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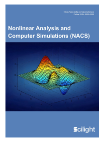

Article

Solutions of Riccati Differential Equations by Galerkin Method Using Fibonacci and Lucas Polynomials

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Abstract: In our present paper, we have obtained approximate solutions of non-linear quadratic Riccati differential equations using the Galerkin method. We considered Fibonacci polynomials and then Lucas polynomials as basis functions i.e., we presented two numerical methods: Fibonacci Galerkin method and Lucas Galerkin method. We have given some applications of this method by solving some differential equations. Then compare the approximate solution with the exact solution and approximate solutions with other methods. From this comparison and error analysis we can say that our method gives better results.

Keywords: Fibonacci polynomials; Lucas polynomials; Galerkin method; Riccati equation

2020 MSC: 11B39; 65A99; 11C08; 11Y99

1. Introduction

Fibonacci and Lucas sequences [1] are defined by second order recurrence relations. First two terms of Fibonacci sequence are $0,1$ and of Lucas sequence are $2,1$. Next terms are obtained by adding the previous two terms. These sequences are of great interest and have applications in many fields [2–10] like coding/decoding, communication, cryptography, music, medicine and Diophantine equations. Fibonacci and Lucas sequences are extended to Fibonacci polynomial and Lucas polynomial. These are continuous polynomials on $\mathbb{R}$.

Fibonacci polynomials [11] are defined by the following recurrence relation

$$f_{n+2}(x) = x f_{n+1}(x) + f_n(x),\ \forall n \geq 1 \quad \text{and} \quad f_0(x) = 1, f_1(x) = x.$$

Lucas polynomials [11] are defined by the following recurrence relation

$$l_{n+2}(x) = x l_{n+1}(x) + l_n(x),\ \forall n \geq 1 \quad \text{and} \quad l_0(x) = 2, l_1(x) = x.$$

Nalli et al. [12] defined generalized forms of Fibonacci and Lucas polynomials.

Galerkin method of weighted residuals is used to solve differential equations with the help of basis functions. Galerkin method is named after the Soviet mathematician Boris Galerkin. It is a well known numerical method for solving differential equations. Many researchers solved a variety of differential equations with different kind of methods and basis functions.

Rannacher [13] solved partial differential equations with this method. In [14] Dogan et al. solved regularised long wave equation

$$U_t + U_x + \varepsilon U U_x - \mu U_{xxt} = 0$$

where ε and μ are positive parameters and the subscripts t and x denote differentiation, with the physical boundary conditions $U \to 0$ as $|x| \to 0$ using Galerkin method. Yan and Shu [15] solved time dependant partial differential equations with local discontinuous Galerkin method.

For some linear and non-linear differential equations it becomes very difficult to find exact solutions, in these cases we use numerical methods to find approximate solutions of these differential equations. Riccati equation is one of these equations. Riccati equation is quadratic non-linear differential equation. It is one of the significant equation from the class of non-linear quadratic equations. In literature several authors solved non-linear Riccati differential equations(RDE) by different numerical methods [16–21].

Batiha [16] solved RDE by multistage variational iteration method (MVIM). Mishra and Rani [22] applied modified Laplace Adomian decomposition method to approximate the solution of non-linear quadratic Riccati differential equation. Roba et al. [23] solved Riccati differential equation by fifth order predictor-corrector method, in which Newton's backward difference interpolation formula is also used. In [24] Kassahun et al. used eighth order predictor-corrector method to solve RDE.

Ghorbani et al. [18] presented piecewise-truncated variational iteration algorithm to solve RDE. Baizar and Eslami [17] introduced differential transform method (DTM) to solve RDE. Allahviranloo et al. [25] solved RDE by Adomian's decomposition method (ADM), modified Adomian's decomposition method (MADM), variational iteration method (VIM), modified variational iteration method (MVIM), homotopy perturbation method (HPM), modified homotopy perturbation method (MHPM) and homotopy analysis method (HAM). Khalid et al. [26] introduced Perturbation iteration algorithm (PIA) to solve RDE.

Koonprasert et al. [27] solved linear and non-linear boundary value problems using Galerkin method with basis functions as Fibonacci polynomials. They named this method Fibonacci Galerkin method.

In the current paper, we will approximate the solution of non-linear quadratic Riccati differential equations by using Galerkin method. We will use Fibonacci polynomials as basis functions. We named this method Fibonacci Galerkin method (FGM). We also approximate solution by using Galerkin method and Lucas polynomials as basis functions, we call this method Lucas Galerkin method (LGM). Further, we give comparisons among exact and our obtained solutions with the help of tables and graphs.

2. Methodology

Non-linear quadratic Riccati differential equation is given as

$$\frac{dy}{dt} = q(t)y(t) + r(t)y^2(t) + p(t),\ y(0) = a \ \text{ on the interval } [0,t],$$

here $y(t)$ is unknown function and $q(t), r(t), p(t)$ are known scalar functions and a is arbitrary constant. Consider the residual function

$$R(t) = \frac{dy}{dt} - (q(t)y(t) + r(t)y^2(t)) - p(t)$$

and $w_j(t),\ j = 1$ to n be the weight function, then by Galerkin method of weighted residuals

$$\int_0^t R(t)w_j(t)dt = 0, \ \text{ for } j = 1 \text{ to } n$$

$$\int_0^t \left(\frac{dy}{dt} - (q(t)y(t) + r(t)y^2(t)) - p(t)\right)w_j(t)dt = 0, \ \text{ for } j = 1 \text{ to } n$$

$$\int_0^t \frac{dy}{dt}w_j(t)dt - \int_0^t [q(t)y(t) + r(t)y^2(t)]w_j(t)dt - \int_0^t p(t)w_j(t)dt = 0, \ \text{ for } j = 1 \text{ to } n$$

$$w_j(t)y(t)|_0^t - \int_0^t w_j'(t)y(t)dt - \int_0^t [q(t)y(t) + r(t)y^2(t)]w_j(t)dt - \int_0^t p(t)w_j(t)dt = 0, \ \text{ for } j = 1 \text{ to } n$$

$$w_j(t)y(t) - w_j(0)y(0) - \int_0^t w_j'(t)y(t)dt - \int_0^t [q(t)y(t) + r(t)y^2(t)]w_j(t)dt - \int_0^t p(t)w_j(t)dt = 0, \ \text{ for } j = 1 \text{ to } n$$

$$\int_0^t w_j'(t)y(t)dt + \int_0^t [q(t)y(t) + r(t)y^2(t)]w_j(t)dt - w_j(t)y(t) + w_j(0)y(0) + \int_0^t p(t)w_j(t)dt = 0, \ \text{ for } j = 1 \text{ to } n$$

Let the approximate solution of $y(t)$

$$y(t) = \sum_{i=1}^{N} c_i\phi_i(t),$$

here $\phi_i(t)$ are basis functions and $c_i's$ are artbitrary constants then

$$\sum_{i=1}^{N} c_i \int_0^t w_j'(t)\phi_i(t)dt + \sum_{i=1}^{N} c_i \int_0^t [q(t)\phi_i(t) + r(t)\phi_i(t) \sum_{k=1}^{N} c_k\phi_k(t)]w_j(t)dt - \sum_{i=1}^{N} w_j(t)c_i\phi_i(t) + aw_j(0) + \int_0^t p(t)w_j(t)dt = 0,$$

where $j = 1$ to n.

This is a system of non-linear equations. We will solve these equations by Newton's method and obtain approximate solution.

