Linear models are easier to implement and often less expensive to solve, but they may neglect important information associated with the material derivative, density-dependent pressure-force scaling, and pressure-dependent compressibility. Recent numerical investigations of Navier–Stokes and related fluid-flow

systems have emphasized the importance of accurate pressure treatment, reliable spatial and temporal approximation, nonlinear coupling, and stability analysis in reproducing physically meaningful flow behaviour [3–6]. In particular, nonlinear effects may become increasingly important near boundaries, under strong forcing, or when the governing parameters produce transitions between stable, oscillatory, and more complex flow regimes. The usefulness of a linearized model therefore depends not only on the structure of the governing equations, but also on the boundary conditions, parameter values, magnitude of the perturbations from the reference state, and duration of the simulation.

The objective of this paper is to compare linear and nonlinear compressible-flow models under common initial and boundary data. The comparison is intentionally numerical and diagnostic rather than a proof of convergence for a new scheme. We examine the pressure response and the pointwise difference between the two formulations for several configurations. The revised presentation distinguishes clearly between what is computed and what is not computed. In particular, the present simulations use one stated grid resolution and do not include a grid-refinement study. Similarly, density is evaluated locally as a diagnostic quantity needed for the pressure update, but total mass is not integrated or monitored over time.

The paper is organized as follows. Section 2 recalls the compressibility relation and the logarithmic density variable. Section 3 gives the governing nonlinear and linearized equations. Section 4 describes the model configurations. Section 5 treats the pressure-dependent compressibility case and clarifies its linearization. Section 6 gives the numerical discretization and computational parameters, including the CFL estimate. Section 7 presents the numerical comparisons, and Section 8 summarizes the conclusions and limitations.

2. Compressible Fluids

We begin with a brief review of compressible fluids, following closely the presentation given in [7]. We assume no temperature dependence in this work. Let p denote the pressure and ρ the density of the fluid. ρ is an increasing function of p , and there is a one-to-one relation between p and ρ :

$$\rho = \rho(p) \quad \text{and} \quad p = p(\rho). \quad (1)$$

Moreover, the specific volume V of an element of fluid decreases as p increases.

The compressibility $\beta(p)$ is determined by the relative change in V with respect to changes in p , that is

$$-\frac{1}{V} \frac{dV}{dp} = \beta(p) \quad (2)$$

or, since the specific volume $V = 1/\rho$, Equation (2) can be rewritten in terms of ρ to get

$$\frac{1}{\rho} \frac{d\rho}{dp} = \beta(p). \quad (3)$$

See [8] for (2) where the minus sign is chosen so that $\beta(p)$ is positive.

Let p_0 and ρ_0 be reference values for the pressure and density, which are chosen by the investigator and depend on the problem at hand. Integrating (3) we find

$$\ln\left(\frac{\rho}{\rho_0}\right) = \int_{p_0}^p \beta(q) dq. \quad (4)$$

In the sequel, it will be convenient to make use of the function

$$\sigma = \ln\left(\frac{\rho}{\rho_0}\right) \quad (5)$$

which was introduced by [9].

Example 1. $\beta(p) = 0$. Then from (4), $\rho = \rho_0$ for all p , that is, ρ is constant and the fluid is incompressible. We shall only be dealing with the case where $\beta(p)$ is not identically zero.

Example 2. $\beta(p) = \beta$, a constant. This is an important special case, and we find for it:

$$\sigma = \beta(p - p_0) \quad \text{and} \quad \rho = \rho_0 e^\sigma.$$

3. Governing Equations

We work in three dimensions, so $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are points in $\mathbb{R}^3$, and t, τ are time variables. The notation is standard: if $\varphi(x, t)$ is a generic function, then

$$\varphi_t(x, t) = \frac{\partial \varphi}{\partial t}$$

is the ordinary partial derivative of φ with respect to t , and

$$\frac{d}{dt}\varphi(x, t) = \varphi_t(x, t) + (v(x, t) \cdot \nabla)\varphi(x, t)$$

is the material derivative of φ , where ∇ is the gradient operator. The velocity field is denoted by $v = (v_1(x, t), v_2(x, t), v_3(x, t))$, and

$$v \cdot \nabla = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3}.$$

The mathematical foundation for continuum mechanics and the derivation of these fundamental operators can be found in [10, 11].

The derivation of equation of conservation of mass follows the usual pattern and we find as before

$$\rho_t + \nabla(\rho v) = 0.$$

Alternatively, we can write

$$\frac{\rho_t}{\rho} + v \cdot \frac{\nabla \rho}{\rho} + \nabla \cdot v = 0.$$

Using σ from (5), we have $\rho_t/\rho = \sigma_t$ and $\nabla \rho/\rho = \nabla \sigma$, leading to

$$\sigma_t + v \cdot \nabla \sigma + \nabla \cdot v = 0, \quad (6)$$

$$\frac{d\sigma}{dt} + \nabla \cdot v = 0, \quad (7)$$

where $d\sigma/dt = \sigma_t + v \cdot \nabla \sigma$ denotes the material derivative.

The stress tensor must be also modified. It is still symmetric representing conservation of momentum but is now

$$T = -p\mathfrak{I} + 2\mu D + \lambda(\nabla \cdot v)\mathfrak{I}$$

See [12] for a complete derivation. $\mathfrak{I}$ is the identity tensor and $2D = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$ is the rate of deformation tensor. There are two viscosities here, the shear viscosity, μ , and the bulk viscosity, λ . $\mu \geq 0$ and $\lambda + \mu \geq 0$. The equation governing the conservation of linear momentum is

$$\frac{dv}{dt} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta v + \frac{\lambda}{\rho}\nabla(\nabla \cdot v) + F, \quad (8)$$

where ρF represents body forces. Typically, $F = \pm g$, with the sign depending on the orientation of the x_3 -axis.

The $\beta(p)$ has to be determined experimentally.

We begin by assuming that $\beta(p)$ is a constant. The relationship between ρ, p, σ are

$$\sigma = \beta(p - p_0), \quad \rho = \rho_0 e^\sigma.$$

The complete systems to be solved are:

Nonlinear system:

$$\frac{d\sigma}{dt} + \nabla \cdot v = 0, \quad (9)$$

$$\frac{dv}{dt} = e^{-\sigma} \left\{ -\frac{1}{\rho_0 \beta} \nabla \sigma + \frac{\mu}{\rho_0} \Delta v + \frac{\lambda}{\rho_0} \nabla(\nabla \cdot v) \right\} + F. \quad (10)$$

Linear system:

$$\sigma_t + \nabla \cdot v = 0, \quad (11)$$

$$v_t = -\frac{1}{\rho_0 \beta} \nabla \sigma + \frac{\mu}{\rho_0} \Delta v + \frac{\lambda}{\rho_0} \nabla (\nabla \cdot v) + F. \quad (12)$$

These equations must be solved with identical initial and/or boundary conditions.

The linear system is obtained by retaining the first-order terms about the reference state $\sigma = 0$ and by replacing the material derivative by the ordinary time derivative. Thus the terms $v \cdot \nabla \sigma$, $(v \cdot \nabla)v$, and the density multiplier $e^{-\sigma}$ are omitted in the linearized model. This observation is important for interpreting the numerical results: the reported differences measure the combined effect of convective transport and density-dependent rescaling, not the error of the finite-difference method alone.

