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Article

Gaussian Fluctuations in the Tunneling Probability of a Closed Universe

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Abstract: We consider the quantum creation of a closed universe within the Euclidean path-integral formalism. An analytical expression for the tunneling probability is derived, including both the exponential suppression and the exact Gaussian prefactor due to quadratic fluctuations around the instanton. The calculation is performed in a fixed-interval minisuperspace formulation, where the Hamiltonian constraint is imposed at the level of the classical instanton, while the full lapse integration is not included beyond the leading semiclassical approximation. The result provides a transparent and self-consistent semiclassical estimate of the nucleation rate, refining previous analyses with the inclusion of Gaussian fluctuations.

Keywords: closed universe; quantum tunneling; Gaussian fluctuations

1. Introduction

Understanding the quantum origin of the universe remains one of the fundamental challenges in theoretical cosmology [1]. In the framework of closed Friedmann-Lemaitre-Robertson-Walker (FLRW) models, the nucleation of a finite-size universe can be described as a quantum tunneling process from "nothing", that is from the classically forbidden region where the scale factor vanishes [2,3]. This scenario provides a concrete and physically motivated mechanism for the quantum creation of the universe. The Euclidean path integral offers a natural framework to describe such a process, expressing the transition amplitude between two three-geometries as a sum over all interpolating Euclidean metrics. In this approach, the semiclassical tunneling probability is dominated by the contribution of the instanton solution that extremizes the Euclidean action, leading to an exponential term governed by the cosmological constant. This picture was first formulated in the seminal works of Atkatz and Pagels [2] and Vilenkin [3]. Quite remarkably, although the leading exponential dependence of the tunneling probability is well understood [2–6], and also generalized including massive scalar fields [7–10], the accurate determination of the prefactor due to Gaussian fluctuations was never performed exactly, because it requires handling a complicated differential operator with non-constant coefficients and carefully treating boundary singularities in the instanton solution. To the best of our knowledge, an explicit closed analytical expression for the Gaussian fluctuation prefactor in this Euclidean closed-FLRW tunneling problem has not been previously derived. The aim of the present work is therefore not to formulate a complete constrained gravitational path integral, but to isolate and evaluate exactly the quadratic fluctuation determinant associated with the standard semiclassical Euclidean instanton.

In this work we adopt the path-integral formulation of quantum cosmology for a closed FLRW universe and derive a fully analytical expression for the tunneling probability, including both the exponential suppression and the exact Gaussian prefactor arising from quadratic fluctuations around the instanton solution. Our derivation extends the analyses of Refs. [2, 3], yielding a transparent and self-consistent semiclassical estimate of the tunneling rate, and clarifying the role of the fluctuation determinant in the Euclidean path integral for quantum cosmology. It is important to emphasize that the issue of fluctuations in quantum cosmology has been discussed in detail in several works in the literature. For example, Ref. [11] analyzes the nature of quadratic fluctuations around homogeneous backgrounds within a Hamiltonian/Wheeler-DeWitt formalism, including implications for cosmological perturbations. Ref. [12] investigates conceptual aspects of the no-boundary wave function and the structure of fluctuations in that context, focusing on the definition of the quantum state rather than on an explicit evaluation of a fluctuation determinant. Moreover, Ref. [13] explores the Lorentzian path integral approach with complex contour deformations as a foundation for quantum cosmology. More recent developments have further investigated Lorentzian and Picard-Lefschetz formulations of quantum cosmology, including truly Lorentzian minisuperspace path integrals, resurgence methods around cosmological saddle points, and extensions involving torsion [9, 14, 15]. In contrast, the present work remains within the standard Euclidean tunneling proposal for a closed FLRW universe and focuses on the explicit analytical evaluation of the Gaussian prefactor associated with quadratic fluctuations around the Euclidean instanton solution. This provides a closed-form expression for the semiclassical tunneling probability and quantifies directly the contribution of quadratic fluctuations to the prefactor, without relying on contour deformations or numerical approximations.

