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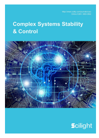

Article

Stabilization Analysis via Discrete-Time Feedback for Hybrid Stochastic Neural Networks with Non-Differentiable Delays and Lévy Noise under Mixed Growth Conditions

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Abstract: This paper endeavors to systematically develop a comprehensive class of hybrid stochastic neural networks (SNNs) that incorporate both time-delay dynamics and Lévy noise. In terms of system settings, Markov switching capturing the complex transitions between different operational modes is used to effectively characterize the inherent hybrid nature, and the activation functions stored in each mode exhibit mixed growth rather than a single linear or polynomial growth. Meanwhile, the time-delay is non-differentiable including piece-wise constant or distributed, which is more general, and Lévy noise as a jump disturbance is processed by Khasminskii-type conditions. Considering the explosion difficulties caused by switching and Lévy noise, proofs of global existence and uniqueness of solutions and moment boundedness of the original system are detailed completed. Furthermore, discrete-time feedback control based on switching-mode observers in Markov chains is introduced to stabilize the original system. By using the Lyapunov functional, the generalized Itô's formula, the M-matrix theory, the sufficient conditions for three stabilization modes are obtained. Moreover, we refined the unclear exponential decay rates near zero in previous studies and expanded the determination methods. Finally, a numerical example illustrating the feasibility of theoretical results is provided. Contrary to previous studies, which employed symmetric Lévy noise, our analysis of existing literature confirms and substantiates the benefits of asymmetric noise.

Keywords: stochastic neural networks; non-differentiable delays; Lévy noise; Markov switching; mixed growth; discrete-time feedback

1. Introduction

Neural networks (NN), since the inception [1,2], have been central to advances machine learning and artificial intelligence, accompanied by widespread application in diverse fields such as computer vision, financial prediction, robotics and medical diagnosis. In contrast to deterministic neural networks (DNNs), where all parameters, inputs, and outputs follow fixed, non-random mappings and are only assumed to hold in an ideal noise-free environment, stochastic neural networks (SNNs) introduce stochasticity into their dynamics, enabling a more accurate characterization of real-world data. Since this stochasticity acts as a natural regularizer, it significantly improves the model's generalization ability and robustness against perturbations. Furthermore, the derivative chain rule employed in the Lyapunov analysis of SNNs is based on Itô's formula, which is fundamentally distinct from the algebraic methods and simple Lyapunov approaches used in DNNs. From the end of 20th century to the beginning of 21th century, some scholars have deeply explored and established the basic framework for stochastic neural networks [3–6], especially, Blythe et al. [6] have paid attention to the stability problem of stochastic time-delay neural networks (STDNNs). In practical systems, time delay is prevalent and inevitable, indicating the uncertainty of signal transmission and simultaneously demonstrating the system's dependence on past states. Specifically, time delays manifest in biological neural networks as the latency of synaptic transmission and axonal signal conduction, which are key sources of neuronal oscillations, rhythmic synchronization, and other emergent behaviors. Meanwhile, in networked control systems, random and abrupt delays are introduced by data packet transmission, queuing, and routing processes. Nowadays, more scholars are involved in the design of STDNNs. However, how to establish a more practical stochastic time-delay neural network with excellent performance remains a challenge.

Switching systems, a key category of hybrid systems, have garnered significant focus, which has the potential to be highly flexible and robust. However, the switching paths of switching systems usually need to be preset, which makes them insufficient in terms of concealment. In [7], Hamilton introduced the Markov switching model to describe economic analysis and made significant contributions in the field of econometrics. Markov switching is initiated by the Markov chain, whose switching transitions are determined by the generator matrix in the sense of probability, which makes the switching information stored in the system stable, while the calling path is concealed. For example, in a large-scale unmanned aerial vehicle (UAV) array, it is applicable in the transfer and operation of the command entity. In [8], this type of switching model is first introduced into the research of SNNs. In [9], the system takes Markov switching and Markov jumping into account simultaneously for the first time, which has greatly inspired the research on hybrid SNNs featuring Markov switching and jump processes.

In practical complex systems such as finance, electric power, ecology and transportation, the evolution of system states usually fails to satisfy the ideal assumptions of smoothness, continuity and non-abrupt changes.Compared with Gaussian noise that can only describe continuous small-amplitude fluctuations, Lévy noise contains the characteristics of jumps, discontinuities and sudden shocks, which can effectively capture extreme events and jump behaviors such as financial crashes, policy interventions, equipment failures and power grid disturbances, making up for the defect that Gaussian noise cannot describe heavy-tailed distribution and non-Gaussian violent fluctuations, so that the stochastic model can better fit the actual system data characteristics. Numerous studies have demonstrated that Lévy noise is of great significance to neuronal stochastic resonance theory. Ref. [10] uses Itô calculus to prove that under specific conditions, white Lévy noise contributes to the detection of subthreshold neuronal signals. Furthermore, in [11], researchers have found that Lévy noise with weak impulse characteristics (called small jumps in this paper) can promote synchronization, while enhancing the impulse characteristics(called big jumps) has the opposite effect. In [12], it has conducted stability analysis on the Cohen-Grossberg STDNNs with Markov switching and Lévy noise. However, the corresponding stabilization problems on highly nonlinear condition still deserves special attention.This is because most existing literature adopts symmetric Lévy noise, an idealized model assuming equal probabilities for positive and negative jumps. In contrast, asymmetric Lévy noise can describe biased and directional random jumps, which conform better to the general form of Lévy processes. For example, sudden load changes and fault impacts in power systems are mostly one-sided asymmetric jumps; data packet losses and burst disturbances in communication and control systems often exhibit skewed distributions; and in financial engineering, the probability of price crashes is typically higher than that of price surges.

The high nonlinearity of a neural network, apart from its net structure, is mainly reflected in the fact that activation function does not grow linearly simply and tends to be fitted with a polynomial. In [13, 14], highly nonlinear hybrid STDNNs featuring a differentiable time-varying delay are valued. Literatures [15–18] propose a version under the condition that time delay is non-differentiable, which meets the requirements for describing state under fast varying. Non-differentiable time delays are widely encountered in practical systems. For example, the piecewise constant sampling delay used in sampled-data control systems is a standard scenario in digital control implementations, while in biological neural networks, discontinuous synaptic transmission delays also follow non-smooth, piecewise dynamical characteristics. Due to their non-smooth nature, conventional derivative-based inequality scaling techniques fail for systems with non-differentiable delays, and this work addresses this issue by employing generalized inequality techniques and non-smooth analysis tools. Except [17], these articles are accustomed to using time-delay feedback control, which will consume a huge amount of energy and make it extremely difficult in practical applications since the controller depends on continuous information. Therefore, while choosing a stabilization strategy, we tend to use discrete-time control based the state observation of complete time duration on Markov chains like [17,19]. It is worth noting that the descriptions of high nonlinearity in these articles are all of a polynomial growth type. Although there is a certain degree of accuracy, this still cannot fully meet the needs of describing reality. Aiming that the stochastic system has a greater preference for global properties, mixed growth is more suitable for describing the characteristics of system over a longer period of time. Mixed growth has already attracted attention in the field of biomedicine [20,21], however, it is rarely mentioned in the research on SNNs. In [22], Liu and Zhu have studied stabilization of one mixed growth SDEs (stochastic differential equations) possessing multiple time-varying delays, regretfully, which did not take into account the huge advantages of mixed growth in terms of descriptive accuracy and model adaptability.

Based on the above elaboration, corresponding stabilization analysis via discrete-time feedback for hybrid stochastic neural networks with non-differentiable delays and Lévy noise under mixed growth conditions is of great significance. To this end, we have selected three stabilization objectives: Exponential stabilization, which provides a quantitative guarantee on the convergence rate and is crucial for practical engineering applications requiring fast and predictable performance; H_∞-stabilization, which captures the system's robustness characteristics; Almost sure stabilization, which establishes convergence along almost all sample paths without requiring moment existence, making it well-suited for the Markov-switching stochastic framework considered in this work. This paper boasts the following strengths in model selection and comprehension:

1. This paper adopts a mixed growth condition under Markov chain switching, relaxing the previously strict assumption of pure polynomial growth. With stronger adaptability, it can be widely used to model the growth laws of various complex dynamic systems in finance, electric power, intelligent manufacturing, transportation, ecology and other fields, breaking through the limitations of traditional single growth condition models.
2. A discrete targeted control strategy based on Markovian spatial nodes is designed, which only controls polynomial growth nodes without intervention on slow linear growth nodes. It optimizes control efficiency and greatly saves control resources, applicable to all kinds of resource-constrained dynamic regulation systems.
3. The proposed discrete feedback control framework permits direct deployment on various data platforms, eliminating the approximation errors induced by continuous emulation in conventional control approaches (eg.time-delay feedback control in [16]). It not only effectively accommodates the complex perturbations induced by non-differentiable delays and jump-type noises but also establishes more rigorous stabilization criteria. Consequently, the accuracy and stability of real-time regulation in complex dynamic systems across multiple disciplines are rigorously guaranteed.
4. The simulation section emphasizes and verifies that the theory is also applicable to systems with asymmetric Lévy noise. Within the existing analytical framework, this paper carries out numerical experiments by drawing on existing research results on asymmetric Lévy noise from [16], which further enriches the validation scenarios for such stochastic systems and provides a reference for subsequent related studies.

The subsequent sections are structured as follows. In Section 2, proposed model and corresponding assumptions and lemmas are presented. In Section 3, we show the boundness of controlled system and the relevant sufficient conditions for stabilization. In Section 4, an example is given to verify our conclusion, and some precautions for system setting are emphasized combining with simulation in other articles. Eventually, some conclusions and outlooks are depicted in Section 5.

2. Model Description and Preliminaries

Notations: The n-dimensional Euclidean space is denoted as $\mathbb{R}^n$. $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^+$ is the non-negative part. It is emphasized that $|\cdot|$ represents the absolute value of an element and 2-norm of a vector in this paper and 0 presents null vector of specific dimensions. For an $m \times n$ -dimension matrix A whose elements are $\{a_{ij}\}_{m,n}$, $\|A\|_{\boldsymbol{F}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}$ is Frobenius norm. Thus, it follows that $\|A\|_{\boldsymbol{F}} = \sqrt{\text{trace}(A^TA)}$, where A^T is the transpose of A. Especially, when Z is a column vector, $\|Z\|_{\boldsymbol{F}} = |Z|$. The family of càdlàg functions $\Psi : [-\tau, 0] \to \mathbb{R}^n$ for $\tau > 0$ known as right-continuous with left limit is denoted by $D([-\tau, 0]; \mathbb{R}^n)$ with its norm defined as $\|\Psi\| = \sup_{-\tau\le t\le 0}|\Psi(t)|$, of which the family of all bounded $\mathcal{F}_0$ -measurable random variables is denoted as $D_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$. Denote $C^{2,1}(\mathbb{R}^n \times \mathsf{S} \times \mathbb{R}^+; \mathbb{R})$ the family of all real-valued functions $V(Z, i, t)$ on $\mathbb{R}^n \times \mathsf{S} \times \mathbb{R}^+$ which is twice continuously differentiable in Z and once in t. For such a V, set $V_t = \frac{\partial V}{\partial t}, V_Z = (\frac{\partial V}{\partial Z_1}, \frac{\partial V}{\partial Z_2}, \cdots, \frac{\partial V}{\partial Z_n})$ and $V_{ZZ} = (\frac{\partial^2 V}{\partial Z_i \partial Z_j})_{n\times n}$. For real numbers x, y, $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$. I_A denotes the indicator function of $A \subseteq \Omega$, defined by $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ otherwise. Suppose $B(t) = (B_1(t), \cdots, B_k(t))$ to be a k-dimension Brownian motion defined in complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ with its filtration $\{\mathcal{F}_t\}_{t\ge0}$ increasing and right-continuous, where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Suppose that $\{\pi(t) \mid t \ge 0\}$ is a right-continuous Markov chain among the state space $\mathsf{S} = \{1, 2, 3, \cdots, N\}$ with generator $\mathit{\Gamma} = (\gamma_{ij})_{N\times N}$ given by a probability generating function

$$\mathbb{P}\{\pi(t+h) = j \mid \pi(t) = i\} = \begin{cases} \gamma_{ij}h + o(h), i \neq j, \\ 1 + \gamma_{ii}h + o(h), i = j, \end{cases} \tag{1}$$

where h signifies an increase in time intervals and γ_{ij} denotes a non-negative average probability of transitioning from state i to state j. Based on the normativity of probability, $\gamma_{ii} = -\sum_{j=1,j\neq i}^{N}\gamma_{ij} \le 0$.

Poisson process is depicted by the Poisson random measure $\mathcal{J}(t,\delta)$ on $\mathbb{R}^+\times\mathbb{R}^n/\{0\}$ and its compensated form $\widetilde{\mathcal{J}}(dt,d\delta)=\mathcal{J}(dt,d\delta)-\vartheta(d\delta)dt$ with ϑ being a Lévy measure that satisfies

$$\int_{\mathbb{R}^n/\{0\}}(1\wedge|\delta|^2)\vartheta(d\delta)<\infty. \tag{2}$$

Typically, the pair $(B,\mathcal{J})$ is commonly referred to as Lévy noise. $\pi(\cdot),B(\cdot),\widetilde{\mathcal{J}}(\cdot\ ,\ \cdot)$ are assuming to be independent of each other.

Remark 1. *Markov chains underpin the memoryless random switching mechanism in Markovian switched systems. They extend deterministic dynamical evolution to a hybrid stochastic framework, where system modes transition probabilistically according to the current state only. This structure significantly influences stability, invariant distributions, and asymptotic behavior, offering a tractable way to model random structural changes and analyze the statistical dynamics of complex systems. Poisson process compared to Gaussian process has a relatively heavy tail. The Lévy measure ϑ is introduced for compensation to ensure achieving appropriate heavy tailed behavior, which makes it easier for extremely large values to occur. In the nervous system, this extreme pattern will seriously interfere with the transmission of nerve impulses, cause abnormal firing of neurons, and even lead to changes in the ability of information processing and encoding. It is apparent that, according to (2), we are unable to determine the boundedness of this measure near the zero point. In addition,the mutual independence assumption here is a standard setup in hybrid stochastic system theory (see [6]). It simplifies the calculation of conditional expectations and characteristic functions, which is crucial for deriving the subsequent stability criteria. In practical terms, this assumption means that the random sampling noise, sudden jumps, and system mode switching occur independently of each other. This idealized control design can serve as a theoretical reference for practical implementations.*

The following mixed growth STDNN with Lévy noise is presented:

$$\begin{aligned}dZ(t)=&\Big[-\mathcal{D}(\pi(t))Z(t^-)+\mathcal{A}(\pi(t))\times f_1\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t\Big)\Big]dt\\&+f_2\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t\Big)dB(t)\\&+\int_{0<|\delta|<l}r\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t,\delta\Big)\widetilde{\mathcal{J}}(dt,d\delta)\\&+\int_{|\delta|\geq l}R\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t,\delta\Big)\mathcal{J}(dt,d\delta),\end{aligned} \tag{3}$$

where $t\in\mathbb{R}^+$, $Z(t)$ represents neuron state, $Z(t^-)=\lim_{s\uparrow t}Z(s)$ represents the neighboring state of this neuron, $\mathcal{D}(\pi(t))=diag\{d_1(\pi(t)),d_2(\pi(t)),\cdots,d_n(\pi(t))\}$ is inhibition matrix, $\mathcal{A}(\pi(t))=(a_{ij}(\pi(t)))_{n\times n}$ is connection weight matrix, therein $d_n(\pi(t))>0,a_{ij}(\pi(t))\in\mathbb{R}$, τ_t is a non-differentiable time-varying delay, $f_1\in C(\mathbb{R}^n\times\mathbb{R}^n\times\mathsf{S}\times\mathbb{R}^+;\mathbb{R}^n)$, $f_2\in C(\mathbb{R}^n\times\mathbb{R}^n\times\mathsf{S}\times\mathbb{R}^+;\mathbb{R}^{n\times m})$, $r,R\in C(\mathbb{R}^n\times\mathbb{R}^n\times\mathsf{S}\times\mathbb{R}^+\times\mathbb{R}^n/\{0\};\mathbb{R}^n)$ encompass specialized activation functions tailored to diverse architectural designs, and $l\in\mathbb{R}^+$ as a constant is introduced as a threshold to distinguish the intensity of jump vector δ. It is noted that Poisson process works on the big jump while its compensated form works on else. In this paper, we just focus on the small jump given the difficulty to stabilize the big jump. Consequently, we obtain a simplified system:

$$\begin{aligned}dZ(t)=&\Big[-\mathcal{D}(\pi(t))Z(t^-)+\mathcal{A}(\pi(t))f_1\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t\Big)\Big]dt\\&+f_2\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t\Big)dB(t)\\&+\int_{0<|\delta|<l}r\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t,\delta\Big)\widetilde{\mathcal{J}}(dt,d\delta)\end{aligned} \tag{4}$$

and its initial condition

$$\left\{\begin{array}{c}\{Z(t):-\tau\leq t\leq 0\}=\Psi\in D_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)\\ \pi(0)=\pi_0\end{array}\right.. \tag{5}$$

In real industrial systems, large jumps that exceed safety limits are inherently truncated by hardware safeguards such as mechanical stops, overvoltage protection circuits, and current limiting mechanisms. At the sensor level, preprocessing techniques—including sliding-window median filtering, clipping filters, Kalman filtering, and robust filtering—are commonly used to eliminate extreme values and suppress outliers. For networked control systems, communication-induced state jumps are further mitigated through packet-loss compensation, time synchronization, and retransmission mechanisms.Such practical scenarios justify the simplification of our system, making it reasonable for us to focus on the detailed investigation of small jumps.