In our present paper, we will use Fibonacci Galerkin method i.e., we will consider Fibonacci plonomials as basis functions and $w_j(t) = F_j(t)$. Further, Lucas Galerkin method is also introduced by using Lucas polynomials as basis functions and $w_j(t) = L_j(t)$.

3. Numerical Examples

Example 1. *To find the approximate solution by Fibonacci Galerkin method of the following non-linear quadratic Riccati differential equation*

$$y'(t) = 1 + y^2(t),\ y(0) = 0$$

in the interval $[0, 0.1]$, *whose exact solution is* $y(t) = \tan t$.

$$y'(t) - y^2(t) = 1,\ y(0) = 0$$

$$\int_o^t [y'(t) - y^2(t)]F_j(t)dt = \int_0^t F_j(t)dt$$

$$F_j(t)y(t)|_0^t - \int_0^t F_j'(t)y(t)dt - \int_0^t y^2(t)F_j(t)dt = \int_0^t F_j(t)dt$$

$$\int_0^t F_j'(t)y(t)dt + \int_0^t y^2(t)F_j(t)dt = F_j(t)y(t) - \int_0^t F_j(t)dt$$

For $n = 3$ *and basis functions as Fibonacci polynomials, let* $F_1 = 1, F_2 = t, F_3 = 1 + t^2$

$$y(t) = \sum_{i=1}^{3} c_iF_i(t),\ c_i's\ \ are\ \ arbitrary\ \ constants$$

$$\sum_{i=1}^{3} c_i \left[\int_0^t F_j'(t)F_i(t)dt + \int_0^t F_i(t)(\sum_{k=1}^{3} C_kF_k(t))dt\right] = F_j(t)y(t) - \int_0^t F_j(t)dt,\ j = 1\ to\ 3$$

We will get the following non-linear equations

$$-c_1 - tc_2 - (1+t^2)c_3 + tc_1^2 + \frac{t^3}{3}c_2^2 + \left(\frac{t^5}{5} + \frac{2t^3}{3} + t\right)c_3^2 + t^2c_1c_2 + \left(\frac{t^4}{2} + t^2\right)c_2c_3 + 2\left(\frac{t^3}{3} + t\right)c_1c_3 + t = 0 \quad (1)$$

$$-\frac{t^2}{2}c_2 - \frac{2t^3}{3}c_3 + \frac{t^2}{2}c_1^2 + \frac{t^4}{4}c_2^2 + \left(\frac{t^6}{6} + \frac{t^4}{2} + \frac{t^2}{2}\right)c_3^2 + \frac{2t^3}{3}c_1c_2 + \left(\frac{2t^5}{5} + \frac{2t^3}{3}\right)c_2c_3 + \left(\frac{t^4}{2} + t^2\right)c_1c_3 + \frac{t^2}{2} = 0 \quad (2)$$

$$\begin{aligned} &-c_1 - \left(t + \frac{t^3}{3}\right)c_2 - \left(1 + t^2 + \frac{t^4}{2}\right)c_3 + \left(\frac{t^3}{3} + t\right)c_1^2 + \left(\frac{t^5}{5} + \frac{t^3}{3}\right)c_2^2 + \left(\frac{3t^5}{5} + t^3 + a\right)c_3^2 + \left(\frac{t^4}{2} + t^2\right)c_1c_2 \\ &+ \left(\frac{t^6}{3} + t^4 + t^2\right)c_2c_3 + 2\left(\frac{t^5}{5} + \frac{2t^3}{3} + t\right)c_1c_3 + \left(\frac{t^3}{3} + t\right) = 0 \end{aligned} \quad (3)$$

We solve above equations for unkowns by Newton method and obtain approximate solution of $y(t)$ *shown in Table 1.*

Now for $n = 4,\ F_1 = 1,\ F_2 = t,\ F_3 = 1 + t^2,\ F_4 = t^3 + 2t$

$$y(t) = \sum_{i=1}^{4} c_iF_i(t)$$

$$\sum_{i=1}^{4} c_i \left[\int_0^t F_j'(t)F_i(t)dt + \int_0^t F_i(t) \sum_{k=1}^{4} C_kF_k(t)dt\right] = F_j(t)y(t) - \int_0^t F_j(t)dt,\ j = 1\ to\ 4$$

Table 1. Comparison of exact and approximate solution of Example 1 for $n = 3$.

t	Approximate Solution	Exact Solution	Error
0.01	0.010000334	0.010000333	9.9×10^{-10}
0.02	0.020002675	0.020002667	8.0×10^{-09}
0.03	0.030009050	0.030009003	4.7×10^{-08}
0.04	0.040021494	0.040021347	1.5×10^{-07}
0.05	0.050042072	0.050041708	3.6×10^{-07}
0.06	0.060072865	0.060072104	7.6×10^{-07}
0.07	0.070115981	0.070114558	1.4×10^{-06}
0.08	0.080173557	0.080171105	2.4×10^{-06}
0.09	0.090247757	0.090243790	3.9×10^{-06}
0.1	0.100340780	0.100334672	6.1×10^{-06}

We will get the following non-linear equations

$$\begin{aligned}&-c_1 - tc_2 - (1+t^2)c_3 - (2t+t^3)c_4 + tc_1^2 + \frac{t^3}{3}c_2^2 + \left(\frac{t^5}{5} + \frac{2t^3}{3} + a\right)c_3^2 + \left(\frac{t^7}{7} + \frac{4t^5}{5} + \frac{4t^3}{3}\right)c_4^2 + t^2 c_1 c_2\\&+2\left(\frac{t^3}{3} + t\right)c_1c_3 + \left(\frac{t^4}{2} + 2t^2\right)c_1c_4 + \left(\frac{t^4}{2} + t^2\right)c_2c_3 + 2\left(\frac{t^5}{5} + \frac{2t^3}{3}\right)c_2c_4 + \left(\frac{t^6}{3} + \frac{3t^4}{2} + 2t^2\right)c_3c_4 + t = 0\end{aligned} \tag{4}$$

$$\begin{aligned}&-\frac{t^2}{2}c_2 - \frac{2t^3}{3}c_3 - \left(\frac{3t^4}{4} + t^2\right)c_4 + \frac{t^2}{2}c_1^2 + \frac{t^4}{4}c_2^2 + \left(\frac{t^6}{6} + \frac{t^4}{2} + \frac{t^2}{2}\right)c_3^2 + \left(\frac{t^8}{8} + \frac{2t^6}{3} + t^4\right)c_4^2 + \frac{2t^3}{3}c_1c_2\\&+\left(\frac{t^4}{2} + a^2\right)c_1c_3 + 2\left(\frac{t^5}{5} + \frac{2t^3}{3}\right)c_1c_4 + 2\left(\frac{t^5}{5} + \frac{t^3}{3}\right)c_2c_3 + \left(\frac{t^6}{3} + t^4\right)c_2c_4 + 2\left(\frac{t^7}{7} + \frac{3t^5}{5} + \frac{2t^3}{3}\right)c_3c_4 + \frac{t^2}{2} = 0\end{aligned} \tag{5}$$