4. Problem Configurations

4.1. Cauchy Problem in $\mathbb{R}^3$

For the Cauchy problem, we specify initial conditions throughout space:

$$v(x, 0) = u(x), \quad x \in \mathbb{R}^3.$$

If one of the variables ρ , p , or σ is given initially, the others can be determined. For example, if $p(x, 0) = q(x)$, then

$$\sigma(x, 0) = \beta(q(x) - p_0) \quad \text{and} \quad \rho(x, 0) = \rho_0 e^{\beta(q(x) - p_0)}.$$

We solve both systems (9)–(10) and (11)–(12) with the same initial conditions.

4.2. Half-Space Problem with Free Surface

Let $G = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$ and $B = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$. This configuration is used as an idealized model for a flow occupying a region above a flat free surface. The imposed velocity on B should be interpreted as a prescribed kinematic forcing at the surface rather than a full dynamic free-surface model. We solve the systems for v and σ in G , $t > 0$, subject to the conditions:

$$\begin{aligned} v(x, 0) &= u(x), \quad x \in G, \\ p(x, 0) &= q(x), \quad x \in G, \\ v(x_1, x_2, 0, t) &= \varphi(x_1, x_2, t), \quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Note that p is not prescribed on B . The analysis of free surface problems and their well-posedness requires careful mathematical treatment, as discussed in [13].

4.3. Space Problem with Bounded Surface

Let H be a bounded, open, simply connected set in $\mathbb{R}^3$. Initially, both v and σ will be given and on the boundary, say, v will be given and σ will not. That is

- Velocity: $v(x, y, z, t) = 0$ for (x, y, z) on the boundary,
- Sigma: $\nabla \sigma \cdot n = 0$ on the boundary where n is the unit normal.

We will use the same initial conditions as in the Half-space case.

5. Pressure-Dependent Compressibility Model

A second important constitutive choice is a compressibility depending explicitly on pressure,

$$\frac{1}{\rho} \frac{d\rho}{dp} = \frac{1}{\gamma p}, \quad \gamma > 1.$$

Integrating from (p_0, ρ_0) gives

$$\sigma = \ln\left(\frac{\rho}{\rho_0}\right) = \frac{1}{\gamma} \ln\left(\frac{p}{p_0}\right), \quad p = p_0 e^{\gamma \sigma} = \left(\frac{p_0}{\rho_0^\gamma}\right) \rho^\gamma.$$

Substitution in the momentum equation yields the nonlinear pressure force

$$-\frac{1}{\rho}\nabla p = -\frac{p_0\gamma}{\rho_0}e^{(\gamma-1)\sigma}\nabla\sigma.$$

Consequently, the nonlinear system may be written as

$$\frac{d\sigma}{dt} + \nabla \cdot v = 0, \quad (13)$$

$$\frac{dv}{dt} = -\frac{p_0\gamma}{\rho_0}e^{(\gamma-1)\sigma}\nabla\sigma + e^{-\sigma}\left\{\frac{\mu}{\rho_0}\Delta v + \frac{\lambda}{\rho_0}\nabla(\nabla \cdot v)\right\} + F. \quad (14)$$

The corresponding linearized model is obtained by expanding about $\sigma = 0$ and retaining only first-order terms. Since $e^{(\gamma-1)\sigma} = 1 + O(\sigma)$ and products such as $\sigma\nabla\sigma$ are second order, the pressure force becomes $-(p_0\gamma/\rho_0)\nabla\sigma$. Therefore,

$$\sigma_t + \nabla \cdot v = 0, \quad (15)$$

$$v_t = -\frac{p_0\gamma}{\rho_0}\nabla\sigma + \frac{\mu}{\rho_0}\Delta v + \frac{\lambda}{\rho_0}\nabla(\nabla \cdot v) + F. \quad (16)$$

This derivation clarifies that the pressure-dependent model is not merely the constant- β model with a different coefficient. The nonlinear pressure-gradient factor is $e^{(\gamma-1)\sigma}$, whereas in the constant-compressibility case the density feedback enters through $e^{-\sigma}$ multiplying the pressure-gradient coefficient. This stronger nonlinear pressure scaling explains why the pressure-dependent case is expected to show a larger discrepancy when the solution departs from the reference state.

The quantity $c^2 = p_0\gamma/\rho_0$ has the units of velocity squared. In the inviscid, unforced, linearized limit, the system reduces to the acoustic wave equation for σ ,

$$\sigma_{tt} = c^2\Delta\sigma,$$

which provides a useful consistency check for the linearization. The analysis of wave propagation in inviscid compressible fluids and the formation of shocks has been extensively studied in [14]; stability issues for hydrodynamic flows are discussed in [15].

6. Numerical Methods

For the numerical solution of the compressible-flow systems, we use an explicit Lax–Wendroff finite-difference method. The method is second-order accurate for smooth solutions of hyperbolic conservation laws, although the present systems also contain viscous and source terms. The purpose of the implementation is to compare the linear and nonlinear models under the same discretization, initial data, boundary data, and time step.

For a generic balance law

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} + \frac{\partial \mathbf{H}(\mathbf{U})}{\partial z} = \mathbf{S}(\mathbf{U}), \quad (17)$$

the one-dimensional predictor-corrector form used in each coordinate direction is

$$\mathbf{U}_{i+1/2}^{n+1/2} = \frac{1}{2}(\mathbf{U}_i^n + \mathbf{U}_{i+1}^n) - \frac{\Delta t}{2\Delta x}(\mathbf{F}_{i+1}^n - \mathbf{F}_i^n) + \frac{\Delta t}{2}\mathbf{S}_i^n, \quad (18)$$

followed by

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}) + \Delta t \mathbf{S}_i^{n+1/2}. \quad (19)$$

The three-dimensional update is implemented by applying the same finite-difference approximations in the x , y , and z directions. Spatial derivatives in the viscous terms are approximated by centered finite differences. The nonlinear convective terms $v \cdot \nabla\sigma$ and $(v \cdot \nabla)v$ are included only in the nonlinear model; the corresponding linear simulations omit them by construction.

The state variable is taken as $\mathbf{U} = (\sigma, v_1, v_2, v_3)^T$. Pressure is recovered from σ by the relevant equation of state. For constant compressibility, $p = p_0 + \sigma/\beta$. For pressure-dependent compressibility, $p = p_0e^{\gamma\sigma}$. The density $\rho = \rho_0e^{\sigma}$ is used locally as a diagnostic quantity in the pressure and momentum updates. The code used for the reported computations does not integrate total mass over the domain and does not track its variation in time;

therefore, no separate mass-conservation diagnostic is claimed.

The computational box is $\Omega = [0, L]^3$ with $L = 1$ in nondimensional plotting variables. The implemented grid contains

$$n_x = 31, \quad n_y = 31, \quad n_z = 21.$$

The simulations reported here use this single resolution. No finer or coarser grid runs were included in the submitted computational data, and hence no grid-refinement or mesh-convergence claim is made.