Now, some related assumptions and lemmas are presented.

Assumption 1. *$\tau_t : \mathbb{R}^+ \to [\underline{\tau}, \tau]$ constitutes a* Borel-*measurable mapping with the property*

$$\widetilde{\tau} := \limsup_{\Delta \to 0+} \left(\sup_{s \geq -\tau} \frac{\sigma\left(K_{s,\Delta}\right)}{\Delta} \right) < \infty, \tag{6}$$

where $\underline{\tau}, \tau \in \mathbb{R}^+$, $K_{s,\Delta} = \{t \in \mathbb{R}^+ \mid t - \tau_t \in [s, s+\Delta)\}$, and $\sigma(\cdot)$ denotes Lebesgue *measure.*

Remark 2. *This property is weaker than usual differentiable time-varying delay with its derivative in $(0,1)$. Actually, most time-varying delays conform to this description, such as time delay of square and sawtooth wave. Moreover, $\widetilde{\tau} > 1$ is proved in [16], which determines the magnitude relationship of some subsequent parameters.*

Lemma 1. *Under Assumption 1, let $\Pi : [-\tau, T - \underline{\tau}] \to \mathbb{R}^+$ is a* càdlàg *function possessing a limited number of discontinuities within any finite time period. Then,*

$$\begin{aligned} \int_0^T \Pi(t - \tau_t) dt &\leq \widetilde{\tau} \int_{-\tau}^{T-\underline{\tau}} \Pi(t) dt \\ &\leq \widetilde{\tau} \Big(\int_{-\tau}^{0} \Pi(t) dt + \int_0^T \Pi(t) dt \Big). \end{aligned} \tag{7}$$

Remark 3. *Due to the introduction of* Lévy *noise, our solution is defined within the* càdlàg *sense, which does not meet the premise of continuity in Lemma 2.2 of [17]. Assumption 1 is equivalent to that for any $\varepsilon > 0$, there exists $\bar{\Delta}$ sufficiently small such that $\sup_{s \geq -\tau} \frac{\sigma(K_{s,\Delta})}{\Delta} \leq \widetilde{\tau} + \varepsilon$ holds for all $\Delta \in (0, \bar{\Delta})$. It is sufficient to prove this problem by utilizing the Riemann-Lebesgue integral under interval division instead of the general Riemann integral.*

Assumption 2. *Suppose that for a constant $\kappa > 0$, and any $Z_{(1)}, Z_{(2)}, \check{Z}_{(1)}, \check{Z}_{(2)} \in \mathbb{R}^n$, $|Z_{(1)}| \vee |Z_{(2)}| \vee |\check{Z}_{(1)}| \vee |\check{Z}_{(2)}| \leq \kappa$, then there exists $L_\kappa > 0$ satisfying*

$$\begin{aligned} |f_1(Z_{(1)}, Y_{(1)}, i, t) - f_1(Z_{(2)}, Y_{(2)}, i, t)| &\leq L_\kappa (|Z_{(1)} - Z_{(2)}| + |Y_{(1)} - Y_{(2)}|), \\ |f_2(Z_{(1)}, Y_{(1)}, i, t) - f_2(Z_{(2)}, Y_{(2)}, i, t)| &\leq L_\kappa (|Z_{(1)} - Z_{(2)}| + |Y_{(1)} - Y_{(2)}|), \end{aligned} \tag{8}$$

where $i \in \mathsf{S}$. Furthermore, without losing generality, divide S into $\mathsf{S}_1 = \{1, 2, 3, \cdots, N_1\}$ and $\mathsf{S}_2 = \{N_1 + 1, N_1 + 2, \cdots, N\}$, then there exists $L > 0$ satisfying

$$\begin{aligned} |f_1(Z, \check{Z}, i, t)| &\leq L(|Z| + |\check{Z}|), \\ |f_2(Z, \check{Z}, i, t)| &\leq L(|Z| + |\check{Z}|), \end{aligned} \tag{9}$$

for $i \in \mathsf{S}_1$, and

$$\begin{aligned} |f_1(Z, \check{Z}, i, t)| &\leq L(|Z| + |\check{Z}| + |Z|^{p_1} + |\check{Z}|^{p_2}), \\ |f_2(Z, \check{Z}, i, t)| &\leq L(|Z| + |\check{Z}| + |Z|^{p_3} + |\check{Z}|^{p_4}), \end{aligned} \tag{10}$$

for $i \in \mathsf{S}_2$, where $p_1, p_3 > 1, p_2, p_4 \geq 1$.

Remark 4. *Assumption 2 depicts a mixed growth condition,thus, activation function does not solely rely on linear or polynomial growth constraints, but rather dynamically adjusts precision limits based on node pairs in the Markov chain. When (9) treated as a special case for (10), p_1, p_2, p_3, p_4 becomes undefined, failing to capture system characteristics, in other words, this construction is not a simple modification of the previous papers about polynomial growth. Meanwhile, this construction is of practical significance, for instance, in multi-drone collaborative forest fire safety detection and evacuation operations, initial fire spread follows linear growth patterns in uniform shrublands, while nonlinear changes under varying slopes and wind directions exhibit polynomial growth. Simply treating all detection scenarios across different terrains as polynomial growth may lead to unnecessary resource waste. Lipschitz continuity (8) is first used to guarantee the existence and uniqueness of system solutions,* *which is a standard requirement for stochastic differential equations driven by Lévy noise and Markovian switching. Beyond this, the Lipschitz condition also plays a critical role in our stability analysis. Specifically, it enables the construction of the Lyapunov function and the derivation of the Itô-type differential operator estimates, which are essential for establishing the mean-square/exponential stabilization criteria.Without this condition, the required bounds for the stochastic Lyapunov analysis cannot be guaranteed.*

Assumption 3. *Suppose that for a constant $\iota > 0$, and any $Z_{(1)}, Z_{(2)}, \check{Z}_{(1)}, \check{Z}_{(2)} \in \mathbb{R}^n$, $|Z_{(1)}| \vee |Z_{(2)}| \vee |\check{Z}_{(1)}| \vee |\check{Z}_{(2)}| \leq \iota$, there exists $\Phi_\iota > 0$ satisfying*

$$\int_{0<|\delta|<l} |r(Z_{(1)}, \check{Z}_{(1)}, i, t, \delta) - r(Z_{(2)}, \check{Z}_{(2)}, i, t, \delta)|\vartheta(d\delta) \leq \Phi_\iota(|Z_{(1)} - Z_{(2)}| + |\check{Z}_{(1)} - \check{Z}_{(2)}|), \tag{11}$$

where $i \in \mathsf{S}$. Furthermore, there exist $\Phi > 0, d \geq 1$, for $0 < |\delta| < l$, satisfying

$$|r(Z, \check{Z}, i, t, \delta)| \leq \Phi|\delta|^d(|Z| + |\check{Z}|). \tag{12}$$

Remark 5. *$r(0, 0, i, t, \delta) \equiv 0$ is a trivial solution. Combining with (2), we can know that $\int_{0<|\delta|<l}(|\delta|^d)\vartheta(d\delta) < \infty$, for $d \geq 2$. It is worth noting that this does imply the measure ϑ is not required to be finite because we may not obtain $\vartheta(\{\delta \in \mathbb{R}^n/\{0\} \mid |\delta| < \varepsilon\}) < \infty$ from $\vartheta(\{\delta \in \mathbb{R}^n/\{0\} \mid |\delta| > \varepsilon\}) < \infty$ for arbitrary $\varepsilon > 0$.*

Under the condition of Assumptions 2 and 3, despite that maximum local solution for (4) are assured, we still need paying attention to the blow-up points. Following Khasminskii-type conditions help to avoid blow-ups within a finite time.

Assumption 4. *There exist some polynomial order estimators $p, q > 2$, where $p > q + p_1 - 1, q \geq 2(p_1 \vee p_2 \vee p_3 \vee p_4) - p_1 + 1$ and $\beta_1, \beta_2, \beta_3$, such that*

$$Z^T[-\mathcal{D}(i)Z + \mathcal{A}(i)f_1(Z, \check{Z}, i, t)] + \frac{p-1}{2}|f_2(Z, \check{Z}, i, t)|^2 \leq \begin{cases} \beta_1|Z|^2 + \beta_1|\check{Z}|^2, i \in \mathsf{S}_1, \\ \beta_1|Z|^2 + \beta_1|\check{Z}|^2 - \beta_2|Z|^q + \beta_3||^q, i \in \mathsf{S}_2, \end{cases} \tag{13}$$

where $\beta_2 > \beta_3\widetilde{\tau}$. Assuming that $q_1 = p\beta_2 - \frac{p(p-2)\beta_3}{p+q-2}$,$q_2 = \frac{pq\beta_3}{p+q-2}$, it can be known that $q_1 > \widetilde{\tau}q_2$ by $\widetilde{\tau} > 1$.

Assumptions 1–4 provide a comprehensive and rigorous characterization of the initial research object. To establish the rationality and demonstrate its potential for stabilizability, it is essential to conduct a systematic verification of two critical aspects: existence and uniqueness of solutions, together with moment boundedness.

Lemma 2. *When Assumptions 1–4 hold,* STDNN (4) *with initial condition* (5) *posses a unique global solution $Z(t)$ on $(-\tau, \infty)$ and for all $t \in \mathbb{R}^+$*

$$\mathbb{E}|Z(t)|^p < \infty \tag{14}$$

and

$$\mathbb{E}\int_0^t |Z(s)|^{p+q-2}ds < \infty. \tag{15}$$

Proof. Existence of solutions is verified first. Denote $r(Z, \check{Z}, i, t, \delta)$ as $r_t(\delta)$ for convenience. For $p > 2$, by mean value theorem, $\xi_1 \in (0, 1)$ can be found to satisfy

$$|Z + r_t(\delta)|^p - |Z|^p = p|Z + \xi_1 r_t(\delta)|^{p-2}\Big(Z + \xi_1 r_t(\delta)\Big)^T r_t(\delta).$$

Similarly, there exists $\xi_2 \in (0, 1)$,

$$p|Z + \xi_1 r_t(\delta)|^{p-2}(Z + \xi_1 r_t(\delta))^T r_t(\delta) - p|Z|^{p-2}Z^T r_t(\delta) \leq \xi_1[p(p-1)|Z + \xi_1\xi_2 r_t(\delta)|^{p-2}|r_t(\delta)|^2].$$

Combined with the above formula, the parameter range and Hölder's inequality, we can derive

$$\begin{aligned}&|Z+r_t(\delta)|^p-|Z|^p-p|Z|^{p-2}Z^Tr_t(\delta)\\ \leq&\xi_1p(p-1)(|Z|+|r_t(\delta)|)^{p-2}|r_t(\delta)|^2\\ \leq&\xi_1p(p-1)2^{p-3}(|Z|^{p-2}+|r_t(\delta)|^{p-2})|r_t(\delta)|^2\\ \leq&\xi_1p(p-1)2^{p-3}(|Z|^{p-2}|r_t(\delta)|^2+|r_t(\delta)|^p).\end{aligned}$$

By (12) and Hölder's inequality,

$$\begin{aligned}|Z|^{p-2}|r_t(\delta)|^2&\leq 2\Phi^2|\delta|^{2d}(|Z|^p+|Z|^{p-2}|\check{Z}|^2)\\ &\leq 2\Phi^2|\delta|^{2d}(\frac{2(p-1)}{p}|Z|^p+\frac{2}{p}|\check{Z}|^p),\end{aligned}$$

$$|r_t(\delta)|^p\leq 2^{p-1}\Phi^p|\delta|^{pd}(|Z|^p+|\check{Z}|^p),$$

$$\begin{aligned}&|Z+r_t(\delta)|^p-|Z|^p-p|Z|^{p-2}Z^Tr_t(\delta)\\ \leq&\xi_1p(p-1)2^{p-3}\Big[\left(2\Phi^2|\delta|^{2d}\frac{2(p-1)}{p}+2^{p-1}\Phi^p|\delta|^{pd}\right)|Z|^p+\left(2\Phi^2|\delta|^{2d}\frac{2}{p}+2^{p-1}\Phi^p|\delta|^{pd}\right)|\check{Z}|^p\Big].\end{aligned}$$

Consequently, there exist $\alpha_1,\alpha_2>0$ such that

$$\int_{0<|\delta|<l}\Big[|Z+r_t(\delta)|^p-|Z|^p-p|Z|^{p-2}Z^Tr_t(\delta)\Big]\vartheta(d\delta)\leq\alpha_1|Z|^p+\alpha_2|\check{Z}|^p. \tag{16}$$

Furthermore, use the stopping time manner to study the blow-up points. Set a finite interval $[0,T]$, a sequence of stopping times $\{\zeta_v\}_{v\geq 0}$, $0=\zeta_0<\zeta_1<\cdots<\zeta_{\bar{v}}=T$ ($\zeta_v=T$, if $v>\bar{v}$), a right-continuous step function $\pi(t)$ jumping at the stopping time, and the truncation functions f_{1k},f_{2k},r_k for f_1,f_2,r in the form of

$$\begin{aligned}f_{1k}(Z,\check{Z},i,t)&=f_1\left(\frac{|Z|\wedge k}{|Z|}Z,\frac{|\check{Z}|\wedge k}{|\check{Z}|}\check{Z},i,t\right),\\ f_{2k}(Z,\check{Z},i,t)&=f_2\left(\frac{|Z|\wedge k}{|Z|}Z,\frac{|\check{Z}|\wedge k}{|\check{Z}|}\check{Z},i,t\right),\\ r_k(Z,\check{Z},i,t)&=r\left(\frac{|Z|\wedge k}{|Z|}Z,\frac{|\check{Z}|\wedge k}{|\check{Z}|}\check{Z},i,t\right),\end{aligned}$$

where noting $\frac{|Z|\wedge k}{|Z|}Z=0$ if $Z=0$, which is the same for $\check{Z}$.

When $t\in[\zeta_v,\zeta_{v+1}]$, we can derive that

$$\begin{aligned}dZ_k(t)=&\Big[-\mathcal{D}(\pi(\zeta_v))Z_k(t^-)+\mathcal{A}(\pi(\zeta_v))f_{1k}\Big(Z_k(t^-),Z_k((t-\tau_t)^-),\pi(\zeta_v),t\Big)\Big]dt\\ &+f_{2k}\Big(Z_k(t^-),Z_k((t-\tau_t)^-),\pi(\zeta_v),t\Big)dB(t)\\ &+\int_{0<|\delta|<l}r_k\Big(Z_k(t^-),Z_k((t-\tau_t)^-),\pi(\zeta_v),t,\delta\Big)\widetilde{\mathcal{J}}(dt,d\delta)\end{aligned}$$

has a unique solution $Z_k(t)$ on $[\zeta_v-\tau,\zeta_{v+1}]$ for any fixed ζ_v.

By induction, the same is true that

$$\begin{aligned}dZ_k(t)=&\Big[-\mathcal{D}(\pi(t))Z_k(t^-)+\mathcal{A}(\pi(t))f_{1k}\Big(Z_k(t^-),Z_k((t-\tau_t)^-),\pi(t),t\Big)\Big]dt\\ &+f_{2k}\Big(Z_k(t^-),Z_k((t-\tau_t)^-),\pi(t),t\Big)dB(t)\\ &+\int_{0<|\delta|<l}r_k\Big(Z_k(t^-),Z_k((t-\tau_t)^-),\pi(t),t,\delta\Big)\widetilde{\mathcal{J}}(dt,d\delta)\end{aligned}$$

on $t \in [0,T]$ with a initial value $Z_k(t) = \Psi(t), t \in [-\tau, 0]$. Devise a predictor $\varpi(t)$ such that $\varpi(t) = \Psi(t)$ on $[-\tau, 0]$ and respective stopping time $\zeta_k(\varpi) := \inf\{t \in [0,T] \mid |\varpi(t)| \vee |\varpi(t-\tau_t)| \geq k\}$, $(\inf \emptyset = \infty)$. Denote $\zeta_k(Z_k)$ as φ_k, specifically, $\varphi_\infty = \lim_{k\to\infty} \varphi_k, \varphi_0 = 0$. Thus, $\varphi_{k-1} \leq \varphi_k$, $Z_k(t) = Z_{k+1}(t)$ on $t \in [0, \varphi_k)$. $Z(t)$ as a local sub function is defined on $[-\tau, \varphi_\infty)$, s.t. $Z(t) = \Psi(t)$ on $[-\tau, 0]$ and $Z(t) = Z_k(t)$ on $[\varphi_{k-1}, \varphi_k)$ for $k \geq 1$ if $\varphi_{k-1} < \varphi_k$.

Then, on $[0, \varphi_k)$, $Z(t) \equiv Z_k(t)$, and for every k, one has

$$\begin{aligned}
&Z((t\wedge\varphi_k)^-) = Z_k((t\wedge\varphi_k)^-)\\
=&Z(0) + \int_0^{(t\wedge\varphi_k)^-} \Big[-\mathcal{D}(\pi(t))Z_k(s^-) + \mathcal{A}(\pi(t)) f_{1k}\Big(Z_k(s^-), Z_k((s-\tau_s)^-), \pi(s), s\Big)\Big] ds\\
&+ \int_0^{(t\wedge\varphi_k)^-} f_{2k}(Z_k(s^-), Z_k((s-\tau_s)^-), \pi(s), s) dB(s)\\
&+ \int_0^{(t\wedge\varphi_k)^-} \int_{0<|\delta|<l} r\Big(Z_k(s^-), Z_k((t-\tau_s)^-), \pi(s), s, \delta\Big) \widetilde{\mathcal{J}}(ds, d\delta)
\end{aligned}$$

on $t \in [0,T]$. If $\varphi_\infty < T$, then

$$\limsup_{t\to\varphi_\infty} |Z(t)| = \limsup_{k\to\infty} |Z(\varphi_k^-)| = \limsup_{k\to\infty} |Z_k(\varphi_k^-)| = \infty,$$

that is, $\{Z(t) \mid t \in [-\tau, \varphi_\infty]\}$ is a maximum local solution on $[-\tau, T]$. Due to arbitrariness of T, set $T = \infty$, we can derive the existence of solutions for (4) with (5). Similarly, using the proof by contradiction, we can derive the uniqueness of solutions for (4) with (5). φ_∞ means the explosion time for blow-up point. Before φ_∞, we can define a more refined sequence of stopping times for proof of boundedness, as followings:

$$\mu_{k'} = \varphi_\infty \wedge \inf\{t \in [0, \varphi_\infty) \mid |Z(t)| \geq k'\} \tag{17}$$

for each $k' \geq \|\Psi\|$. Similarly, we set $\mu_\infty = \lim_{k'\to\infty} \mu_{k'}$.

