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Finite-Time Stability and Fixed-Time Stability of Stochastic Delay System with Lévy Noise

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Abstract: This paper investigates the finite-time and fixed-time stability of stochastic delay differential equations (SDDEs) driven by Lévy noise, employing Itô's formula and the stopping-time technique. First, a novel inequality inspired by the Bihari inequality is derived, and a sufficient condition for finite-time stability of SDDEs under Lévy noise is established. Second, by integrating the Razumikhin method with Lyapunov function theory, we establish Razumikhin-type theorems that guarantee fixed-time stability and provide an explicit upper bound for the settling time. Finally, an example is given to verify the correctness of the theoretical results.

Keywords: finite-time stability; fixed-time stability; stochastic delay differential equation; time-varying delay; Lévy noise

1. Introduction

Over the past several decades, the stability research of stochastic differential equations (SDEs) have already become a hot spot and have received extensive attention from mathematical researchers [1,2]. Among the various types of stability, finite-time stability has drawn significant attention [3–5]. Unlike the stability in probability, and the asymptotic stability, which primarily describe the asymptotic behavior of stochastic systems as time approaches infinity, Finite-time stability focuses on whether the state of a system can reach and remain within an equilibrium or a desired region within a finite period. This type of stability offers practical advantages—including predictability and robustness to uncertainties—making it valuable for both theoretical research and real-world applications [6–8].

The finite-time stability of SDEs and the corresponding Lyapunov criteria were systematically introduced in [3,4], providing a new theoretical path for solving the stability problems of stochastic systems. Subsequently, Yin et al. [9] extended the research to the dynamic characteristics of SDEs and further improved the theoretical system of finite-time stability for SDEs. Later, Yu et al. [10,11] proposed a new theoretical perspective in this field, presenting an innovative method for stability analysis and expanding the application conditions of the differential operator $\mathcal{L}V$. Recently, Yuan et al. [12] proposed a fast finite-time stability theorem, significantly reducing the convergence time by improving the Lyapunov criteria, and achieving good application results in the control of stochastic nonlinear systems.

Fixed-time stability, introduced by Andrieu et al. [13], represents a stronger version of finite-time stability. In this framework, the settling time function is characterized by having a uniformly bounded upper limit. In contrast to finite-time stability, the upper bound estimate of the settling time function here is independent of the system's initial conditions, leading to improved convergence speed and enhanced control precision. Further advancements in the areas of finite-time stability and fixed-time stability can be found in studies including [5] and the references cited therein.

The presence of time delays is an unavoidable characteristic in real-world applications, particularly in engineering systems Time-delay is frequently encountered in theoretical research and practical issues. Usually, Stochastic delay differential equations (SDDEs) are widely applied in domains such as communication networks and biomedical engineering. In numerous stability analyses, assumed that $\tau(t)$ is differentiable and satisfies the derivative constraint $0 < \tau'(t) < 1$ (see references [14–16]). However, this condition is overly strict in practical applications, as many time-delay functions do not satisfy the differentiability condition. To address this issue, Dong and Mao et al. proposed new assumptions for the time-delay function in [17], which is weaker than the traditional differentiability assumption. This implies that more non-differentiable time-delay functions can be applied.

To make our research more proximate to the real models, it is essential to consider the impact of Lévy noise [18,19]. When there exist considerable external disturbances like the fluctuations in the financial market, random noise might present jumping phenomena, which cannot be precisely depicted by Brown motion [2,20]. Lévy noise is commonly employed to model random disturbances characterized by jumps [2,21]. However, the discontinuity of its sample paths [22] significantly increases the complexity of the analysis.

Although finite-time stability results for SDEs have been established, see [23]. Stability results that account for the effects of Lévy noise and time delays have yet to be developed. This situation results in a limited availability of mathematical tools for conducting stability analysis on certain stochastic systems that exhibit delay effects in engineering control applications.

This study aims to tackle this challenge and establishes new criteria for finite-time and fixed-time stability. In comparison to earlier works, the contributions of this paper are highlighted in the following three key aspects.

(i) Compared with the SDDEs without jump terms, the SDDEs with Lévy noise have stronger universality and a broader application background. Therefore, extending the finite-time stability theory from SDDEs without Lévy noise to the case including Lévy noise is of great theoretical significance.

(ii) Inspired by the works of [1,24], the time-delay function we consider does not need to satisfy the differentiability condition. In [2, 17], Dong and Mao highlighted that the time-delay function, despite being the simplest piecewise function, fails to satisfy the conventional time-delay requirements. (that is, the time-delay function is differentiable and its derivative is less than one).

(iii) This paper presents two criteria for finite-time stability. The first criterion can ensure that the SDDEs is finite-time stable. The second criterion improves the existing Razumikhin-type theorem and generalizes it to address the fixed-time stability of SDDEs.

To clearly illustrate the limitations of existing methods and highlight our contributions, Table 1 contrasts representative stochastic finite/fixed-time stability criteria with our proposed result.

Table 1. Comparison of stochastic finite/fixed-time stability criteria.

Study	**Assumption on $\mathcal{L}V(\cdot)$**	**Result Type**	**Convergence Time Bound**
Yin et al. [3]	The Lyapunov function $V(x)$ satisfies $\mathcal{L}V \leq -c(V(x))^{\gamma}, \gamma \in (0,1)$	Finite-time stability in probability	Exists a sequence $\mathbb{E}[T_k] \leq \frac{V(x_0)^{1-\gamma}}{C_k(1-\gamma)}$
Min et al. [6]	Exist a continuous differentiable function $\gamma(v) > 0$ and $V(x)$ satisfies $\mathcal{L}V \leq -\gamma(V(x))$	Generalized Fixed-time stability in probability	$\mathbb{E}[T] \leq M$, where M satisfies $\int_0^r \frac{1}{\gamma(s)} ds \leq M$ for any $0 < r < \infty$
Wang et al. [25]	The continuously differentiable function U_2 satisfies $\mathcal{L}_\sigma U_2(x) \leq -k_\sigma$, where $k_\sigma > 0$ is a finite constant.	Finite-time stability in probability	The settling-time satisfies $\mathbb{E}^x(\xi_0^x) \leq \frac{U_2(x_0)}{k}$, where $k = \min_{\sigma \in \mathcal{N}} k_\sigma$.
This paper (Theorem 1)	The Lyapunov function $V(x)$ satisfies $\mathcal{L}V(x,y,t) \leq -a(V(x,t))^d - bV(y, t+\tau(t))$	Finite-time stability in probability	The settling-time $\mathbb{E}[T_\xi] \leq \frac{\ln(1+\frac{b\overline{\tau}}{ac_1^{d-1}}\|\|\xi\|\|^{p(1-d)})}{(1-d)b\overline{\tau}}$

The structure of the remaining part of this paper is as follows: Section 2 presents the necessary symbol definitions and basic assumptions for the research; Section 3 proposes two core judgment criteria for evaluating finite-time stability and fixed-time stability; Section 4 verifies the main theoretical results through a numerical simulation example; finally, Section 5 summarizes and concludes the research content of the entire paper.