$$\begin{aligned}&-c_1 - \left(\frac{t^3}{3} + t\right)c_2 - \left(\frac{t^4}{2} + t^2 + 1\right)c_3 - \left(\frac{3t^5}{5} + \frac{5t^3}{3} + 2t\right)c_4 + \left(\frac{t^3}{3} + t\right)c_1^2 + \left(\frac{t^5}{5} + \frac{t^3}{3}\right)c_2^2\\&+\left(\frac{t^7}{7} + \frac{3t^5}{5} + t^3 + a\right)c_3^2 + \left(\frac{t^9}{9} + \frac{5t^7}{7} + \frac{8t^5}{5} + \frac{4t^3}{3}\right)c_4^2 + \left(\frac{t^4}{2} + t^2\right)c_1c_2 + 2\left(\frac{t^5}{5} + \frac{2t^3}{3} + a\right)c_1c_3\\&+\left(\frac{t^6}{3} + \frac{3t^4}{2} + 2a^2\right)c_1c_4 + \left(\frac{t^6}{3} + t^4 + t^2\right)c_2c_3 + 2\left(\frac{t^7}{7} + \frac{3t^5}{5} + \frac{2t^3}{3}\right)c_2c_4\\&+2\left(\frac{t^8}{8} + \frac{4t^6}{6} + \frac{5t^4}{4} + t^2\right)c_3c_4 + \left(\frac{t^3}{3} + t\right) = 0\end{aligned} \tag{6}$$

$$\begin{aligned}&-\left(\frac{t^4}{4} + t^2\right)c_2 - \left(\frac{2t^5}{5} + \frac{4t^3}{3} + 2t\right)c_3 - \left(\frac{t^6}{2} + 2t^4 + 2t^2\right)c_4 + \left(\frac{t^4}{4} + t^2\right)c_1^2 + \left(\frac{t^6}{6} + \frac{t^4}{2}\right)c_2^2\\&+\left(\frac{t^8}{8} + t^5 + \frac{5t^4}{4} + t^2\right)c_3^2 + \left(\frac{t^{10}}{10} + \frac{3t^8}{4} + 2t^6 + 2t^4\right)c_4^2 + 2\left(\frac{t^5}{5} + \frac{2t^3}{3}\right)c_1c_2 + 2\left(\frac{t^6}{6} + \frac{3t^4}{4} + t^2\right)c_1c_3\\&+2\left(\frac{t^7}{7} + \frac{4t^5}{5} + \frac{4t^3}{3}\right)c_1c_4 + 2\left(\frac{t^7}{7} + \frac{3t^5}{5} + \frac{2t^3}{3}\right)c_2c_3 + \left(\frac{t^8}{4} + \frac{4t^6}{3} + 2t^4\right)c_2c_4\\&+2\left(\frac{t^9}{9} + \frac{5t^7}{7} + \frac{8t^5}{5} + \frac{4t^3}{3}\right)c_3c_4 - \left(\frac{t^4}{4} + t^2\right) = 0\end{aligned} \tag{7}$$

In Table 2 and Figure 1, comparison between exact and approximate solution is shown by FGM and LGM. Table 2 also shows comparison of errors by presented methods and Method in [22].

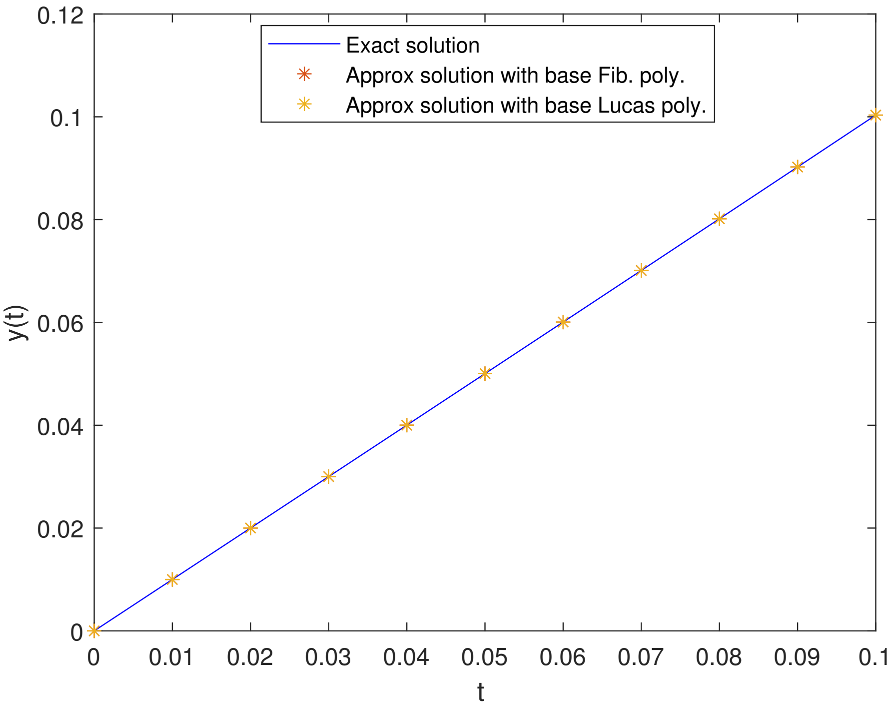

Figure 1. Comparison of presented approximate solution with exact solution for Example 1.

Table 2. Comparison of exact and approximate solution of Example 1 for $n = 4$.

t	Exact Solution	$n=4$ for Fibonacci Polynomials		$n=4$ for Lucas Polynomials		Method [22]
		Approximate Solution	Error	Approximate Solution	Error	
0.01	0.010000333346667	0.010000333346667	0	0.010000333346668	9.9×10^{-16}	2.5×10^{-07}
0.02	0.020002667093402	0.020002667093403	9.9×10^{-16}	0.020002667093424	2.2×10^{-14}	2.0×10^{-06}
0.03	0.030009003241181	0.030009003241180	1.0×10^{-15}	0.030009003241309	1.3×10^{-13}	6.8×10^{-06}
0.04	0.040021346995515	0.040021346995514	1.0×10^{-15}	0.040021346995832	3.2×10^{-13}	1.6×10^{-05}
0.05	0.050041708375539	0.050041708375539	2.1×10^{-16}	0.050041708376024	4.8×10^{-13}	3.1×10^{-05}
0.06	0.060072103831297	0.060072103831296	9.9×10^{-16}	0.060072103827494	3.8×10^{-12}	5.4×10^{-05}
0.07	0.070114557872003	0.070114557872002	9.9×10^{-16}	0.070114557886155	1.4×10^{-11}	8.6×10^{-05}
0.08	0.080171104708073	0.080171104708076	2.9×10^{-15}	0.080171104683900	2.4×10^{-11}	1.3×10^{-04}
0.09	0.090243789909785	0.090243789909813	2.8×10^{-14}	0.090243789825535	8.4×10^{-11}	1.8×10^{-04}
0.1	0.100334672085451	0.100334672085594	1.4×10^{-13}	0.100334672012072	7.3×10^{-11}	2.5×10^{-04}

Example 2. *Find the approximate solution of*

$$y'(t) = \frac{-1}{1+t} + y(t) - y^2(t),\ y(0) = 1,\ 0 \le t \le 1$$

and exact solution is $y(t) = \dfrac{1}{1+t}$

$$\int_0^t y'(t)F_j(t)dt = \int_0^t \frac{-1}{1+t}F_j(t)dt + \int_0^t y(t)F_j(t)dt - \int_0^t y^2(t)F_j(t)dt$$

$$F_j(t)y(t)|_0^t - \int_0^t F_j'(t)y(t)dt = \int_0^t \frac{-1}{1+t}F_j(t)dt + \int_0^t y(t)F_j(t)dt - \int_0^t y^2(t)F_j(t)dt$$

$$F_j(t)y(t) - F_j(0) - \int_0^t F_j'(t)y(t)dt = \int_0^t \frac{-1}{1+t}F_j(t)dt + \int_0^t y(t)F_j(t)dt - \int_0^t y^2(t)F_j(t)dt$$