2. Closed Friedmann-Lemaitre-Robertson-Walker Universe

Let us consider the action functional of a closed Friedmann-Lemaitre-Robertson-Walker universe [1]

$$S[a(t)] = -A \int dt \left[a\, \dot{a}^2 - V(a) \right] , \tag{1}$$

where

$$A = \frac{3\pi c^3}{4G\Lambda} \tag{2}$$

with c the speed of light in vacuum, G the gravitational constant, and Λ the cosmological constant. Here $a(t)$ is the scale factor of this minisuperspace approximation of the universe with

$$V(a) = a \left(1 - \frac{1}{3} a^2 \right) \tag{3}$$

a sort of effective potential energy, and $\dot{a} = da/dt$. Please note that a is dimensionless and it must be multiplied by $a_0 = 1/\sqrt{\Lambda}$ to get the dimensional one. Similarly, t is the dimensionless time and it must be multiplied by $t_0 = a_0/c$ to get the dimensional one. The constant of motion during the time evolution is

$$E = -A \left[a\, \dot{a}^2 + V(a) \right] . \tag{4}$$

Notice that both A and E have the dimensional units of an action (Joule $\times$ seconds). In quantum cosmology the lapse function is the Lagrange multiplier enforcing the Hamiltonian constraint and reflects the invariance under time reparametrizations. In the present reduced formulation we work in a fixed proper-time gauge, so that the lapse is not treated as an independent dynamical variable in the fluctuation determinant. Equivalently, the zero-energy Hamiltonian constraint is imposed on the classical instanton solution, while the Gaussian determinant is evaluated for fluctuations of the scale factor with fixed endpoints over the corresponding Euclidean time interval. This allows us to compute Gaussian fluctuations directly for this fixed-interval system, providing a well-defined semiclassical prefactor within the chosen gauge-fixed minisuperspace setting, although it does not represent the full Faddeev-Popov or BRST quantization of the gravitational path integral.

3. Quantum Tunneling from "Nothing"

In the description of a closed FLRW universe, invariance under time reparametrization leads to the constraint $E = 0$ [3, 16]. Under this constraint the region with $0 \leq a \leq \sqrt{3}$ is classically forbidden. One can calculate the probability rate P_T of quantum tunneling from $a = 0$ to $a = \sqrt{3}$ by using the quantum propagator $\langle a_F = \sqrt{3}, t_F | a_I = 0, t_I = 0 \rangle$ as follows

$$P_T = \left| \langle a_F = \sqrt{3}, t_F | a_I = 0, t_I = 0 \rangle \right|^2 . \tag{5}$$

The ratio is needed to have a dimensionless quantity with initial probability P_T equal to one. The quantum propagator can be written as a Feynman path integral [17, 18]

$$\langle a_F = \sqrt{3}, t_F | a_I = 0, t_I = 0 \rangle = \int \mathcal{D}[a(t)]\, e^{\frac{i}{\hbar} S[a(t)]} . \tag{6}$$

Performing a Wick rotation
$$t = i\tau \tag{7}$$
we have
$$\langle a_F = \sqrt{3}, \tau_F | a_I = 0, \tau_I = 0 \rangle = \int \mathcal{D}[a(\tau)] \, e^{-S_E[a(\tau)]/\hbar} \tag{8}$$
with
$$S_E[a(\tau)] = A \int_0^{\tau_F} d\tau \, [a \, a'^2 + V(a)] \tag{9}$$
the Euclidean action and $a' = da/d\tau$. Quite remarkably
$$a_c(\tau) = \sqrt{3} \, \sin\left(\frac{\tau}{\sqrt{3}} \right) \tag{10}$$
is the instanton solution that extremizes $S_E[a(\tau)]$ with boundary conditions $a_c(0) = 0$ and $a_c(\tau_F) = \sqrt{3}$ provided that
$$\tau_F = \frac{\pi}{2}\sqrt{3} \; . \tag{11}$$

