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Exponential Stability in the p-th Moment for Caputo Fractional Stochastic Delay Differential Systems

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Abstract: This paper investigates the dynamic behavior of Caputo-type fractional stochastic delay differential equations. We construct an Itô-Caputo differential formula associated with the Mittag-Leffler type matrix polynomial function. Utilizing this formula along with tools such as Hölder's inequality, the Burkholder-Davis-Gundy (BDG) inequality, and Gronwall's inequality, the p-th moment estimation bounds for the system's solution are rigorously derived. Furthermore, by constructing stochastic differential operators and imposing several assumptions, a criterion for the p-th moment exponential stability of the system's solution is established. Finally, numerical examples are provided to verify the effectiveness of the proposed theoretical results.

Keywords: caputo fractional derivative; stochastic differential equation; stability

1. Introduction

Fractional calculus finds extensive applications across numerous scientific and engineering domains, including mechanics, physics, biology, and cybernetics. In recent years, fractional derivatives have been utilized to analyze the memory and hereditary properties of complex systems within diverse application areas. For example, Sadek [1] defined the trigonometric q-derivative and q-integral in the quantum framework, combined them with the Caputo fractional derivative, and proposed a new trigonometric Caputo q-fractional calculus; the corresponding q-fractional differential equations are solved using the q–Laplace transform. For related research, one can refer to the work of Yang et al. in references [2]. For more detailed information on fractional differential equations, monographs [3–6] can be consulted.

In practical problems, systems may be subject to various internal or external disturbances, leading to delays in system responses and complicating the relationship between system inputs and outputs. To more accurately describe and analyze such systems with time-delay characteristics, the introduction of time-delay systems has become inevitable. Time-delay differential systems can more truly reflect the dynamic behavior of actual systems, reveal their inherent complex features, and facilitate an in-depth understanding of the laws governing the dynamic evolution of systems. For example, Yong-Ki et al. [7] investigated the approximate controllability of fractional delay integrodifferential systems of order $1 < r < 2$ using multivalued functions, Sobolev-type operators, cosine families, fractional analysis, and Martelli's fixed point theorem. Li et al. [8] constructed fractional delay Mittag-Leffler matrix polynomial functions and systematically studied their related properties. Using these fractional delay Mittag-Leffler matrix polynomial functions, combined with the method of variation of constants and the superposition principle, they derived the exact expression of the solution for Caputo-type fractional delay linear homogeneous differential systems and proved the finite-time stability of the system solutions. In [9], the authors obtained relative controllability results for fractional delay differential systems by constructing Gram matrices and corresponding control functions. Furthermore, within the Banach space framework, they studied the existence, uniqueness, and finite-time stability of solutions for nonlinear fractional delay differential systems using the contraction mapping theorem [10]. In addition, delay differential systems have important theoretical significance in engineering fields such as communication systems, power systems, and network transmission systems. For related research, please refer to the references [11–14].

Stochastic differential equations play a pivotal role in numerous branches of science and industry, such as biology, physics, economics, engineering, and financial markets. In fact, these applications rely heavily on the stability of the system, and system stability is the primary factor to be considered in the process of system performance analysis and synthesis, relevant research can be found in references [15–18]. For example, in reference [19], Itô's and Nishimura developed a tool based on Lyapunov functions to explore the stability and robustness of cascaded non-linear stochastic systems. Zhao and Deng discussed the stochastic stability and instability of stochastic systems in reference [20]. In reference [21], Zhu adopted the proof by contradiction to investigate the p-th moment exponential stability of a class of stochastic delay differential equations driven by Lévy processes, thereby deriving several novel stability theorems. In reference [22], Ruili Song et al. By devising new Lyapunov functions, employing M-matrix techniques, and addressing non-differentiable delay functions, the existence, uniqueness, asymptotic boundedness, and exponential stability of nonlinear mixed neutral stochastic delay differential systems wit Lévy jumps and distinct structures were successfully established. Li [23] proposed an averaging inequality with fractional conditions, and used this inequality to handle the singular integral kernel problem in Caputo-type fractional stochastic differential systems. Finally, he obtained the averaging principle for the solution of the following system:

$$\begin{cases} ({}^C D^{\alpha}_{-\vartheta+} y)(t) = Ay(t) + By(t-h) + f(t, y(y)) + \delta(t, y(t))\dfrac{dw(t)}{ds},\ t \in [0, T], \\ y(t) = \Psi(t),\ -h \le t \le 0, \end{cases} \tag{1}$$

where $\alpha \in [0.5, 1],\ h > 0,\ A, B \in \mathbb{R}^{n\times n},\ f \in C([0,T], \mathbb{R}^n), \delta \in C([0,T], \mathbb{R}^{n\times n})$. Based on the research of Li [23], we conducted exponential stability studies for system (1). In reference [24], Xiao et al. Explored the stochastic stability as well as stochastic asymptotic stability of Caputo fractional stochastic differential equations. Through the establishment of the Caputo-adapted Itô's formula, they successfully obtained the results regarding almost exponential stability and p-th moment exponential stability. For more research results, readers are recommended to refer to the books in [25, 26] and the review paper [27].