2. Preliminaries

In this paper, $\mathbb{R}$ denotes the set of all realnumbers, $\mathbb{R}^d$ represents the d-dimensional Euclidean space, $\mathbb{R}_+$ represents the set of nonnegative real-numbers, $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^d$. $\mathbb{R}^{m\times n}$ denotes the space of $m \times n$ matrices with real entries. For a matrix X, $X^{\mathbb{T}}$denotes the transpose of a vector or matrix, $|X| = [trace(X^{\mathbb{T}}X)]^{\frac{1}{2}}$. $\mathcal{K}$ denotes s the set of all functions that are strictly increasing, continuous and vanish at the origin. $\mathbb{C}([-\tau, 0]; \mathbb{R}^d)$ represents the set of all continuous functions ϕ mapping from the interval $[-\tau, 0]$ to $\mathbb{R}^d$

equipped with the uniform norm defined as $||\phi|| = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$. $L^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^d)$ refers to the family of all $\mathcal{F}_0$ measurable, $\mathbb{C}([-\tau,0];\mathbb{R}^d)$-valued stochastic variables ϕ with $\mathbb{E}||\phi||^P < \infty$. $L^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^d)$ refers to the collection of all $\mathcal{F}_0$-measurable stochastic processes φ taking values in $C([-\tau,0];\mathbb{R}^d)$, such that $\mathbb{E}\|\varphi\|^p < \infty$.

As with the works in [26–28], we also consider the SDDEs without large jumps,

$$\begin{aligned} dx(t) =& f(x(t), x(t+\tau(t)), t)dt + g(x(t), x(t+\tau(t)), t)dB(t) \\ &+ \int_{|z|<c} H_1(x(t)^-, x(t+\tau(t))^-, t, z)\tilde{N}(dt, dz) \;\; x_{t_0} = \xi, \;\; t \in [0,\infty]. \end{aligned} \tag{1}$$

where $-\tau \leq \tau(t) \leq 0$, $x(t)^- = \lim_{s\to t^-} x(s)$, $x_0 = \xi = \{\xi(\theta), -\tau \leq \theta \leq 0\} \in L^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^d)$, the mappings $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$, denote the space of all real-valued $d \times m$ matrices, $H_1 : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d$, and the constant $c \in (0,\infty]$ represents the upper bound on the allowable jump magnitude. $\tau(t)$ satisfies $-\tau \leq \tau(t) \leq 0$. $B(\cdot)$ represents a Brownian motion. Let N be a Poisson random measure with the compensating measure $\tilde{N}$ and intensity measure ν. Here, ν is a Lévy measure satisfying the relation $\tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)$ and the integral condition $\int_{\mathbb{R}^d\setminus\{0\}}(|z|^p \wedge 1)\nu(dz) < \infty$. Usually, the pair (B,N) is called a Lévy noise, the functions f, g are called drift and diffusion coefficients respectively. $\int_{|z|<c} H_1(x(t)^-, x(t+\tau(t))^-, t, z)\tilde{N}(dt,dz)$ is called "small jump". More detail see [18].

Assume f, g, H_1 are continuous coefficients. Moreover, $f(0,0,t) = 0$, $g(0,0,t) = \mathbf{0}$, $H_1(0,0,t,z) = 0 \forall t \in \mathbb{R}_+$. This indicates that the stochastic differential equation (SDE) (1) admits a trivial solution at zero. As noted in Yin et al. [4], a stochastic nonlinear system that exhibits finite-time stability typically does not possess both locally Lipschitz continuous coefficient functions f and g. Therefore, when investigating finite-time stochastic stability, it is sufficient to assume the existence of solutions for such systems. For further details, refer to [29].

For simplicity, denote $x(t)$ as x and $x(t+\tau(t))$ as y. Let $\mathbb{C}^{2,1}(\mathbb{R}^d \times [-\tau,\infty);\mathbb{R}_+)$ represent the class of all nonnegative functions $V(x,t)$ defined on $\mathbb{R}^d \times [-\tau,\infty)$, such that they are twice continuously differentiable with respect to x and once differentiable with respect to t. For Lyapunov function $V(x,t) \in \mathbb{C}^{2,1}(\mathbb{R}^d \times [-\tau,\infty);\mathbb{R}_+)$, the operator $\mathcal{L}V(x,y,t)$, from $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$ to $\mathbb{R}$

$$\begin{aligned} \mathcal{L}V(x,y,t) &= V_t(x,t) + V_x(x,t)f(x,y,t) + \frac{1}{2}trace[g^{\mathbb{T}}(x,y,t)V_{xx}(x,t)g(x,y,t)] \\ &\quad + \int_{|z|<c}[V(x + H_1(x,y,t,z),t) - V(x,t) - H_1(x,y,t,z)V_x(x,t)]\nu(dz), \end{aligned}$$

where

$$\begin{aligned} &V_t(x,t) = \frac{\partial V(x,t)}{\partial t}, V_x(x,t) = (\frac{\partial V(x,t)}{\partial x_1}, \cdots, \frac{\partial V(x,t)}{\partial x_d}), \\ &V_{xx}(x,t) = (\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j})_{d\times d}. \end{aligned}$$

(A1) Assume the function$\tau(\cdot)$ is Borel measurable,

$$-\tau' := \inf_{0\leq t<\infty}(t+\tau(t)) > -\infty,$$

and

$$\overline{\tau} := \limsup_{\triangle\to 0^+} \sup_{s\geq -\tau'} \frac{Leb(I_{s,\triangle})}{\triangle} < \infty,$$

where $Leb(I_{s,\triangle})$ is the Lebesgue measure of the set $I_{s,\triangle} = \{t \in \mathbb{R}_+ : t+\tau(t) \in [s, s+\triangle)\}$.

(A2) $\mathbb{E}|x(t;\xi)|^p$ is continuous with respect to $t \geq 0$.