By taking $F_1 = 1, F_2 = t, F_3 = 1+t^2, F_4 = t^3+2t$

$$y(t) = \sum_{i=1}^{4} c_i F_i(t),\ c_i's\ \text{are arbitrary constants}$$

$$\sum_{i=1}^{4} c_i\left(F_i(t)F_j(t)-\int_0^t F_j'(t)F_i(t)dt\right)-F_j(0)=\int_0^t \frac{-1}{1+t}F_j(t)dt+\sum_{i=1}^{4} c_i\left(\int_0^t F_i(t)F_j(t)dt-\sum_{k=1}^{4} c_k\int_0^t F_i(t)F_j(t)F_k(t)dt\right),$$

here $j=1$ *to* 4

For $j=1$

$$\begin{aligned}&(1-t)c_1+\left(t-\frac{t^2}{2}\right)c_2+\left(t^2+1-\frac{t^3}{3}-t\right)c_3+\left(t^3+2t-\frac{t^4}{4}-t^2\right)c_4-1+\log(1+t)+tc_1^2+\frac{t^3}{3}c_2^2\\&+\left(\frac{t^5}{5}+\frac{2t^3}{3}+t\right)c_3^2+\left(\frac{t^7}{7}+\frac{4t^5}{5}+\frac{4t^3}{3}\right)c_4^2+t^2c_1c_2+2\left(\frac{t^3}{3}+t\right)c_1c_3+2\left(\frac{t^4}{4}+t^2\right)c_1c_4\\&+2\left(\frac{t^4}{4}+\frac{t^2}{2}\right)c_2c_3+2\left(\frac{t^5}{5}+\frac{2t^3}{3}c_2c_4\right)+2\left(\frac{t^6}{6}+\frac{3t^4}{4}+t^2\right)c_3c_4=0\end{aligned} \tag{8}$$

For $j=2$

$$\begin{aligned}&\frac{-t^2}{2}c_1+\left(\frac{t^2}{2}-\frac{t^3}{3}\right)c_2+\left(\frac{2t^3}{3}-\frac{t^4}{4}-\frac{t^2}{2}\right)c_3+\left(\frac{3t^4}{4}t^4+t^2-\frac{t^5}{5}-\frac{2t^3}{3}\right)c_4+t-\log(1+t)+\frac{t^2}{2}c_1^2\\&+\frac{t^4}{4}c_2^2+\left(\frac{t^6}{6}+\frac{t^4}{2}+\frac{t^2}{2}\right)c_3^2+\left(\frac{t^8}{8}+\frac{2t^6}{3}+t^4\right)c_4^2+\frac{2t^3}{3}c_1c_2+2\left(\frac{t^4}{4}+\frac{t^2}{2}\right)c_1c_3+2\left(\frac{t^5}{5}+\frac{2t^3}{3}\right)c_1c_4\\&+2\left(\frac{t^5}{5}+\frac{t^3}{3}\right)c_2c_3+2\left(\frac{t^6}{6}+\frac{t^4}{2}\right)c_2c_4+2\left(\frac{t^7}{7}+\frac{3t^5}{5}+\frac{2t^3}{3}\right)c_3c_4=0\end{aligned} \tag{9}$$

For $j=3$

$$\begin{aligned}&\left(1-t-\frac{t^3}{3}\right)c_1+\left(t+\frac{t^3}{3}-\frac{t^4}{4}-\frac{t^2}{2}\right)c_2+\left(\frac{t^4}{2}+t^2+1-\frac{t^5}{5}-\frac{2t^3}{3}-t\right)c_3\\&+\left(\frac{3t^5}{5}+\frac{5t^3}{3}+2t-\frac{t^6}{6}-\frac{3t^4}{4}-t^2\right)c_4-1+\frac{t^2}{2}-t+2\log(t+1)+\left(\frac{t^3}{3}+t\right)c_1^2+\left(\frac{t^5}{5}+\frac{t^3}{3}\right)c_2^2\\&+\left(\frac{t^7}{7}+\frac{3t^5}{5}+t^3+t\right)c_3^2+\left(\frac{t^9}{9}+\frac{5t^7}{7}+\frac{8t^5}{5}+\frac{t^3}{3}\right)c_4^2+2\left(\frac{t^4}{4}+\frac{t^2}{2}\right)c_1c_2+2\left(\frac{t^5}{5}+\frac{2t^3}{3}+t\right)c_1c_3\\&+2\left(\frac{t^6}{6}+\frac{3t^4}{4}+t^2\right)c_1c_4+2\left(\frac{t^6}{6}+\frac{t^4}{2}+\frac{t^2}{2}\right)c_2c_3+2\left(\frac{t^7}{7}+\frac{3t^5}{5}+\frac{2t^3}{3}\right)c_2c_4\\&+2\left(\frac{t^8}{8}+\frac{2t^6}{3}+\frac{3t^4}{4}+t^2\right)c_3c_4=0\end{aligned} \tag{10}$$

For $j=4$

$$\begin{aligned}&\left(\frac{-t^4}{4}-t^2\right)c_1+\left(\frac{t^4}{4}+t^2-\frac{t^5}{5}-\frac{2t^3}{3}\right)c_2+\left(\frac{2t^5}{5}+\frac{4t^3}{3}t^3-\frac{t^6}{6}-\frac{3t^4}{4}-t^2\right)c_3\\&+\left(\frac{t^6}{2}+2t^4+2t^2-\frac{t^7}{7}-\frac{4t^5}{5}-\frac{4t^3}{3}\right)c_4+\frac{t^3}{3}-\frac{t^2}{2}+3t-3\log(t+1)+\left(\frac{t^4}{4}+t^2\right)c_1^2+\left(\frac{t^6}{6}+\frac{t^4}{2}\right)c_2^2\\&+\left(\frac{t^8}{8}+\frac{2t^6}{3}+\frac{5t^4}{4}+t^2\right)c_3^2+\left(\frac{t^{10}}{10}+\frac{3t^8}{4}+2t^6+2t^4\right)c_4^2+2\left(\frac{t^5}{5}+\frac{2t^3}{3}\right)c_1c_2+2\left(\frac{t^6}{6}+\frac{3t^4}{4}+t^2\right)c_1c_3\\&+2\left(\frac{t^7}{7}+\frac{4t^5}{5}+\frac{4t^3}{3}\right)c_1c_4+2\left(\frac{t^7}{7}+\frac{3t^5}{5}+\frac{2t^3}{3}\right)c_2c_3+2\left(\frac{t^8}{8}+\frac{2t^6}{3}+t^4\right)c_2c_4\\&+2\left(\frac{t^9}{9}+\frac{5t^7}{7}+\frac{8t^5}{5}+\frac{4t^3}{3}\right)c_3c_4=0\end{aligned} \tag{11}$$

See Table 3 and Figure 2.