The Euclidean action for this solution can be evaluated in closed form:
$$S_E[a_c(\tau)] = 2A = \frac{3\pi c^3}{2G\Lambda} \; . \tag{12}$$

It follows immediately that within the saddle point approximation, where
$$P_T = |e^{-S_E[a_c(\tau)]/\hbar}|^2 \; , \tag{13}$$
the tunneling probability reads
$$P_T = \exp\left(-\frac{3\pi c^3}{G\Lambda\hbar} \right) \; . \tag{14}$$

This result was obtained for the first time by Atkatz and Pagels [2] and Vilenkin [3] in 1982, see also Refs. [4–8]. The exponentially decreasing behavior obtained here arises from the Wick rotation $t = i\tau$, that is the Vilenkin tunneling proposal. In the Hartle-Hawking no-boundary proposal [19], the opposite sign in the exponent originates from the Wick rotation $t = -i\tau$ [20], leading to an opposite sign in the exponential and a wave function that favors smaller initial universes, in contrast with the Vilenkin tunneling approach considered here, which tends to select initial expansion. The question of which proposal is the correct approach is still debated in quantum cosmology, and there is no definitive consensus [10]. However, the Vilenkin tunneling proposal generally leads to initial conditions that make inflation more probable, since a sufficiently large initial value of the cosmological constant Λ is required to drive an inflationary phase, whereas small values of Λ would not lead to inflation and are exponentially suppressed in the tunneling probability [8,20]. Recent analyses of modified tunneling wave functions and Lorentzian path integrals show that this question remains active, especially when quantum-gravity corrections or different integration contours are included [9,14,21].

4. Gaussian Fluctuations

Before proceeding with the calculation of Gaussian fluctuations, it is important to note that, in a full quantum gravitational treatment, the wave function of the universe ψ must satisfy the Hamiltonian constraint $\hat{H}\psi = 0$, that is the Wheeler-DeWitt equation [16], which encodes time-reparametrization invariance. Expanding the Euclidean action around the classical instanton and computing the prefactor directly, as done in the present work, does not strictly enforce this constraint beyond leading semiclassical order. Nevertheless, within the minisuperspace approximation and the standard Euclidean tunneling approach, this procedure provides a well-defined and fully analytical estimate of the contribution of quadratic fluctuations to the semiclassical tunneling probability. From this point of view, our strategy is consistent with previous semiclassical treatments in the literature [2–4], and offers a simplified, but analytically tractable, perspective compared to approaches [11,12] that enforce time-reparametrization invariance at the quantum level.

We also stress that the Euclidean formulation of quantum gravity is known to suffer from the conformal-factor problem, namely from the fact that the Euclidean gravitational action is not bounded from below in the space of all metrics. The present minisuperspace calculation does not aim to solve this general problem. Rather, it should be understood as a semiclassical calculation performed around the closed-FLRW tunneling instanton, in the same spirit as the original Euclidean tunneling treatments. Alternative Lorentzian contour formulations provide a more fundamental way of addressing some of these issues, but they lead to a different path-integral prescription and are not the framework adopted here [9,13–15].

Let us now expand quadratically the Euclidean action $S_E[a(\tau)]$ around the instanton classical solution $a_c(\tau)$ using

$$a(\tau) = a_c(\tau) + \delta a(\tau) \tag{15}$$

with $\delta a(\tau)$ representing a small fluctuation such that $\delta a(0) = \delta a(\tau_F) = 0$, i.e. with Dirichlet boundary conditions. In particular, the quadratic contribution from the kinetic term reads $a_c(\delta a')^2 + 2a_c'\delta a \delta a'$, while the potential contributes $\frac{1}{2}V''(a_c)(\delta a)^2$. The mixed term $2a_c'\delta a\delta a'$ can be integrated by parts, and the term $a_c(\delta a')^2$ can be written in symmetric form using integration by parts. After these manipulations we find