The nondimensional parameters used in the displayed simulations are

$$\rho_0 = 1.0, \quad p_0 = 1.0, \quad \mu = 0.01, \quad \lambda = 0.01, \quad \gamma = 1.4, \quad T = 0.1.$$

The constant-compressibility tests use representative values of β , including 1.0, 0.1, and 0.01, as shown in the error plots. These values are not intended to represent a single physical fluid; rather, they are chosen to test the sensitivity of the linearization error to the strength of the pressure-density response.

For the dimensional scaling used in the computational setup, the maximum velocity magnitude is approximately 0.5 m/s, the time increment is $\Delta t = 600$ s, and the representative grid spacing is $\Delta x = 6667$ m. The corresponding Courant number is

$$\text{CFL} = \frac{\max |v| \Delta t}{\Delta x} = \frac{0.5 \times 600}{6667} \approx 0.045. \quad (20)$$

This value is well below one and is consistent with a stable explicit update for the advective part of the computation. The dimensional initial ranges used in the numerical setup are 293.0–305.1 K for temperature and 1.01×10^5 – 2.02×10^7 Pa for pressure. Temperature is not evolved as an independent variable in the present model; it is part of the initialization and physical interpretation, while the pressure-density relation is advanced through σ .

The pointwise error plotted in the comparisons is

$$E_p(x, t) = |p_{\text{NL}}(x, t) - p_{\text{L}}(x, t)|, \quad (21)$$

where p_{NL} and p_{L} denote the pressure obtained from the nonlinear and linearized models, respectively. Along a line $x = x_0, y = y_0$, the displayed profiles show $E_p(x_0, y_0, z, t)$ at selected time levels. When interpreting these figures, boundary and interior behavior should be understood as pointwise pressure-difference behavior along the plotted line, not as a separately computed global L^2 or L^∞ norm over the full grid.

7. Numerical Results and Comparison

The following comparisons use the same numerical scheme and the same data for the linear and nonlinear models. The figures should be read as diagnostic comparisons between two model formulations. Since only one grid resolution was used, the plots do not constitute a mesh-convergence study. The emphasis is on the spatial distribution of the model discrepancy, especially the contrast between the interior and the boundary region.

7.1. Dirichlet Problem

For the Dirichlet problem, the system is initialized with:

$$\begin{aligned} v_1(x, 0) &= \frac{10}{1 + x_1^2 + x_2^2 + x_3^2} e^{-(x_1^2 + x_2^2 + x_3^2)}, \\ v_2(x, 0) &= 10 e^{-(x_1^2 + x_2^2 + x_3^2)}, \\ v_3(x, 0) &= 10 e^{-(x_1^2 + x_2^2 + x_3^2)}. \end{aligned}$$

Figures 1 and 2 present the three-dimensional pressure distributions for the Dirichlet linear and nonlinear models, respectively, at the reported time levels for $\beta = 1.0$. Both solutions exhibit comparable large-scale propagation patterns, while the nonlinear solution retains the density-dependent and convective contributions omitted by the linearized formulation. Such differences are expected to become more visible near boundary-influenced regions, where sharp gradients and numerical boundary treatment can materially affect compressible-flow calculations [16–20].

Figure 3 compares the corresponding pressure profiles along the z -axis at $x = 0.5$ and $y = 0.5$. The two models remain close over much of the interior, indicating that the linearized approximation captures the principal pressure evolution when the state remains close to the reference configuration. This behavior is consistent with the use of linearized or preconditioned formulations in weakly compressible and low-Mach-number regimes [21].

Figure 4 reports the pointwise pressure difference E_p along the same line. The error is spatially nonuniform: it remains comparatively small through much of the interior but increases near the boundary layers generated by the imposed data. This localization is consistent with the recognized sensitivity of high-order compressible-flow computations to boundary closure and boundary reconstruction [19]. The dependence on β can be interpreted through the equation of state and the associated density feedback. In the constant-compressibility case, $\sigma = \beta(p - p_0)$, and the nonlinear momentum equation contains the multiplier $e^{-\sigma}$. Hence, the difference between the nonlinear coefficient and its linearized value is controlled by $|e^{-\sigma} - 1|$. For small $|\sigma|$, $e^{-\sigma} - 1 = -\sigma + O(\sigma^2)$, so the linear and nonlinear models are expected to remain close when the computed state stays near the reference state. The larger boundary errors indicate that this small-perturbation assumption is less accurate in the boundary-influenced portion of the domain. More generally, robust high-order and invariant-domain-preserving methods are designed to control numerical behavior in regions containing strong gradients or restrictive physical constraints [18,20].

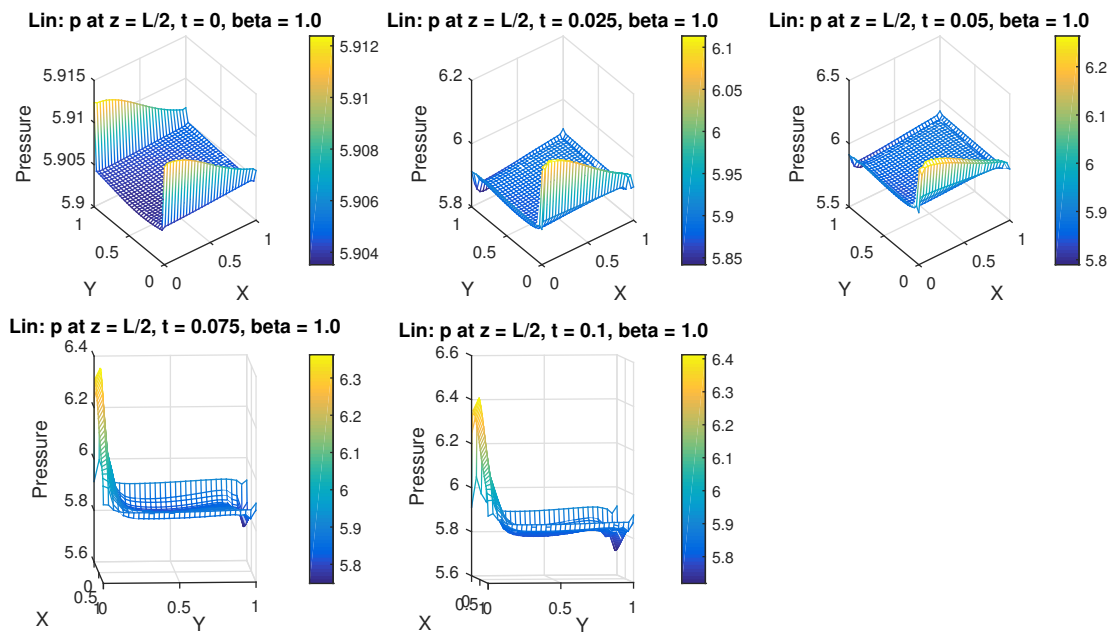


Figure 1. 3-D plot of pressure for the Dirichlet linear case along the z -axis at different time levels with $\beta = 1.0$.