Lemma 1. *(see [24], Lemma 2.7) If the Assumption$(A1)$ hold, and that exhibits at most a finite number of jumps within any bounded time interval, there exist $\Psi(t) : [-\tau',\infty) \to \mathbb{R}_+$ be a càdlàg function with at most a finite number of jumps over any finite time interval, satisfies*

$$\int_0^T \Psi(T+\tau(t))dt < \overline{\tau}\int_{-\tau'}^T \Psi(t)dt, \; \forall T > 0.$$

Lemma 2. *(see[12], [Lemma 3]) If exists a right-continuous function $h(\cdot)$, the following condition holds:*

$$h(t) - h(u) \leq -\int_u^t ah(s)^d + bh(s)ds, \ \forall 0 \leq u \leq t. \tag{2}$$

where a, $b > 0$ and $0 < d < 1$. Then there exists $T > 0$, such that

$$h(t) \leq [h(0)^{1-d} + \frac{a}{b}(1 - e^{(1-d)bt})]^{\frac{1}{1-d}} e^{-bt}, \ \ \forall t \in [0, T].$$

Remark 1. *In [12], it was demanded that the $h(\cdot)$ in inequality (2) satisfy the continuous condition. However, through the analysis of the proof process, we noticed that the continuous condition could be weakened to the right-continuous condition, and the result still remained valid.*

Definition 1. *(Finite-time stability [3]). The SDDEs (1) is said to be finite-time stability, if $\forall\ \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d)$, have:*

(i) Finite-time attractiveness in probability: $\forall\ \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d)$, $T_\xi = \inf\{t \geq 0; x(t; \xi) = 0\}$ is finite a.s., meaning that $P\{T_\xi < \infty\} = 1$.

(ii) Stability in probability: There exists a positive constant $\delta = \delta(\varepsilon, r)$ for all$\varepsilon \in (0, 1)$ and $r > 0$, it follows that $P\{|x| < r, \ \forall t \geq 0\} \geq 1 - \varepsilon$, where $||\xi|| < \delta$.

Definition 2. *(Fixed-time stability [30]). The solution of SDDEs (1) is said to be fixed-time stable if it is finite-time stable and there exist a positive constant $T_{\max}$ such that $\sup_{\xi \in L^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^d)} T(\xi) \leq T_{\max} < \infty$.*

Now, we introduce a few notations for the next lemma regarding probability stability. Choose a constant $h > 0$ satisfying $h \geq 2c$, and define the set $S_h = \{x \in \mathbb{R}^d : |x| < h\}$ Fix $h > 0$ such that $h \geq 2c$, and $S_h = \{x \in \mathbb{R}^d : |x| < h\}$.

Lemma 3. *If there exist $V(x, t)$, such that*

$$\mathcal{L}V(x, y, t) \leq 0,$$

is satisfied for all $(x, t) \in S_h \times [-\tau, \infty)$, then the SDDEs (1) is stable in probability.

Proof. Note that $V(x, t)$ is positive definite function, There exists a function $\mu(\cdot)$,

$$V(x, t) \geq \mu(|x|), \ \forall (x, t) \in S_h \times [-\tau, \infty).$$

Let $\varepsilon \in (0, 1)$, $r > 0$. Without loss of generality, Suppose that $0 < r < \frac{h}{2}$. Since V is continuous and satisfies $V(0, 0) \equiv 0$, it follows that for any $\varepsilon > 0$, there exists a corresponding $\delta = \delta(\varepsilon, r) > 0$, $\forall \varepsilon > 0$, such that

$$\sup_{x \in S_\delta} V(x, 0) \leq \mu(r)\varepsilon. \tag{3}$$

Now fix the initial value ξ arbitrarily, with $||\xi|| \in S_\delta$, Define the stopping time

$$\alpha = \inf\{t \geq 0 : |x(t)| \geq r\},$$

which is the first exit time of $x(t)$ from S_r. Applying Itô's formula,

$$\begin{aligned}
& V(x(t \wedge \alpha), t \wedge \alpha) \\
= \ & V(x(0), 0) + \int_0^{t\wedge\alpha} \mathcal{L}V(x(s)^-, s)ds + \int_0^{t\wedge\alpha} V_x(x(s), s)g(x(s), x(s + \tau(s)), s)dB(s) \\
& + \int_0^{t\wedge\alpha} \int_{|z|<c} [V(x(s)^- + H(x(s)^-, x(s + \tau(s))^-, s, z), s) - V(x(s)^-, s)]d\tilde{N}(ds, dz).
\end{aligned}$$

Obviously,$\int_0^{t\wedge\alpha} V_x(x(s), s)g(x(s), x(s+\tau(s)), s)dB(s)$ and $\int_0^{t\wedge\alpha} \int_{|z|<c}[V(x(s)^- + H(x(s)^-, x(s+\tau(s))^-, s, z), s) - V(x(s)^-, s)]d\tilde{N}(ds, dz)$ are two martingales.

From condition $\mathcal{L}V \leq 0$ and taking the expection,

$$\mathbb{E}[V(x(t \wedge \alpha), t \wedge \alpha)] \leq V(||\xi||, 0). \tag{4}$$

The solution $x(\cdot)$ of the SDDEs possesses càdlàg paths, $r \leq |x(\tau)| \leq r + c \leq \frac{h}{2} + \frac{h}{2} = h$. Since $V(x,t) \geq \mu(|x|)$, it follows that

$$V(x(\alpha), \alpha) \geq \mu(|x(\tau)|) \geq \mu(r).$$

From the previous relations we deduce that

$$V(||\xi||, 0) \geq \mathbb{E}[V(x(t \wedge \alpha), t \wedge \alpha)] \geq \mathbb{E}[I_{\alpha \leq t} V(x(\alpha), \alpha)] \geq P(\alpha \leq t)\mu(r).$$

Together with (3) and (4), we have

$$P(\alpha \leq t)\mu(r) \leq V(||\xi||, 0) \leq \sup_{x \in (a,b)} \leq \mu(r)\varepsilon \leq \mu(r)\varepsilon,$$

which implies

$$P(\alpha \leq t) \leq \varepsilon.$$

Taking the limit as $t \to \infty$, we obtain $P(\alpha \leq \infty) \leq \varepsilon$, which implies

$$P\{|x| < r, \ \forall t \geq 0\} \geq 1 - \varepsilon.$$

Therefore, the SDDEs (1) is stable in probability. □

To prove the subsequent theorems, we present the stable in probability in Lyapunov-Razumikhin approach.