$$\begin{aligned} S_E[a(\tau)] &= S_E[a_c(\tau)] \\ &+ \frac{1}{2}\int_0^{\tau_F} d\tau \int_0^{\tau_F} d\tau' \delta a(\tau')\, \frac{\delta^2 S_E}{\delta a(\tau)\delta a(\tau')}\, \delta a(\tau) , \end{aligned} \tag{16}$$

where

$$\begin{aligned} \frac{\delta^2 S_E}{\delta a(\tau)\delta a(\tau')} &= \delta(\tau-\tau')\Big[-2A\frac{d}{d\tau}\left(a_c(\tau)\frac{d}{d\tau}\right) \\ &+ A\left(V''(a_c(\tau)) - 2a_c''(\tau)\right)\Big] . \end{aligned} \tag{17}$$

Including these Gaussian fluctuations the propagator becomes [17,18]

$$\langle a_F = \sqrt{3}, \tau_F | a_I = 0, \tau_I = 0\rangle = F_E\, e^{-S_E[a_c(\tau)]/\hbar} \tag{18}$$

where

$$F_E = \mathcal{C}\,\det\left[\frac{1}{\hbar}\frac{\delta^2 S_E}{\delta a(\tau)\delta a(\tau')}\right]^{-1/2} \tag{19}$$

with $\mathcal{C}$ the appropriate (infinite) constant related to the measure of the path integral.

In contrast to ordinary vacuum decay in quantum field theory, where the Euclidean fluctuation operator has one negative eigenvalue leading to an imaginary part of the energy [22,23], the present model describes a fixed-endpoint tunneling amplitude in minisuperspace rather than a decay rate per unit spacetime volume. The Dirichlet boundary conditions remove the translational zero mode that is present in bounce calculations with time-translation invariance. Moreover, possible sign ambiguities of the reduced fluctuation determinant do not have the same interpretation as in false-vacuum decay, because the quantity computed here is not an imaginary part of an energy but the absolute semiclassical tunneling probability between two fixed boundary geometries. For this reason we take the absolute value of the determinant of quadratic fluctuations, ensuring that the prefactor remains real and positive. With this prescription,

$$P_T = |F_E|^2\, e^{-2S_E/\hbar} \tag{20}$$

represents the absolute semiclassical probability for tunneling in this minisuperspace, rather than a real-time decay rate.

4.1. Approximate Prefactor

To compute the prefactor F_E of Equation (20) we approximate the classical bounce $a_c(\tau)$ near the barrier maximum (inverted potential minimum) $a_m = \sqrt{3}/2$, where the Euclidean action can be expanded quadratically around a_m. Namely, for the calculation of Gaussian fluctuations instead of (9) we use the simplified action

$$S_E[a(\tau)] = A\int_0^{\tau_F} d\tau \left[a_m {a'}^2 + V(a_m) + \frac{1}{2}V''(a_m)(a-a_m)^2\right] \tag{21}$$

with $V''(a_m) = -\sqrt{3}$. In this approximation we get

$$\frac{\delta^2 S_E}{\delta a(\tau)\delta a(\tau')} = \delta(\tau-\tau')\Big[-\sqrt{3}A\frac{d^2}{d\tau^2} - \sqrt{3}A\Big] . \tag{22}$$

The Euclidean propagator for this quadratic problem is exactly that of an inverted harmonic oscillator of effective mass $m = \sqrt{3}A$ and effective frequency $\omega = 1$. For the harmonic oscillator of frequency ω, in real time the prefactor of the propagator is [17,18]

$$F = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t_F)}} \; . \tag{23}$$

For the Euclidean (imaginary time) inverted harmonic oscillator we have $t_F = -i\tau_F$ and consequently, taking into account that $\sin(-ix) = -i\sinh(x)$, the Euclidean prefactor is

$$F_E = \sqrt{\frac{m\omega}{2\pi\hbar \sinh(\omega\tau_F)}} \; . \tag{24}$$

In particular, for our problem with $m = \sqrt{3}A$, $\omega = 1$, and $\tau_F = \pi\sqrt{3}/2$ we obtain

$$F_E = \sqrt{\frac{\sqrt{3}A}{2\pi\hbar \sinh(\frac{\sqrt{3}\pi}{2})}} \; . \tag{25}$$