Let a Lyapunov function $V_1 = |Z|^p$.

By Itô's formula,

$$dV_1 = \mathcal{L}V_1(Z(t^-), Z((t-\tau_t)^-), \pi(t), t)dt + \mathcal{M}dB(t),$$

where

$$\begin{aligned}
&\mathcal{L}V_1(Z, \check{Z}, i, t)\\
=&p|Z|^{p-2}Z^T\Big[-\mathcal{D}(i)Z + \mathcal{A}(i)f_1(Z,\check{Z},i,t)\Big] + \frac{p}{2}|Z|^{p-2}|f_2(Z,\check{Z},i,t)|^2\\
&+ \frac{p(p-2)}{2}|Z|^{p-4}|Z^T f_2(Z,\check{Z},i,t)|^2\\
&+ \int_{0<|\delta|<l} \Big[|Z + r_t(\delta)|^p - |Z|^p - p|Z|^{p-2}Z^T r_t(\delta)\Big]\vartheta(d\delta)\\
\leq&p|Z|^{p-2}\Big(Z^T[-\mathcal{D}(i)Z + \mathcal{A}(i)f_1(Z,\check{Z},i,t)]\\
&+ \frac{p-1}{2}|f_2(Z,\check{Z},i,t)|^2\Big) + \alpha_1|Z|^p + \alpha_2|\check{Z}|^p,
\end{aligned}$$

$$\mathcal{M} = p|Z|^{p-2}Z^T f_2\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\Big).$$

By Assumption 4, we can conclude that

$$\begin{aligned}
\mathcal{L}V_1(Z,\check{Z},i,t) \leq& \beta_1 p|Z|^p + \beta_1 p|Z|^{p-2}|\check{Z}|^2 - \beta_2 p|Z|^{p+q-2}\\
&+ \beta_3 p|Z|^{p-2}|\check{Z}|^q + \alpha_1|Z|^p + \alpha_2|\check{Z}|^p,
\end{aligned}$$

for $i \in \mathsf{S}_2$.

Choose a constant ε_1, ϱ_0 such that

$$0<\beta_1 p\varepsilon_1<\min\{\frac{2}{\widetilde{\tau}},\frac{2-\frac{1}{\tau}\ln\frac{q_1-\varrho_0}{q_2\widetilde{\tau}}}{1+\frac{q_1-\varrho_0}{q_2}}\}, \tag{18}$$

where $0<\varrho_0<q_1-\widetilde{\tau}q_2$, $2-\frac{1}{\tau}\ln\frac{q_1-\varrho_0}{q_2\widetilde{\tau}}>0$.

According to Young's inequality,

$$|Z|^{p-2}|\check{Z}|^2\le C(\varepsilon_1,p,2)|Z|^p+\varepsilon_1|\check{Z}|^p,\Big(C(\varepsilon_1,p,2)=\frac{p-2}{p}(\frac{p}{2}\varepsilon_1)^{-\frac{2}{p-2}}\Big),$$
$$|Z|^{p-2}|\check{Z}|^q\le\frac{p-2}{p+q-2}|Z|^{p+q-2}+\frac{q}{p+q-2}|\check{Z}|^{p+q-2}.$$

After sorting, it can be obtained that

$$\begin{aligned}&\mathcal{L}V_1(Z,\check{Z},i,t)\\ \le&(\beta_1p+\beta_1pC(\varepsilon_1,p,2)+\alpha_1)|Z|^p+(\alpha_2+\beta_1p\varepsilon_1)|\check{Z}|^p-q_1|Z|^{p+q-2}+q_2|\check{Z}|^{p+q-2}\\ \le&U_0^1-2|Z|^p+\beta_1p\varepsilon_1|\check{Z}|^p-(q_1-\varrho_0)|Z|^{p+q-2}+q_2|\check{Z}|^{p+q-2},\end{aligned} \tag{19}$$

in which $U_0^1=\sup_{s\ge0}[(2+\beta_1pC(\varepsilon_1,p,2)+\beta_1p+\alpha_1+\alpha_2)|s|^p-\varrho_0|s|^{p+q-2}]$.

Together with (17) and (19), by generalized Itô's formula,

$$\begin{aligned}&\mathbb{E}|Z(t\wedge\mu_{k'})|^p\\ \le&|\Psi(0)|^p+\mathbb{E}\int_0^{t\wedge\mu_{k'}}\mathcal{L}V_1(Z(s^-),Z((s-\tau_s)^-),\pi(s),s)ds\\ \le&|\Psi(0)|^p+U_0^1t-(q_1-\varrho_0)\mathbb{E}\int_0^{t\wedge\mu_{k'}}|Z(s)|^{p+q-2}ds\\ &-2\mathbb{E}\int_0^{t\wedge\mu_{k'}}|Z(s)|^p+\beta_1p\varepsilon_1\mathbb{E}\int_0^{t\wedge\mu_{k'}}|Z((s-\tau_s)^-)|^pds\\ &+q_2\mathbb{E}\int_0^{t\wedge\mu_{k'}}|Z((s-\tau_s)^-)|^{p+q-2}ds.\end{aligned} \tag{20}$$

According to Lemma 1 , it is deduced that

$$\mathbb{E}\int_0^{t\wedge\mu_{k'}}|Z((s-\tau_s)^-)|^pds\le\widetilde{\tau}\Big(\mathbb{E}\int_0^{t\wedge\mu_{k'}}|Z(s)|^pds+\mathbb{E}\int_{-\tau}^0|Z(s)|^pds\Big),$$

$$\mathbb{E}\int_0^{t\wedge\mu_{k'}}|Z((s-\tau_s)^-)|^{p+q-2}ds\le\widetilde{\tau}\Big(\mathbb{E}\int_0^{t\wedge\mu_{k'}}|Z(s)|^{p+q-2}ds+\mathbb{E}\int_{-\tau}^0|Z(s)|^{p+q-2}ds\Big).$$

Substituting the above inequalities into (20) and combining with (18), one has $\mathbb{E}|Z(t\wedge\mu_{k'})|^p\le U_1^1+U_0^1t$, where $U_1^1=|\Psi(0)|^p+\beta_1p\varepsilon_1\widetilde{\tau}\tau\|\Psi\|^p+q_2\widetilde{\tau}\tau\|\Psi\|^{p+q-2}$.

According to the Markov's inequality, it yields that

$$\mathbb{P}(t\ge\mu_{k'})=\mathbb{P}(|Z(t)|\ge k')\le\frac{\mathbb{E}|Z(t\wedge\mu_{k'})|^p}{k'^p}\le\frac{U_1^1+U_0^1t}{k'^p}.$$

Let $k'\to\infty$, then $\mathbb{P}(\mu_\infty\le t)=0$. According to the normativity of probability, it can be inferred that $\mathbb{P}(\mu_\infty>t)=1$. And making $t\to\infty$, we have $\mathbb{P}(\mu_\infty=\infty)=1, i.e.\mu_\infty=\infty$ almost surely.

Let $\varrho_1>0$ be the sole root of the equation:

$$q_1-\varrho_0-q_2\widetilde{\tau}\exp(\varrho_1\tau)=0. \tag{21}$$

By generalized Itô's formula,

$$\begin{aligned}&\exp(\varrho_1 t)\mathbb{E}|Z(t)|^p\\ \leq&|\Psi(0)|^p+\mathbb{E}\int_0^t\exp(\varrho_1 s)\Big(\varrho_1|Z(s)|^p+\mathcal{L}V_1(Z(s^-),Z((s-\tau_s)^-),\pi(s),s)\Big)ds\\ \leq&|\Psi(0)|^p+\mathbb{E}\int_0^t\exp(\varrho_1 s)\Big(\varrho_1|Z(s)|^p+U_0^1-2|Z(s)|^p+\beta_1 p\varepsilon_1|Z(s-\tau_s)|^p+q_2|Z(s-\tau_s)|^{p+q-2}\\ &-(q_1-\varrho_0)|Z(s)|^{p+q-2}\Big)ds. \qquad (22)\end{aligned}$$

According to Lemma 1, it yields that

$$\begin{aligned}&\mathbb{E}\int_0^t\exp(\varrho_1 s)|Z(s-\tau_s)|^p ds\\ \leq&\exp(\varrho_1\tau)\mathbb{E}\int_0^t\exp(\varrho_1(s-\tau_s))|Z(s-\tau_s)|^p ds\\ \leq&\widetilde{\tau}\exp(\varrho_1\tau)\Big(\mathbb{E}\int_{-\tau}^0\exp(\varrho_1 s)|Z(s)|^p ds+\mathbb{E}\int_0^t\exp(\varrho_1 s)|Z(s)|^p ds\Big),\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\int_0^t\exp(\varrho_1 s)|Z(s-\tau_s)|^{p+q-2}ds\\ \leq&\exp(\varrho_1\tau)\mathbb{E}\int_0^t\exp(\varrho_1(s-\tau_s))|Z(s-\tau_s)|^{p+q-2}ds\\ \leq&\widetilde{\tau}\exp(\varrho_1\tau)\Big(\mathbb{E}\int_{-\tau}^0\exp(\varrho_1 s)|Z(s)|^{p+q-2}ds+\mathbb{E}\int_0^t\exp(\varrho_1 s)|Z(s)|^{p+q-2}ds\Big).\end{aligned}$$

Combining with (21), we can further conclude that

$$\exp(\varrho_1 t)\mathbb{E}|Z(t)|^p\leq U_2^1+\frac{U_0^1}{\varrho_1}\exp(\varrho_1 t)+\mathbb{E}\int_0^t\exp(\varrho_1 s)\Big[\big(\varrho_1-2+\beta_1 p\varepsilon_1\widetilde{\tau}\exp(\varrho_1\tau)\big)|Z(s)|^p\Big]ds,$$

where

$$U_2^1=|\Psi(0)|^p+\beta_1 p\varepsilon_1\widetilde{\tau}\exp(\varrho_1\tau)\mathbb{E}\int_{-\tau}^0\exp(\varrho_1 s)|Z(s)|^p ds+q_2\widetilde{\tau}\exp(\varrho_1\tau)\mathbb{E}\int_{-\tau}^0\exp(\varrho_1 s)|Z(s)|^{p+q-2}ds.$$

To apply Gronwall's inequality,

$$\exp(\varrho_1 t)\mathbb{E}|Z(t)|^p\leq\left(U_2^1+\frac{U_0^1}{\varrho_1}\exp(\varrho_1 t)\right)\exp\Big[\big(\varrho_1-2+\beta_1 p\varepsilon_1\widetilde{\tau}\exp(\varrho_1\tau)\big)t\Big].$$

Hence, it follows that

$$\mathbb{E}|Z(t)|^p\leq U_2^1\exp\Big[(-2+\beta_1 p\varepsilon_1\widetilde{\tau}\exp(\varrho_1\tau))t\Big]+\frac{U_0^1}{\varrho_1}\exp\Big[(\varrho_1-2+\beta_1 p\varepsilon_1\widetilde{\tau}\exp(\varrho_1\tau))t\Big].$$

By (18) and (21), it yields that

$$-2+\beta_1 p\varepsilon_1\widetilde{\tau}\exp(\varrho_1\tau)<0,\ \varrho_1-2+\beta_1 p\varepsilon_1\widetilde{\tau}\exp(\varrho_1\tau)<0,$$

which means $\mathbb{E}|Z(t)|^p<\infty$, for $i\in\mathsf{S}_2$.

By Assumption 4, we can conclude that

$$\mathcal{L}V_1(Z,\check{Z},i,t)\leq(\beta_1 p+\alpha_1)|Z|^p+\beta_1 p|\check{Z}|^2|Z|^{p-2}+\alpha_2|\check{Z}|^p,$$

if $i\in\mathsf{S}_1$. Similarly, $\mathcal{L}V_1(Z,\check{Z},i,t)\leq(\beta_1 p+\alpha_1+\beta_1 pC(\varepsilon_2,p,2))|Z|^p+(\beta_1 p\varepsilon_2+\alpha_2)|\check{Z}|^p$. Deleting all the terms containing q_1 and q_2 in all the inequalities and equations appropriately, we can choose a fit ε_2 and ϱ_2 to verify $\mathbb{E}|Z(t)|^p<\infty$ for $i\in\mathsf{S}_1$. Consequently, by using the indicator function technique, we can find the assertion (14) holds. Considering that (15) can be deduced from (14), Lemma 2 is proofed. □

Next, we introduce the following lemma to depict the case of non-node time on the Markov chain (1).

Lemma 3. *Ref. [16] For $s \geq 0$, $h > 0$, $i \in \mathsf{S}$, one has*

$$\mathbb{P}\Big(\pi(s) \neq i, s \in (t, t+h] \mid \pi(t) = i\Big) \leq 1 - \exp(-\bar{\gamma} h),$$

where $\bar{\gamma} = \max_{i \in \mathsf{S}}(-\gamma_{ii})$.

Despite the unique solution and boundedness, the stability for system (4) cannot be firmly ensured. A discrete observer-based feedback control function without time-delay is expected to be devised for stabilizing the original system. We design the following controller by sampling the switching nodes in S space:

$$u(Z(t_m^-), \pi(t_m), t) : \mathbb{R}^n \times \mathsf{S} \times \mathbb{R}^+ \to \mathbb{R}^n,$$

where u is Borel measurable. Observation time is clearly shown as $t_m = [t/h]h$, in which $[t/h] = m, (m = 0, 1, 2, \cdots,)$ represents the integer of t/h, distinguished from the dimension m defined above. $Z(t_m^-) = \lim_{s \uparrow t_m^-} Z(s)$. Set $u \equiv 0$, if $\pi(t_m) \in \mathsf{S}_1$. Apparently, h is the duration between two observation times. As a result, the system post-control can be represented as

$$\begin{aligned} dZ(t) =& \Big[-\mathcal{D}(\pi(t))Z(t^-) + \mathcal{A}(\pi(t)) f_1\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\Big) + u(Z(t_m^-), \pi(t_m), t)\Big] dt \\ &+ f_2\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\Big) dB(t) \\ &+ \int_{0<|\delta|<l} r\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t, \delta\Big) \widetilde{\mathcal{J}}(dt, d\delta). \end{aligned} \tag{23}$$

In order to handle u conveniently, we put forward the following assumption.

Assumption 5. *Considering all $Z_{(1)}, Z_{(2)} \in \mathbb{R}^n$, there exists a constant $\rho > 0$ satisfying*

$$|u(Z_{(1)}, i, t) - u(Z_{(2)}, i, t)| \leq \rho |Z_{(1)} - Z_{(2)}|, \tag{24}$$

where $i \in \mathsf{S}_2$, $t \in \mathbb{R}^+$. $u(0, i, t) = 0$ is a trivial solution.

Remark 6. *The condition $u(0, i, t) = 0$ is a standard persistency condition for neural networks, ensuring that the zero input corresponds to the zero equilibrium point. It is essential for existence,uniqueness and boundedness. Without this condition, the system might have multiple equilibria or the solution trajectory could diverge even without external excitation. Furthermore, it is worth emphasizing that the Lyapunov functional approach in the stochastic sense is adopted in this paper, which is due to the fact that unlike methods such as the Razumikhin approach (which typically yields more conservative results for systems with multiple delays) or linear matrix inequality (LMI)-based methods (which require solving high-dimensional LMIs with high computational cost for large-scale systems), the Lyapunov functional approach provides a systematic, unified framework that can simultaneously handle time delays, stochastic disturbances, and nonlinearities, which is essential for the generality of our results.*

3. Main Results

Before studying the stabilization conditions, we need to provide additional clarification regarding boundedness of controlled systems.

Theorem 1. *According to Assumptions 1–5 for any initial condition (5), the controlled system (23) yields global unique solution satisfying*

$$\sup_{t \in (-\tau, \infty)} \mathbb{E}|Z(t)|^p < \infty. \tag{25}$$

Proof. To elaborate sampling content of the observer, we regard the time that is a multiple of h as the matching time, then the part that does not match the actual time can be regarded as a time-delay function $\Delta h : \mathbb{R}^+ \to [0, h]$, and specifically $\Delta h(t) = t - mh$, for $mh \leq t < (m+1)h$.