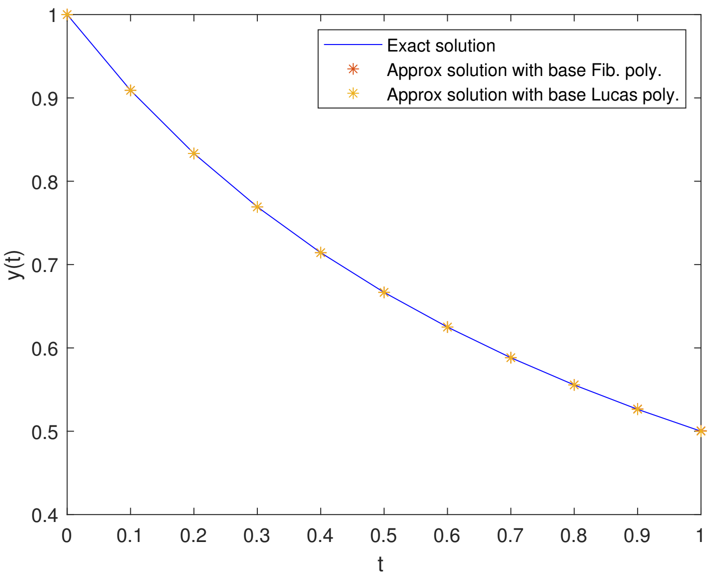

Figure 2. Comparison of presented approximate solution with exact solution for Example 2.

Table 3. Comparison of exact and approximate solution of Example 2.

t	Exact Solution	$n=4$ for Fibonacci Polynomials		$n=4$ for Lucas Polynomials	
		Approximate Solution	Error	Approximate Solution	Error
0.1	0.909090924268709	0.909090909090909	1.5×10^{-08}	0.909090903094811	5.9×10^{-09}
0.2	0.833333699041515	0.833333333333333	3.7×10^{-07}	0.833333180955830	1.5×10^{-07}
0.3	0.769232922713790	0.769230769230769	2.2×10^{-06}	0.769229859137361	9.1×10^{-07}
0.4	0.714292936198081	0.714285714285714	7.2×10^{-06}	0.714282715494715	2.9×10^{-06}
0.5	0.666684609942635	0.666666666666667	1.8×10^{-05}	0.666659549297927	7.1×10^{-06}
0.6	0.625037081201994	0.625000000000000	3.7×10^{-05}	0.624986319721110	1.4×10^{-05}
0.7	0.588303026158657	0.588235294117647	6.8×10^{-05}	0.588212676612227	2.3×10^{-05}
0.8	0.555668867223434	0.555555555555556	1.1×10^{-04}	0.555522317262287	3.3×10^{-05}
0.9	0.526771423454695	0.526315789473684	4.6×10^{-04}	0.526271647535036	4.4×10^{-05}
1	0.500573787270132	0.500000000000000	5.7×10^{-04}	0.499946841991314	5.3×10^{-05}

Example 3. *Find the approximate solution of*

$$u'(t) = 16t^2 - 5 + 8tu(t) + u^2(t),\ u(0) = 1,\ 0 \le t \le 1$$

and exact solution is $u(t) = 1 - 4t$

$$\int_0^t u'(t)F_j(t)dt = \int_0^t \left(16t^2 - 5 + 8tu(t) + u^2(t)\right) F_j(t)dt$$

$$F_j(t)u(t)|_0^t - \int_0^t F_j'(t)u(t)dt = \int_0^t \left(16t^2 - 5 + 8tu(t) + u^2(t)\right) F_j(t)dt$$

$$F_j(t)u(t) - F_j(0) - \int_0^t F_j'(t)u(t)dt = \int_0^t \left(16t^2 - 5 + 8tu(t) + u^2(t)\right) F_j(t)dt$$

By taking $F_1 = 1, F_2 = t, F_3 = 1 + t^2, F_4 = t^3 + 2t$

$$u(t) = \sum_{i=1}^{4} c_i F_i(t)$$

$$\sum_{i=1}^{4} c_i F_i(t)F_j(t) - F_j(0) - \sum_{i=1}^{4} c_i \int_0^t F_j'(t)F_i(t)dt = \int_0^t (16t^2 - 5)F_j(t)dt + \sum_{i=1}^{4} c_i \int_0^t \left(8tF_i(t) + \sum_{i=k}^{4} c_k F_i(t)F_k(t)\right) F_j(t)dt$$

where $j = 1$ *to* 4

For j=1

$$\begin{aligned}&-1-\frac{16t^3}{3}+5t+(1-4t^2)c_1+\left(t-\frac{8t^3}{3}\right)c_2+(1-3t^2-2t^4)c_3+\left(2t-\frac{13}{3}t^3-\frac{8t^5}{5}\right)c_4\\&-tc_1^2-\frac{t^3}{3}c_2^2-\left(\frac{t^5}{5}+\frac{2t^3}{3}+t\right)c_3^2-\left(\frac{t^7}{7}+\frac{4t^5}{5}+\frac{4t^3}{3}\right)c_4^2-t^2c_1c_2-\left(\frac{t^4}{2}+t^2\right)c_2c_3\\&-\left(\frac{t^6}{3}+\frac{3t^4}{2}+2t^2\right)c_3c_4-\left(\frac{t^4}{2}+2t^2\right)c_1c_4-2\left(\frac{t^3}{3}+t\right)c_1c_3-2\left(\frac{t^5}{5}+\frac{2t^3}{3}\right)c_2c_4=0\end{aligned} \tag{12}$$

For $j=2$

$$\begin{aligned}&-4t^4+\frac{5t^2}{2}-\frac{8t^3}{3}c_1+\left(\frac{t^2}{2}-2t^4\right)c_2+\left(-2t^3-\frac{8t^5}{5}\right)c_3+\left(t^2-\frac{13t^4}{4}-\frac{4t^6}{3}\right)c_4\\&\frac{-t^2}{2}c_1^2-\frac{t^4}{4}c_2^2-\left(\frac{t^6}{6}+\frac{t^4}{2}+\frac{t^2}{2}\right)c_3^2-\left(\frac{t^8}{8}+\frac{2t^6}{3}+t^4\right)c_4^2-\frac{2t^3}{3}c_1c_2-2\left(\frac{t^5}{5}+\frac{t^3}{3}\right)c_2c_3\\&-2\left(\frac{t^7}{7}+\frac{3t^5}{5}+\frac{2t^3}{3}\right)c_3c_4-2\left(\frac{t^5}{5}+\frac{2t^3}{3}\right)c_1c_4-\left(\frac{t^4}{2}+\frac{t^2}{2}\right)c_1c_3-\left(\frac{t^6}{3}+t^4\right)c_2c_4=0\end{aligned} \tag{13}$$

For $j=3$

$$\begin{aligned}&-1-\frac{16t^5}{5}-\frac{11t^3}{3}+5t+(1-2t^4-4t^2)c_1+\left(t-\frac{7t^3}{3}-\frac{8t^5}{5}\right)c_2+\left(1-3t^2-\frac{7t^4}{2}-\frac{4t^6}{3}\right)c_3\\&+\left(2t-\frac{11t^3}{3}-\frac{21t^5}{5}-\frac{8t^7}{7}\right)c_4-\left(\frac{t^3}{3}+t\right)c_1^2-\left(\frac{t^5}{5}+\frac{t^3}{3}\right)c_2^2-\left(\frac{t^7}{7}+\frac{3t^5}{5}+t^3+t\right)c_3^2\\&-\left(\frac{t^9}{9}+\frac{5t^7}{7}+\frac{8t^5}{5}+\frac{4t^3}{3}\right)c_4^2-\left(\frac{t^4}{2}+t^2\right)c_1c_2-2\left(\frac{t^6}{6}+\frac{t^4}{2}+\frac{t^2}{2}\right)c_2c_3-\left(\frac{t^8}{4}+\frac{4t^6}{3}+\frac{5t^4}{2}+2t^2\right)c_3c_4\\&-\left(\frac{t^6}{3}=\frac{3t^4}{2}+2t^2\right)c_1c_4-2\left(\frac{t^5}{5}+\frac{2t^3}{3}+t\right)c_1c_3-2\left(\frac{t^7}{7}+\frac{3t^5}{5}+\frac{2t^3}{3}\right)c_2c_4=0\end{aligned} \tag{14}$$