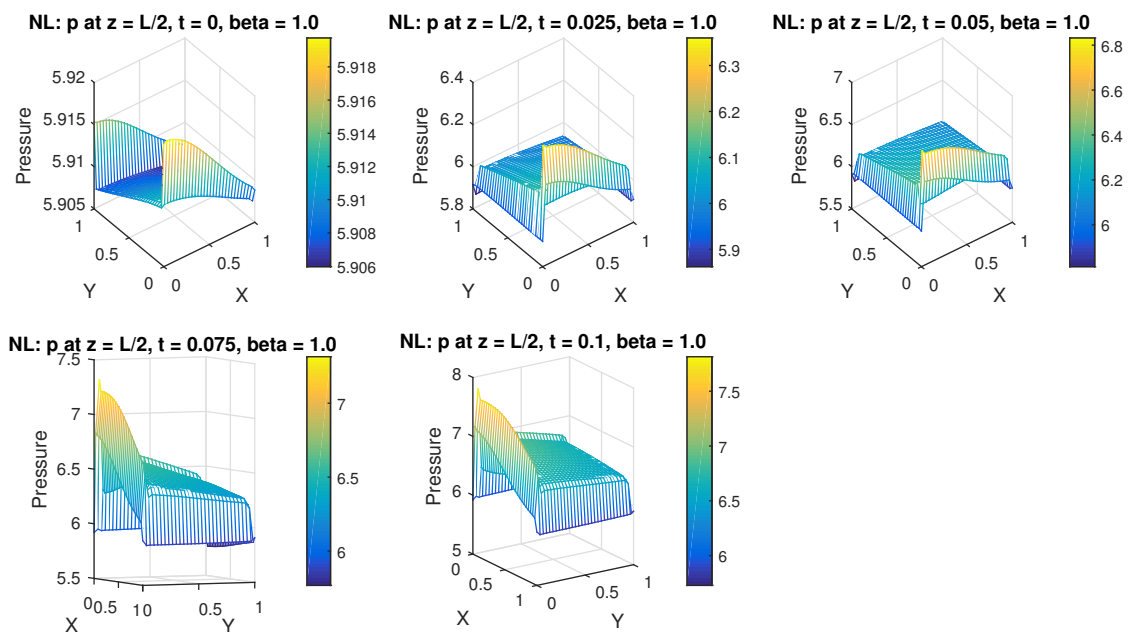
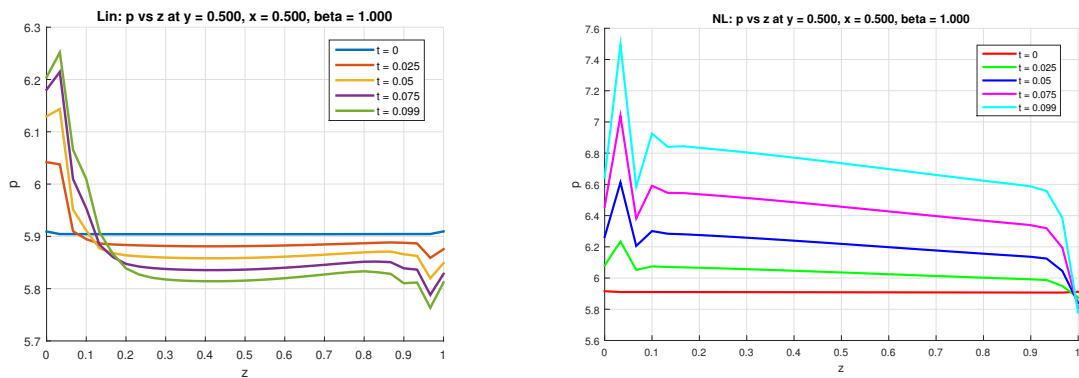


Figure 2. 3-D plot of pressure for the Dirichlet nonlinear case along the z -axis at different time levels with $\beta = 1.0$.



(a) Pressure along z -axis at different time levels with $x = 0.5$ and $y = 0.5$ for the linear case. (b) Pressure along z -axis at different time levels with $x = 0.5$ and $y = 0.5$ for the nonlinear case.

Figure 3. Comparison of pressure evolution along the z -axis for Dirichlet boundary conditions. Both cases show similar wave propagation characteristics with $x = 0.5$ and $y = 0.5$ fixed.

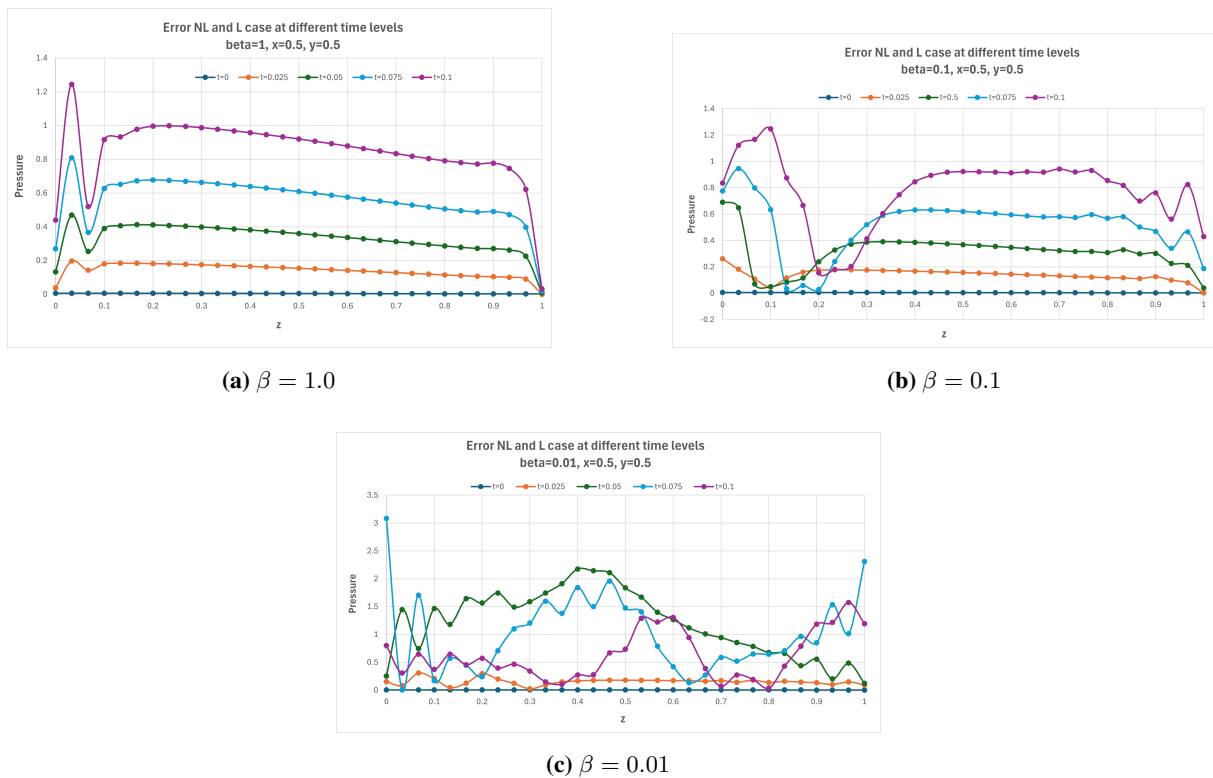


Figure 4. Dirichlet model: Error between linear and nonlinear solutions along the z -axis at different time levels for various β values.

