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Vector Duality for Set-Semidefinite Multiobjective Optimization Problems

Sorin-Mihai Grad [1,2,*] and Lidia Höpfner [3]

[1] Faculty of Mathematics, University of Duisburg-Essen, Thea-Leymann-Str. 9, 45127 Essen, Germany
[2] Department of Mathematics, Babeş-Bolyai University Cluj-Napoca, Kogălniceanu Street, No. 1, 400084 Cluj-Napoca, Romania
[3] Faculty of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, Germany
* Correspondence: smgrad.work@gmail.com

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Abstract: In this note we extend a vector duality approach for set-semidefinite multiobjective optimization problems consisting in the vector minimization with respect to a given convex cone of matrices of a matrix-valued function subject to both geometric and set-semidefinite cone-inequality constraints. Our contribution generalizes and improves earlier results from the literature.

Keywords: multiobjective optimization; matrix optimization; vector duality; weakly efficient solution

1. Introduction and Preliminaries

Optimization problems involving matrix-valued functions appear in various research fields ranging from elasticity theory to biology (see [1,2] for more on this) and their study gave birth to matrix optimization, among whose branches we mention here only semidefinite and copositive programming, respectively. One can find in the literature contributions to both scalar and multiobjective optimization where matrix-valued functions play important roles. Among the works presenting duality investigations for multiobjective matrix optimization problems we mention [1–8].

In this note we extend the vector duality approach for semidefinite multiobjective problems introduced in [3, 4] (following ideas from [4, 9–11]) to the more general framework of set-semidefinite multiobjective optimization problems that consist in the vector minimization with respect to a given convex cone of matrices of a matrix-valued function subject to both geometric and set-semidefinite inequality constraints. The concept of set-semidefinite optimization was introduced in [1, 2] and represents an umbrella for both semidefinite [12] and copositive programming [13], having other relevant special cases such as second-order optimization or semi-infinite optimization. This motivated us to propose a vector duality scheme for very general set-semidefinite multiobjective optimization problems, that can be adapted for any relevant special case. The proposed extension of the multiobjective dual problem introduced in [3] to the considered framework is not merely a cosmetic or a trivial generalization, because, as the entries of a dual variable matrix can be also negative, a one-to-one transfer of that construction is not possible, and the multiobjective dual problem proposed in this work corrects thus some inaccuracies in the one from [3].

Let X and Y be locally convex Hausdorff vector spaces, whose topological dual spaces are X^* and Y^*, respectively. By $\mathcal{L}(X, Y)$ we denote the vector space of the linear continuous mappings from X to Y. A set $K \subseteq X$ is said to be a *cone* if $tx \in K$ for all $x \in K$ and all $t \geq 0$. The *dual cone* to the cone $K \subseteq X$ is $K^* = \{x^* \in X^* : \langle x^*, x\rangle \geq 0\ \forall x \in K\}$, where by $\langle x^*, x\rangle$ we denote the value at $x \in X$ of the linear continuous functional $x^* \in X^*$. A convex cone $K \subseteq X$ induces on X the partial ordering "$\leqq_K$", defined by $x \leqq_K y$ when $y - x \in K$, where $x, y \in X$. Given a subset U of X, by $\operatorname{int} U$ and δ_U we denote its *interior* and *indicator function*, respectively.

For simplicity, we consider in this note only with symmetric matrices, noting that our investigations can be considered (like in [1,2]) for general real matrices. We denote the set of the *symmetric* $k \times k$ real matrices by $\mathcal{S}^k$.