Lemma 4. *Consider the SDDEs* (1)*, if there exist positive definite function $V(x,t) \in \mathbb{C}^{2,1}(S_h \times [-\tau, \infty); \mathbb{R}_+)$ such that*

$$\begin{aligned}
&(i)\ a_1(x) \leq V(x,t),\\
&(ii)\ \mathcal{L}V(x,y,t) \leq 0,\\
&whenever\ \mathbb{E}V(x(t+\tau(t)), t+\tau(t)) < \mathbb{E}V(x(t),t),\ for\ all\ t \in \mathbb{R}_+,
\end{aligned}$$

then the SDDEs (1) *is stable in probability.*

Proof. Define t_1 (possilbly infinite) by

$$t_1 = \inf\{t \geq 0 : \mathbb{E}V(x(t+\tau(t)), t+\tau(t)) < \mathbb{E}V(x(t),t)\},$$

This implies that

$$\mathbb{E}V(x(\theta), \theta) \leq \mathbb{E}V(x(\theta+\tau(\theta)), \theta+\tau(\theta)),\ \forall \theta \in [0, t_1).$$

Recursively applying this estimate, we can obtain

$$\mathbb{E}V(x(\theta+\tau(\theta)), \theta+\tau(\theta)) \leq \mathbb{E}V(x(\theta+\tau(\theta)+\tau(\theta+\tau(\theta))), t+\tau(\theta)+\tau(\theta+\tau(\theta))),$$

and

$$\mathbb{E}V(x(\theta), \theta) \leq \mathbb{E}V(x(\theta+\tau(\theta)+\tau(\theta+\tau(\theta))), \theta+\tau(\theta)+\tau(\theta+\tau(\theta))).$$

By constant recursion and induction, it could be sum $\theta + \tau(\theta) + \tau(\theta+\tau(\theta)) + \cdots < 0$ for $\theta \in [0, t_1)$.i.e.,

$$\mathbb{E}V(x(\theta), \theta) \leq \mathbb{E}V(\xi, 0), \tag{5}$$

for all $\theta \in [0, t_1)$.

On the other hand, define t_2 (possilbly infinite) by

$$t_2 = \inf\{t \geq t_1 : \mathbb{E}V(x(t+\tau(t)), t+\tau(t)) \geq \mathbb{E}V(x(t), t)\},$$

This implies that for all $t \in [t_1, t_2)$, have

$$\mathbb{E}V(x(\theta+\tau(\theta)), \theta+\tau(\theta)) < \mathbb{E}V(x(\theta), \theta).$$

Under the assumptions stated in the lemma,

$$\mathcal{L}V(x, y, t) \leq 0.$$

By Lemma 3, the following result holds:

$$\mathbb{E}V(x(\theta), \theta) \leq \mathbb{E}V(x(t_1), t_1),$$

for all $\theta \in [t_1, t_2)$. Consequently, it follows that estimate (5) holds for all $t \in [0, t_2)$. The analysis above can be iterated for all $t \geq 0$.

Let us demonstrate by contradiction that the estimate (5) is actually valid for all $t \geq 0$. Let us define the t_3 as follows:

$$t_3 = \inf\{t \geq t_2 : \mathbb{E}V(x, t) > \mathbb{E}V(\xi, 0)\}.$$

This implies that $\mathbb{E}V(x, t) \leq \mathbb{E}V(\xi, 0)$ $\forall t \in [0, t_3)$ holds. In particular, it holds that $\mathbb{E}V(x(t_3+\tau(t_3)), t_3+\tau(t_3)) < \mathbb{E}V(x(t_3), t_3)$. Then $\mathbb{E}V(x(t), t) \leq \mathbb{E}V(\xi, 0)$. This contradicts the definition of t_3. Therefore, the estimate (5) is valid for all $t \geq 0$. According to the Lemma 3, the SDDEs (1) is stable in probability. □

3. Main Result

In this section, the stable criterion theorems for stochastic SDDEs (1) are given.

Theorem 1. *Under Assumption $(A1)$, let c_1, c_2, p, a, b be positive constants and $0 < d < 1$. If there exists a positive difinite Lyapunov function $V(x, t) \in \mathbb{C}^{2,1}(\mathbb{R}^d \times [-\tau, \infty); \mathbb{R}_+)$, such that*

$$c_1|x|^p \leq V(x, t) \leq c_2|x|^p, \tag{6}$$

$$\mathcal{L}V(x, y, t) \leq -a(V(x, t))^d - bV(y, t+\tau(t)),\ t \geq 0, \tag{7}$$

then the SDDEs (1) is finite-time stable in probability.

Proof. Since the functions f, g, H_1 satisfy $f(0, 0, t) = 0$, $g(0, 0, t) = \mathbf{0}$, $H_1(0, 0, t, z) = \mathbf{0}$ hold for all $t \in \mathbb{R}_+$ and $|z| < c$, which indicating that the SDDEs (1) has the trivial zero solution $x(t) \equiv 0$.

Note that $\mathcal{L}V(x, y, t) \leq 0$, from Lemma 3, one can shows that for $\forall \xi \in \mathbb{C}^P_{\mathcal{F}_0}([-\tau, 0]; \mathcal{R}^d)$, the SDDEs (1) is stable in probability. We next prove that the SDDEs (1) is finite-time stable in probability.

When $\xi = 0$, it follows that $x(t) \equiv 0,\ \forall t \geq -\tau$. When $\xi \neq 0$, there exists an integer $k \in \{2, 3, \cdots\}$ such that $||\xi|| \in (\frac{1}{k}, k)$. Let us introduce the $\beta_k = \inf\{t \geq 0 : |x(t;\xi)| \notin (\frac{1}{k}, k)\}$, This indicates the moment when the absolute value of the system state first exceeds the interval $(\frac{1}{k}, k)$. Define the stopping time sequence β_k

$$\beta_k = \inf\{t \geq 0 : |x(t;\xi)| \notin (\frac{1}{k}, k)\},$$

which is the first exit time of $x(t;\xi)$ from $(\frac{1}{k}, k)$. where $k \in \{2, 3, \cdots\}$. It is easy to prove that β_k is an increasing stopping time, $\beta_\infty = \lim_{k\to\infty} \beta_k$. Applying the Itô's formula to V for arbitrary $t \geq u \geq 0$, the following expression is obtained:

$$\begin{aligned} V(x(t\wedge\beta_k), t\wedge\beta_k) =& V(x(u\wedge\beta_k), u\wedge\beta_k) + \int_{u\wedge\beta_k}^{t\wedge\beta_k} \mathcal{L}V(x, y, s)ds + \int_{u\wedge\beta_k}^{t\wedge\beta_k} V_x(x, s)dB(s) \\ &+ \int_{u\wedge\beta_k}^{t\wedge\beta_k} V((x)^- + H(x^-, y^-, s, z), s) - V(x(s)^-, s)\tilde{N}(ds, dz). \end{aligned}$$

Let $= t \wedge \beta_k$,taking expectation for both sides, one can verify that

$$\mathbb{E}[V(x(t \wedge \beta_k), t \wedge \beta_k)] - \mathbb{E}[V(x(u \wedge \beta_k), u \wedge \beta_k)] \leq \int_u^t \mathbb{E}I_{(s \leq \beta_k)} \mathcal{L}V(x(s), y(s), s) ds. \tag{8}$$