7.2. Half-Space Problem

For the half-space case, the initial conditions are

$$p(x, 0) = p_0 + \rho_0 g x_3, \quad v(x, 0) = 0,$$

and the prescribed velocity components on the free surface are

$$\begin{aligned} v_1(x_1, x_2, 0, t) &= \sin(x_2 t), \\ v_2(x_1, x_2, 0, t) &= \sin(x_1 t), \\ v_3(x_1, x_2, 0, t) &= \cos((x_1^2 + x_2^2)t). \end{aligned}$$

Figures 5 and 6 show the three-dimensional pressure evolution for the half-space linear and nonlinear models, respectively, at $\beta = 1.0$. The prescribed free-surface velocity drives the response from the boundary into the interior.

The two solutions exhibit similar bulk propagation patterns, but differences develop near the driven surface because the nonlinear model retains convective transport and density-dependent coefficients. Accurate boundary treatment is particularly important in this setting because errors introduced at the surface can propagate into the computational domain [19].

As illustrated in Figure 7, the discrepancy between the linear and nonlinear numerical solutions depends on β and on the distance from the driven surface. The largest pointwise pressure differences occur near the part of the domain most directly affected by the imposed surface motion, whereas the interior differences remain smaller over the reported time interval. Thus, for these data, the linearized model reproduces the bulk pressure response more accurately than the near-boundary response. This conclusion is consistent with the small-perturbation character of linearized compressible-flow models and with the need for specialized treatment in low-Mach-number or boundary-dominated calculations [19,21].

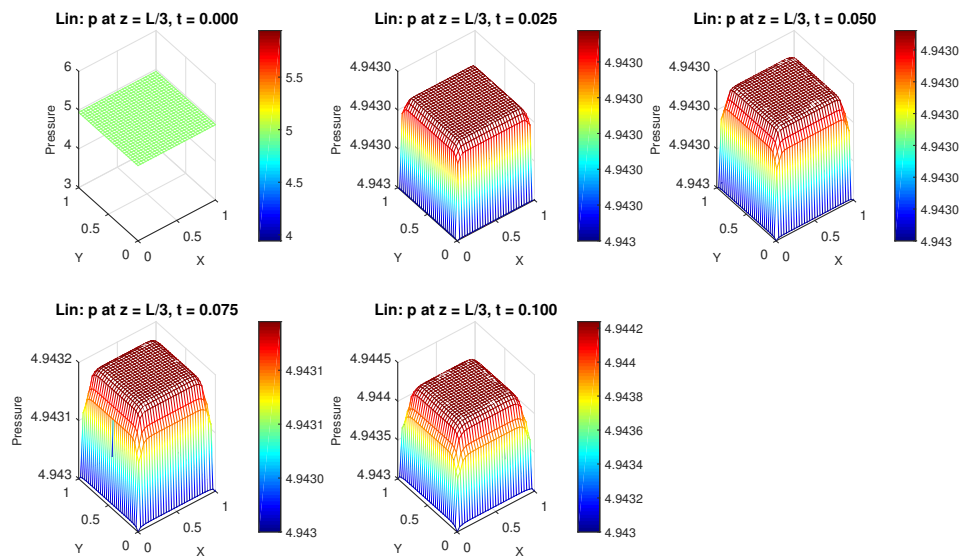


Figure 5. 3-D plot of pressure for the Half-Space linear case along the z -axis at different time levels with $\beta = 1.0$.

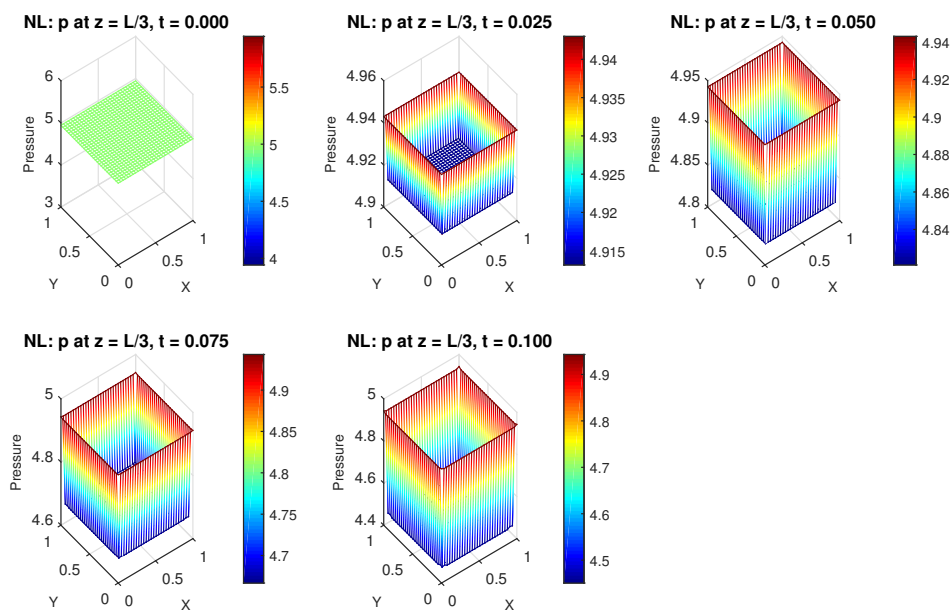


Figure 6. 3-D plot of pressure for the Half-Space nonlinear case along the z -axis at different time levels with $\beta = 1.0$.