The *entries* of a matrix $A \in \mathbb{R}^{k\times k}$ will be denoted by A_{ij}, $i,j = 1,\dots,k$, while its *trace* by $\operatorname{Tr} A$. The cone of the *positive semidefinite* symmetric $k\times k$ real matrices is $\mathcal{S}^k_+$ and it is self dual, while the cone of the *copositive* symmetric $k\times k$ real matrices is $\mathcal{S}^k_{++}$ and its dual is the set of the *completely positive* symmetric $k\times k$ real matrices. The *partial ordering* induced by a convex cone $C \subseteq \mathcal{S}^k$ on $\mathcal{S}^k$ is denoted by "$\leqq_C$", and when $A \leqq_C B$ and $A \neq B$, where $A, B \in \mathcal{S}^k$, we write "$A \leq_C B$". A matrix-valued function $H : \mathbb{R}^n \to \mathcal{S}^k$ is said to be *C-convex* if $H(tx + (1-t)y) \leqq_C tH(x) + (1-t)H(y)$ for all $x, y \in \mathbb{R}^n$ and all $t \in [0,1]$. The *Frobenius inner product* of two matrices $A, B \in \mathcal{S}^k$ is defined as $\langle A, B\rangle = \operatorname{Tr}(AB)$.

For a function $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we use the classical notations for its *domain* $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ and *epigraph* $\operatorname{epi} f = \{(x,r) \in X \times \mathbb{R} : f(x) \leq r\}$. The *conjugate function* of f is $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f^*(y) = \sup\{y^\top x - f(x) : x \in \mathbb{R}^n\}$. Between a function and its conjugate there is the *Young-Fenchel inequality* $f^*(y) + f(x) \geq y^\top x$ for all $x, y \in \mathbb{R}^n$. For a vector function $G : X \to Y$ and $Q \in Y^*$ we denote by $(QG) : X \to \mathbb{R}$ the function defined by $(QG)(x) = \langle Q, G(x)\rangle$.

Following [1,2], for a set $K \subseteq Y$ consider the *K-semidefinite cone* $C^K_{\mathcal{L}} = \{A \in \mathcal{L}(Y, Y^*) : \langle Ay, y\rangle \geq 0\ \forall y \in K\}$. For any $K \subseteq Y$, $C^K_{\mathcal{L}}$ is a convex cone and encompasses as special cases the classical cones of matrices, since when $Y = \mathbb{R}^k$ one has $C^{\mathbb{R}^k}_{\mathcal{S}^k} = \mathcal{S}^k_+$, $C^{\mathbb{R}^k_+}_{\mathcal{S}^k} = \mathcal{S}^k_{++}$ and $C^{\{0\}}_{\mathcal{S}^k} = \mathcal{S}^k$. In [1,2] one can find formulae for characterizing the interior and the dual cone of $C^K_{\mathcal{L}}$.

2. Multiobjective Set-Semidefinite Duality

Let the nonempty sets $S \subseteq X$ and $W \subseteq \mathbb{R}^k$, the matrix-valued function $F : X \to \mathcal{S}^k$ and the vector function $G : X \to \mathcal{L}(Y, Y^*)$. For $i,j \in \{1,\dots,k\}$, denote by $f_{ij} : \mathbb{R}^n \to \mathbb{R}$ the function defined as $f_{ij}(x) = (F(x))_{ij}$. The primal *multiobjective semidefinite optimization problem* we consider is (the vector minimization is done with respect to the convex cone $C^W_{\mathcal{L}} \subseteq \mathcal{S}^k$, a more general setting than considered in [3], where it was done with respect to $\mathcal{S}^k_+$)

$$(PVS) \qquad \operatorname*{Min}_{x\in\mathcal{A}} F(x),$$

where

$$\mathcal{A} = \left\{x \in S : G(x) \in -C^K_{\mathcal{L}}\right\}.$$

An element $\bar{x} \in \mathcal{A}$ is said to be an *efficient solution* to (PVS) if there exists no $x \in \mathcal{A}$ such that $F(x) \leq_{C^W_{\mathcal{L}}} F(\bar{x})$, and the set of all the efficient solutions to (PVS) is denoted by $\mathcal{E}(PVS)$. An element $\bar{x} \in \mathcal{A}$ is said to be a *properly efficient solution* to (PVS) (in the sense of linear scalarization) if there exists a $\Lambda \in \operatorname{int}(C^W_{\mathcal{L}})^*$ such that $\operatorname{Tr}(\Lambda F(\bar{x})) \leq \operatorname{Tr}(\Lambda F(x))$ for all $x \in \mathcal{A}$, and the set of all the properly efficient solutions to (PVS) (in the sense of linear scalarization) is denoted by $\mathcal{PE}_{LS}(PVS)$. A properly efficient solution $\bar{x}$ to (PVS) is also efficient to (PVS), but the opposite implication fails to hold in general. The corresponding notions for vector maximization problems can be defined in a similar manner.