Thus, by using Equations (12) and (25) the tunneling probability (20) is given by

$$\begin{aligned} P_T &= \frac{\sqrt{3}}{4\sinh(\pi\sqrt{3}/2)} \left(\frac{3\pi c^3}{G\Lambda\hbar}\right) \exp\left(-\frac{3\pi c^3}{G\Lambda\hbar}\right) \\ &\simeq 0.018 \left(\frac{3\pi c^3}{G\Lambda\hbar}\right) \exp\left(-\frac{3\pi c^3}{G\Lambda\hbar}\right) . \end{aligned} \tag{26}$$

4.2. Exact Prefactor

To obtain an exact formula for the prefactor of the Gaussian fluctuations term we notice that Equation (17) can be re-written as

$$\frac{\delta^2 S_E}{\delta a(\tau)\delta a(\tau')} = \delta(\tau - \tau')\, 2A\, \mathcal{L} \; , \tag{27}$$

where

$$\mathcal{L} = \left[-\frac{d}{d\tau}\left(a_c(\tau)\frac{d}{d\tau}\right) + \frac{1}{2}\left(V''(a_c(\tau)) - 2a_c''(\tau)\right) \right] \tag{28}$$

is a quite nasty differential operator. Our strategy is to derive another operator, $\tilde{\mathcal{L}}$, such that $\det[\mathcal{L}] = \det[\tilde{\mathcal{L}}]$. The new operator $\tilde{\mathcal{L}}$ will be simpler and for it we can use the Gel'fand-Yaglom theorem [24,25], which also helps us to fix the constant $\mathcal{C}$ which appears in the definition of F_E, see Equation (19). This procedure, discussed in the Appendix A, gives the exact Euclidean prefactor of Gaussian fluctuations:

$$F_E = \sqrt{\frac{A}{2\pi\hbar}\frac{\Gamma(5/4)}{\Gamma(3/4)}} \; . \tag{29}$$

Therefore, recalling Equations (12) and (20) the tunneling probability reads

$$\begin{aligned} P_T &= \frac{\Gamma(5/4)^2}{2\Gamma(3/4)^2} \left(\frac{3\pi c^3}{G\Lambda\hbar}\right) \exp\left(-\frac{3\pi c^3}{G\Lambda\hbar}\right) \\ &\simeq 0.318 \left(\frac{3\pi c^3}{G\Lambda\hbar}\right) \exp\left(-\frac{3\pi c^3}{G\Lambda\hbar}\right) . \end{aligned} \tag{30}$$

This result provides a meaningful evaluation of the quantum tunneling probability, improving on the estimate obtained using the inverted harmonic oscillator approximation, Equation (26). Notice that the difference between Equations (26) and (30) is only in the numerical value of the prefactor. This is not fully surprising: the dominant contribution to the fluctuation determinant comes from the region near the instanton peak, where the potential is accurately described by the harmonic approximation.