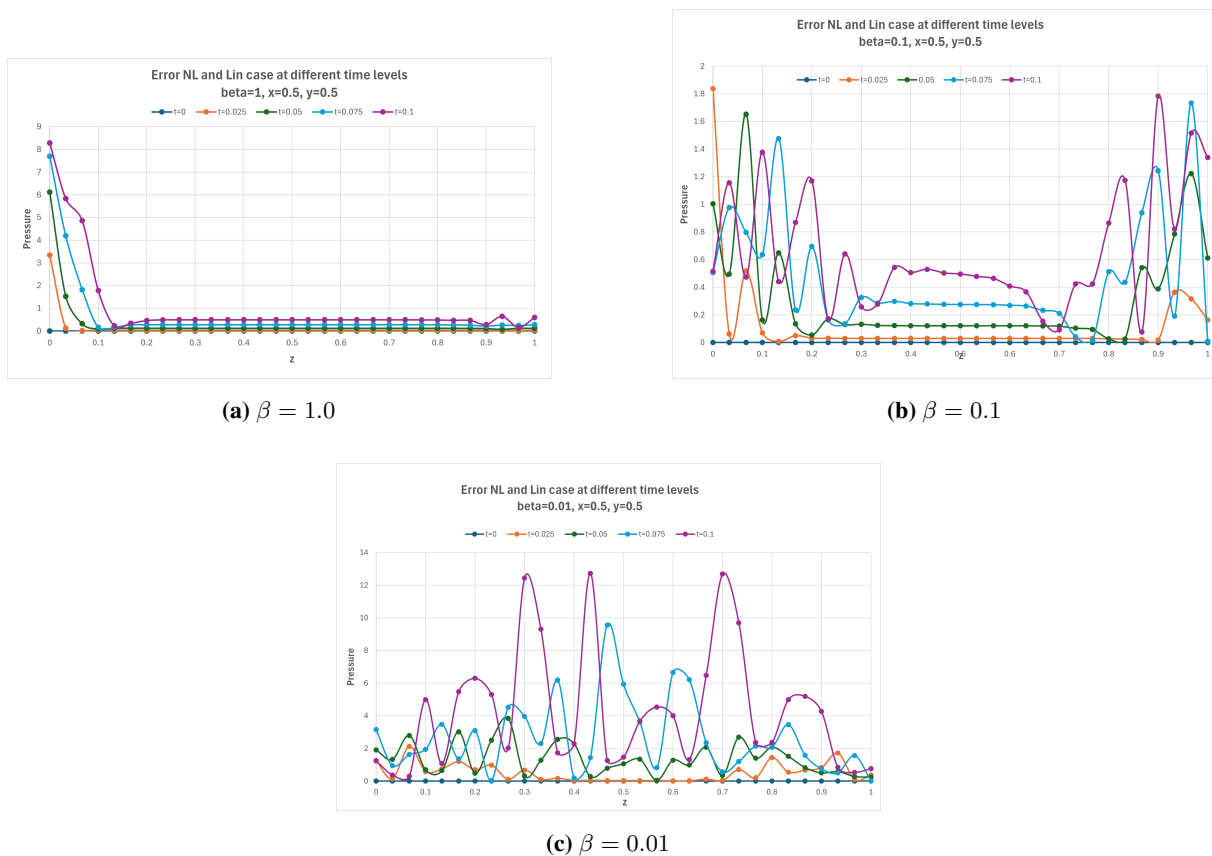


Figure 7. Half-Space model: Error between linear and nonlinear solutions along the z -axis at different time levels for various β values.

7.3. Bounded Domain

This section presents numerical results for the bounded-domain configuration using the same initial conditions as in the half-space problem.

Figures 8 and 9 display the three-dimensional pressure evolution for the bounded-domain linear and nonlinear models, respectively, at $\beta = 1.0$. The confinement imposed by the boundary conditions modifies the propagation and reflection of the pressure response. Although the two formulations remain qualitatively similar in the interior, the nonlinear model incorporates the full density feedback and convective effects and therefore develops more visible differences near the boundary-constrained region.

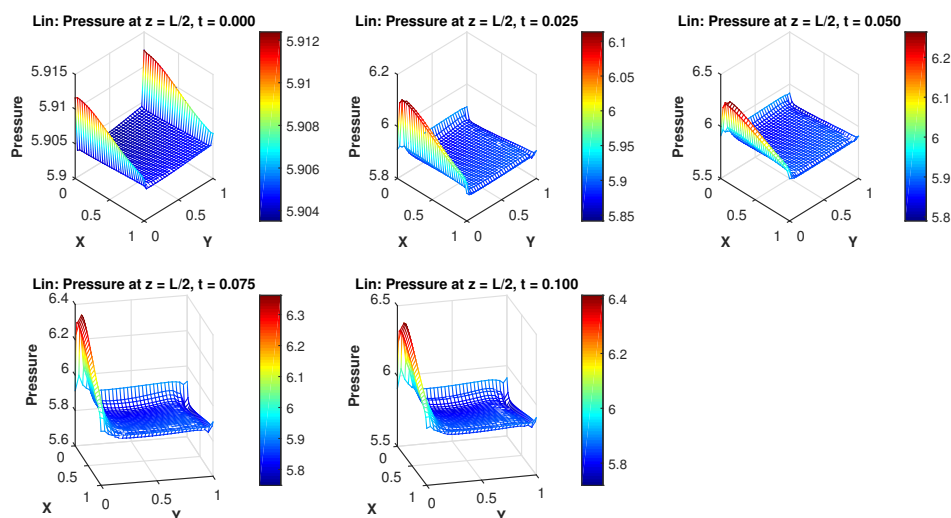


Figure 8. 3-D plot of pressure for the bounded domain linear case along the z -axis at different time levels with $\beta = 1$.

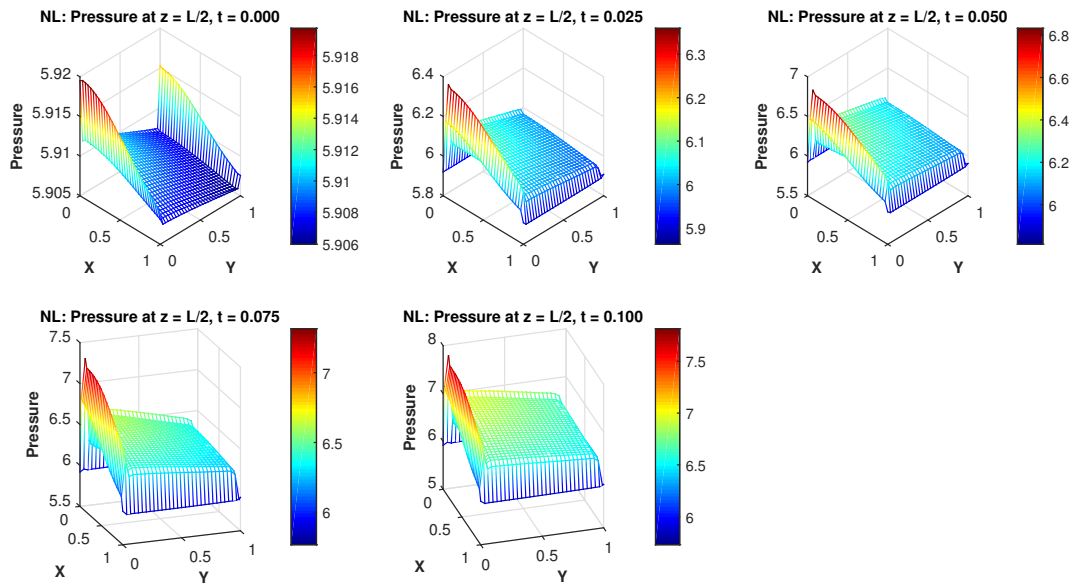


Figure 9. 3-D plot of pressure for the bounded domain nonlinear case along the z -axis at different time levels with $\beta = 1$.

Figure 10 confirms this boundary sensitivity. The homogeneous velocity condition and the normal-gradient condition for σ restrict the response near ∂H , where the pointwise pressure discrepancy becomes most pronounced. The interior remains comparatively well approximated by the linearized model during the simulated interval. This observation should not be interpreted as a global conservation result, because total mass was not evaluated as a separate diagnostic in the present implementation. In future extensions, invariant-domain-preserving and conservative discretizations may provide useful additional safeguards for compressible Euler-type calculations [20].

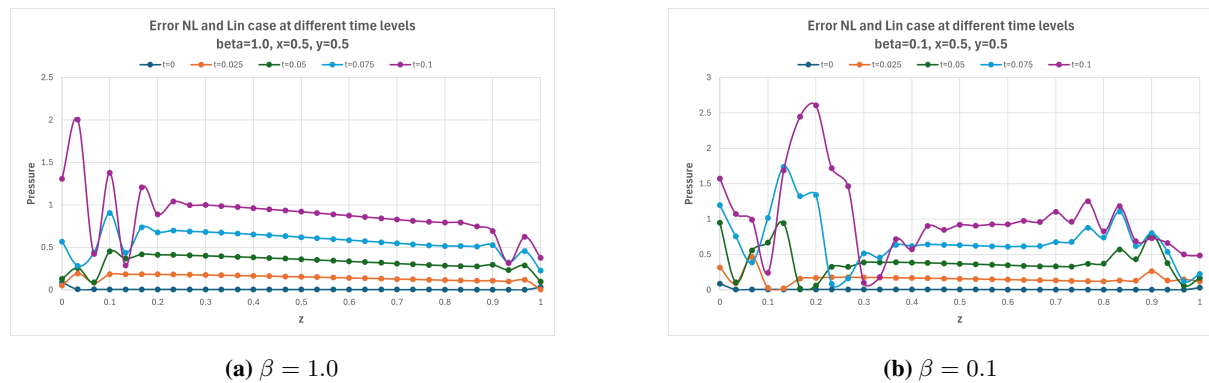


Figure 10. Bounded Domain model: Error between linear and nonlinear solutions along the z -axis at different time levels for various β values.