Remark 1. *Similar vector optimization problems were considered in [3–7] for the case $Y = \mathbb{R}^m$ and with different additional hypotheses imposed on F, S and H.*

The vector dual problem we assign to (PVS) is inspired by the ones proposed in [3] (see also [4,9–11,14] for more on this duality concept), being

$$(DVS) \qquad \operatorname*{Max}_{(\Lambda,Q,P,V)\in\mathcal{B}} H(\Lambda, Q, P, V),$$

where

$$\mathcal{B} = \Big\{(\Lambda, Q, P, V) \in \operatorname{int}(C^W_{\mathcal{L}})^* \times (C^K_{\mathcal{L}})^* \times (X^*)^{k\times k} \times \mathbb{R}^{k\times k} : P = (p_{ij})_{i,j=1,\dots,k},$$
$$p_{ij} \in \operatorname{dom}(\Lambda_{ij} f_{ij})^*\ \forall i,j \in \{1,\dots,k\},\ -\sum_{\substack{i,j=1,\\ \Lambda_{ij}\neq 0}}^{k} p_{ij} \in \operatorname{dom}(QG)^*_S,\ \operatorname{Tr}(\Lambda V) = 0\Big\}$$

and, for $i,j = 1,\dots,k$,

$$(H(\Lambda, Q, P, V))_{ij} = V_{ij} - \begin{cases} \frac{1}{\Lambda_{ij}}(\Lambda_{ij} f_{ij})^*(p_{ij}) + \frac{1}{z(\Lambda)\Lambda_{ij}}(QG)^*_S\left(-\sum\limits_{\substack{i,j=1,\\ \Lambda_{ij}\neq 0}}^{k} p_{ij}\right), & \text{if } \Lambda_{ij} \neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

where $z(\Lambda)$ denotes the *number of nonzero entries* of the matrix Λ.

Remark 2. *The vector dual problem* (DVS) *is not merely a trivial generalization of the multiobjective dual problem introduced in [3] for a semidefinite multiobjective problem. The differences between these duals are not only cosmetic, and appeared due to the fact that the entries of the matrix* Λ *(in* (DVS)*) can also be negative. Not taking this into consideration could have affected the corresponding duality investigations. In this way we extend the mentioned duality concept towards a larger class of matrices, correcting moreover some inaccuracies observed in the multiobjective dual problem proposed in [3] concerning the possible negative entries of the matrix* Λ*. Note also that one can consider more general vector duality schemes for* (PVS) *(see [14] for an overview, also [4,15]), however, as specified, for instance, in [10], the vector dual problems specifically developed for certain classes of multiobjective optimization problems bring certain advantages.*

Remark 3. *One can replace in* $\mathcal{B}$ *the constraint equality* $\mathrm{Tr}(\Lambda V) = 0$ *by* $\mathrm{Tr}(\Lambda V) \leq 0$*, obtaining thus another vector dual problem to* (PVS) *with a larger feasible set and, consequently, image set, than* (DVS)*. Because of this feature, it could turn out to be more suitable for (primal-)dual numerical approaches, as the feasibility of the iterates would be easier to establish. However, we do not treat it here separately because the duality investigations regarding it follow analogously and its efficient solutions coincide with the ones of* (DVS)*. Another modification of the vector dual problem* (DVS) *could be by replacing in its constraints the interior of* $(C_{\mathcal{L}}^W)^*$ *with its quasi-interior, following the ideas explored in [4].*

Remark 4. *If* $(\Lambda, Q, P, V) \in \mathcal{B}$*, one can easily note that* $V \notin (C_{\mathcal{L}}^W \cup (-C_{\mathcal{L}}^W)) \setminus \{0\}$*.*

The weak duality statement for (PVS) and (DVS) follows.