Correspondingly, (23) can be further refined into

$$
\begin{aligned}
dZ(t) =& \Big[-\mathcal{D}(\pi(t))Z(t^-) + \mathcal{A}(\pi(t)) f_1\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\Big) \\
&+ u\left(Z((t-\Delta h(t))^-), \pi(t-\Delta h(t)), t\right) \Big] dt \\
&+ f_2\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\Big) dB(t) \\
&+ \int_{0<|\delta|<l} r\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t, \delta\Big) \widetilde{\mathcal{J}}(dt, d\delta).
\end{aligned} \tag{26}
$$

As compared to the proof of Lemma 2, we just process the input item:

$$
\begin{aligned}
\mathcal{L}V_1(Z, \check{Z}, \tilde{Z}, i, \tilde{i}, t) &= p|Z|^{p-2} Z^T u(\tilde{Z}, \tilde{i}, t) + \mathcal{L}V_1(Z, \check{Z}, i, t) \\
&\leq p\rho |Z|^{p-1} |\tilde{Z}| + \mathcal{L}V_1(Z, \check{Z}, i, t).
\end{aligned}
$$

According to Young's inequality,

$$
|Z|^{p-1} |\tilde{Z}| \leq C(\varepsilon_1, p, 1)|Z|^p + \varepsilon_1 |\tilde{Z}|^p,
$$

where $C(\varepsilon_1, p, 1) = \frac{p-1}{p}(p\varepsilon_1)^{-\frac{1}{p-1}}$.

Then,

$$
\begin{aligned}
&\mathcal{L}V_1(Z, \check{Z}, \tilde{Z}, i, \tilde{i}, t) \\
\leq& \Big(\beta_1 p + \beta_1 p C(\varepsilon_1, p, 2) + p\rho C(\varepsilon_1, p, 1) + \alpha_1\Big)|Z|^p \\
&+ (\alpha_2 + \beta_1 p \varepsilon_1)|\check{Z}|^p + p\rho\varepsilon_1 |\tilde{Z}|^p - q_1 |Z|^{p+q-2} + q_2 |\check{Z}|^{p+q-2} \\
\leq& U_0^2 - 2|Z|^p + \beta_1 p \varepsilon_1 |\check{Z}|^p + p\rho\varepsilon_1 |\tilde{Z}|^p - (q_1 - \varrho_0)|Z|^{p+q-2} + q_2 |\check{Z}|^{p+q-2},
\end{aligned}
$$

for $i \in \mathsf{S}_2$, where $U_0^2 = \sup_{s\geq 0}[(2 + \beta_1 p C(\varepsilon_1, p, 2) + \beta_1 p + p\rho C(\varepsilon_1, p, 1) + \alpha_1 + \alpha_2)|s|^p - \varrho_0 |s|^{p+q-2}]$.

Simply replacing U_0^1 with U_0^2 in the subsequent proof of Lemma 2,

$$
\begin{aligned}
&\mathbb{E}|Z(t \wedge \mu_{k'})|^p \\
\leq& U_1^1 + U_0^2 t + p\rho\varepsilon_1 \int_0^{t\wedge\mu_{k'}} \mathbb{E}|Z(s - \Delta h(s))|^p ds \\
\leq& U_1^1 + U_0^2 t + p\rho\varepsilon_1 \int_0^{t} \mathbb{E}|Z(s - \Delta h(s))|^p I_{[0,\tau_k]} ds,
\end{aligned}
$$

$$
\mathbb{E}|Z(s - \Delta h(s))|^p I_{[0,\tau_k]} \leq \sup_{0\leq \bar{s}\leq s} \mathbb{E}|Z(\bar{s})|^p I_{[0,\tau_k]} \leq \sup_{0\leq \bar{s}\leq s} \mathbb{E}|Z(\bar{s}) \wedge \mu_{k'}|^p.
$$

To apply Gronwall's inequality,

$$
\sup_{0\leq \bar{s}\leq s} \mathbb{E}|Z(\bar{s} \wedge \mu_{k'})|^p \leq (U_1^1 + U_0^2 t)\exp(p\rho\varepsilon_1 t). \tag{27}
$$

The relevant conclusion are consistent with Lemma 2.
Similarly, by comparing above indirect conclusion with aforementioned, one has

$$
\sup_{0\leq \bar{s}\leq t} \exp(\varrho_1 \bar{s})\mathbb{E}|Z(\bar{s})|^p \leq \left(U_2^1 + \frac{U_0^2}{\varrho_1}\exp(\varrho_1 t)\right) \exp\Big[\big(\varrho_1 - 2 + \beta_1 p \varepsilon_1 \widetilde{\tau}\exp(\varrho_1 \tau)\big)t + p\rho\varepsilon_1\Big].
$$

Therefore,

$$\begin{aligned}&\mathbb{E}|Z(t)|^p\\ \leq&U_2^1\exp\Big[(-2+\beta_1 p\varepsilon_1\widetilde{\tau}\exp(\varrho_1\tau))t+p\rho\varepsilon_1\Big]+\frac{U_0^2}{\varrho_1}\exp\Big[(\varrho_1-2+\beta_1 p\varepsilon_1\widetilde{\tau}\exp(\varrho_1\tau))t+p\rho\varepsilon_1\Big].\end{aligned}\tag{28}$$

Hence, we have

$$\sup_{t\in(-\tau,\infty)}\mathbb{E}|Z(t)|^p<\infty,$$

for $i\in\mathsf{S}_2$. When $i\in\mathsf{S}_1$, since $u=0$, the case reduces to the local conclusion of Lemma 2, which is therefore trivial. Consequently, conclusion (25) can be established. □

In order to study the stabilization characteristics, we put forward two reasonable assumptions on the basis of above-mentioned.

Assumption 6. *There exist $a_i,\bar{a}_i,\tilde{a}_i\in\mathbb{R}, b_i,\bar{b}_i,\tilde{b}_i,\hat{a}_i,\hat{b}_i,\grave{a}_i,\grave{b}_i\in\mathbb{R}^+$ for $i\in\mathsf{S}_1$, satisfying*

$$\begin{aligned}&2\Big(Z^T\Big[-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)+u(Z,i,t)\Big]+\frac{1}{2}|f_2(Z,\check{Z},i,t)|^2\Big)\\&+\int_{0<|\delta|<l}\Big[|Z+r_t(\delta)|^2-|Z|^2-2Z^Tr_t(\delta)\Big]\vartheta(d\delta)\leq a_i|Z|^2+b_i|\check{Z}|^2,\end{aligned}$$

$$Z^T\Big[-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)+u(Z,i,t)\Big]+\frac{p_1}{2}|f_2(Z,\check{Z},i,t)|^2\leq\bar{a}_i|Z|^2+\bar{b}_i|\check{Z}|^2,$$

$$Z^T\Big[-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)+u(Z,i,t)\Big]+\frac{p_1+q-2}{2}|f_2(Z,\check{Z},i,t)|^2\leq\tilde{a}_i|Z|^2+\tilde{b}_i|\check{Z}|^2,$$

$$\begin{aligned}&\int_{0<|\delta|<l}\Big[|Z+r_t(\delta)|^{p_1+1}-|Z|^{p_1+1}-(p_1+1)|Z|^{p_1-1}Z^Tr_t(\delta)\Big]\vartheta(d\delta)\\ \leq&\hat{a}_i|Z|^{p_1+1}+\hat{b}_i|\check{Z}|^{p_1+1}\end{aligned}$$

$$\begin{aligned}&\int_{0<|\delta|<l}\Big[|Z+r_t(\delta)|^{p_1+q-1}-|Z|^{p_1+q-1}-(p_1+q-1)|Z|^{p_1+q-3}Z^Tr_t(\delta)\Big]\vartheta(d\delta)\\ \leq&\grave{a}_i|Z|^{p_1+q-1}+\grave{b}_i|\check{Z}|^{p_1+q-1},\end{aligned}$$

where $u\equiv 0$.

Assumption 7. *There exist $a_i,\bar{a}_i\in\mathbb{R}$ and $\hat{a}_i,\hat{b}_i,b_i,c_i,w_i,\bar{b}_i,\bar{c}_i,\bar{w}_i\in\mathbb{R}^+$ for $i\in\mathsf{S}_2$, satisfying*

$$\begin{aligned}&2\Big(Z^T\Big[-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)+u(Z,i,t)\Big]+\frac{1}{2}|f_2(Z,\check{Z},i,t)|^2\Big)\\&+\int_{0<|\delta|<l}\Big[|Z+r_t(\delta)|^2-|Z|^2-2Z^Tr_t(\delta)\Big]\vartheta(d\delta)\\ \leq&a_i|Z|^2+b_i|\check{Z}|^2-c_i|Z|^p+w_i|\check{Z}|^p,\end{aligned}\tag{29}$$

$$Z^T\Big[-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)+u(Z,i,t)\Big]+\frac{p_1}{2}|f_2(Z,\check{Z},i,t)|^2\leq\bar{a}_i|Z|^2+\bar{b}_i|\check{Z}|^2-\bar{c}_i|Z|^p+\bar{w}_i|\check{Z}|^p,\tag{30}$$

$$\begin{aligned}&\int_{0<|\delta|<l}\Big[|Z+r_t(\delta)|^{p_1+1}-|Z|^{p_1+1}-(p_1+1)|Z|^{p_1-1}Z^Tr_t(\delta)\Big]\vartheta(d\delta)\\ \leq&\hat{a}_i|Z|^{p_1+1}+\hat{b}_i|\check{Z}|^{p_1+1}.\end{aligned}\tag{31}$$

Based on Assumption 6 and (29)–(31), it yields that

$$M_1 := -diag(a_1, a_2, a_3, \cdots, a_N) - \Gamma,$$

$$M_2 := -diag((p_1+1)\bar{a}_1 + \hat{a}_1, \cdots, (p_1+1)\bar{a}_N + \hat{a}_N) - \Gamma,$$

$$M_3 := -(p_1+q-1)diag(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_{N_1}) - diag(\grave{a}_1, \grave{a}_2, \cdots, \grave{a}_{N_1}) - (\gamma_{ij})_{i,j\in \mathsf{S}_1}$$

are non-singular M-matrices.

In addition,

$$\epsilon_1\tilde{\tau} < 1, \epsilon_3\tilde{\tau} < \epsilon_2, (2\tilde{\tau} + p_1 - 1)\epsilon_4 + \epsilon_7\tilde{\tau} < 1, \frac{\epsilon_6(p_1 - 1 + q\tilde{\tau})}{p_1+q-1} < \epsilon_5, \tag{32}$$

in which $\epsilon_1 = \max_{i\in\mathsf{S}} \eta_i b_i$, $\epsilon_2 = \min_{i\in\mathsf{S}_2} \eta_i c_i$, $\epsilon_3 = \max_{i\in\mathsf{S}_2} \eta_i w_i$, $\epsilon_4 = \max_{i\in\mathsf{S}} \bar{b}_i\bar{\eta}_i$, $\epsilon_5 = \min_{i\in\mathsf{S}_2} \bar{\eta}_i\bar{c}_i$, $\epsilon_6 = \max_{i\in\mathsf{S}_2} \bar{\eta}_i\bar{w}_i$, $\epsilon_7 = \max_{i\in\mathsf{S}} \bar{\eta}_i\hat{b}_i$ and $(\eta_1, \eta_2, \cdots, \eta_N)^T = M_1^{-1}(1,1,\cdots,1)^T$, $(\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_N)^T = M_2^{-1}(1,1,\cdots,1)^T$.

For convenience, denote $\varsigma := \max_{i\in\mathsf{S}_1} \varsigma_i$ and ς_i represents the i-th row sum of ${M_3}^{-1}$, then we have

$$Q = \frac{(p_1+1)\epsilon_5 - \frac{(p_1-1)(p_1+1)\epsilon_6}{p_1+q-1}}{\max_{i\in\mathsf{S}_2}(\sum_{j\in\mathsf{S}_1} \gamma_{ij})\varsigma + 1} \tag{33}$$

and $(\tilde{\eta}_1, \tilde{\eta}_2, \cdots, \tilde{\eta}_{N_1}) = Q{M_3}^{-1}(1,1,\cdots,1)^T$.

Furthermore, it is assumed that the following condition can be met:

$$(p+q-3+2\tilde{\tau})\epsilon_8 + \tilde{\tau}\epsilon_9 < Q, \tag{34}$$

where $\epsilon_8 = \max_{i\in\mathsf{S}_1} \tilde{\eta}_i\tilde{b}_i$, $\epsilon_9 = \max_{i\in\mathsf{S}_1} \tilde{\eta}_i\grave{b}_i$.

According to the properties of $M-$matrix, η_i, $\bar{\eta}_i$, $\tilde{\eta}_i$ are positive. Then, we can define a new Lyapunov function $V_2(Z,i) = \eta_i|Z|^2 + \bar{\eta}_i|Z|^{p_1+1} + \tilde{\eta}_i|Z|^{p_1+q-1}I_{i\in\mathsf{S}_1}$.

If $\Delta h(t) \equiv 0$ in (26), for the drift term $\mathcal{L}V_2(Z,\check{Z},i,t)$, one has

$$\begin{aligned}
&\mathcal{L}V_2(Z,\check{Z},i,t)\\
=&2\eta_i\Big(Z^T\Big[-\mathcal{D}(i)Z + \mathcal{A}(i)f_1(Z,\check{Z},i,t) + u(Z,i,t)\Big] + \frac{1}{2}|f_2(Z,\check{Z},i,t)|^2\Big)\\
&+ (p_1+1)\bar{\eta}_i\Big(|Z|^{p_1-1}Z^T\Big[-\mathcal{D}(i)Z + \mathcal{A}(i)f_1(Z,\check{Z},i,t) + u(Z,i,t)\Big] + \frac{1}{2}|Z|^{p_1-1}|f_2(Z,\check{Z},i,t)|^2\\
&+ \frac{p_1-1}{2}|Z|^{p_1-3}|Z^Tf_2(Z,\check{Z},i,t)|^2\Big) + \sum_{j=1}^{N}\gamma_{ij}(\eta_j|Z|^2 + \bar{\eta}_j|Z|^{p_1+1})\\
&+ \int_{0<|\delta|<l}\eta_i\Big[|Z+r_t(\delta)|^2 - |Z|^2 - 2Z^Tr_t(\delta)\Big]\vartheta(d\delta)\\
&+ \int_{0<|\delta|<l}\bar{\eta}_i\Big[|Z+r_t(\delta)|^{p_1+1} - |Z|^{p_1+1} - (p_1+1)|Z|^{p_1-1}Z^Tr_t(\delta)\Big]\vartheta(d\delta)\\
&+ (p_1+q-1)\tilde{\eta}_i(|Z|^{p_1+q-3}Z^T\Big[-\mathcal{D}(i)Z + \mathcal{A}(i)f_1(Z,\check{Z},i,t) + u(Z,i,t)\Big]\\
&+ \frac{1}{2}|Z|^{p_1+q-3}|f_2(Z,\check{Z},i,t)|^2 + \frac{p_1+q-3}{2}|Z|^{p_1+q-5}|Z^Tf_2(Z,\check{Z},i,t)|^2\Big)I_{i\in\mathsf{S}_1}\\
&+ \sum_{j=1}^{N_1}\gamma_{ij}\tilde{\eta}_j|Z|^{p_1+q-1} + \int_{0<|\delta|<l}\tilde{\eta}_i\Big[|Z+r_t(\delta)|^{p_1+q-1} - |Z|^{p_1+q-1} - (p_1+q-1)|Z|^{p_1+q-3}Z^Tr_t(\delta)\Big]\vartheta(d\delta)I_{i\in\mathsf{S}_1}.
\end{aligned}$$

According to Assumptions 6 and 7, it yields that

$$
\begin{aligned}
&\mathcal{L}V_2(Z,\check{Z},i,t)\\
\leq&\eta_i\Big[a_i|Z|^2+b_i|\check{Z}|^2+\left(-c_i|Z|^q+w_i|\check{Z}|^q\right)I_{i\in\mathsf{S}_2}\Big]\\
&+(p_1+1)\,\bar{\eta}_i|Z|^{p_1-1}\Big[\bar{a}_i|Z|^2+\bar{b}_i|\check{Z}|^2+(-\bar{c}_i|Z|^q+\bar{w}_i|\check{Z}|^q)I_{i\in\mathsf{S}_2}\Big]\\
&+\bar{\eta}_i\Big(\hat{a}_i|Z|^{p_1+1}+\hat{b}_i|\check{Z}|^{p_1+1}\Big)+\sum_{j=1}^{N}\gamma_{ij}\left(\eta_j|Z|^2+\bar{\eta}_j|Z|^{p_1+1}\right)\\
&+\Big[(p_1+q-1)\,\tilde{\eta}_i|Z|^{p_1+q-3}\left(\tilde{a}_i|Z|^2+\tilde{b}_i|\check{Z}|^2\right)+\tilde{\eta}_i\left(\grave{a}_i|Z|^{p_1+q-1}+\grave{b}_i|\check{Z}|^{p_1+q-1}\right)\Big]I_{i\in\mathsf{S}_1}\\
&+\sum_{j=1}^{N_1}\gamma_{ij}\tilde{\eta}_j|Z|^{q+p_1-1}.
\end{aligned}
$$

By Young's inequality, one has

$$
\begin{aligned}
|Z|^{p_1-1}|\check{Z}|^2&\leq\frac{p_1-1}{p_1+1}|Z|^{p_1+1}+\frac{2}{p_1+1}|\check{Z}|^{p_1+1},\\
|Z|^{p_1-1}|\check{Z}|^q&\leq\frac{p_1-1}{p_1+q-1}|Z|^{p_1+q-1}+\frac{q}{p_1+q-1}|\check{Z}|^{p_1+q-1},\\
|Z|^{p_1+q-3}|\check{Z}|^2&\leq\frac{p_1+q-3}{p_1+q-1}|Z|^{p_1+q-1}+\frac{2}{p_1+q-1}|\check{Z}|^{p_1+q-1}.
\end{aligned}
$$