7.4. Compressibility Pressure Dependence Flow Model

This section presents numerical results for the pressure-dependent compressibility model using the same initial and boundary conditions as in the half-space configuration.

Figures 11 and 12 present the three-dimensional pressure fields for the linear and nonlinear pressure-dependent compressibility models, respectively. The qualitative difference between the two fields is stronger than in the constant-compressibility cases because the nonlinear pressure-gradient term contains the factor $e^{(\gamma-1)\sigma}$, whereas the linear model replaces this factor by unity and omits the convective terms.

As shown in Figure 13, this model produces the largest linear–nonlinear discrepancy among the reported configurations. The boundary amplification should therefore be attributed to the combined influence of nonlinear pressure scaling, density feedback, boundary-driven transport, and the numerical treatment of boundary gradients. Similar sensitivity motivates the development of high-order adaptive schemes and stable boundary procedures for compressible flows [18,19]. For this case, the linearized formulation is reliable only as a small-perturbation

approximation near the reference state. Once stronger boundary-induced pressure variations develop, the nonlinear formulation is required to represent the computed dynamics more faithfully.

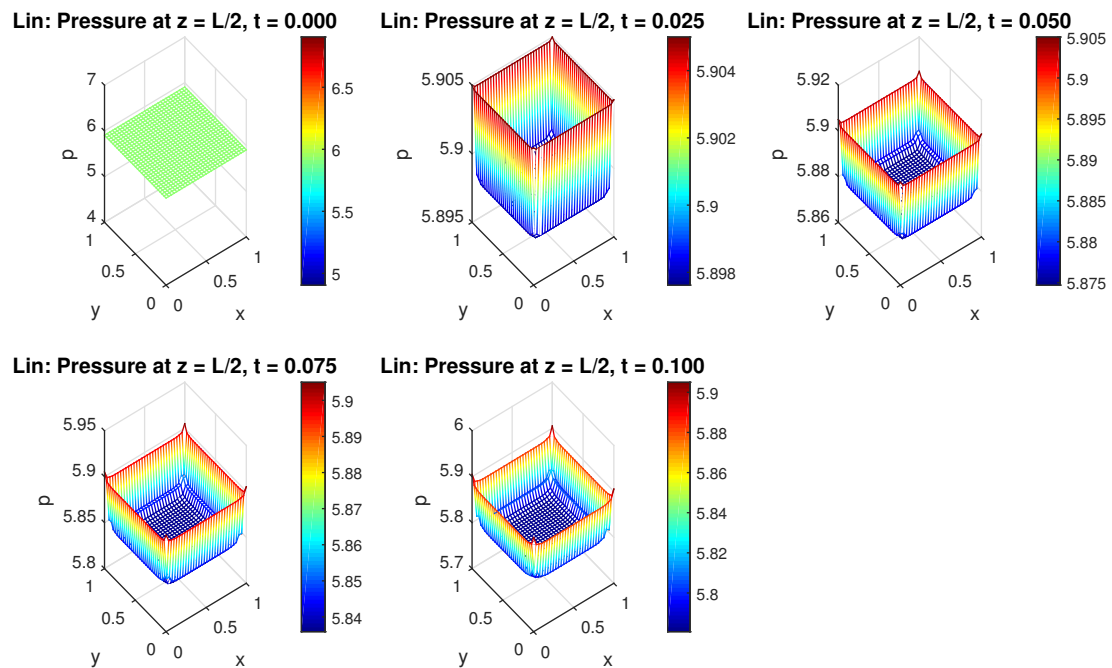


Figure 11. 3-D plot of pressure for the pressure-dependent compressibility model linear case along the z -axis at different time levels.

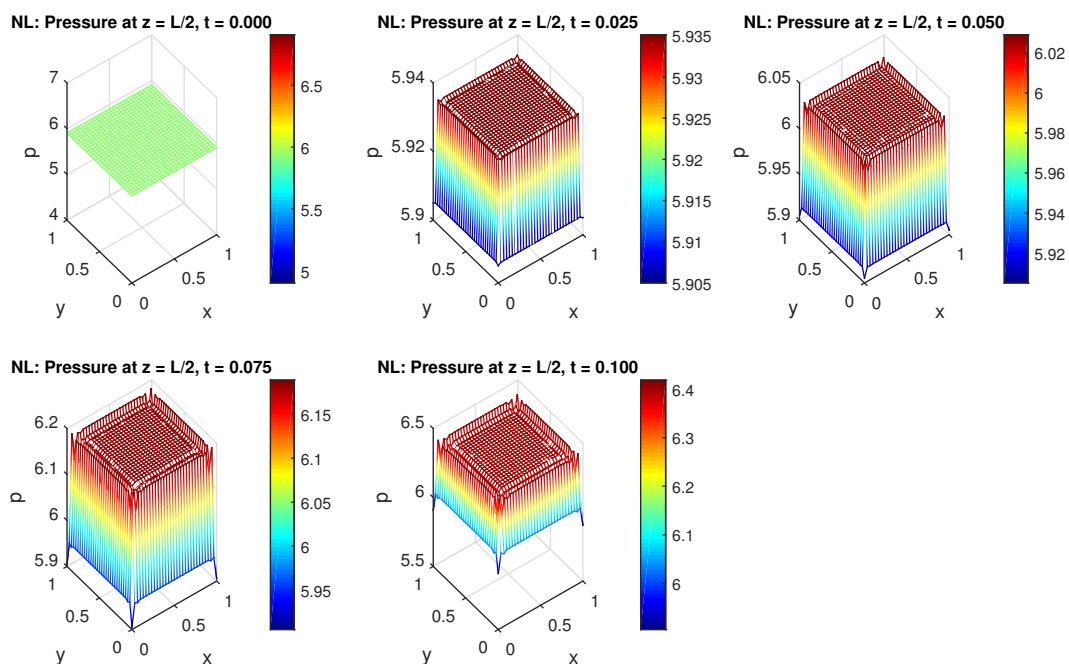


Figure 12. 3-D plot of pressure for the pressure-dependent compressibility model nonlinear case along the z -axis at different time levels.