Theorem 1. *There are no* $x \in \mathcal{A}$ *and* $(\Lambda, Q, P, V) \in \mathcal{B}$ *such that* $F(x) \leq_{C_{\mathcal{L}}^W} H(\Lambda, Q, P, V)$*.*

Proof. If for $x \in \mathcal{A}$ and $(\Lambda, Q, P, V) \in \mathcal{B}$ holds $F(x) \leq_{C_{\mathcal{L}}^W} H(\Lambda, Q, P, V)$, then $0 > \mathrm{Tr}\left(\Lambda(F(x) - H(\Lambda, Q, P, V))\right) = \sum_{i,j=1,\Lambda_{ij}\neq 0}^{k} (\Lambda_{ij} f_{ij}(x) + (\Lambda_{ij} f_{ij})^*(p_{ij})) + (QG)_S^*\big(-\sum_{i,j=1,\Lambda_{ij}\neq 0}^{k} p_{ij}\big) \geq (\sum_{i,j=1,\Lambda_{ij}\neq 0}^{k} p_{ij})^\top x - (QG)(x) - \delta_S(x) - (\sum_{i,j=1,\Lambda_{ij}\neq 0}^{k} p_{ij})^\top x \geq 0$ because $x \in \mathcal{A}$. As this cannot happen, the assumption we made is false. □

In order to prove strong duality for the primal-dual pair of multiobjective optimization problems $(PVS) - (DVS)$ one needs additional hypotheses. One can consider several types of *regularity conditions* (cf. [4,14]) or a variant of the Kurcyusz-Robinson-Zowe regularity condition (see [1] (13)), however, in order not to overcomplicate this investigation we consider here only a *generalized Slater* type one, namely

$$(RC) \quad \Big| \quad \exists x' \in S \text{ such that } F \text{ is continuous at } x' \text{ and } G(x') \in -\operatorname{int} C_{\mathcal{L}}^K.$$

The continuity part of this hypothesis can be verified in the usual manner (for instance, when the matrix-valued objective function is everywhere continuous), and the interior of such a cone is described in ([1] Theorem 2.29).

Theorem 2. *If* S *is a convex set,* F *a* $C_{\mathcal{L}}^W$*-convex matrix-valued function,* H *a* $C_{\mathcal{L}}^K$*-convex vector function,* $\bar{x} \in \mathcal{PE}_{LS}(PVS)$ *and the regularity condition* (RC) *is fulfilled, there exists* $(\bar{\Lambda}, \bar{Q}, \bar{P}, \bar{V}) \in \mathcal{E}(DVS)$ *such that* $F(\bar{x}) = H(\bar{\Lambda}, \bar{Q}, \bar{P}, \bar{V})$*.*

Proof. Since $\bar{x}$ is properly efficient to (PVS), there exists a $\bar{\Lambda} \in \operatorname{int}(C_{\mathcal{L}}^W)^*$ such that $\mathrm{Tr}(\bar{\Lambda}F(\bar{x})) \leq \mathrm{Tr}(\bar{\Lambda}F(x))$ for all $x \in \mathcal{A}$. The fulfillment of (RC) yields (cf. [14] Theorem 3.2.12) strong duality for the scalarized optimization problem attached to (PVS)

$$\inf_{x\in\mathcal{A}} \mathrm{Tr}(\bar{\Lambda}F(x))$$

and its Fenchel-Lagrange dual

$$\sup_{\substack{Q\in(C_{\mathcal{L}}^K)^*,\\ T\in X^*}} \Big\{ -(\bar{\Lambda}F)^*(T) - (QG)_S^*(-T) \Big\},$$

thus the latter has the optimal solutions $\bar{Q}$ and $\bar{T}$ that fulfill

$$\mathrm{Tr}(\bar{\Lambda}F(\bar{x})) = -(\bar{\Lambda}F)^*(\bar{T}) - (\bar{Q}G)_S^*(-\bar{T}). \tag{1}$$