It is straightforward to observe that

$$\begin{aligned} & \mathbb{E}[V(x(t \wedge \beta_k), t \wedge \beta_k)] - \mathbb{E}[V(x(u \wedge \beta_k), u \wedge \beta_k)] \\ = \ & \mathbb{E}[I_{(t \leq \beta_k)} V(x(t), t)] - \mathbb{E}[I_{(u \leq \beta_k)} V(x(u), u)] + \mathbb{E}[I_{(t \leq \beta_k) - (u \leq \beta_k)} V(x(\beta_k), \beta_k)] \\ \geq \ & \mathbb{E}[I_{(t \leq \beta_k)} V(x(t), t)] - \mathbb{E}[I_{(u \leq \beta_k)} V(x(u), u)] \\ \geq \ & c_1 \mathbb{E}[I_{(t \leq \beta_k)} |x(t)|^p] - c_1 \mathbb{E}[I_{(u \leq \beta_k)} |x(u)|^p]. \end{aligned}$$

On the other hand, applying (7) together with Hölder's inequality, we can get

$$\begin{aligned} \mathbb{E}[I_{(s \leq \beta_k)} \mathcal{L}V(x(s), y(s), s)] &\leq -aE[I_{(s \leq \beta_k)} V(x(s), s)^d] - bE[I_{(s \leq \beta_k)} V(y(s), s + \tau(s))] \\ &\leq -a\{\mathbb{E}[I_{(s \leq \beta_k)} V(x(s), s)]\}^d - b\mathbb{E}[I_{(s \leq \beta_k)} V(y(s), s + \tau(s))] \\ &\leq -ac_1^d \{\mathbb{E}[I_{(s \leq \beta_k)} |x|^p]\}^d - bc_1 \mathbb{E}[I_{(s \leq \beta_k)} |y|^p]. \end{aligned}$$

Combining this with Lemmas 1 and 2, the following result is derived:

$$\begin{aligned} \int_u^t \mathbb{E}I_{(s \leq \beta_k)} \mathcal{L}V(x(s), y(s), s) ds \leq & - ac_1^d \int_u^t \{\mathbb{E}[I_{(s \leq \beta_k)} |x|^p]\}^d ds \\ & - bc_1 \overline{\tau} \int_{u-\tau'}^t \mathbb{E}[I_{(s \leq \beta_k)} |x|^p] ds \\ \leq & - \int_u^t ac_1^d \{\mathbb{E}[I_{(s \leq \beta_k)} |x|^p]\}^d \\ & + bc_1 \overline{\tau} \mathbb{E}[I_{(s \leq \beta_k)} |x|^p] ds. \end{aligned} \tag{9}$$

According to (8) and (9), shows that

$$c_1 \mathbb{E}[I_{(t \leq \beta_k)} |x(t)|^p] - c_1 \mathbb{E}[I_{(u \leq \beta_k)} |x(u)|^p] \leq - \int_u^t ac_1^d \{\mathbb{E}[I_{(s \leq \beta_k)} |x|^p]\}^d + bc_1 \overline{\tau} \mathbb{E}[I_{(s \leq \beta_k)} |x|^p] ds \tag{10}$$

Let $h(t) = \mathbb{E}[I_{(t \leq \beta_k)} |x(t)|^p]$, shows that

$$h(t) - h(u) \leq - \int_u^t ac_1^{d-1} h(s)^d + b\overline{\tau} h(s) ds.$$

Next, we proceed to demonstrate the right-continuity of $h(t)$. Recall that

$$h(t) = \mathbb{E}[|x(t \wedge \beta_k)|^p] - \mathbb{E}\big[I_{(t > \beta_k)} |x(\beta_k)|^p\big].$$

It is evident that $\mathbb{E}[|x(t \wedge \beta_k)|^p]$ is a right-continuous function. Based on the definition of β_k, we can express $\mathbb{E}\big[I_{(t > \beta_k)} |x(\beta_k)|^p\big]$ as $\mathbb{P}(t > \beta_k) |x(t)|^p$, where $|x(t)|$ takes the value $\frac{1}{k}$ or k. Since $\mathbb{P}(t > \beta_k)$ is right-continuous, it follows that $h(t)$ is also right-continuous. Applying Lemma 2, we derive the following inequality:

$$h(t) \leq [h(0)^{1-d} + \frac{ac_1^{d-1}}{b\overline{\tau}} (1 - e^{(1-d) b \overline{\tau} t})]^{\frac{1}{1-d}} e^{-b\overline{\tau} t}, \ \ \forall t \in [0, \overline{T}].$$

where

$$\overline{T} = \frac{\ln(1 + \frac{b\overline{\tau}}{ac_1^{d-1}} ||\xi||^{p(1-d)})}{(1-d) b\overline{\tau}}.$$

Then,

$$h(\overline{T}) = \mathbb{E}[I_{(\overline{T} \leq \beta_k)} |x(\overline{T})|^p] = 0,$$

which also implies$P(\beta_k \leq \overline{T}) = 1$. Let $k \to \infty$, by the dominated convergence theorem, we have $P(\beta_\infty \leq \overline{T}) = 1$.

Define a stopping time $T_\xi = \inf\{t \geq 0; x(t;\xi) = 0\}$, then we have

$$P(T_\xi \leq \overline{T}) = P(\beta_\infty \leq \overline{T}) = 1.$$

Then the SDDEs (1) is finite-time stable in probability. □

Remark 2. *In our results, it is not required that the time-varying delay $\tau(t)$ satisfies the assumptions of being differentiable and having a derivative less than a certain value.*

The Theorem 1 can obtain the finite-time stability of the SDDEs (1), The settling time T_ξ of Theorem 1 is related to the initial condition ξ. Although the conditions of the theorem that we will present next is stricter than the Theorem 1, it can be shown that the SDDEs (1) is fixed-time stable, while the settling time is independent of the initial state ξ of the SDDEs (1).

Theorem 2. *Assume that there exists $V(x,t)$, along with $k(s) \in (\mathbb{R}_+; \mathbb{R}_+)$ and a_1, $a_2 \in \mathcal{K}$, satisfying the following conditions:*

$$\begin{aligned}
&(i)\ a_1(x) \leq V(x,t) \leq a_2(x), && (11)\\
&(ii)\ \mathcal{L}V(x,y,t) \leq -c_1 k(V(x,t)) + c_2 k(V(y,t+\tau(t))), && (12)\\
&(iii)\ k(V(y,t+\tau(t))) < qk(V(x,t)),\ \forall t \geq 0, && (13)\\
&(iv)\ k'(s) \geq 0, \forall s \geq 0, && (14)\\
&(v)\ \int_0^p \frac{1}{k(s)} ds \leq M < \infty,\ \forall p \in (0,\infty), && (15)
\end{aligned}$$

where $c_1,\ c_2,\ M > 0$ and $c_1 - qc_2 > 0$, then the SDDEs (1) is fixed-time stable in probability.