The validity of the semiclassical approximation employed here relies on the saddle-point expansion of the Euclidean path integral. In the present model this requires the Euclidean action of the instanton to be large compared to $\hbar$, which translates into the condition $3\pi c^3/(2G\Lambda\hbar) \gg 1$. Under this assumption, higher-order (non-Gaussian) fluctuations are parametrically suppressed, and the Gaussian prefactor computed here provides the leading quantum correction to the tunneling probability. The prefactor changes the leading result only by an algebraic factor, while the dominant dependence on the cosmological constant remains exponentially controlled by the instanton action. Thus the correction is important for a consistent semiclassical normalization, but it does not alter the exponential hierarchy of nucleation probabilities. The results presented here provide a reliable semiclassical estimate of the tunneling probability and the associated prefactor, but do not capture fully nonperturbative quantum effects, which could be accessed only through numerical integration of the full path integral or direct solution of the Wheeler-DeWitt equation. Indeed, one possible beyond-semiclassical approach is the direct numerical solution of the Wheeler-DeWitt equation in minisuperspace, which fully enforces the Hamiltonian constraint and captures nonperturbative quantum effects [8,9]. Alternatively, one could try a numerical evaluation of the Euclidean path integral using discretization, lattice techniques, or Monte Carlo methods can account for higher-order fluctuations beyond the Gaussian level. Among these numerical alternatives, a direct numerical solution of the Wheeler-DeWitt equation appears more suitable for the present one-dimensional minisuperspace model, since it implements the Hamiltonian constraint exactly and reduces the problem to a well-defined ordinary differential equation. A numerical evaluation of the path integral would be technically more involved and would not provide additional advantages for a system with a single degree of freedom. However, these numerical methods are typically computationally intensive and less transparent than the analytical semiclassical approach presented here. Thus, the analytical calculation of the Gaussian prefactor remains valuable, as it gives a clear and tractable estimate of the first-order quantum corrections to the tunneling probability.

5. Conclusions

We have presented an analytical evaluation of the quantum tunneling probability for a closed FLRW universe, including both the exponential suppression and the prefactor due to Gaussian fluctuations around the instanton solution. The prefactor has been calculated in two ways: an approximated procedure based on a harmonic expansion near the barrier maximum, and a fully analytical calculation employing an isospectral transformation and the Gel'fand-Yaglom theorem. The main novelty of the present work is the explicit analytical evaluation of the Gaussian determinant associated with the reduced Euclidean fluctuation operator. The calculation clarifies how the quadratic prefactor modifies the leading tunneling probability: it changes the overall normalization by an algebraic factor proportional to $3\pi c^3/(G\Lambda\hbar)$, while the dominant dependence remains governed by the exponential instanton action. The result is therefore meaningful only in the semiclassical regime $S_E/\hbar \gg 1$ and within the minisuperspace truncation. Beyond the Gaussian approximation, the tunneling probability could in principle be obtained numerically either from the full Euclidean path integral, using discretization and Monte Carlo or lattice methods to capture higher-order quantum fluctuations, or directly by solving the Wheeler-DeWitt equation in minisuperspace, providing a fully nonperturbative evaluation of the nucleation probability. The tunneling probability calculated here should be interpreted as a semiclassical measure of the likelihood for universe nucleation in minisuperspace. The approach follows the Vilenkin tunneling proposal [3] and differs from the Hartle-Hawking no-boundary prescription [19], which would invert the sign of the exponential and favor smaller initial universes. Finally, although the present study is concerned with quantum cosmology rather than compact objects, recent investigations of charged black holes and electromagnetic effects in alternative theories of gravity provide complementary examples of how strong-gravity systems may be used to test gravitational dynamics beyond the simplest Einstein-Hilbert setting [26,27]. These topics are outside the scope of the present minisuperspace calculation, but they illustrate the broader relativistic context in which semiclassical and strong-field gravitational phenomena are currently being explored.

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Conflict of Interest

The author declares no conflict of interest. Given the role as Editorial Board Member, Luca Salasnich had no involvement in the peer review of this paper and had no access to information regarding its peer-review process. Full responsibility for the editorial process of this paper was delegated to another editor of the journal.

Use of AI and AI-Assisted Technologies

During the preparation of this work, the author used ChatGPT and Claude AI to have a triple check (author plus two AIs) of the analytical calculations, and also to prepare the graphical abstract. After using these tools, the author reviewed and edited the content as needed and take full responsibility for the content of the published article.

Appendix A

In this Appendix we provide the technical details of the calculation of the fluctuation determinant entering the prefactor F_E. Starting from the operator $\mathcal{L}$ of Equation (28), obtained from the second variation of the Euclidean action, we perform an isospectral transformation to a Schrödinger-like operator $\tilde{\mathcal{L}}$ and apply the Gel'fand-Yaglom theorem [24,25] to evaluate the corresponding functional determinant.