Employing ([14] Theorem 3.2.7), the continuity of F at a point $x' \in S$ secured by (RC) yields the existence of some $\bar{p}_{ij} \in X^*$, $i,j = 1,\dots,k$, such that $\sum_{i,j=1}^k \bar{p}_{ij} = \bar{T}$ and

$$(\bar{\Lambda}F)^*(\bar{T}) = \sum_{i,j=1}^{k} (\bar{\Lambda}_{ij}f_{ij})^*(\bar{p}_{ij}) = \sum_{\substack{i,j=1,\\ \bar{\Lambda}_{ij}\neq 0}}^{k} (\bar{\Lambda}_{ij}f_{ij})^*(\bar{p}_{ij}),$$

because $\bar{\Lambda}_{ij} = 0$ implies $\bar{p}_{ij} = 0$. Then $\sum_{i,j=1,\bar{\Lambda}_{ij}\neq 0}^k \bar{p}_{ij} = \bar{T}$ and one has $\bar{p}_{ij} \in \operatorname{dom}(\bar{\Lambda}_{ij}f_{ij})^*$ for all $i,j \in \{1,\dots,k\}$ and $-\sum\limits_{i,j=1,\Lambda_{ij}\neq 0}^{k} p_{ij} \in \operatorname{dom}(QG)_S^*$. For $i,j \in \{1,\dots,k\}$ take

$$\bar{V}_{ij} = f_{ij}(\bar{x}) + \frac{1}{\bar{\Lambda}_{ij}}(\bar{\Lambda}_{ij}f_{ij})^*(\bar{p}_{ij}) + \frac{1}{z(\bar{\Lambda})\bar{\Lambda}_{ij}}(\bar{Q}G)_S^*\left(-\sum_{\substack{i,j=1,\\ \bar{\Lambda}_{ij}\neq 0}}^{k} \bar{p}_{ij}\right)$$

if $\bar{\Lambda}_{ij} \neq 0$, and $\bar{V}_{ij} = f_{ij}(\bar{x})$ otherwise. Then, employing also (1), one gets

$$\operatorname{Tr}(\bar{\Lambda}\bar{V}) = \sum_{\substack{i,j=1,\\ \bar{\Lambda}_{ij}\neq 0}}^{k} \bar{\Lambda}_{ij}\left(f_{ij}(\bar{x}) + \frac{1}{\bar{\Lambda}_{ij}}(\bar{\Lambda}_{ij}f_{ij})^*(\bar{p}_{ij}) + \left(\frac{1}{z(\bar{\Lambda})\bar{\Lambda}_{ij}}\right)(\bar{Q}G)_S^*\left(-\sum_{\substack{i,j=1,\\ \bar{\Lambda}_{ij}\neq 0}}^{k} \bar{p}_{ij}\right)\right) = 0.$$

Consequently, after denoting $\bar{P} = (\bar{p}_{ij})_{i,j=1,\dots,k}$, one notices that $(\bar{\Lambda},\bar{Q},\bar{P},\bar{V}) \in \mathcal{B}$. Assuming that $(\bar{\Lambda},\bar{Q},\bar{P},\bar{V}) \notin \mathcal{E}(DVS)$, Theorem 1 yields a contradiction, therefore $(\bar{\Lambda},\bar{Q},\bar{P},\bar{V}) \in \mathcal{E}(DVS)$. □

Last but not least, let us give necessary and sufficient optimality conditions for the primal-dual pair of multiobjective optimization problems $(PVS) - (DVS)$.

Theorem 3.