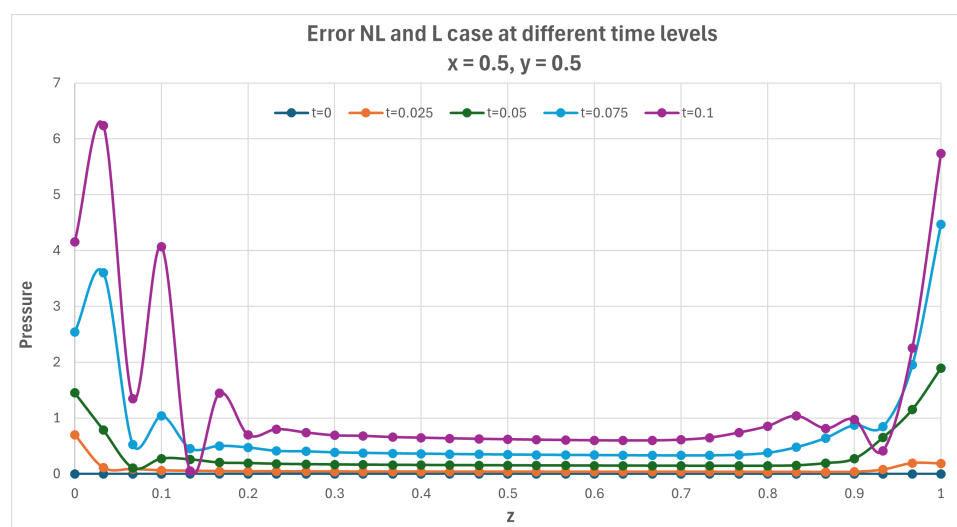


Figure 13. Dependence flow model: Error between linear and nonlinear numerical solutions along the z -axis at different time levels

8. Conclusions

This paper compared linearized and nonlinear compressible-flow models under identical computational settings. The simulations indicate that the linearized equations can approximate the nonlinear pressure response in the interior of the domain for the reported regimes, especially when the solution remains close to the reference state. The largest discrepancies occur near boundaries, where imposed or constrained boundary data interact with convective transport and density-dependent pressure scaling.

The analysis also clarifies the role of the nonlinear coefficients. In the constant-compressibility model, the main density feedback enters through the multiplier $e^{-\sigma}$ in the momentum equation. In the pressure-dependent compressibility model, the nonlinear pressure-gradient scaling is stronger, through $e^{(\gamma-1)\sigma}$. This explains why the pressure-dependent case exhibits a more pronounced mismatch between the linear and nonlinear computations.

The conclusions are intentionally limited to the diagnostics actually computed. The present code uses density locally for the pressure and momentum updates but does not monitor total mass. The simulations were performed at one grid resolution, $31 \times 31 \times 21$, and therefore no grid-refinement or convergence-rate claim is made. Future work should include systematic grid refinement, global error norms, conservation diagnostics, and a more detailed comparison between the factor $e^{-\sigma}$ or $e^{(\gamma-1)\sigma}$ and its linearized value over the full computational domain.

Author Contributions

M.M.: conceptualization, methodology, theoretical development, formal analysis, software, numerical analysis, validation, visualization, writing—original draft preparation, writing—review and editing, supervision, and project administration; N.S.: methodology, formal analysis, numerical investigation, validation, and writing—review and editing; A.K.: methodology, theoretical analysis, validation, and writing—review and editing; R.B.G.: theoretical analysis, methodology, validation, and writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Funding

This research received no external funding.

Institutional Review Board Statement

Not applicable.

Informed Consent Statement

Not applicable.

Data Availability Statement

Not applicable.

Acknowledgments

The first and third authors acknowledge computational resources provided by Zayed University.

Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

References

1. Batchelor, G.K. *An Introduction to Fluid Dynamics*; Cambridge University Press: Cambridge, UK, 2000.
2. Landau, L.D.; Lifshitz, E.M. *Fluid Mechanics*, 2nd ed.; Pergamon Press: Oxford, UK, 1987.
3. Mohammad, M.; Trounev, A. An Advanced Algorithm for Solving Incompressible Fluid Dynamics: From Navier–Stokes to Poisson Equations. *Eur. Phys. J. Spec. Top.* **2025**, *234*, 2191–2208.
4. Mohammad, M.; Trounev, A.; Kumar, S. High-Precision Euler Wavelet Methods for Fractional Navier–Stokes Equations and Two-Dimensional Fluid Dynamics. *Phys. Fluids* **2024**, *36*, 127143.
5. Mohammad, M.; Trounev, A. Numerical Simulation and Nonlinear Dynamics in Rotating Magnetoconvection: Chaos, Attractors, and Stability Transitions. *Chaos Solitons Fractals* **2025**, *201*, 117219.
6. Mohammad, M.; Trounev, A.; Alshbool, M.; et al. An Innovative Algorithm Based on Wavelets for Solving Three-Dimensional Nanofluid Bio-Convection Model Near a Stagnation Point. *Results Phys.* **2022**, *41*, 105889.
7. Guenther, R.B.; Roetman, E.L. *Introduction to the Potential Theory for the Time Dependent Stokes System*; Chapter 15; CRC Press/Taylor & Francis Group: Boca Raton, FL, USA, 2026.
8. Greiner, W.; Neise, L.; Stöcker, H. *Thermodynamics and Statistical Mechanics*; Springer-Verlag: New York, NY, USA, 1995.
9. Weizel, W. *Lehrbuch der Theoretischen Physik*, 2nd improved ed.; Springer-Verlag: Berlin, Germany, 1955.
10. Gurtin, M.E. *An Introduction to Continuum Mechanics*; Academic Press: Boston, MA, USA, 1982.
11. Temam, R. *Navier-Stokes Equations: Theory and Numerical Analysis*; American Mathematical Society: Providence, RI, USA, 2001.
12. Serrin, J. Mathematical Principles of Classical Fluid Mechanics. In *Handbuch der Physik*; Flügge, S., Ed.; Springer-Verlag: Berlin, Germany, 1959; pp. 125–263.
13. Beale, J.T. Large-Time Regularity of Viscous Surface Waves. *Arch. Ration. Mech. Anal.* **1984**, *84*, 307–352.
14. Flügge, S. *Practical Quantum Mechanics*; Springer: Berlin/Heidelberg, Germany, 1999.
15. Drazin, P.G. *Introduction to Hydrodynamic Stability*; Cambridge University Press: Cambridge, UK, 2002.
16. Majda, A. *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*; Springer-Verlag: New York, NY, USA, 2012.
17. Lions, P.L. *Mathematical Topics in Fluid Mechanics: Volume 2: Compressible Models*; Oxford University Press: Oxford, UK, 1996.
18. Ziegler, J.L.; Deiterding, R.; Shepherd, J.E.; et al. An adaptive high-order hybrid scheme for compressive, viscous flows with detailed chemistry. *J. Comput. Phys.* **2011**, *230*, 7598–7630.
19. Li, T.; Shu, C.-W.; Zhang, M. Stability analysis of the inverse Lax–Wendroff boundary treatment for high-order upwind-biased finite-difference schemes. *J. Comput. Appl. Math.* **2016**, *299*, 140–158.
20. Clayton, B.; Guermond, J.-L.; Popov, B. Invariant Domain-Preserving Approximations for the Euler Equations with Tabulated Equation of State. *SIAM J. Sci. Comput.* **2022**, *44*, A444–A470.
21. Tweedt, D.L.; Chima, R.V.; Turkel, E. Preconditioning for Numerical Simulation of Low Mach Number Three-Dimensional Viscous Turbomachinery Flows. In Proceedings of the 28th AIAA Fluid Dynamics Conference, Snowmass Village, CO, USA, 29 June–2 July 1997.