After some proper arrangements, there is

$$
\begin{aligned}
&\mathcal{L}V_2(Z,\check{Z},i,t)\\
\leq&\eta_ia_i|Z|^2+\eta_ib_i|\check{Z}|^2+(-\eta_ic_i|Z|^q+\eta_iw_i|\check{Z}|^q)I_{i\in S_2}+\bar{\eta}_i\Big((p_1+1)\bar{a}_i+\hat{a}_i\Big)|Z|^{p_1+1}\\
&+(p_1+1)\bar{\eta}_i\bar{b}_i\Big(\frac{p_1-1}{p_1+1}|Z|^{p_1+1}+\frac{2}{p_1+1}|\check{Z}|^{p_1+1}\Big)\\
&+\Big[-(p_1+1)\bar{\eta}_i\bar{c}_i|Z|^{p_1+q-1}+(p_1+1)\bar{\eta}_i\bar{w}_i\Big(\frac{p_1-1}{p_1+q-1}|Z|^{p_1+q-1}+\frac{q}{p_1+q-1}|\check{Z}|^{p_1+q-1}\Big)\Big]I_{i\in\mathsf{S}_2}\\
&+\Big[\left((p_1+q-1)\tilde{a}_i+\grave{a}_i\right)\tilde{\eta}_i|Z|^{p_1+q-1}+(p_1+q-1)\tilde{\eta}_i\tilde{b}_i\Big(\frac{p_1+q-3}{p_1+q-1}|Z|^{p_1+q-1}\\
&+\frac{2}{p_1+q-1}|\check{Z}|^{p_1+q-1}\Big)+\tilde{\eta}_i\grave{b}_i|\check{Z}|^{p_1+q-1}\Big]I_{i\in\mathsf{S}_1}+\sum_{j=1}^{N}\gamma_{ij}(\eta_j|Z|^2+\bar{\eta}_j|Z|^{p_1+1})+\sum_{j=1}^{N_1}\gamma_{ij}\tilde{\eta}_j|Z|^{q+p_1-1}.
\end{aligned}
$$

According to the definition of parameters $\eta_i,\bar{\eta}_i,\tilde{\eta}_i$, it is not difficult for us to obtain

$$
\begin{aligned}
&\eta_ia_i+\sum_{j=1}^{N}\gamma_{ij}\eta_j=-1,i\in I_{i\in\mathsf{S}},\\
&\bar{\eta}_i\left((p_1+1)\bar{a}_i+\hat{a}_i\right)+\sum_{j=1}^{N}\gamma_{ij}\bar{\eta}_j=-1,i\in I_{i\in\mathsf{S}},\\
&\tilde{\eta}_i((p_1+q-1)\tilde{a}_i+\grave{a}_i)+\sum_{j=1}^{N_1}\gamma_{ij}\tilde{\eta}_j=-Q,i\in I_{i\in\mathsf{S}_1}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\mathcal{L}V_2(Z,\check{Z},i,t)\\
\leq&-|Z|^2+\epsilon_1|\check{Z}|^2+[-\epsilon_2|Z|^q+\epsilon_3|\check{Z}|^q]I_{i\in\mathsf{S}_2}-(1-(p_1-1)\epsilon_4)|Z|^{p_1+1}+(2\epsilon_4+\epsilon_7)|\check{Z}|^{p_1+1}\\
&+\Big[-\Big((p_1+1)\epsilon_5-\frac{(p_1-1)(p_1+1)}{q+p_1-1}\epsilon_6\Big)|Z|^{p_1+q-1}+\frac{\epsilon_6(p_1+1)q}{q+p_1-1}|\check{Z}|^{p_1+q-1}\Big]I_{i\in\mathsf{S}_2}\\
&+[-(Q-(p_1+q-3)\epsilon_8)|Z|^{p_1+q-1}+(2\epsilon_8+\epsilon_9)|\check{Z}|^{p_1+q-1}]I_{i\in\mathsf{S}_1}\\
\leq&-|Z|^2+\epsilon_1|\check{Z}|^2+[-\epsilon_2|Z|^q+\epsilon_3|\check{Z}|^q]I_{i\in\mathsf{S}_2}-(1-(p_1-1)\epsilon_4)|Z|^{p_1+1}+(2\epsilon_4+\epsilon_7)|\check{Z}|^{p_1+1}\\
&-(Q-(p_1+q-3)\epsilon_8)|Z|^{p_1+q-1}+(2\epsilon_8+\epsilon_9)|\check{Z}|^{p_1+q-1}.
\end{aligned}
\tag{35}
$$

Remark 7. *According to (33), the ratio of coefficients for $|Z|$-terms to the coefficient of $|\check{Z}|$-terms of the same order is controlled by $-\widetilde{\tau}$, in light of the previous design, which is conductive to the application of Lemma 1. Here, we can clearly observe the preliminary impact of noise on the stability conditions:n our stochastic system model, the diffusion term captures the noise effect, and the correlation between different noise sources introduces additional coupling terms in the stability analysis. These coupling terms are reflected in the off-diagonal elements of the M-matrix conditions, which generally makes the stability criteria more conservative compared to the uncorrelated noise case.*

Assumption 8. *Nine positive constants $\psi_i(i = 1, 2, \cdots, 9)$ and an auxiliary function $\mathbb{W} \in C(\mathbb{R}^n, \mathbb{R}^+)$ can be seized to make*

$$\begin{aligned}&\mathcal{L}V_2(Z, \check{Z}, i, t) + \psi_1\Big(2\eta_i|Z| + (p_1 + 1)\bar{\eta}_i|Z|^{p_1}\Big)^2 \\ &\quad + \psi_2| - \mathcal{D}(i)Z + \mathcal{A}(i)f_1(Z, \check{Z}, i, t)|^2 + \psi_3|f_2(Z, \check{Z}, i, t)|^2 + \psi_4\int_{0<|\delta|<l}|r_t(\delta)|^2\vartheta(d\delta) \\ &\le - \psi_5|Z|^2 + \psi_6|\check{Z}|^2 - \mathbb{W}(Z) + \psi_7\mathbb{W}(\check{Z})\end{aligned} \tag{36}$$

and

$$\psi_8|Z|^{p_1+q-1} \le \mathbb{W}(Z) \le \psi_9(1 + |Z|^{p_1+q-1}), \tag{37}$$

where $\psi_5 > \psi_6\widetilde{\tau}, \psi_7 \in (0, 1/\widetilde{\tau})$.

Remark 8. *By (10), the left-hand-side terms of (36)$\le$ $\mathcal{L}V_2(Z, \check{Z}, i, t) + 8\psi_1\eta_i^2|Z|^2 + 2\psi_1(p_1 + 1)^2\bar{\eta}_i|Z|_1^p + \psi_4(|Z|^2 + |\check{Z}|^2) + 4\psi_2L^2(|Z|^2 + |\check{Z}|^2 + |Z|^{2p_1} + |\check{Z}|^{2p_2}) + 4\psi_3L^2(|Z|^2 + |\check{Z}|^2 + |Z|^{2p_3} + |\check{Z}|^{2p_4}$. According to Assumption 4, $q + p_1 - 1 \ge 2(p_1 \vee p_2 \vee p_3 \vee p_4)$, it can be said that*

$$\begin{aligned}|Z|^{2p_i} &\le |Z|^2 + |Z|^{q+p_1-1}, (i = 1, 3),\\ |\check{Z}|^{2p_i} &\le |\check{Z}|^2 + |\check{Z}|^{q+p_1-1}, (i = 2, 4).\end{aligned}$$

Furthermore, the left-hand-side terms of (36)$\le -\psi_5|Z|^2 - \bar{\psi}_1|Z|^q - \bar{\psi}_3|Z|^{p_1+1} - \bar{\psi}_5|Z|^{p_1+q-1} + \psi_6|\check{Z}|^2 + \bar{\psi}_2|\check{Z}|^q + \bar{\psi}_4|\check{Z}|^{p_1+1} + \bar{\psi}_6|\check{Z}|^{p_1+q-1}$, where $\psi_5, \psi_6, \bar{\psi}_i(i = 1, 2, \cdots, 6)$ are positive numbers with $\psi_5 > \psi_6\widetilde{\tau}$ and $\bar{\psi}_{i-1} > \bar{\psi}_i\widetilde{\tau}, (i = 2, 4, 6)$. Let $\mathbb{W}(Z) = \bar{\psi}_1|Z|^q + \bar{\psi}_3|Z|^{p_1+1} + \bar{\psi}_5|Z|^{p_1+q-1}$ and $\psi_7 = \max_{i=2,4,6}\frac{\bar{\psi}_i}{\bar{\psi}_{i-1}}$, $\psi_8 = \bar{\psi}_5$, $\psi_9 = \bar{\psi}_1 + \bar{\psi}_3 + \bar{\psi}_5$. Then, it is easy to see that $\psi_7 \in (0, 1/\widetilde{\tau})$ and the left-hand-side terms of (36)$\le -\psi_5|Z|^2 + \psi_6|\check{Z}|^2 - \mathbb{W}(Z) + \psi_7\mathbb{W}(\check{Z})$ and $\psi_8|Z|^{p_1+q-1} \le \mathbb{W}(Z) \le \psi_9(1 + |Z|^{p_1+q-1})$, that is, the assumption is reasonable. Moreover, it is consistent on $i \in \mathsf{S}_1$ and $i \in \mathsf{S}_2$ so that the subsequent proof can be carried out on $i \in \mathsf{S}$.

The proofs presented earlier in this paper mainly employ the state-space method. In order to solve the subsequent stabilization problem, we introduce a new Lyapunov functional to describe the effect of càdlàg system based on observation time and dispose of $\Delta h(t)$ in the same way as handling time-delay in other articles: $V^*(\hat{Z}_t, \hat{\pi}_t, t) = V_2(Z, i) + \theta\int_0^h\int_{t-\Delta h}^t G(s')ds'd\Delta h$ for $t \in \mathbb{R}^+$ with

$$\begin{aligned}&G(s')\\ =&h\Big| - \mathcal{D}(\pi(s'))Z(s'^-) + \mathcal{A}(\pi(s'))f_1\Big(Z(s'^-), Z((s' - \tau_{s'})^-), \pi(s'), s'\Big) + u\Big(Z((s' - \Delta h)^-), \pi(s' - \Delta h), s'\Big)\Big|^2 \\ &+ \Big|f_2\Big(Z(s'^-), Z((s' - \tau_{s'})^-), \pi(s'), s'\Big)\Big|^2 + \int_{0<|\delta|<l}\Big|r\Big(Z(s'^-), Z((s' - \tau_{s'})^-),\pi(s'), s', \delta\Big)\Big|^2\vartheta(d\delta),\end{aligned} \tag{38}$$

where $\hat{Z}_t = \{Z(t - \Delta h) : \Delta h \in [0, 2\tau]\}$, $\hat{\pi}_t = \{\pi(t - \Delta h) : \Delta h \in [0, 2\tau]\}$, $\theta > 0$ is a undetermined gain. Set $x(-\Delta h) = \Psi(-\tau)$ for $\Delta h \in (\tau, 2\tau]$ and $\pi(-\Delta h) = \pi_0$ for $\Delta h \in (0, 2\tau]$, and for $s' \in [-2\tau, 0)$, let $f_1(Z, \check{Z}, i, s') = f_1(Z, \check{Z}, i, 0), f_2(Z, \check{Z}, i, s') = f_2(Z, \check{Z}, i, 0), r_{s'}(\delta) = r_0(\delta), u(Z, i, s') = u(Z, i, 0)$ to make $\hat{Z}_t$ and $\hat{\pi}_t$ well defined on $t \in [0, 2\tau]$ and the system with a proper initial condition. Moreover, we can obtain

$$d\Big[\theta\int_0^h\int_{t-\Delta h}^t G(s')ds'd\Delta h\Big] = \Big[\theta hG(t) - \theta\int_{t-h}^t G(s')ds'\Big]dt \tag{39}$$

and

$$\begin{aligned}&\mathcal{L}V_2(Z,\check{Z},\tilde{Z},i,\tilde{i},t)\\=&\mathcal{L}V_2(Z,\check{Z},i,t)-[2\eta_i+(p_1+1)\bar{\eta}_i|Z|^{p_1-1}]Z^T[u(Z,i,t)-u(\tilde{Z},\tilde{i},t)].\end{aligned}\tag{40}$$

Combining (39) with (40), the drift term can be obtained as

$$\begin{aligned}&\mathcal{L}V^*(\hat{Z}_{t^-},\hat{\pi}_t,t)\\=&\mathcal{L}V_2\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t\Big)-\Big[2\eta_{\pi(t)}+(p_1+1)\bar{\eta}_{\pi(t)}|Z(t^-)|^{p_1-1}\Big]Z(t^-)^T\\&\times\Big[u\Big(Z(t^-),\pi(t),t\Big)-u\Big(Z((t-\Delta h)^-),\pi(t-\Delta h),t\Big)\Big]+\theta hG(t)-\theta\int_{t-h}^{t}G(s')ds'.\end{aligned}\tag{41}$$

Separate mixed terms using the mean inequality to get

$$\begin{aligned}&\mathcal{L}V^*(\hat{Z}_{t^-},\hat{\pi}_t,t)\\\le&\mathcal{L}V_2\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t\Big)+\theta hG(t)-\theta\int_{t-h}^{t}G(s')ds'+\psi_1\Big(2\eta_{\pi(t)}+(p_1+1)\bar{\eta}_{\pi(t)}|Z(t^-)|^{p_1}\Big)^2\\&+\frac{1}{4\psi_1}\Big|u\Big(Z(t^-),\pi(t),t\Big)-u\Big(Z((t-\Delta h)^-),\pi(t-\Delta h),t\Big)\Big|^2.\end{aligned}\tag{42}$$

Denote the right-hand-side terms as $\mathbf{B}\mathcal{L}V^*(\hat{Z}_{t^-},\hat{\pi}_t,t)$, by Theorem 1 and (24), we can assert that

$$\mathbb{E}|\mathbf{B}\mathcal{L}V^*(\hat{Z}_{t^-},\hat{\pi}_t,t)|<\infty,$$

for any $t\in\mathbb{R}^+$.

Next, we will put forward some theorems aimed at different stabilization effects.

Theorem 2. *Let Assumptions 1–8 hold, and let*

$$\theta=\frac{6\rho^2}{\psi_1}\Big(1+8\big(1-\exp(-\frac{\bar{\gamma}}{4\rho})\big)\Big)\tag{43}$$

with the time duration h satisfying

$$h\le\sqrt{\frac{\psi_2}{2\theta}}\wedge\frac{\psi_3}{\theta}\wedge\frac{\psi_4}{\theta}\wedge\frac{1}{4\rho}\tag{44}$$

and

$$\psi_5-\psi_6\widetilde{\tau}-4\theta h^2\rho^2-\frac{4\rho^2}{\psi_1}\big(1-\exp(-\bar{\gamma}h)\big)>0,\tag{45}$$

then the solution of (23) with any initial condition (5) satisfies

$$\int_0^\infty\mathbb{E}|Z(t)|^{\bar{p}}dt<\infty,\bar{p}\in[2,p_1+q-1],$$

known as the H_∞-stability in $L^{\bar{p}}$, for $\bar{p}\in[2,p_1+q-1]$.

Proof. Setting $\varphi_\infty=\infty$, by generalized Itô's formula, one has

$$\mathbb{E}V^*(\hat{Z}_{t\wedge\mu_{k'}},\hat{\pi}_{t\wedge\mu_{k'}},t\wedge\mu_{k'})\le V^*(\hat{Z}_0,\hat{\pi}_0,0)+\mathbb{E}\int_0^{t\wedge\mu_{k'}}\mathbf{B}\mathcal{L}V^*(\hat{Z}_{s^-},\hat{\pi}_s,s)ds.\tag{46}$$

Letting $k'\to\infty$, by the dominated convergence theorem and Fubini theorem, we can obtain

$$0\le\mathbb{E}V^*(\hat{Z}_t,\hat{\pi}_t,t)\le V^*(\hat{Z}_0,\hat{\pi}_0,0)+\int_0^t\mathbb{E}(\mathbf{B}\mathcal{L}V^*(\hat{Z}_{s^-},\hat{\pi}_s,s))ds.\tag{47}$$

According to (24), (36), (42), (44), after rearrangement, we get

$$\begin{aligned}
&\mathbb{E}(\mathbf{B}\mathcal{L}V^*(\hat{Z}_{t-},\hat{\pi}_t,t))\\
\le&-\psi_5\mathbb{E}|Z(t^-)|^2+\psi_6\mathbb{E}|Z((t-\tau_t)^-)|^2-\mathbb{E}\mathbb{W}(Z(t^-))+\psi_7\mathbb{E}\mathbb{W}(Z((t-\tau_t)^-))\\
&+2\theta h^2\rho^2\mathbb{E}|Z((t-\Delta h)^-)|^2+\frac{1}{4\psi_1}\mathbb{E}|u(Z(t^-),\pi(t),t)-u\Big(Z((t-\Delta h)^-),\pi(t-\Delta h),t\Big)|^2\\
&-\theta\mathbb{E}\int_{t-h}^{t}G(s')ds'\\
\le&-\psi_5\mathbb{E}|Z(t^-)|^2+\psi_6\mathbb{E}|Z((t-\tau_t)^-)|^2-\mathbb{E}\mathbb{W}(Z(t^-))+\psi_7\mathbb{E}\mathbb{W}(Z((t-\tau_t)^-))\\
&+4\theta h^2\rho^2\mathbb{E}|Z((t-\Delta h)^-)-Z(t^-)|^2+4\theta h^2\rho^2\mathbb{E}|Z(t^-)|^2+\frac{\rho^2}{2\psi_1}\mathbb{E}|Z((t-\Delta h)^-)-Z(t^-)|^2\\
&+\frac{1}{2\psi_1}\mathbb{E}|u\Big(Z((t-\Delta h)^-),\pi(t),t\Big)-u\Big(Z((t-\Delta h)^-),\pi(t-\Delta h),t\Big)|^2-\theta\mathbb{E}\int_{t-h}^{t}G(s')ds'.
\end{aligned}\tag{48}$$