(a) *If S is a convex set, F a $C_{\mathcal{L}}^W$-convex matrix-valued function, H a $C_{\mathcal{L}}^K$-convex vector function, $\bar{x} \in \mathcal{PE}_{LS}(PVS)$ and the regularity condition (RC) is fulfilled, there exists $(\bar{\Lambda},\bar{Q},\bar{P},\bar{V}) \in \mathcal{E}(DVS)$ such that*

- *(i)* $F(\bar{x}) = H(\bar{\Lambda},\bar{Q},\bar{P},\bar{V})$;
- *(ii)* $\bar{\Lambda}_{ij}f_{ij}(\bar{x}) + (\bar{\Lambda}_{ij}f_{ij})^*(\bar{p}_{ij}) = \langle \bar{p}_{ij}, \bar{x}\rangle$ *for all* $i,j \in \{1,\dots,k\}$;
- *(iii)* $(\bar{Q}G)_S^*\left(-\sum\limits_{i,j=1}^{k} \bar{p}_{ij}\right) = -\left\langle \sum\limits_{i,j=1}^{k} \bar{p}_{ij}, \bar{x}\right\rangle$;
- *(iv)* $\langle \bar{Q}, G(\bar{x})\rangle = 0$;
- *(v)* $\operatorname{Tr}(\bar{\Lambda}\bar{V}) = 0$.

(b) *Assume that $\bar{x} \in \mathcal{A}$ and $(\bar{\Lambda},\bar{Q},\bar{P},\bar{V}) \in \operatorname{int}(C_{\mathcal{L}}^W)^* \times (C_{\mathcal{L}}^K)^* \times (X^*)^{k\times k} \times \mathbb{R}^{k\times k}$ fulfill the relations (i)–(v). Then $\bar{x} \in \mathcal{PE}_{LS}(PVS)$ and $(\bar{\Lambda},\bar{Q},\bar{P},\bar{V}) \in \mathcal{E}(DVS)$.*

Proof.

(a) The existence of a $(\bar{\Lambda},\bar{Q},\bar{P},\bar{V}) \in \mathcal{E}(DVS)$ such that $F(\bar{x}) = H(\bar{\Lambda},\bar{Q},\bar{P},\bar{V})$ is guaranteed by Theorem 2. The relations (i) and (v) are thus satisfied. Moreover, for the so-obtained $\bar{V}$ it holds

$$0 = \operatorname{Tr}(\bar{\Lambda}\bar{V}) = \sum_{\substack{i,j=1,\\ \bar{\Lambda}_{ij}\neq 0}}^{k}\left(\bar{\Lambda}_{ij}f_{ij}(\bar{x}) + (\bar{\Lambda}_{ij}f_{ij})^*(\bar{p}_{ij}) + \frac{1}{z(\bar{\Lambda})}(\bar{Q}G)_S^*\left(-\sum_{\substack{i,j=1,\\ \bar{\Lambda}_{ij}\neq 0}}^{k} \bar{p}_{ij}\right)\right). \quad (2)$$

On the other hand, the Young-Fenchel inequality yields $\bar{\Lambda}_{ij}f_{ij}(\bar{x}) + (\bar{\Lambda}_{ij}f_{ij})^*(\bar{p}_{ij}) \geq \bar{p}_{ij}^\top \bar{x}$ for all $i,j \in \{1,\dots,k\}$, and

$$(\bar{Q}G)_S^*\left(-\sum_{\substack{i,j=1,\\ \bar{\Lambda}_{ij}\neq 0}}^{k} \bar{p}_{ij}\right) + \langle \bar{Q}, G(\bar{x})\rangle \geq \left(-\sum_{\substack{i,j=1,\\ \bar{\Lambda}_{ij}\neq 0}}^{k} \bar{p}_{ij}\right)^\top \bar{x},$$

which, taking into consideration (2), imply $\langle \bar{Q}, G(\bar{x}) \rangle \geq 0$. But $\langle \bar{Q}, G(\bar{x}) \rangle \leq 0$, because $\bar{Q} \in (C^K_{\mathcal{L}})^*$ and $G(\bar{x}) \in -C^K_{\mathcal{L}}$, and since $\bar{\Lambda}_{ij} = 0$ implies $\bar{p}_{ij} = 0$, the Young-Fenchel inequalities given above are fulfilled as equalities and $\langle \bar{Q}, G(\bar{x}) \rangle = 0$, hence relations $(ii) - (iv)$ are fulfilled, too.