Let us consider the operator $\mathcal{L}$ of Equation (28) which satisfies the differential problem

$$0 = \mathcal{L}\, \delta a(\tau) \; . \tag{A1}$$

An isospectral transformation

$$\delta a(\tau) = \frac{\chi(\tau)}{\sqrt{a_c(\tau)}} \tag{A2}$$

with $a_c(\tau) = \sqrt{3}\sin(\tau/\sqrt{3})$ gives a Schrödinger-like (more precisely, Sturm-Liouville type) differential equation

$$0 = \tilde{\mathcal{L}}\, \chi(\tau) = \left[-\frac{d^2}{d\tau^2} + U(\tau) \right] \chi(\tau) \tag{A3}$$

with

$$U(\tau) = -\frac{3}{4} - \frac{1}{12} \frac{1}{\sin^2\left(\frac{\tau}{\sqrt{3}}\right)} \; . \tag{A4}$$

This transformation rewrites the original Sturm-Liouville problem in a Schrödinger-like form with standard second-derivative structure, so that the Gel'fand-Yaglom theorem [24, 25] can be applied directly. Since the transformation is isospectral, the functional determinants of $\mathcal{L}$ and $\tilde{\mathcal{L}}$ coincide.

In our problem the Gel'fand-Yaglom theorem [24,25] implies that

$$F_E = \sqrt{\frac{A}{\pi\hbar}} \left(\frac{\det \tilde{\mathcal{L}}}{\det \tilde{\mathcal{L}}_0} \right)^{-1/2} = \sqrt{\frac{A}{\pi\hbar}} \left(\frac{\chi(\tau_F)}{\chi_0(\tau_F)} \right)^{-1/2} , \tag{A5}$$

where the term $\sqrt{A/(\pi\hbar)}$ originates from the path integral measure and from the factor $2A$ appearing in the second variation of the Euclidean action. This ensures that the prefactor has the correct dimension and is consistent with the standard semiclassical limit. The reference operator $\tilde{\mathcal{L}}_0$ is introduced solely to define the determinant ratio appearing in the Gel'fand-Yaglom formula. Any regular operator with known solution satisfying the same boundary conditions can be used for this purpose. The final prefactor depends only on the ratio $\chi(\tau_F)/\chi_0(\tau_F)$ and is therefore independent of the specific choice of $\tilde{\mathcal{L}}_0$.

The function $\chi(\tau)$ satisfies Equation (A3) while $\chi_0(\tau)$ solves the equation

$$0 = \tilde{\mathcal{L}}_0\, \chi_0(\tau) \; , \tag{A6}$$

where $\tilde{\mathcal{L}}_0$ is the reference operator. In general, as said before, the choice of this reference operator is rather arbitrary, as any operator with a regular, known solution satisfying the same boundary conditions can be used; the physical prefactor is determined by the ratio of the solutions at the final time, which is invariant under such a choice.

According to the Gel'fand-Yaglom theorem [24,25], the functions $\chi(\tau)$ and $\chi_0(\tau)$ must satisfy Dirichlet boundary conditions at the lower endpoint:

$$\chi(0) = \chi_0(0) = 0 \; , \qquad \chi'(0) = \chi_0'(0) = 1 \tag{A7}$$

These conditions ensure the regularity and a unique normalization of the determinant ratio. Physically, the condition $\chi(0) = 0$ corresponds to vanishing fluctuations $\delta a(0) = 0$ at the starting point of the instanton trajectory, while $\chi'(0) = 1$ fixes the overall scale of the mode function. The reference operator $\tilde{\mathcal{L}}_0$ of Equation (A6) is chosen as

$$\tilde{\mathcal{L}}_0 = -\frac{d^2}{d\tau^2} + \frac{1}{4} \; , \tag{A8}$$

and its normalized solution, such that $\chi_0(0) = 0$ and $\chi_0'(0) = 1$, is given by