According to Lemma 3 and (24), we further conclude that

$$\begin{aligned}
&\mathbb{E}|u(Z(t-\Delta h)^-,\pi(t),t)-u(Z((t-\Delta h)^-),\pi(t-\Delta h),t)|^2\\
=&\mathbb{E}\Big[\mathbb{E}|u(Z((t-\Delta h)^-),\pi(t),t)-u(Z((t-\Delta h)^-),\pi(t-\Delta h),t)|^2\mid\mathcal{F}_{t-\Delta h}\Big]\\
\le&\mathbb{E}[4\rho^2|Z((t-\Delta h)^-)^2\mathbb{E}(I_{\pi(t-\Delta h)\neq\pi(t)}\mid\mathcal{F}_{t-\Delta h})]\\
=&\mathbb{E}[4\rho^2|Z((t-\Delta h)^-)^2\mathbb{P}\Big(\pi(t)\neq i|\pi(t-\Delta h)=i\Big)]\\
\le&\mathbb{E}[4\rho^2|Z((t-\Delta h)^-)^2(1-\exp(-\bar{\gamma}h))]\\
\le&8\rho^2\big(1-\exp(-\bar{\gamma}h)\big)\mathbb{E}|Z(t^-)|^2+8\rho^2\big(1-\exp(-\bar{\gamma}h)\big)\mathbb{E}|Z(t^-)-Z((t-\Delta h)^-)|^2.
\end{aligned}\tag{49}$$

Moreover, by Cauchy-Schwarz inequality, the integral mean value theorem and $\Delta h\le h$, one has

$$\mathbb{E}|Z(t^-)-Z((t-\Delta h)^-)|^2\le 3\mathbb{E}\int_{t-h}^{t}G(s')ds'.\tag{50}$$

Substituting (49), (50) into (48), and by (47), we derive that

$$\begin{aligned}
&\mathbb{E}V^*(\hat{Z}_t,\hat{\pi}_t,t)\\
\le&V^*(\hat{Z}_0,\hat{\pi}_0,0)-\Big[\psi_5-4\theta h^2\rho^2-\frac{4\rho^2}{\psi_1}(1-\exp(-\bar{\gamma}h))\Big]\mathbb{E}\int_0^t|Z(s^-)|^2ds\\
&-\mathbb{E}\int_0^t\mathbb{W}(Z(s^-))ds+\psi_6\mathbb{E}\int_0^t|Z((s-\tau_s)^-)|^2ds+\psi_7\mathbb{E}\int_0^t\mathbb{W}(Z((s-\tau_s)^-))ds\\
&+\Big(\frac{3\rho^2}{2\psi_1}+12\theta h^2\rho^2+\frac{12\rho^2}{\psi_1}\big(1-\exp(-\bar{\gamma}h)\big)-\theta\Big)\mathbb{E}\int_0^t\int_{s-h}^{s}G(s')ds'ds.
\end{aligned}\tag{51}$$

According to (43), (44),

$$(1-12h^2\rho^2)\theta\ge\frac{1}{4}\theta\ge\frac{3\rho^2}{2\psi_1}\Big(1+8\big(1-\exp(-\bar{\gamma}h)\big)\Big),$$

then the final term in (51) ≤ 0 and can be eliminated through scaling.

By Lemma 1,

$$\begin{aligned}
\mathbb{E}\int_0^t|Z((s-\tau_s)^-)|^2ds\tilde{\tau}\Big(\int_{-\tau}^{0}\mathbb{E}|Z(s^-)|^2ds+\int_0^t\mathbb{E}|Z(s^-)|^2ds\Big),\\
\mathbb{E}\int_0^t\mathbb{W}(Z((s-\tau_s)^-))ds\le\tilde{\tau}\Big(\int_{-\tau}^{0}\mathbb{E}\mathbb{W}(Z(s^-))ds+\int_0^t\mathbb{E}\mathbb{W}(Z(s^-))ds\Big).
\end{aligned}$$

Furthermore, we can obtain

$$\begin{aligned}
0\le&\mathbb{E}V^*(\hat{Z}_t,\hat{\pi}_t,t)\\
\le&U_1^*-\Big[\psi_5-\psi_6\tilde{\tau}-4\theta h^2\rho^2-\frac{4\rho^2}{\psi_1}\big(1-\exp(-\bar{\gamma}h)\big)\Big]\\
&\times\int_0^t\mathbb{E}|Z(s^-)|^2ds-(1-\psi_7\tilde{\tau})\int_0^t\mathbb{E}\mathbb{W}(Z(s^-))ds<\infty,
\end{aligned}$$

for any $t\in\mathbb{R}^+$.

Here, $U_1^* = V^*(\hat{Z}_0, \hat{\pi}_0, 0) + \widetilde{\tau}\tau \sup_{-\tau \le s < 0} \Big[\psi_6 \mathbb{E}|Z(s)|^2 + \psi_7 \mathbb{E}\mathbb{W}(Z(s))\Big]$.

By integrating (37), (45), along with $\psi_7 \in (0, 1/\widetilde{\tau})$ and taking the limit $t \to \infty$, one has

$$\int_0^\infty \mathbb{E}|Z(t)|^2 dt \le \frac{U^*}{\psi_5 - \psi_6\widetilde{\tau} - 4\theta h^2\rho^2 - \frac{4\rho^2}{\psi_1}(1 - \exp(-\bar{\gamma}h))} < \infty,$$
$$\int_0^\infty \mathbb{E}|Z(t)|^{p_1+q-1} dt \le \frac{U^*}{1 - \psi_7\widetilde{\tau}} < \infty.$$

Eventually, considering that power functions with powers greater than 1 have convexity, the assertion $\int_0^\infty \mathbb{E}|Z(t)|^{\bar{p}} dt < \infty, \bar{p} \in [2, p_1 + q - 1]$ holds. □

Remark 9. *In fact, it is not difficult to see in the process of proof that* (43), (44) *can take a more general form as* $\theta = \frac{3\rho^2\hat{k}^2}{2\psi_1(\hat{k}^2-12)}\Big(1 + 8\big(1 - \exp(-\frac{\bar{\gamma}}{\hat{k}\rho})\big)\Big)$ *and* $h \le \sqrt{\frac{\psi_2}{2\theta} \wedge \frac{\psi_3}{\theta} \wedge \frac{\psi_4}{\theta} \wedge \frac{1}{\hat{k}\rho}}$. *It is noted that* $\hat{k} > 2\sqrt{3}$ *here and in [19],* $\hat{k} = 6$. *Here, we only use this specific case to facilitate the presentation of subsequent proofs.*

Theorem 3. *Let Assumptions 1–8 hold, then the solution of* (23) *with any initial condition* (5) *satisfies*

$$\lim_{t \to \infty} \mathbb{E}|Z(t)|^{\bar{p}} = 0, \bar{p} \in [2, p), \tag{52}$$

known as the asymptotic stability in $L^{\bar{p}}$, $\bar{p} \in [2, p)$.

Proof. By (25), we set $E_1 = \sup_{t \in (-\tau, \infty)} \mathbb{E}|Z(t)|^p$.

By (16), there exist $\bar{\alpha}_1$ and $\bar{\alpha}_2$ such that

$$\begin{aligned}
&\int_{0<\delta<l} \Big(|Z(t^-) + r(Z(t^-), Z((t-\tau_t)^-), \pi(t), t)|^2 \\
&\qquad - |Z(t^-)|^2 - 2Z(t^-)^T r\left(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\right)\Big)\vartheta(d\delta) \\
&\le \bar{\alpha}_1 |Z(t^-)|^2 + \bar{\alpha}_2 |Z((t-\tau_t)^-)|^2.
\end{aligned}$$

Selecting any $t_1 < t_2$ on $[0, \infty)$, by generalized Itô's formula, combining (10) with (24), one has

$$\begin{aligned}
&|\mathbb{E}|Z(t_2)|^2 - \mathbb{E}|Z(t_1)|^2| \\
\le& \Big|\mathbb{E}\int_{t_1}^{t_2} \Big[2|Z(t^-)|\Big(- \mathcal{D}(\pi(t))Z(t^-) + \mathcal{A}(\pi(t)) f_1(Z(t^-), Z((t-\tau_t)^-), \pi(t), t) \\
&+ u(Z((t-\Delta h)^-), \pi(t-\Delta h), t)\Big) + |f_2(Z(t^-), Z((t-\tau_t)^-), \pi(t), t)|^2 \\
&+ \int_{0<|\delta|<l} \Big(|Z(t^-) + r(Z(t^-), Z((t-\tau_t)^-), \pi(t), t)|^2 \\
&- |Z(t^-)|^2 - 2Z(t^-)^T r\left(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\right)\Big)\vartheta(d\delta)\Big] dt\Big| \\
\le& \mathbb{E}\int_{t_1}^{t_2} \Big[2|Z(t^-)|\Big(\max_{1\le i\le n} d_i |Z(t^-)| + L \max_{1\le i,j\le n} a_{ij}\Big(|Z(t^-)| \\
&+ |Z((t-\tau_t)^-)| + |Z(t^-)|^{p_1} + |Z((t-\tau_t)^-)|^{p_2}\Big) \\
&+ \rho|Z((t-\Delta h)^-)|\Big) + L^2\Big(|Z(t^-)| + |Z((t-\tau_t)^-)| \\
&+ |Z(t^-)|^{p_3} + |Z((t-\tau_t)^-)|^{p_4}\Big)^2 + \bar{\beta}_1|Z(t^-)|^2 + \bar{\beta}_2|Z((t-\tau_t)^-)|^2\Big] dt \\
\le& \int_{t_1}^{t_2} \Xi_1\Big(1 + \mathbb{E}|Z(t^-)|^p + \mathbb{E}|Z((t-\tau_t)^-)|^p + \mathbb{E}|Z(t-\Delta h)|^p\Big) dt \\
\le& \Xi_1(1 + 3E_1)(t_2 - t_1),
\end{aligned}$$

where Ξ_1 is an upper bound of four coefficients, which suggests uniform continuity of $\mathbb{E}|Z(t)|^2$ for all $t \in [0, \infty)$.

Combining with $\int_0^\infty \mathbb{E}|Z(t)|^2 dt < \infty$, we can conclude that

$$\lim_{t\to\infty} \mathbb{E}|Z(t)|^2 = 0, \tag{53}$$

which implies that the assertion holds when $\bar{p} = 2$. Fixing a $\bar{p} \in (2, p)$, by Hölder's inequality, we further derive

$$\begin{aligned}\mathbb{E}|Z(t)|^{\bar{p}} \le& (\mathbb{E}|Z(t)|^2)^{\frac{p-\bar{p}}{p-2}} (\mathbb{E}|Z(t)|^p)^{\frac{\bar{p}-2}{p-2}} \\ \le& (\mathbb{E}|Z(t)|^2)^{\frac{p-\bar{p}}{p-2}} (E_1)^{\frac{\bar{p}-2}{p-2}}.\end{aligned} \tag{54}$$

By (53), the assertion (52) holds. □

Theorem 4. *Let Assumptions 1–8 hold, and let*

$$\theta = \frac{6\rho^2}{\psi_1}\Big(1 + 8\big(1 - \exp(-\frac{\bar{\gamma}}{6\rho})\big)\Big) \tag{55}$$

with the time duration h satisfying

$$h \le \sqrt{\frac{\psi_2}{2\theta}} \wedge \frac{\psi_3}{\theta} \wedge \frac{\psi_4}{\theta} \wedge \frac{1}{6\rho} \tag{56}$$

and

$$\psi_5 - 2\psi_6\widetilde{\tau} - 4\theta h^2\rho^2 - \frac{4\rho^2}{\psi_1}\big(1 - \exp(-\bar{\gamma}h)\big) > 0, \tag{57}$$

then the solution of (23) with any initial condition (5) satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\ln(\mathbb{E}|Z(t)|^{\bar{p}}) < 0 \tag{58}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\ln(\mathbb{E}|Z(t)|) < 0 \; a.s. \tag{59}$$

for $\bar{p} \in [2, p)$, known as the exponential stability in $L^{\bar{p}}$, $\bar{p} \in [2, p)$ and the almost sure exponential stability, respectively.

Proof. Setting a positive constant λ to measure the rate of exponential decline, one has

$$\begin{aligned}&\exp(\lambda(t\wedge\mu_{k'}))\mathbb{E}V^*(\hat{Z}_{t\wedge\mu_{k'}}, \hat{\pi}_{t\wedge\mu_{k'}}, t\wedge\mu_{k'}) \\ \le& V^*(\hat{Z}_0, \hat{\pi}_0, 0) + \int_0^{t\wedge\mu_{k'}} \exp(\lambda s)\mathbb{E}\Big(\lambda V^*(\hat{Z}_s, \hat{\pi}_s, s) + \mathbf{B}\mathcal{L}V^*(\hat{Z}_{s^-}, \hat{\pi}_s, s)\Big)ds.\end{aligned}$$

Recalling the form of V^* and setting $\chi_1 = \min_{i\in\mathsf{S}} \eta_i$, $\chi_2 = \max_{i\in\mathsf{S}} \eta_i$, $\chi_3 = \max_{i\in\mathsf{S}} \bar{\eta}_i$, it yields that

$$\begin{aligned}\chi_1 \exp(\lambda t)\mathbb{E}|Z(t)|^2 \le& V^*(\hat{Z}_0, \hat{\pi}_0, 0) + \int_0^t \exp(\lambda s)\Big(\lambda\chi_2\mathbb{E}|Z(t^-)|^2 + \lambda\chi_3\mathbb{E}|Z(t^-)|^{p_1+1}\Big)ds + \lambda\theta G_1(t) \\ &+ \int_0^t \exp(\lambda s)\mathbb{E}(\mathbf{B}\mathcal{L}V^*(\hat{Z}_{s^-}, \hat{\pi}_s, s))ds,\end{aligned} \tag{60}$$

where

$$G_1(t) = \mathbb{E}\int_0^t \exp(\lambda s)\Big(\int_0^h \int_{s-\Delta h}^s G(s')ds'd\Delta h\Big)ds.$$

Combining the proof of Theorem 2 and using the dominated convergence theorem and Fubini theorem, we can obtain

$$\begin{aligned}&\int_0^t \exp(\lambda s)\mathbb{E}(\mathbf{B}\mathcal{L}V^*(\hat{Z}_{s^-},\hat{\pi}_s,s))ds\\
\le& -\Big[\psi_5-4\theta h^2\rho^2-\frac{4\rho^2}{\psi_1}\big(1-\exp(-\bar{\gamma}h)\big)\Big]\int_0^t \exp(\lambda s)\mathbb{E}|Z(s^-)|^2ds-\int_0^t \exp(\lambda s)\mathbb{E}\mathbb{W}(Z(s^-))ds\\
&+\psi_6\int_0^t \exp(\lambda s)\mathbb{E}|Z((s-\tau_s)^-)|^2ds+\psi_7\int_0^t \exp(\lambda s)\mathbb{E}\mathbb{W}(Z((s-\tau_s)^-))ds\\
&+\Big(\frac{3\rho^2}{2\psi_1}+12\theta h^2\rho^2+\frac{12\rho^2}{\psi_1}\big(1-\exp(-\bar{\gamma}h)\big)-\theta\Big)\mathbb{E}\int_0^t \exp(\lambda s)\int_{s-h}^s G(s')ds'ds. \qquad (61)\end{aligned}$$

According to Lemma 1 ,

$$\begin{aligned}&\int_0^t \exp(\lambda s)\mathbb{E}|Z((s-\tau_s)^-)|^2ds\\
\le&\exp(\lambda\tau)\int_0^t \exp(\lambda(s-\tau_s)^-)\mathbb{E}|Z((s-\tau_s)^-)|^2ds\\
\le&\exp(\lambda\tau)\widetilde{\tau}\Big(\int_{-\tau}^0 \exp(\lambda s)\mathbb{E}|Z(s^-)|^2ds+\int_0^t \exp(\lambda s)\mathbb{E}|Z(s^-)|^2ds\Big),\end{aligned}$$

$$\begin{aligned}&\int_0^t \exp(\lambda s)\mathbb{E}\mathbb{W}(Z((s-\tau_s)^-))ds\\
\le&\exp(\lambda\tau)\int_0^t \exp(\lambda(s-\tau_s)^-)\mathbb{E}\mathbb{W}(Z((s-\tau_s)^-))ds\\
\le&\exp(\lambda\tau)\widetilde{\tau}\Big(\int_{-\tau}^0 \exp(\lambda s)\mathbb{E}\mathbb{W}(Z(s^-))ds+\int_0^t \exp(\lambda s)\mathbb{E}\mathbb{W}(Z(s^-))ds\Big). \qquad (62)\end{aligned}$$

In addition, for $p_1>1, q>2$, we can obtain $p_1+1>2, p_1+1<p_1+q-1$ and from the properties of power functions, we know that exactly one of $\mathbb{E}|Z(t^-)|^{p_1+1}\le\mathbb{E}|Z(t^-)|^2$ and $\mathbb{E}|Z(t^-)|^{p_1+1}\le\mathbb{E}|Z(t^-)|^{p_1+q-1}$ holds. So we can further get

$$\begin{aligned}\mathbb{E}|Z(t^-)|^{p_1+1}\le&\mathbb{E}|Z(t^-)|^2+\mathbb{E}|Z(t^-)|^{p_1+q-1}\\
\le&\mathbb{E}|Z(t^-)|^2+{\psi_8}^{-1}\mathbb{E}\mathbb{W}(Z(s^-)). \qquad (63)\end{aligned}$$

Substitute (61), (62), (63) into (60) to obtain

$$\begin{aligned}&\chi_1\exp(\lambda t)\mathbb{E}|Z(t)|^2\\
\le&U_2^*-\Big[\psi_5-4\theta h^2\rho^2-\frac{4\rho^2}{\psi_1}\big(1-\exp(-\bar{\gamma}h)\big)-\lambda\chi_2-\lambda\chi_3-\psi_6\widetilde{\tau}\exp(\lambda\tau)\Big]\int_0^t \exp(\lambda s)\mathbb{E}|Z(s^-)|^2ds\\
&-\Big(1-\lambda\chi_3{\psi_8}^{-1}-\psi_7\widetilde{\tau}\exp(\lambda\tau)\Big)\int_0^t \exp(\lambda s)\mathbb{E}\mathbb{W}(Z(s^-))ds+\lambda\theta G_1(t)\\
&-\Big(\theta-\frac{3\rho^2}{2\psi_1}-12\theta h^2\rho^2-\frac{12\rho^2}{\psi_1}\big(1-\exp(-\bar{\gamma}h)\big)\Big)G_2(t), \qquad (64)\end{aligned}$$

where

$$U_2^*=V^*(\hat{Z}_0,\hat{\pi}_0,0)+\widetilde{\tau}\exp(\lambda\tau)\Big(\psi_6\int_{-\tau}^0 \exp(\lambda s)\mathbb{E}|Z(s^-)|^2ds+\psi_7\int_{-\tau}^0 \exp(\lambda s)\mathbb{E}\mathbb{W}(Z(s^-))ds\Big)$$

and $G_2(t)=\mathbb{E}\int_0^t \exp(\lambda s)\int_{s-h}^s G(s')ds'ds$. It is obvious that $G_1(t)\le hG_2(t)$.