Proof. By Lemma 3 and condition (12), the SDDEs (1) is stable in probability. When $\xi = 0$, it holds that $x(t) \equiv 0\ \forall t \geq -\tau$. When $\xi \neq 0$, there exists an integer $k \in \{2,3,\cdots\}$ such that $||\xi|| \in (\frac{1}{k}, k)$. We define $\alpha_{1k},\ \alpha_{2k},\ \alpha_{3k}$ as follows:

$$\begin{aligned}
\alpha_{1k} &= \inf\{t \geq 0 : |x(t;\xi)| \notin (\frac{1}{k}, k)\},\\
\alpha_{2k} &= \inf\{t \geq 0 : |x(t;\xi)| \notin [0, \frac{1}{k}]\},\\
\alpha_{3k} &= \inf\{t \geq 0 : |x(t;\xi)| \notin [k, \infty]\}.
\end{aligned}$$

The stopping times α_{1k}, α_{2k}, α_{3k} are all non-decreasing and $\alpha_{1k} = \alpha_{2k} \wedge \alpha_{3k}$. Since exist a solution $x(t)$ of SDDEs (1) is defined on $[0,\infty)$, $\alpha_{3\infty} = \lim_{k\to\infty} \alpha_{3k} = \infty$. Note that $\alpha_{1\infty} = \lim_{k\to\infty} \alpha_{1k} = \lim_{k\to\infty} \alpha_{2k} = \alpha_{2\infty}$.

According to (14) and (15), the function is defined as follows:

$$F(x,t) := \int_0^{V(x,t)} \frac{1}{k(s)} ds. \tag{16}$$

Applying Itô's formula, we have

$$\begin{aligned}
F(x(t\wedge\alpha_{1k}), t\wedge\alpha_{1k}) =& F(x(0),0) + \int_0^{t\wedge\alpha_{1k}} \frac{V_t}{k(V(x,s))} ds + \int_0^{t\wedge\alpha_{1k}} \frac{V_x}{k(V(x,s))} f(x,y,s) ds\\
&+ \frac{1}{2}\int_0^{t\wedge\alpha_{1k}} \frac{1}{k(V(x,s))} trace(g^T(x,y,s)V_x x(x,s) g(x,y,s)) ds\\
&- \frac{1}{2}\int_0^{t\wedge\alpha_{1k}} \frac{k'(V(x,s))}{k(V(x,s))^2} trace(|V_x(x,s)g(x,y,s)|^2) ds\\
&+ \int_0^{t\wedge\alpha_{1k}} \int_{|z|<c} F(x(s)^- + H(x(s)^-, y(s)^-, s, z), s) - F(x(s)^-, s)\\
&- H(x(s)^-, y(s)^-, s, z) F_x(x(s)^-, s) v(dz) ds\\
&+ C(t),
\end{aligned} \tag{17}$$

where

$$C(t) = \int_0^{t\wedge\alpha_{1k}} \frac{V_x}{k(V(x,s))} g(x,y,s)dB(s) + \int_0^{t\wedge\alpha_{1k}} \int_{|z|<c} F(x(s)^- + H(x(s)^-, y(s)^-, s, z), s) - F(x(s)^-, s)\tilde{N}(ds, dz).$$

Where we have used Taylor's formula,

$$\begin{aligned}
&\int_{|z|<c} F(x+H(x,y,s,z),s) - F(x,s) - H(x,y,s,z)F_x(x,s)v(dz) \\
&= \int_{|z|<c} H(x,y,s,z)^T\Big(\int_0^1 (1-\theta)F_{xx}(x+\theta H(x,y,s,z))d\theta\Big)H(x,y,s,z)v(dz) \\
&= \int_{|z|<c} \Big\{\int_0^1 (1-\theta)\Big[\frac{1}{k(V(x,s))}H(x,y,s,z)^T V_{xx}(x+\theta H(x,y,s,z),s)H(x,y,s,z) \\
&\quad - \frac{k'(V(x,s))}{k^2V(x,s)}|V_x(x+\theta H(x,y,s,z),s)H(x,y,s,z)|^2\Big]d\theta\Big\}v(dz) \\
&= \frac{1}{k(V(x,s))}\int_{|z|<c}[V(x+H(x,y,s,z),s) - V(x,s) - H(x,y,s,z)V_x(x,s)]v(dz) \\
&\quad - \frac{k'(V(x,s))}{k^2V(x,s)}\int_{|z|<c}\int_0^1 (1-\theta)|V_x(x+\theta H(x,y,s,z),s)H(x,y,s,z)|^2 d\theta v(dz).
\end{aligned} \tag{18}$$

According to $k'(V(x,s)) \geq 0$, it follows that the second term on the right side of (18) and the fifth term on the right side of (17) are both less than 0. Substitute (18) into (17), and then take expectation for both sides of (17), we have

$$\mathbb{E}F(x(t\wedge\alpha_{1k}), t\wedge\alpha_{1k}) \leq \mathbb{E}F(||\xi||, 0) + \mathbb{E}\int_0^{t\wedge\alpha_{1k}} \frac{\mathcal{L}V(x,y,s)}{k(V(x,s))}ds.$$

From (12) and (13),

$$\mathbb{E}F(x(t\wedge\alpha_{1k}), t\wedge\alpha_{1k}) \leq \mathbb{E}F(||\xi||, 0) - (c_1 - qc_2)E(t\wedge\alpha_{1k}). \tag{19}$$

Given that $F(x,t) \geq 0$, we have

$$\mathbb{E}(t\wedge\alpha_{1k}) \leq \frac{\mathbb{E}F(||\xi||, 0)}{c_1 - qc_2}.$$

Let $t\to\infty$ and $k\to\infty$, from the Fatou Lemma and (15), we have $\mathbb{E}(\alpha_{2\infty}) = \mathbb{E}(\alpha_{1\infty}) \leq \frac{\mathbb{E}F(||\xi||,0)}{c_1-qc_2} < \frac{M}{c_1-qc_2}$. We now define $T_\xi = \inf\{t\geq 0; x(t;\xi) = 0\}$, It is clear that $T_\xi = \alpha_{2\infty}$. Hence

$$\mathbb{E}(T_\xi) < \frac{M}{c_1 - qc_2},$$

which also implies $P\{T_\xi < \infty\} = 1$.

Finally, we will prove that $x(t;\xi) = 0 \;\forall t \geq T_\xi$. Writing $T_\xi = T$ simply, define an stopping time sequences $\alpha_n = \inf\{t \geq; ||x(t;\xi)|| \geq n\}$. Clearly, this is an increasing stopping time sequence and $\lim_{n\to\infty}\alpha_n = \infty$. Combining (12), (13), $V(x(T), T) = 0$, we obtain

$$\begin{aligned}
\mathbb{E}V(x((t+T)\wedge\alpha_n), (t+T)\wedge\alpha_n) &\leq \mathbb{E}V(x(T),T) + \mathbb{E}\int_T^{(t+T)\wedge\alpha_n}\mathcal{L}V(x,y,s)ds \\
&\leq 0.
\end{aligned}$$

Given that $V(x,t)$ is positive definite, it follows that $V(x((t+T)\wedge\alpha_n), (t+T)\wedge\alpha_n) = 0\; a.s.$, which implies that $x((t+T)\wedge\alpha_n) = 0$. Taking the limit as $n\to\infty$, we have $x(t+T) = 0\;\forall t\geq 0$.