For $j=4$

$$\begin{aligned}&\frac{-8t^6}{3}-\frac{27t^4}{4}+5t^2+\left(\frac{-8t^5}{5}-\frac{16t^3}{3}\right)c_1+\left(t^2-\frac{15}{4}t^4-\frac{4t^6}{3}\right)c_2+\left(\frac{-22t^5}{5}-4t^3-\frac{8t^7}{7}\right)c_3\\&+\left(2t^2-6t^4-\frac{29t^6}{6}-t^8-\frac{16t^5}{5}\right)c_4-\left(\frac{t^4}{4}+t^2\right)c_1^2-\left(\frac{t^6}{6}+\frac{t^4}{2}\right)c_2^2-\left(\frac{t^8}{8}+\frac{2t^6}{3}+\frac{5t^4}{4}+t^2\right)c_3^2\\&-\left(\frac{t^{10}}{10}+\frac{3t^8}{4}+2t^6+2t^4\right)c_4^2-2\left(\frac{t^5}{5}+\frac{2t^3}{3}\right)c_1c_2-2\left(\frac{t^7}{7}+\frac{3t^5}{5}+\frac{2t^3}{3}\right)c_2c_3\\&-2\left(\frac{t^9}{9}+\frac{5t^7}{7}+\frac{8t^5}{5}+\frac{4t^3}{3}\right)c_3c_4-2\left(\frac{t^7}{7}+\frac{4t^5}{5}+\frac{4t^3}{3}\right)c_1c_4-\left(\frac{t^6}{3}+\frac{3t^4}{2}+2t^2\right)c_1c_3-\left(\frac{t^8}{4}+\frac{4t^6}{3}+2t^4\right)c_2c_4=0\end{aligned} \tag{15}$$

See Table 4 and Figure 3. In Table 4 comparison of errors in our methods and method in [21] is shown.

Table 4. Comparison of exact and approximate solution of Example 3.

t	Exact Solution	$n=4$ for Fibonacci Polynomials		$n=4$ for Lucas Polynomials		Method [21]
		Approximate Solution	Error	Approximate Solution	Error	
0.1	0.6	0.600000000000001	9.9×10^{-16}	0.600000000000000	0	2.3×10^{-04}
0.2	0.2	0.199999999999999	9.9×10^{-16}	0.199999999999994	6.0×10^{-15}	3.7×10^{-12}
0.3	−0.2	−0.200000000000000	0	−0.200000000000052	5.2×10^{-14}	4.5×10^{-04}
0.4	−0.6	−0.600000000001501	1.5×10^{-12}	−0.600000012115453	1.2×10^{-08}	4.7×10^{-04}
0.5	−1.0	−1.000000013559188	1.4×10^{-08}	−1.000000000000322	3.2×10^{-13}	9.4×10^{-11}
0.6	−1.4	−1.400000000000000	0	−1.400000000046840	4.7×10^{-11}	4.7×10^{-04}
0.7	−1.8	−1.800349707136145	3.5×10^{-04}	−1.799999730208330	2.7×10^{-07}	4.5×10^{-04}
0.8	−2.2	−2.201167083285902	1.1×10^{-03}	−2.199999167376733	8.3×10^{-07}	1.6×10^{-10}
0.9	−2.6	−2.599999999229953	7.7×10^{-10}	−2.600000000323980	3.2×10^{-10}	2.3×10^{-04}
1	−3.0	−3.000000000268688	2.7×10^{-10}	−2.999994452834936	5.5×10^{-06}	0

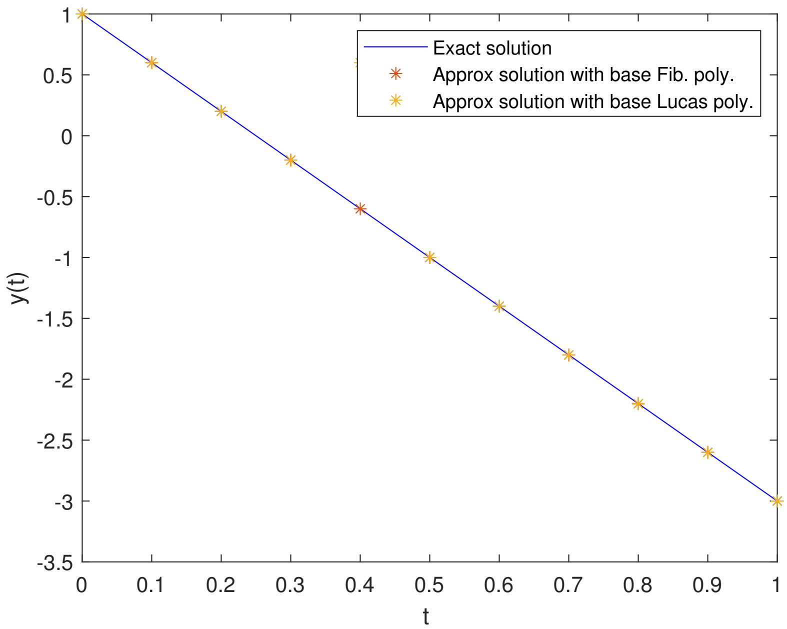

Figure 3. Comparison of presented approximate solution with exact solution for Example 3.

Example 4. *Find the approximate solution of*

$$y'(t) = 2y(t) - y^2(t) + 1,\ 0 \le t \le 1,\ y(0) = 0$$

and the exact solution is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$$

$$\int_0^t \left(y'(t) - 2y(t) + y^2(t) - 1\right) F_j(t)dt = 0$$

$$F_j(t)y(t)|_0^a - \int_0^t F'(j)y(t)dt - \int_0^t 2y(t)F_j(t)dt + \int_0^t y^2(t)F_j(t)dt - \int_0^t F_j(t)dt = 0$$

$$F_j(t)y(t) - \int_0^t F'(j)y(t)dt - \int_0^t 2y(t)F_j(t)dt + \int_0^t y^2(t)F_j(t)dt - \int_0^t F_j(t)dt = 0$$

$$y(t) = \sum_{j=1}^{4} c_j F_j(t)$$

We will get the following equations

$$\begin{aligned}&(1-2t)c_1 + (t-t^2)c_2 + \left(1+t^2-2t-\frac{2t^3}{3}\right)c_3 + \left(t^3+2t-\frac{t^4}{2}-2t^2\right)c_4 + tc_1^2 + \frac{t^3}{3}c_2^2 + \left(\frac{t^5}{5}+\frac{2t^3}{3}+t\right)c_3^2\\ &+\left(\frac{t^7}{7}+\frac{4t^5}{5}+\frac{4t^3}{3}\right)c_4^2 + t^2c_1c_2 + \left(\frac{t^4}{2}+t^2\right)c_2c_3 + 2\left(\frac{t^3}{3}+t\right)c_1c_3 + 2\left(\frac{t^6}{6}+\frac{3t^4}{4}+t^2\right)c_3c_4\\ &+2\left(\frac{t^4}{4}+t^2\right)c_1c_4 + 2\left(\frac{t^5}{5}+\frac{2t^3}{3}\right)c_2c_4 - t = 0\end{aligned} \tag{16}$$