Now, we need to select a proper λ to satisfy

$$1-\lambda\chi_3{\psi_8}^{-1}-\psi_7\widetilde{\tau}\exp(\lambda\tau)\ge 0, \qquad (65)$$

$$\psi_5 - 4\theta h^2\rho^2 - \frac{4\rho^2}{\psi_1}\big(1-\exp(-\bar{\gamma}h)\big) - \lambda\chi_2 - \lambda\chi_3 - \psi_6\tilde{\tau}\exp(\lambda\tau) \geq 0, \tag{66}$$

$$\theta - \frac{3\rho^2}{2\psi_1} - 12\theta h^2\rho^2 - \frac{12\rho^2}{\psi_1}\big(1-\exp(-\bar{\gamma}h)\big) \geq \lambda\theta h. \tag{67}$$

Considering $h < \frac{1}{6\rho}$ instead of $h < \frac{1}{4\rho}$, from(67), we need $\lambda h \leq \frac{5}{12}$. When λ is sufficiently small, the condition is ensured. Furthermore, comparing (57) with target (66), $\lambda > 0$ can be seized to make (65), (66) hold. Therefore, we can ensure

$$\chi_1 \exp(\lambda t)\mathbb{E}|Z(t)|^2 \leq U_2^*,$$

that is,

$$\mathbb{E}|Z(t)|^2 \leq \frac{U_2^*}{\chi_1}\exp(-\lambda t), \tag{68}$$

for any $t \in \mathbb{R}^+$. According to the result (54), one has

$$\mathbb{E}|Z(t)|^{\bar{p}} \leq (\mathbb{E}|Z(t)|^2)^{\frac{p-\bar{p}}{p-2}}(E_1)^{\frac{\bar{p}-2}{p-2}} \leq (E_1)^{\frac{\bar{p}-2}{p-2}}(\frac{U_2^*}{\chi_1})^{\frac{p-\bar{p}}{p-2}}\exp(-\frac{p-\bar{p}}{p-2}\lambda t) \tag{69}$$

for any $\bar{p} \in (2,p)$. Combining (68) with (69), the assertion (57) holds for any $\bar{p} \in [2,p)$.

Recalling that $t_m = mh$, by Cauchy-Schwarz inequality and Doob′s martingale theorem,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t_m\leq t\leq t_{m+1}}|Z(t)|^2\Big)\\
\leq& 4\mathbb{E}|Z(t_m)|^2 + 4h\mathbb{E}\int_{t_m}^{t_{m+1}}|-\mathcal{D}(\pi(t))Z(t^-) + \mathcal{A}(\pi(t))f_1\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\Big)\\
&+ u\Big(Z(t_m^-), \pi(t_m), t\Big)|^2dt + 16\mathbb{E}\int_{t_m}^{t_{m+1}}|f_2\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\Big)|^2dt\\
&+ 16\mathbb{E}\int_{t_m}^{t_{m+1}}\int_{0<|\delta|<l}|r\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t, \delta\Big)|^2\vartheta(d\delta)dt.
\end{aligned}$$

Based on Assumptions 2–5, using the dominated convergence theorem and Fubini theorem, it is concluded that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t_m\leq t\leq t_{m+1}}|Z(t)|^2\Big)\\
\leq& 4\mathbb{E}|Z(t_m)|^2 + \Xi_2\int_{t_m}^{t_{m+1}}\mathbb{E}\Big(|Z(t^-)|^2 + |Z((t-\tau_t)^-)|^2 + |Z(t_m^-)|^2 + |Z(t^-)|^{\bar{p}^*} + |Z((t-\tau_t)^-)|^{\bar{p}^*}\Big)dt,
\end{aligned}$$

in which $\bar{p}^* = 2(p_1 \vee p_2 \vee p_3 \vee p_4) \in (2,p)$, and Ξ_2 is an upper bound of five coefficients.

Therefore, we can apply (68), (69) to get

$$\mathbb{E}\Big(\sup_{t_m\leq t\leq t_{m+1}}|Z(t)|^2\Big) \leq \Xi_3\exp(-\breve{\varepsilon}t_m).$$

where Ξ_3 is a constant and $\breve{\varepsilon} = \frac{p-\bar{p}^*}{p-2}\lambda$.

Pick a proper sequence of events $\mathbf{A}_m = \{\pi(t) \mid \sup_{t_m\leq t\leq t_{m+1}}|Z(t)| > \exp(-0.25\breve{\varepsilon}t_m)\}$, where $\pi(t) \in \Omega$ represents almost all sample paths.

Then, according to Chebyshev′s inequality, one has

$$\mathbb{P}(\mathbf{A}_m) \leq \frac{\mathbb{E}\Big(\sup_{t_m\leq t\leq t_{m+1}}|Z(t)|^2\Big)}{[\exp(-0.25\breve{\varepsilon}t_m)]^2} \leq \Xi_3\exp(-0.5\breve{\varepsilon}t_m).$$

Therefore,

$$\sum_{m=0}^{\infty} \mathbb{P}(\mathbf{A}_m) \leq \Xi_3 \sum_{m=0}^{\infty} \exp(-0.5\check{\varepsilon} t_m) < \infty,$$

which means that the series $\sum_{m=0}^{\infty} \mathbb{P}(\mathbf{A}_m)$ converges.

By Borel-Cantelli theorem, we can know

$$\mathbb{P}(\lim \sup_{m\to\infty} \mathbf{A}_m) = 0,$$

that is, there almost surely exists $N(\pi(t))$ such that when $m \geq N(\pi(t))$,

$$\sup_{t_m \leq t \leq t_{m+1}} |Z(t)| \leq \exp(-0.25\check{\varepsilon} t_m).$$

Thus, for $m \geq N(\pi(t))$, we have

$$\frac{1}{t} \ln(|Z(t)|) \leq -\frac{0.25 t_m \check{\varepsilon}}{t_{m+1}} = -\frac{0.25 m\check{\varepsilon}}{m+1},$$

for $t \in [t_m, t_{m+1}]$, which implies

$$\lim \sup_{t\to\infty} \frac{1}{t} \ln(|Z(t)|) \leq -0.25\check{\varepsilon} < 0 \; a.s.$$

In this regard, it can be asserted that (59) holds. □

Remark 10. *θ is merely an auxiliary parameter used to determine the range of h, so it can be adjusted and changed in above theorems. The condition in Theorem 3.7 in [17] is radical, which makes the conditions in articles cited subsequently all used in rather radical way. In this paper, we draw on the methodological ideas of [17] and present conservative and reliable conditions (55)–(57). In fact, the coefficient before $\psi_6\widetilde{\tau}$ only needs to be larger than 1, and setting to 2 enables more systems to meet the condition and reduces determination costs to some extent, which partly resolves the issue of indeterminate exponential decay rates as the solution approaches 0. (54) is used in the proof, which means that Theorem 4 further demonstrates the decreasing rate of asymptotic stability. Moreover, it is noted that, in general, almost surly exponential stabilization cannot be obtained from exponential stabilization in the $p-$th moment, which means the relevant proof in Theorem 4 is essential.*

In order to deeply explore the control strategy of the system, combining the result in Remark 9, we give a more general corollary according to the above proof.

Corollary 1. *Let Assumptions 1–8 hold, and let*

$$\theta = \frac{3\rho^2 \hat{k}^2}{2\psi_1(\hat{k}^2 - 12)} \Big(1 + 8\big(1 - \exp(-\frac{\bar{\gamma}}{\check{k}\rho})\big)\Big) \tag{70}$$

with the time duration h satisfying

$$h \leq \sqrt{\frac{\psi_2}{2\theta}} \wedge \frac{\psi_3}{\theta} \wedge \frac{\psi_4}{\theta} \wedge \frac{1}{\check{k}\rho} \tag{71}$$

and

$$\psi_5 - 2\psi_6\widetilde{\tau} - 4\theta h^2\rho^2 - \frac{4\rho^2}{\psi_1}\big(1 - \exp(-\bar{\gamma}h)\big) > 0, \tag{72}$$

where $\check{k} > \hat{k} > 2\sqrt{3}$ and $\check{k}, \hat{k}$ are regulation parameter, the assertion (58) and (59) hold.

Remark 11. *For saving the cost of calculation and verification, $\check{k}, \hat{k}$ are preferable to be chosen as positive integers, for example, in this paper, $\check{k} = 6$, $\hat{k} = 4$. According to the previous information, when the activation function* *is determined, the appropriate $\psi_i, i = 1, 2, \cdots, 6$ will be optimized first. According to the monotonicity, gain θ is increasing concerning the control gain ρ. When h is extremely small, there is no requirement for the control gain, however, which means that the Markov switching speed is extremely fast so as to consume a large amount of resources. To this end, we hope to train the control gain according to the condition in Theorems 2–4 to gain a proper h, which not only enables the original system to be stabilized (which requires h sufficiently small), but also helps us extend the overall duration of the process and cut down costs. Apparently, aimed at different $\psi_i, i = 1, 2, \cdots, 6$, setting a variety of $\check{k}, \hat{k}$ will influence the training effect. Further issues of optimizing the control energy consumption will not be discussed in this paper.*

4. Numerical Simulation

An example will be given throughout this section to demonstrate the validity of theoretical results.

Example 1. *Consider a nonlinear hybrid* STDNN *with Lévy noise:*

$$\begin{aligned} dZ(t) =& \Big[-\mathcal{D}(\pi(t))Z(t^-) + \mathcal{A}(\pi(t))f_1\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\Big)\Big]dt \\ &+ f_2\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t\Big)dB(t) \\ &+ \int_{0<|\delta|<l} r\Big(Z(t^-), Z((t-\tau_t)^-), \pi(t), t, \delta\Big)\widetilde{\mathcal{J}}(dt, d\delta) \end{aligned} \tag{73}$$

on $t \geq 0$ and $Z(t) = \Psi = 1 + \sin(t)$, for $t \in [-0.3, 0]$, which means $\tau = 0.3$. The Markov *chain $\pi(t)$ is over the state space $\mathsf{S} = \{1, 2, 3\}$ with the generator matrix $\Gamma = \{-2, 1, 1; 1, -2, 1; 1, 1, -2\}$, and the time duration $h = 7 \times 10^{-6}$. Set $N_1 = 1$, then $\mathsf{S}_1 = \{1\}$, $\mathsf{S}_2 = \{2, 3\}$. For inhibition coefficients and connection weights, one has $\mathcal{D}(1) = 0.3$, $\mathcal{D}(2) = 0.2$, $\mathcal{D}(3) = 0.4$, $\mathcal{A}(1) = 0.1$, $\mathcal{A}(2) = 0.3$, $\mathcal{A}(3) = 0.6$. The non-differentiable time-varying time delay is configured as $\tau_t = 0.1|\sin t| + 0.2$, which means that $\tau = 0.3, \underline{\tau} = 0.2, \widetilde{\tau} = \frac{1}{1-0.1} = 1.1111$.*

Set

$$f_1(Z, \check{Z}, i, t) = \begin{cases} 0.2\sin Z + 0.7\tanh \check{Z}, i = 1, \\ -1.5Z^3 + 0.8Z\check{Z}, i = 2, \\ -1.1Z^3 + 1.2Z\check{Z}, i = 3, \end{cases}$$

$$f_2(Z, \check{Z}, i, t) = \begin{cases} 0.1\sin Z + 0.2\tanh \check{Z}, i = 1, \\ 0.2Z\check{Z}, i = 2, \\ 0.3Z\check{Z}, i = 3, \end{cases}$$

which means in Assumption 4, $p_1 = 3, p_2 = p_3 = p_4 = 2$, $p > 6, q \geq 4$. Set $p = 7, q = 4$, by (13), one has $\beta_1 = 0.36, \beta_2 = 0.165, \beta_3 = 0.135$, satisfying $\beta_2 > \widetilde{\tau}\beta_3$. Therefore, $q_1 = 0.63, q_2 = 0.42$, satisfying $q_1 > \widetilde{\tau}q_2$. Set jump intensity threshold $l = 5$ and

$$r_t(\delta) = r(Z, \check{Z}, i, t, \delta) = \begin{cases} 0.5\check{Z}\delta - 0.5Z\delta, i = 1, \\ 0.2\check{Z}\delta - 0.5Z\delta, i = 2, \\ 0.3\check{Z}\delta - 0.5Z\delta, i = 3. \end{cases}$$

The Lévy measurable ϑ is characterized by $\vartheta(d\delta) = \boldsymbol{\lambda}\mathbb{F}(d\delta) = 0.5 \times \exp(-2|\delta|)d\delta$, where $\boldsymbol{\lambda} = 0.5$ represents the rate and $\mathbb{F}(\cdot)$ depicts the distribution whose corresponding probability density function is $\exp(-2|\delta|)$, which ensures the fulfillment of (2). Moreover, to simplify it a little bit, we only consider occurring the positive jump here, that is $0 \leq \delta \leq 5$ in subsequent calculations.

Remark 12. *In fact, our strategy is effective for any jump within a bounded range. The Lévy noise in our model description is symmetric, and related similar tests have already been conducted in other articles (e.g., [16, 19]). In [11], it is proved that enhancing the asymmetry of Lévy noise is beneficial to the synchronization under subthreshold drive, which makes us interested in conducting more experimental verifications in the subsequent work.*

Figure 1 indicates that the original system (73) is unstable and there is a huge oscillation relative to the step size.

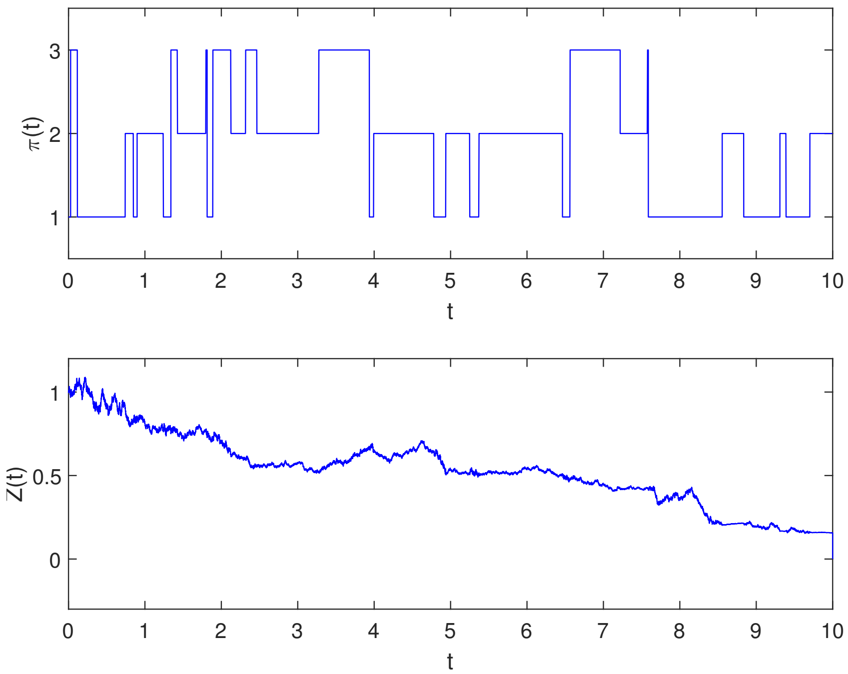

Figure 1. Sample path of Markov chain and state of original mixed growth STDNNs (73) with Lévy noise .