$$\chi_0(\tau) = 2 \sinh\left(\frac{\tau}{2}\right) \; . \tag{A9}$$

Notice that the reference operator (A8) is chosen for its constant coefficients and for the fact that its solution with Dirichlet boundary conditions, $\chi_0(\tau)$, is simple and regular. As previously stressed, the Gel'fand-Yaglom theorem [24, 25] states that the determinant ratio, and hence the prefactor, is completely determined by $\chi(\tau_F)/\chi_0(\tau_F)$, independent of the absolute normalization of either $\chi(\tau)$ or $\chi_0(\tau)$. In this way, the functional determinant is correctly normalized and provides a physically meaningful estimate of the tunneling prefactor, while keeping the calculation analytically tractable.

The determination of $\chi(\tau)$ and $\chi(\tau_F)$ is much more complicated. It is useful to introduce the variable

$$y = \cos\left(\frac{\tau}{\sqrt{3}}\right) \tag{A10}$$

and the function

$$\psi(y) = \chi(\tau(y)) \tag{A11}$$

such that

$$\chi(\tau_F) = \psi(y_F) \tag{A12}$$

with $\tau_F = \pi\sqrt{3}/2$ and $y_F = 0$. In terms of $\psi(y)$, Equation (A3) becomes

$$0 = \left[(1-y^2)\frac{d^2}{dy^2} - y\frac{d}{dy} + \frac{9}{4} - \frac{1}{4(1-y^2)} \right] \psi(y) \tag{A13}$$

where trigonometric functions do not appear explicitly. The problem becomes a bit simpler setting

$$\psi(y) = (1-y^2)^{1/2}\, u(y) \; . \tag{A14}$$

In this way, from Equation (A13) we find

$$0 = \left[(1-y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy} + 2 \right] u(y) \; . \tag{A15}$$

This is a Legendre equation, whose general solution reads [28]

$$u(y) = C_1\, P_1^{1/2}(y) + C_2\, Q_1^{1/2}(y) \tag{A16}$$

with $P_\nu^\mu(y)$ the Legendre polynomials of the first kind and $Q_\nu^\mu(y)$ the Legendre polynomials of the second kind. Here $\nu = 1$ and $\mu = 1/2$.

Regularity at $\tau = 0$ (i.e. $y = 1$) requires the solution to remain finite, which eliminates the divergent term proportional to $P_1^{1/2}(y)$, thus setting $C_1 = 0$. The normalization $\chi'(0) = 1$ then fixes C_2 uniquely. These conditions imply that $u(y)$ is regular and properly normalized at $y = 1$ with

$$C_1 = 0 \qquad \text{and} \qquad C_2 = \frac{8\sinh(\frac{\tau_F}{2})}{\sqrt{\pi}} \frac{\Gamma(3/4)}{\Gamma(5/4)} \; . \tag{A17}$$

Thus, the compact final formula of $\chi(\tau)$ is

$$\chi(\tau) = \frac{8\sinh(\frac{\pi\sqrt{3}}{4})}{\sqrt{\pi}} \frac{\Gamma(3/4)}{\Gamma(5/4)} \sin\left(\frac{\tau}{\sqrt{3}}\right) Q_1^{1/2}\left(\cos\left(\frac{\tau}{\sqrt{3}}\right)\right) . \tag{A18}$$

The value of $Q_1^{1/2}(0)$ is known explicitly:

$$Q_1^{1/2}(0) = \frac{\sqrt{\pi}}{2} \frac{\Gamma(3/4)}{\Gamma(5/4)}. \tag{A19}$$

Then it follows that

$$\chi(\tau_F) = \psi(y_F) = C_2\, Q_1^{1/2}(0) = 4\sinh\left(\frac{\tau_F}{2}\right) \frac{\Gamma(3/4)^2}{\Gamma(5/4)^2} . \tag{A20}$$

In conclusion, from Equations (A5), (A9), and (A20) with $\tau_F = \pi\sqrt{3}/2$ (i.e., $y_F = 0$), we obtain the prefactor F_E of Equation (29).

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