(b) From (ii) it follows that $\bar{p}_{ij} \in \mathrm{dom}(\bar{\Lambda}_{ij} f_{ij})^*$ for all $i, j \in \{1, \ldots, k\}$. Moreover, whenever $\bar{\Lambda}_{ij} = 0$ one immediately obtains $\bar{p}_{ij} = 0$. Further, (iii) yields $-\sum_{i,j=1, \bar{\Lambda}_{ij} \neq 0}^{k} \bar{p}_{ij} = -\sum_{i,j=1 \neq 0}^{k} \bar{p}_{ij} \in \mathrm{dom}(QG)^*_S$. Because of (v), it follows that $(\bar{\Lambda}, \bar{Q}, \bar{P}, \bar{V}) \in \mathcal{B}$.

Summing up (ii) for all $i, j \in \{1, \ldots, k\}$ and adding also (iii) one obtains (recalling that $\bar{p}_{ij} = 0$ whenever $\bar{\Lambda}_{ij} = 0$)

$$\sum_{i,j=1}^{k} \left(\bar{\Lambda}_{ij} f_{ij}(\bar{x}) + (\bar{\Lambda}_{ij} f_{ij})^*(\bar{p}_{ij}) \right) + (\bar{Q}G)^*_S \left(- \sum_{\substack{i,j=1, \\ \bar{\Lambda}_{ij} \neq 0}}^{k} \bar{p}_{ij} \right) = 0,$$

that yields strong duality for the scalarized optimization problem attached to (PVS)

$$\inf_{x \in \mathcal{A}} \mathrm{Tr}\,(\bar{\Lambda} F(x))$$

and its Fenchel-Lagrange dual problem. This yields immediately $\bar{x} \in \mathcal{PE}_{LS}(PVS)$ (due to the definition of properly efficient solutions). Assuming $(\bar{\Lambda}, \bar{Q}, \bar{P}, \bar{V}) \notin \mathcal{E}(DVS)$, i.e. the existence of a $(\tilde{\Lambda}, \tilde{Q}, \tilde{P}, \tilde{V}) \in \mathcal{B}$ such that $H(\bar{\Lambda}, \bar{Q}, \bar{P}, \bar{V}) \leq_{C^w_{\mathcal{L}}} H(\tilde{\Lambda}, \tilde{Q}, \tilde{P}, \tilde{V})$, (i) would yield a contradiction to the weak duality statement from Theorem 1. Consequently, the efficiency of $(\bar{\Lambda}, \bar{Q}, \bar{P}, \bar{V})$ to (DVS) follows immediately. □

Remark 5. *Of interest and subject to further research are a converse duality statement regarding the primal-dual pair of multiobjective optimization problems* $(PVS) - (DVS)$ *and the development of an* ε*-duality framework for investigating* (PVS) *beyond the convex setting by exploiting ideas from [4,16].*

Remark 6. *Specializing the spaces and cones considered in this note, one can rediscover or improve different results from the literature, especially the duality investigations from [3]. Additionally, one could try to compare the optimality conditions obtained in this work with those derived via Lagrange duality in [1] for vector-optimization problems with similar constraints as in our investigations but with a different type of objective function, under additional Fréchet differentiability hypotheses on the involved vector and matrix functions.*

Remark 7. *Further topics to be addressed in subsequent work concern potential applications of the proposed duality approach in actually solving set-semidefinite multiobjective optimization problems, for instance via primal-dual methods.*

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