To stabilize it, a saturation controller dominated by a sign function is introduced here [19]:

$$u(Z,1)=0, u(Z,2)=-2(|Z|\wedge 1.5)Z/|Z|,$$
$$u(Z,3)=-2.2(|Z|\wedge 2)Z/|Z|,$$

clearly $\rho = 2.2$. Controlled system is presented that

$$\begin{aligned} dZ(t) =& \Big[-\mathcal{D}(\pi(t))Z(t^-)+\mathcal{A}(\pi(t))f_1\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t\Big) \\ &+u(Z(t_m^-),\pi(t_m^-),t)\Big]dt+f_2\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t\Big)dB(t) \\ &+\int_{0<|\delta|<l} r\Big(Z(t^-),Z((t-\tau_t)^-),\pi(t),t,\delta\Big)\widetilde{\mathcal{J}}(dt,d\delta). \end{aligned} \tag{74}$$

Certainly, controlled system (74) satisfies the corresponding property in Theorem 1.
Since $Z^T u(Z,i) \leq 0.1317Z^4 - (2I_2(i) + 2.2I_3(i))Z^2$, we can get

$$\begin{aligned} &2\Big(Z^T\Big[-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)+u(Z,i,t)\Big] \\ &+\frac{1}{2}|f_2(Z,\check{Z},i,t)|^2\Big)+\int_{0<|\delta|<l}\Big[|Z+r_t(\delta)|^2-|Z|^2-2Z^Tr_t(\delta)\Big]\vartheta(d\delta) \\ \leq& \begin{cases} -0.4076Z^2+0.2124\check{Z}^2, i=1, \\ -4.3563Z^2+0.2575\check{Z}^2-0.3766Z^4+0.02\check{Z}^4, i=2, \\ -5.1202Z^2+0.72\check{Z}^2-0.2916Z^4+0.045\check{Z}^4, i=3, \end{cases} \end{aligned}$$

$$\begin{aligned} &Z^T\Big[-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)+u(Z,i,t)\Big]+\frac{3}{2}|f_2(Z,\check{Z},i,t)|^2 \\ \leq& \begin{cases} -0.215Z^2+0.155\check{Z}^2, i=1, \\ -2.2Z^2+0.12\check{Z}^2-0.1683Z^4+0.03\check{Z}^4, i=2, \\ -2.6Z^2+0.36\check{Z}^2-0.1008Z^4+0.0675\check{Z}^4, i=3, \end{cases} \end{aligned}$$

$$Z^T\Big[-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)+u(Z,i,t)\Big]+\frac{5}{2}|f_2(Z,\check{Z},i,t)|^2 \leq -0.195|Z|^2+0.235|\check{Z}|^2, i=1,$$

$$\int_{0<|\delta|<l}\Big[|Z+r_t(\delta)|^6-|Z|^6-6|Z|^4Z^Tr_t(\delta)\Big]\vartheta(d\delta)\leq 0.4903|Z|^6+0.1294|\check{Z}|^6, i=1$$

$$\int_{0<|\delta|<l}\Big[|Z+r_t(\delta)|^4-|Z|^4-4|Z|^2Z^Tr_t(\delta)\Big]\vartheta(d\delta)\leq\begin{cases}0.2640Z^4+0.1058\check{Z}^4, i=1,\\0.1718Z^4+0.0277\check{Z}^4, i=2,\\0.2031Z^4+0.0448\check{Z}^4, i=3.\end{cases}$$

Hence, (29)–(32) hold when $a_1=-0.4076, b_1=0.2124, \bar{a}_1=-0.215, \bar{b}_1=0.155, a_2=-4.3563,$ $b_2=0.2575, \bar{a}_2=-2.2, \bar{b}_2=0.12, c_2=0.3766, w_2=0.02, \bar{c}_2=0.1683, \bar{w}_2=0.03,$ $a_3=-5.1202, b_3=0.72, \bar{a}_3=-2.6, \bar{b}_3=0.36, c_3=0.2916, w_3=0.045, \bar{c}_3=0.1008, \bar{w}_3=0.0675,$ $\tilde{a}_1=-0.195, \tilde{b}_1=0.235, \grave{a}_1=0.4903, \grave{b}_1=0.1294, \hat{a}_1=0.2640, \hat{b}_1=0.1058, \hat{a}_2=0.1718, \hat{b}_2=0.0277,$ $\hat{a}_3=0.2031, \hat{b}_3=0.0448$, as well as $M_1=\begin{pmatrix}2.4076 & -1 & -1\\ -1 & 6.3563 & -1\\ -1 & -1 & 7.1202\end{pmatrix}$, $M_2=\begin{pmatrix}2.5960 & -1 & -1\\ -1 & 10.6282 & -1\\ -1 & -1 & 12.1969\end{pmatrix}$, $M_3=2.6797$, which are both $M-$matrices.

Remark 13. *It is worth noting that for the nodes storing a linearly growing activation function, the corresponding connection weight should not too large compared with the inhibition coefficient, for example, while setting* $\mathcal{A}(1)=0.5$, *we can find* $a_1=0.0324, \bar{a}_1=0.005$ *are positive, which makes* M_1, M_2 *not* M*-matrices so that our stabilization scheme is not applicable.*

Then, $\eta_1=0.6559, \eta_2=0.3038, \eta_3=0.2752, \bar{\eta}_1=0.4965, \bar{\eta}_2=0.1535, \bar{\eta}_3=0.1353$.So $\epsilon_1=0.1981, \epsilon_4=0.0770, \epsilon_7=0.0525$, for $i\in\mathsf{S}$, $\epsilon_2=0.0802, \epsilon_3=0.0124, \epsilon_5=0.0136, \epsilon_6=0.0091$, for $i\in\mathsf{S}_2$, which satisfies (32). Moreover, by (33), $Q=0.0423$. Then, we can further get $\tilde{\eta}_1=0.0158$, $\epsilon_8=0.0037, \epsilon_9=0.0020$, which satisfies (34). It is obvious that

$$V_2(Z,i)=\begin{cases}0.6559Z^2+0.4965Z^4, i=1,\\0.3038Z^2+0.1535Z^4, i=2,\\0.2752Z^2+0.1353Z^4, i=3.\end{cases}$$

As a result of (35), we obtain

$$\mathcal{L}V_2(Z,\check{Z},i,t)\leq -Z^2+0.1981\check{Z}^2+[-0.0802Z^4+0.0124\check{Z}^4]I_{i\in\mathsf{S}_2}-0.8460Z^4+0.2065\check{Z}^4-0.0275Z^6+0.0094\check{Z}^6.$$

In addition, we have

$$\Big(2\eta_i|Z|+(p_1+1)\bar{\eta}_i|Z|^{p_1}\Big)^2\leq 1.7208Z^2+5.2105Z^4+3.9442Z^6,$$

$$|-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)|^2\leq 0.48Z^2+0.0098\check{Z}^2+0.7776Z^4+0.7776\check{Z}^4+1.3068Z^6,$$

$$|f_2(Z,\check{Z},i,t)|^2\leq 0.02Z^4+0.02\check{Z}^4+0.02Z^2+0.08\check{Z}^2,$$

$$\int_{0<|\delta|<l}|r_t(\delta)|^2\vartheta(d\delta)\leq 0.0624Z^2+0.0624\check{Z}^2.$$

Selecting $\psi_1=0.002, \psi_2=0.001, \psi_3=0.3, \psi_4=0.5$, we obtain

$$\begin{aligned}&\mathcal{L}V_2(Z,\check{Z},i,t)+\psi_1\Big(2\eta_i|Z|+(p_1+1)\bar{\eta}_i|Z|^{p_1}\Big)^2\\&+\psi_2|-\mathcal{D}(i)Z+\mathcal{A}(i)f_1(Z,\check{Z},i,t)|^2+\psi_3|f_2(Z,\check{Z},i,t)|^2+\psi_4\int_{0<|\delta|<l}|r_t(\delta)|^2\vartheta(d\delta)\\\leq&-0.9588Z^2+0.2534\check{Z}^2-0.8828Z^4+0.2257\check{Z}^4-0.0183Z^6+0.0094\check{Z}^6\\\leq&--0.9588Z^2+0.2534\check{Z}^2-\mathbb{W}(Z)+0.1219\mathbb{W}(\check{Z}),\end{aligned}$$

in which $\mathbb{W}(Z)=0.8828Z^4+0.0183Z^6, \psi_5=0.9588, \psi_6=0.2534, \psi_7=0.5137, \psi_8=0.0183, \psi_9=0.8471$. Based on Theorem 2, it can be calculated that $\theta=38,135$ and $h\leq 7.8\times 10^{-6}$. Based on Theorem 4, similarly, we can obtain $\theta = 30,852$ and $h \leq 9.7 \times 10^{-6}$. Recalling that $h = 7 \times 10^{-6}$, in view of Theorems 2–4, it follows that the controlled mixed growth hybrid STDNN with Lévy noise (74) is H_∞-, asymptotically, and exponentially stable in $L^{\bar{p}}$ for any $\bar{p} \in [2, 7)$. And the results also show that even if the system has a good exponential decay trend, its robustness may not necessarily be strong, for example, if $h = 8 \times 10^{-6}$, the H_∞-stability can not be ensured, which means the controlled system still has weak robustness. Although the conditions in [19] are radical, when (4.32) in [19] is changed to the form in this paper, we can get $h \leq 0.00302$, which makes the setting $h = 0.002$ in [19] is still correct. This paper only improve a universal judgment condition.

The image only presents outcomes based on the probabilities of a single pathway, failing to fully demonstrate the superior characteristics that remain stable under the expectation of various scenarios. To ensure the accuracy of our conclusions, we conducted numerous simulation experiments across different paths. Here, we provide only two illustrative results for reference. Under $\pi(0) = 1$, Figures 2 and 3 display controlled scenarios under two distinct paths.

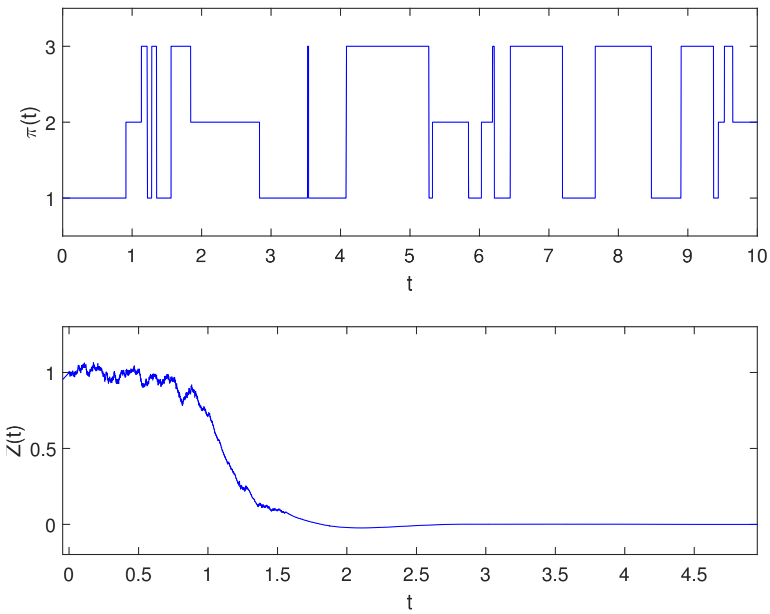

Figure 2. Sample path 1 of Markov chain and state of controlled mixed growth hybrid STDNNs (73) with Lévy noise.

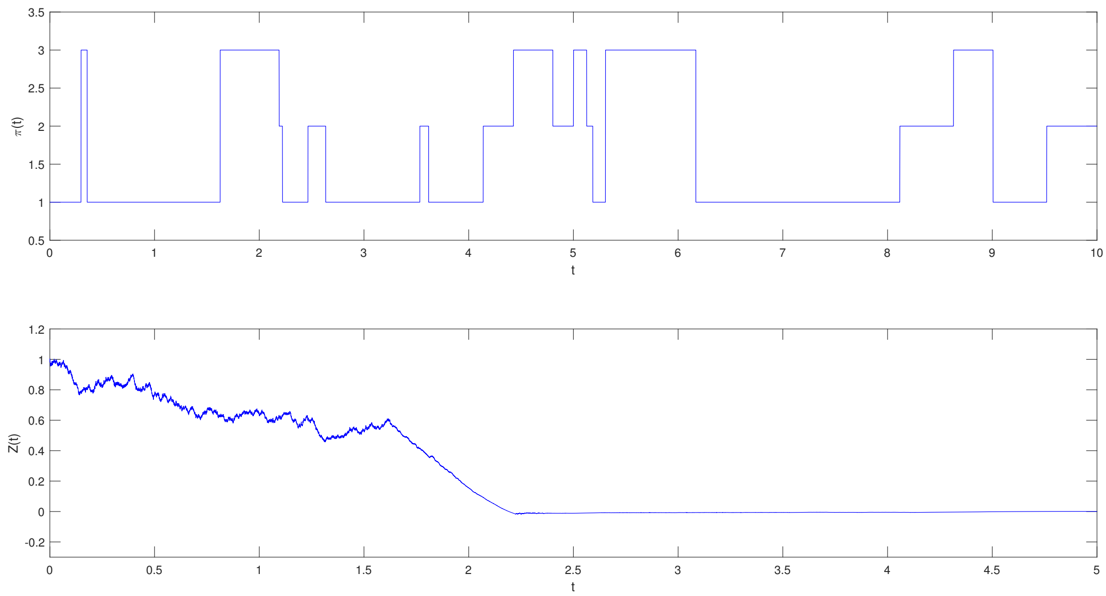

Figure 3. Sample path 2 of Markov chain and state of controlled mixed growth hybrid STDNNs (73) with Lévy noise.

Despite that the sample paths of the system have been changed, the stabilizing effect is still remarkable. To sum up, the simulation results apparently support our theoretical result.

Remark 14. *In this example, Z is a one-dimensional vector, so the 2-norm is equivalent to the absolute value. This is done merely to consider the aesthetic presentation of the stochastic system, just as in other papers. In practical applications, the determination should be made according to the sufficient conditions. Moreover, compared with the Figure 4.2 in [17], there are still oscillatory behaviors in the early stage of the Figures 2 and 3, which is because our definition of exponential stability only focuses on the local trend of the tail, rather than global exponential stability dependent on initial values. It is worth noting that the core computational complexity of this paper lies in verifying the M-matrix conditions derived from the Lyapunov functional approach. For an n-dimensional system, the complexity of verifying these conditions is $O(n^3)$. The subsequent validation steps are consistent across systems of any dimension and involve only extremely low-dimensional computations, which is significantly lower than that of conventional LMI methods commonly used for mean-square exponential stability, which typically exhibit a complexity of $O(n^6)$. For large-scale systems, the M-matrix conditions can be verified in a distributed or block-wise manner, which further reduces the computational burden and improves scalability. Naturally, as the scale of the system expands, sparse matrix information capture techniques need to be considered to enhance the system's performance.*

5. Conclusions

In this paper, we conduct stabilization analysis of a hybrid neural network system. We introduce a type of jump interference Lévy noise which has a significant impact on the neural potential and describe a STDNN on càdlàg family. Unlike differentiable time-delays only appearing in continuous functions by [15], we focus on a general non-differentiable time-delays and demonstrate its properties on càdlàg family. Meanwhile, we use Markov chains to describe the switching mode in order to reflect the hybrid characteristics of the system and improve the system via discrete-time observation methods. Since the activation function is highly nonlinear and features an alternating pattern of linear growth and polynomial growth (called mixed growth), sacrificing a certain degree of accuracy to unify these two major types of models within one framework is of great significance for highly nonlinear analysis field. Mainly by constructing the Lyapunov functional, utilizing the M-matrix, combination with generalized Itô's formula, we give the sufficient conditions to complete the realization of stabilization aiming at H_∞-, asymptotical, and exponential stability, respectively. Moreover, we provide the optimization direction for the sufficient conditions, and analyze some characteristics of the system by simulation.It is important to emphasize that, while the mathematical conditions are strictly derived from Lyapunov theory, the intermittent discrete control design philosophy itself is heuristic: it is motivated by practical application scenarios such as stock market regulation (active intervention during market peaks, inaction during market downturns) and brain wave stabilization in brain-inspired science, where intermittent, event-triggered control is naturally suited.

Some limitations of the current study are as follows: the framework imposes strict requirements that the Markov switching, Lévy noise, and Gaussian white noise are mutually independent; there is insufficient case study on asymmetric Lévy noise; the numerical examples used are relatively simple and do not distinguish the additional effects under different types of Lévy noise; furthermore, the adopted method exhibits a certain degree of conservativeness. Based on the limitations of the current study, future work can be carried out in the following aspects: First, the independence assumptions among the Markov switching, Lévy noise, and Gaussian white noise can be relaxed to investigate the stability of systems with correlated noise and switching processes, establishing a more general theoretical framework. Second, the study of asymmetric Lévy noise will be deepened to derive stability criteria applicable to general Lévy noise, and the effectiveness of the results will be verified through more practical engineering examples. Third, advanced techniques such as delay-partitioning methods and relaxed inequalities will be adopted to improve the construction of Lyapunov functionals, thereby reducing the conservativeness of the stability conditions. Finally, the proposed method will be extended to complex practical scenarios such as power systems and networked control systems to explore the system dynamics under different combinations of noise and delays, further verifying the engineering application value of the theoretical approach. In recent years, the field of hybrid stochastic systems has seen active exploration. For example, [23,24] propose schemes for specific types of induced objects (neutral-type), while [25] combines the setup of discrete-time observers with intermittent control. These works, along with this paper, take almost-sure exponential stability as the core performance metric. However, given exponential stability is an overly idealized criterion, its feasibility and accuracy can be challenging in systems with more pronounced nonlinear characteristics. Consequently, some researchers have begun exploring polynomial stability patterns [26]. By incorporating non-differentiable time delays and mixed growth conditions, this paper demonstrates stronger potential for applications to real-world systems compared to the aforementioned works, though further refinement of the strategy is still needed. Considering $\widetilde{\tau} > 1$, it is of great theoretical significance to design neutral-type neural network system for non-differentiable time delays. In the existing literature, recent contributions have advanced the field along two complementary directions: some focus on improving control strategies from the perspective of control methodologies [27], while others develop novel stability criteria for different types of stability [28,29]. All these works provide important references and valuable insights for our future research.

Author Contributions

X.W.: Writing—original draft; S.L.: Supervision and auxiliary experiment. All authors have read and approved the final version of the manuscript for publication.

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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