$$\begin{aligned}&-t^2c_1 + \left(\frac{t^2}{2}-\frac{2t^3}{3}\right)c_2 + \left(\frac{2t^3}{3}-\frac{t^4}{2}-t^2\right)c_3 + \left(t^2+\frac{3t^4}{4}-\frac{2t^5}{5}-\frac{4t^3}{3}\right)c_4 + \frac{t^2}{2}c_1^2 + \frac{t^4}{4}c_2^2 + \left(\frac{t^6}{6}+\frac{t^4}{2}+\frac{t^2}{2}\right)c_3^2\\ &+\left(\frac{t^8}{8}+\frac{2t^6}{3}+t^4\right)c_4^2 + \frac{2t^3}{3}c_1c_2 + 2\left(\frac{t^5}{5}+\frac{t^3}{3}\right)c_2c_3 + 2\left(\frac{t^4}{4}+\frac{t^2}{2}\right)c_1c_3 + 2\left(\frac{t^7}{7}+\frac{3t^5}{5}+\frac{2t^3}{3}\right)c_3c_4\\ &+2\left(\frac{t^5}{5}+\frac{2t^3}{3}\right)c_1c_4 + 2\left(\frac{t^6}{6}+\frac{t^4}{2}\right)c_2c_4 - \frac{t^2}{2} = 0\end{aligned} \tag{17}$$

$$\left(1-2t-\frac{2t^3}{3}\right)c_1+\left(t-t^2+\frac{t^3}{3}-\frac{t^4}{2}\right)c_2+\left(1-2t+\frac{t^4}{2}+t^2-\frac{2t^5}{5}-\frac{4t^3}{3}\right)c_3+\left(\frac{3t^5}{5}+\frac{5t^3}{3}+2t-\frac{t^6}{3}-\frac{3t^4}{2}-2t^2\right)c_4$$
$$+\left(t+\frac{t^3}{3}\right)c_1^2+\left(\frac{t^5}{5}+\frac{t^3}{3}\right)c_2^2+\left(\frac{t^7}{7}+\frac{3t^5}{5}+t^3+t\right)c_3^2+\left(\frac{t^9}{9}+\frac{5t^5}{7}+\frac{8t^5}{5}+\frac{4t^3}{3}\right)c_4^2+\left(\frac{t^4}{4}+t^2\right)c_1c_2$$
$$+\left(\frac{a^6}{3}+a^4+a^2\right)c_2c_3+2\left(\frac{t^5}{5}+\frac{2t^3}{3}+t\right)c_1c_3+\left(\frac{t^8}{4}+\frac{4t^6}{3}+\frac{5t^4}{2}+2t^2\right)c_3c_4+\left(\frac{t^6}{3}+\frac{3t^4}{2}+2t^2\right)c_1c_4 \tag{18}$$
$$+2\left(\frac{t^7}{7}+\frac{3t^5}{5}+\frac{2t^3}{3}\right)c_2c_4-t-\frac{t^3}{3}=0$$

$$-\left(\frac{t^4}{2}+2t^2\right)c_1+\left(t^2-\frac{4t^3}{3}+\frac{t^4}{4}-\frac{2t^5}{5}\right)c_2+\left(-2t^2+\frac{4t^3}{3}-\frac{3t^4}{2}+\frac{2t^5}{5}-\frac{t^6}{3}\right)c_3$$
$$+\left(2t^2-\frac{8t^3}{3}+2t^4-\frac{8t^5}{5}+\frac{t^6}{2}-\frac{2t^7}{7}\right)c_4+\left(\frac{t^4}{4}+t^2\right)c_1^2+\left(\frac{t^6}{6}+\frac{t^4}{2}\right)c_2^2+\left(\frac{t^8}{8}+\frac{2t^6}{3}+\frac{5t^4}{4}+t^2\right)c_3^2$$
$$+\left(\frac{t^{10}}{10}+\frac{3t^8}{4}+2t^6+2t^4\right)c_4^2+\left(\frac{2t^5}{5}+\frac{4t^3}{3}\right)c_1c_2+2\left(\frac{t^7}{7}+\frac{3t^5}{5}+\frac{2t^3}{3}\right)c_2c_3+2\left(\frac{t^9}{9}+\frac{5t^7}{7}+\frac{8t^5}{5}+\frac{4t^3}{3}\right)c_3c_4 \tag{19}$$
$$+2\left(\frac{t^7}{7}+\frac{4t^5}{5}+\frac{4t^3}{3}\right)c_1c_4+2\left(\frac{t^6}{6}+\frac{3t^4}{4}+t^2\right)c_1c_3+2\left(\frac{t^8}{8}+\frac{2t^6}{3}+t^4\right)c_2c_4-\frac{t^4}{4}-t^2=0$$

See Tables 5 and 6 and Figure 4.

In Table 5 comparison of errors in our methods and method in [22] is shown.

Table 5. Comparison of exact and approximate solution of Example 4 for t = [0.01, 0.1].

t	Exact Solution	$n=4$ for Fibonacci Polynomials		$n=4$ for Lucas Polynomials		Method [22]
		Approximate Solution	Error	Approximate Solution	Error	
0.01	0.010100329953180	0.010100329953180	0	0.010100329953176	2.9×10^{-15}	2.5×10^{-07}
0.02	0.020402611830266	0.020402611830267	9.9×10^{-16}	0.020402611830263	3.0×10^{-15}	2.1×10^{-06}
0.03	0.030908718550429	0.030908718550430	9.9×10^{-16}	0.030908718550431	2.0×10^{-15}	7.1×10^{-06}
0.04	0.041620431605239	0.041620431605241	2.0×10^{-15}	0.041620431605239	0	1.7×10^{-05}
0.05	0.052539435209824	0.052539435209838	1.4×10^{-14}	0.052539435209834	9.9×10^{-15}	3.4×10^{-05}
0.06	0.063667310371906	0.063667310371965	5.9×10^{-14}	0.063667310371964	5.8×10^{-14}	6.0×10^{-05}
0.07	0.075005528890499	0.075005528890711	2.1×10^{-13}	0.075005528890710	2.1×10^{-13}	9.7×10^{-05}
0.08	0.086555447297052	0.086555447297694	6.4×10^{-13}	0.086555447297694	6.4×10^{-13}	1.5×10^{-04}
0.09	0.098318300752784	0.098318300754494	1.7×10^{-12}	0.098318300754494	1.7×10^{-12}	2.1×10^{-04}
0.1	0.110295196916962	0.110295196921082	4.1×10^{-12}	0.110295196921086	4.1×10^{-12}	2.9×10^{-04}

Table 6. Comparison of exact and approximate solution of Example 4 for t = [0.2, 2.0].

t	Exact Solution	FGM	LGM	Method [16]	Method [28]
0.2	0.2419767996	0.2419768010	0.2419768010	0.2396149017	0.2419778327
0.4	0.5678121663	0.5678126833	0.5678126823	0.5626231618	0.5678455132
0.6	0.9535662165	0.9535785307	0.9535785307	0.9468409011	0.9536660329
0.8	1.3463636554	1.3464357962	1.3464357969	1.3405640980	1.3463791062
1.0	1.6894983916	1.6896183957	1.6896183957	1.6863821450	1.6860271032
1.2	1.9513601180	1.9511451475	1.9511451469	1.9509491870	1.9150510260
1.4	2.1313266100	2.1299124125	2.1299123963	2.1325827440	2.1791315021
1.6	2.2462859590	2.2426924355	2.2426924356	2.2481414290	-50.98229780
1.8	2.3163247370	2.3098953077	2.3098953191	2.3181237490	-5338.782860
2.0	2.3577716530	2.3483247584	2.348324754	2.3592420980	-286352.7325