In conclusion, the SDDEs (1) is finite-time stable and $\mathbb{E}(T_\xi) < \frac{M}{c_1-qc_2}$. where M is a positive constant that does not depend on the initial conditions. According to Definitions 2 and 1, we prove that the SDDEs (1) is fixed-time stable. □

Remark 3. *Let us clarify that numerous functions exist that satisfy the requirements imposed on $k(\cdot)$. For example, let $k(s) = as^{\alpha} + b,\ \alpha > 1, a, b \in \mathbb{R}_+$, it is easy to prove that $k(s) = as^{\alpha} + b$ satisfies (14) and (15). According to the approach used in Corollary 3.2 of Ref. [5], we can obtain: $\mathbb{E}(T_\xi) < \frac{1}{c_1 - qc_2}\int_0^{V(x,t)} \frac{1}{k(s)} ds < \frac{1}{c_1 - qc_2}\frac{1}{b}(\frac{b}{a})^{\frac{1}{\alpha}}(1 + \frac{1}{\alpha - 1})$*

Remark 4. *To ensure that $\mathcal{L}V$ remains negative, we have to impose very strict constraints on the equation Such conditions would limit the applicability of the results primarily to equations resembling stochastic diffusion equations. Motivated by the Razumikhin-type theorem for SDDEs, we observe that it is not necessary to require $\mathcal{L}V(x, y, t)$ to be negative for all $t \geq 0$. Instead, negativity of the operator can be imposed under specific relationships between $V(x(t + \tau(t)), t + \tau(t))$ and $V(x(t), t)$.*

Theorem 3. *Let d, M, $\lambda > 0$ Suppose there exists $V(x, t)$, along with $k(s) \in (\mathbb{R}_+; \mathbb{R}_+)$ and $a_1,\ a_2 \in \mathcal{K}$, such that:*

$$
\begin{aligned}
&(i)\ a_1(x) \leq V(x,t) \leq a_2(x),\\
&(ii)\ k'(s) \geq 0, \forall s \geq 0 \text{ and } \int_0^p \frac{1}{k(s)} ds \leq M < \infty,\ \forall p \in (0,\infty),\\
&(iii)\ \mathcal{L}V(x,y,t) \leq -\lambda k(V(x,t)),\\
&whenever\ \mathbb{E}\int_{V(x,t)}^{V(x(t+\tau(t)),t+\tau(t))} \frac{1}{k(s)} ds < d\tau,\ for\ all\ t \in \mathbb{R}_+,
\end{aligned}
$$

then the SDDEs (1) is fixed-time stable, and $T_\xi \leq \frac{M}{min\{\lambda, d\}}$.

Proof. Step 1: According to the condition of theorem, the SDDEs (1) is stable in probability. Since we have an imolication

$$
\begin{aligned}
&\mathbb{E}V(x(t+\tau(t)), t+\tau(t)) < \mathbb{E}V(x,t)\\
\Rightarrow &\mathbb{E}\int_0^{V(x(t+\tau(t)),t+\tau(t))} \frac{1}{k(s)} ds < \mathbb{E}\int_0^{V(x,t)} \frac{1}{k(s)} ds + d\tau\\
\Rightarrow &\mathbb{E}\mathcal{L}V(x,y,t) \leq -\lambda \mathbb{E}k(V(x,t)),
\end{aligned}
$$

then the conditions of lemma 4 for stable in probability are satisfied.

Step 2: Construct a function $Q(t) = \mathbb{E}\int_0^{V(x(t),t)} \frac{1}{k(s)} ds$. Let us define t_1(possilbly infinite) by

$$
t_1 = \inf\{t \geq 0 : Q(t + \tau(t)) < Q(t) + d\tau\},
$$

Consequently, for every $\theta \in [0, t_1)$, the following inequality holds:

$$
\begin{aligned}
&Q(t+\tau(t)) \geq Q(t) + d\tau\\
\Rightarrow &Q(\theta) \leq Q(\theta + \tau(\theta)) + d\tau(\theta).
\end{aligned}
$$

Recursively applying this estimate, we can obtain

$$
Q(\theta + \tau(\theta)) \leq Q(\theta + \tau(\theta) + \tau(\theta + \tau(\theta))) + d\tau(\theta + \tau(\theta)),
$$

and

$$
Q(\theta) \leq Q(\theta + \tau(\theta) + \tau(\theta + \tau(\theta))) + d(\tau(\theta) + \tau(\theta + \tau(\theta + \tau(\theta)))).
$$

By constant recursion and induction, it could be sum $\theta + \tau(\theta) + \tau(\theta + \tau(\theta)) + \cdots < 0$ for $\theta \in [0, t_1)$, since $\tau(\theta),\ \tau(\theta + \tau(\theta)),\ \cdots < 0$. Eventually, it yields

$$
\begin{aligned}
Q(\theta) &\leq \max\{0, Q(0) - c\theta\}\\
&\leq \max\{0, M - \min\{\lambda, d\}\theta\}.
\end{aligned}
\tag{20}
$$

for all $\theta \in [0, t_1)$. It should be noted that this estimation indicates that once $x(\eta) = 0$ for some $\eta \in [0, t_1)$, then $x(t) = 0\ \forall t \in [\eta, t_1)$. In addition, when $t_1 = \infty$, the settling time T_ξ must exist. In fact, assuming that $T_\xi = +\infty$, we have $x(t) \neq 0$ for any large enough t. Conversely, we observe that $Q(\theta) \leq \max\{0, M - \min\{\lambda, d\}\theta\} \leq 0$ with

$t \to +\infty$. It creates a contradiction. Therefore, T_ξ must exist and satisfied $T_\xi = \frac{M}{\min\{\lambda,d\}}$.

Step 3: Define the stopping time t_2 (possilbly infinite) by

$$t_2 = \inf\{t \geq t_1 : Q(t+\tau(t)) \geq Q(t) + d\tau\},$$

This implies that

$$Q(\theta + \tau(\theta)) < Q(\theta) + d\tau. \ \forall \theta \in [t_1, t_2).$$

Under the assumptions of the theorem, it holds that

$$\mathcal{L}V(x,y,t) \leq -\lambda k(V(x,t)).$$

According to the Theorem 2, we have

$$Q(t) \leq \max\{0, M - \min\{\lambda, d\}(t - t_1)\}, \ \forall t \in [t_1, t_2).$$

Therefore, it follows that estimate (20) holds for all $t \in [0, t_2)$. The analysis above can be iterated for all $t \geq 0$.