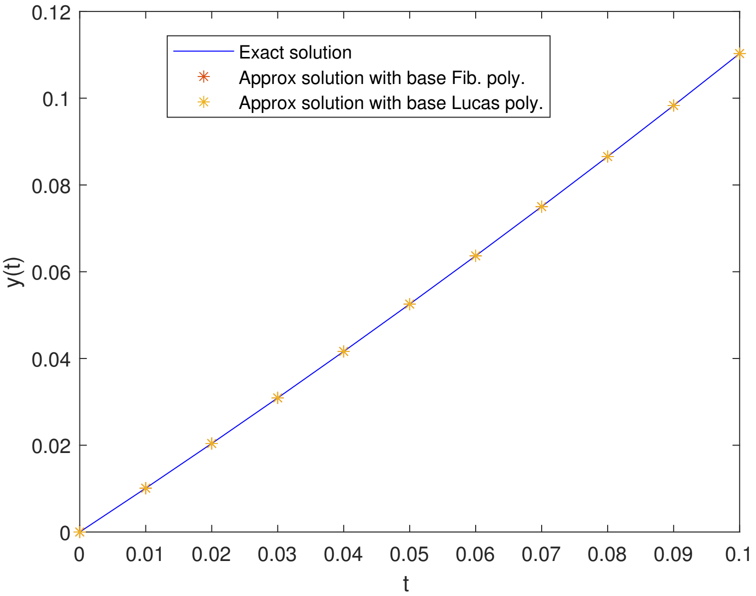

Figure 4. Comparison of presented approximate solution with exact solution for Example 4.

4. Conclusions

In the present paper some non-linear quadratic Riccati differential equations are solved. We have given numerical algorithms using Fibonacci and Lucas polynomials as basis functions and weight functions based on Galerkin method. Some numerical examples are solved for $n = 3$ or 4 and tables and figures show comparison of exact and approximate solutions for FGM and LGM. We see that results are highly accurate as errors are less. From Tables 1 and 2, we observe that results for $n = 4$ are more accurate than result for $n = 3$. Accuracy increases as value of n increases.

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No AI tools were utilized for this paper.

References

1. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2019.
2. Fischer, B.; Modersitzki, J. Fast inversion of matrices arising in image processing. *Numer. Algorithms* **1999**, *22*, 1–11.

3. Olson, B.J.; Shaw, S.W.; Shi, C.; et al. Circulant matrices and their application to vibration analysis. *Appl. Mech. Rev.* **2014**, *66*, 040803.
4. Tas, N.; Ucar, S.; Ozgur, N.Y.; et al. A new coding/decoding algorithm using Fibonacci numbers. *Discrete Math. Algorithms Appl.* **2018**, *10*, 1850028.
5. Ucar, S.; Tas, N.; Ozgur, N.Y. A new application to coding theory via Fibonacci and Lucas numbers. *Math. Sci. Appl. E-Notes* **2019**, *7*, 62–70.
6. Prasad, K.; Mahato, H. Cryptography using generalized Fibonacci matrices with Affine-Hill cipher. *J. Discrete Math. Sci. Cryptogr.* **2022**, *25*, 2341–2352.
7. Prasad, K.; Mahato, H.; Kumari, M. A novel public key cryptography based on generalized Lucas matrices. *arXiv* **2022**, arXiv:2202.08156.
8. Bellini, E.; Marcolla, C.; Murru, N. An application of p-Fibonacci error-correcting codes to cryptography. *Mathematics* **2021**, *9*, 789.
9. Bala, R.; Mishra, V. Solutions of equations $x^2-(p^2q^2\pm 3p)y^2=\pm k^t$. *Examples Counterexamples* **2022**, *2*, 100043.
10. Bala, R.; Mishra, V. On the circulant matrices with Ducci sequence and Gaussian Fibonacci numbers. *AIP Conf. Proc.* **2021**, *2352*, 100043.
11. Hoggatt, V.E., Jr.; Bicknell, M. Roots of Fibonacci polynomials. *Fibonacci Q.* **1973**, *11*, 271–274.
12. Nalli, A.; Haukkanen, P. On generalized Fibonacci and Lucas polynomials. *Chaos Solitons Fractals* **2009**, *42*, 3179–3186.
13. Rannacher, R. Adaptive Galerkin finite element methods for partial differential equations. *J. Comput. Appl. Math.* **2001**, *128*, 205–233.
14. Dogan, A. Numerical solution of RLW equation using linear finite elements within Galerkin's method. *Appl. Math. Model.* **2002**, *26*, 771–783.
15. Yan, J.; Shu, C.W. Local discontinuous Galerkin methods for partial differential equations with higher order derivatives. *J. Sci. Comput.* **2002**, *17*, 27–47.
16. Batiha, B. A new efficient method for solving quadratic Riccati differential equation. *Int. J. Appl. Math. Res.* **2015**, *4*, 24.
17. Biazar, J.; Eslami, M. Differential transform method for quadratic Riccati differential equation. *Int. J. Nonlinear Sci.* **2010**, *9*, 444–447.
18. Ghorbani, A.; Momani, S. An effective variational iteration algorithm for solving Riccati differential equations. *Appl. Math. Lett.* **2010**, *23*, 922–927.
19. Mustapha, R.A.; Idowu, G.A.; Ajetunmobi, M.O. A Comparative Study of Numerical Methods for Solving the Riccati Equation. *Int. J. Math. Sci. Optim. Theory Appl.* **2018**, *4*, 356–363.
20. Yang, C.; Hou, J.; Qin, B. Numerical solution of Riccati differential equations by using hybrid functions and tau method. *Int. J. Math. Comput. Sci.* **2012**, *6*, 871–874.
21. Ghomanjani, F.; Khorram, E. Approximate solution for quadratic Riccati differential equation. *J. Taibah Univ. Sci.* **2017**, *11*, 246–250.
22. Mishra, V.; Rani, D. Newton-Raphson based modified Laplace Adomian decomposition method for solving quadratic Riccati differential equations. *MATEC Web Conf.* **2016**, *57*, 05001.
23. Kiltu, G.G.; Roba, G.; Hailu, K. Fifth order predictor-corrector method for solving quadratic Riccati differential equations. *Int. J. Eng. Appl. Sci.* **2017**, *9*, 51–64.
24. Kassahun, W.; Shiferaw, A.; Gebregiorgis, S. Eighth order Predictor-Corrector method to solve quadratic Riccati differential equations. *Momona Ethiop. J. Sci.* **2021**, *13*, 213–224.
25. Allahviranloo, T. Application of iterative methods for solving general Riccati equation. *Int. J. Ind. Math.* **2012**, *4*, 389–404.
26. Khalid, M.; Sultana, M.; Zaidi, F.; et al. An effective perturbation iteration algorithm for solving Riccati differential equations. *Int. J. Comput. Appl.* **2015**, *111*, 1–5.
27. Koonprasert, S.; Sirisubtawee, S.; Srisutat, P.; et al. A Fibonacci Galerkin method for solving certain types of boundary value problems: Annual Meeting in Mathematics 2023. *Thai J. Math.* **2024**, *22*, 179–196.
28. Batiha, B.; Noorani, M.S.M.; Hashim, I. Application of variational iteration method to a general Riccati equation. *Int. Math. Forum* **2007**, *2*, 2759–2770.