Step 4: Let us demonstrate by contradiction that the estimate (20) is actually valid for all $t \geq 0$. We employ a proof by contradiction to establish that the estimation formula (20) is valid $\forall t \geq 0$. Let us define t_3 as follows:

$$t_3 = \inf\{t \geq t_2 : Q(t) > \max\{0, M - \min\{\lambda, d\}t\}\}.$$

As a result, it follows that $\forall t \in [0, t_3)$, the inequality $Q(t) \leq \max\{0, M - \min\{\lambda, d\}t\}$ holds. This implies that for all $t \in [0, t_3)$, we have $Q(t) \leq \max\{0, M - \min\{\lambda, d\}t\}$. In particular, it holds that $Q(t_3 + \tau(t_3)) < Q(t_3) + d\tau$. From the Step 3, we have $Q(t_3) \leq \max\{0, M - \min\{\lambda, d\}t_3\}$. This contradicts the definition of t_3. Therefore, the estimate (20) is valid for all $t \geq 0$. According to the estimate (20) and $Q(t) \geq 0$, we have $T_\xi = \frac{M}{\min\{\lambda,d\}}$. □

Remark 5. *The Theorem 3 is inspired by Ref. [31], the objective of the proof is to show that the $V(x,t)$ will fixed-time stable regardless of whether the relations between $V(x(t+\tau(t)), t+\tau(t))$ and $V(x(t),t)$ given in condition (iii) are satisfied or not.*

4. Example

In this section, we show the finite-time stability for following SDDEs with Lévy noise.

Example 1. *Consider the following SDDEs (1) with $c = 1$,*

$$\begin{aligned} dx = & \Big[-|x|^{\frac{1}{2}} - \Big(1 + \frac{\pi}{4}\Big)x - 2y(t)\Big]dt + xdB(t) \\ & + \int_{|z|<1} (y^- z - x^-)\tilde{N}(dt, dz), \quad t \geq 0. \end{aligned} \tag{21}$$

where $y = x(t+\tau(t))$, initial data $\xi = \{x(t) : t \in (-0.1, 0)\} = 2$ and $\tau(t) = -0.1|\sin(t)|$ with $-\tau' = 0$ and $\overline{\tau} = 1.1111$. Let Lévy measure $\nu(\cdot)$ satisfy $\nu(dz) = \frac{dz}{1+|z|^2}$, so $\int_{|z|<1} \frac{dz}{1+|z|^2} = \frac{\pi}{2}$. Therefore, one can compute

$$\begin{aligned} \int_{|z|<1} \frac{1}{2}|yz|^2 \frac{dz}{1+|z|^2} &= (1 - \frac{\pi}{4})|y|^2, \\ \int_{|z|<1} \frac{1}{2}|x|^2 \frac{dz}{1+|z|^2} &= \frac{\pi}{4}|x|^2, \\ \int_{|z|<1} x(yz - x)\frac{dz}{1+|z|^2} &= -\frac{\pi}{2}|x|^2. \end{aligned}$$

Now, choose Lyapunov function $V(x,t)=\frac{1}{2}|x|^2$*. Using Itô formula,*

$$\begin{aligned}\mathcal{L}V(x) &= x(-|x|^{\frac{1}{2}}-(1+\frac{\pi}{4})x-2y)+\frac{1}{2}|x|^2\\ &\quad+\int_{|z|<1}\frac{1}{2}|yz|^2-\frac{1}{2}|x|^2-(yz-x)xv(dz)\\ &\leq -2^{\frac{3}{4}}(V(x,t))^{\frac{3}{4}}-\frac{\pi}{2}V(y,t),\end{aligned}$$

which implies that the inequality (7) *holds with* $a=2^{\frac{3}{4}},\ b=\frac{\pi}{2},\ d=\frac{3}{4},\ c_1=c_2=\frac{1}{2}$.

Based on Theorem 1, it holds that $V(x)=0,\ \forall t\geq\frac{72ln(1+\frac{5\pi}{18})}{10\pi}$*. Note that* $V(x)=\frac{1}{2}|x|^2$*, we can get* $x(t)=0,\ t\geq\frac{72ln(1+\frac{5\pi}{18})}{10\pi}$*. Therefore, the SDDEs* (21) *is finite-time stable.*

Computer simulations are conducted using the Euler-Maruyama method. (refer to, for instance, [32]) with a step size of 10^{-3}*. For the numerical experiments, the time-varying delay is set as* $\tau(t)=-0.1|\sin(t)|$*, and the initial condition is given by* $\xi=\{x(t):t\in(-0.1,0)\}=2$*.*

Figure 1 displays the state trajectory corresponding to the SDDE (21)*.*

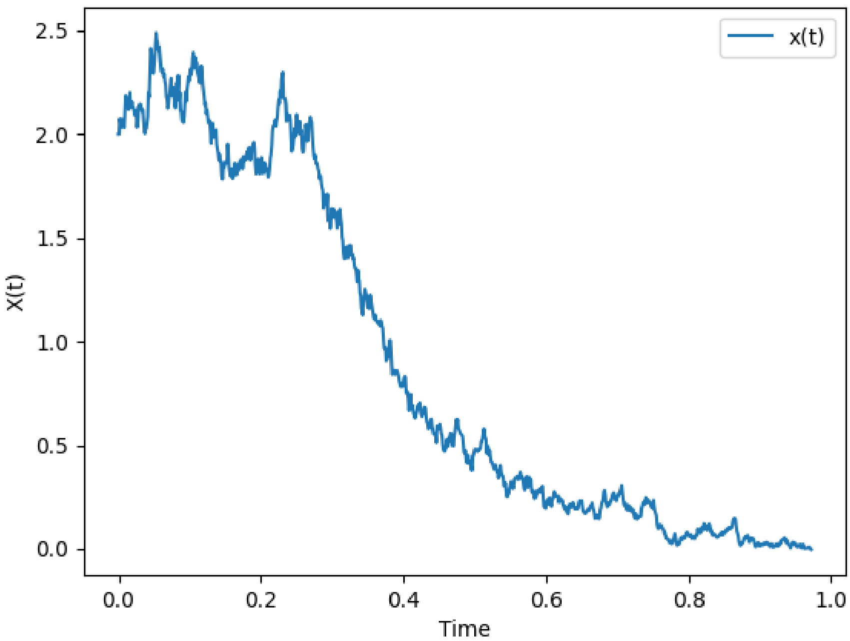

Figure 1. The trajectories of solution of SDDEs (21)

5. Conclusions

This paper investigates the stability criteria for SDDEs with Lévy noise. The obtained results provide an effective stability analysis method for the finite-time control. Based on this research and in combination with the literature [33], the predefined-time stability problem of the SDDEs can be further studied in the future.

Author Contributions

J.W. and G.X.: formal analysis, writing—original draft preparation; G.X. and Y.G.: writing—review and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare that they have no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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