It is worth noting that the research works in references [21, 22] merely focus on the stability of integer-order stochastic differential equations. However, time delay is also a crucial factor in determining the stability of a system. Although the stability of fractional stochastic differential systems was studied in reference [24], the discussion of delay effects was lacking. This paper conducts an in-depth study on a class of Caputo-type fractional stochastic differential equations with pure time delay, and explores their p-th moment estimation and p-th exponential stability. The main innovations are as follows:

1. We established an Itô's differential formula associated with the Mittag-Leffler type matrix polynomial function. 2. We rigorously derived the p-th moment estimation for the system solution, providing a theoretical basis for the boundedness and non-explosion of the system. 3. By constructing the operator $\mathfrak{L}V$, we established the conditions under which the system solution satisfies p-th moment exponential stability.

The remaining part of this paper is structured as follows. In Section 2, the symbols and relevant concepts will be introduced. In Section 3, will demonstrate the derivation process of the Itô's formula involving the Mittag-Leffler type matrix polynomial function. In Section 4, presents our main research results. Under several sufficient conditions, we derive the p-th order estimation formula of the system and prove the exponential stability of the system in the sense of the p-th order. Finally, in Section 5, we illustrate the effectiveness of the obtained conclusions through examples.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_\varsigma\}_{\varsigma\ge 0}$ satisfying the usual conditions, $\mathbb{E}(\cdot)$ denotes the expectation regarding the measure P, $L^2(\Omega, \mathcal{F}_\varsigma, \mathcal{R}^n)$ denotes the Hilbert space of $\mathcal{F}_\varsigma$-measurable square integrable random variables with values in $\mathcal{R}^n$. We are concerned with Caputo fractional stochastic delay differential equations:

$$\begin{cases} ({}^C D^{\alpha}_{0+} \Upsilon)(\varsigma) = \mathcal{A}\Upsilon(\varsigma) + \mathcal{B}\Upsilon(\varsigma - \upsilon) + \varphi(\varsigma, \Upsilon(\varsigma)) + \psi(\varsigma, \Upsilon(\varsigma))\dfrac{dw(\varsigma)}{d\varsigma},\ \varsigma \in [0, \mathcal{T}], \\ \Upsilon(\varsigma) = \Psi(\varsigma),\ \varsigma \in [-\upsilon, 0], \end{cases} \tag{2}$$

where $\alpha \in [\frac{1}{2}, 1]$, $\upsilon > 0$ denotes the time delay and $\mathcal{A}, \mathcal{B} \in \mathcal{R}^{n\times n}$ are any matrices, $\Upsilon(\varsigma) \in \mathcal{R}^n$ is a stochastic process. $\varphi : [0, \mathcal{T}] \times \mathcal{R}^n \to \mathcal{R}^n$, $\psi : [0, \mathcal{T}] \times \mathcal{R}^n \to \mathcal{R}^{n\times n}$. $\{w(\varsigma), \varsigma \in [0, \mathcal{T}]\}$ is the given n-dimensional standard Brown motion on a complete probability space $(\Omega, \mathcal{F}, P)$. The initial function $\Psi \in C^2([-\upsilon, 0], \mathcal{R}^n)$ satisfies $\mathbb{E}(|\Psi(\varsigma)|^2) < \infty$.

Definition 1 (see [28]). *The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ for a function $\Upsilon : [0, \infty) \to \mathcal{R}$ can be written as*

$$({}^{RL}D^{\alpha}_{0+}\Upsilon)(\varsigma) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\varsigma} \int_0^{\varsigma} (\varsigma - s)^{-\alpha} \Upsilon'(s) ds, \ \varsigma > 0.$$

Definition 2 (see [28]). *The Caputo derivative of order $0 < \alpha < 1$ for a function $\Upsilon : [0, \infty) \to \mathcal{R}$ can be written as*

$$({}^{C}D^{\alpha}_{0+}\Upsilon)(\varsigma) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\varsigma} (\varsigma - s)^{-\alpha} \Upsilon'(s) ds, \ \varsigma > 0.$$

Definition 3 (see [28]). *The fractional integral of order $0 < \alpha < 1$ for a function $\Upsilon : [0, \infty) \to \mathcal{R}$ can be written as*

$$(I^{\alpha}_{0+}\Upsilon)(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma} (\varsigma - s)^{\alpha-1} \Upsilon(s) ds, \ \varsigma > 0.$$

Definition 4 (see [11]). *The coefficient matrices $\mathbf{Q}_i(\varsigma_1, \cdots, \varsigma_n)$, $i = 1, 2, \cdots$, satisfies the following multivariate determining matrix equation*

$$\begin{aligned} \mathbf{Q}_{i+1}(\varsigma) &= \mathcal{A}\mathbf{Q}_i(\varsigma) + \mathcal{B}\mathbf{Q}_i(\varsigma - \upsilon), \\ \mathbf{Q}_0(\varsigma) &= \mathbf{Q}_i(-\upsilon) = \mathbf{\Theta}, \ \mathbf{Q}_1(0) = \mathbf{I}, \\ i &= 0, \ 1, \ 2, \cdots, \ \varsigma = 0, \ h, \ 2h, \cdots, \end{aligned}$$

where $\mathbf{\Theta}$ is a zero matrix, $\mathbf{I}$ is an identity matrix.

Remark 1. *The specific values of the coefficient matrix $\mathbf{Q}_{i+1}(j\upsilon)$ are given in the following Table 1:*

Table 1. Specific forms of the coefficient matrices $\mathbf{Q}_{i+1}(j\upsilon)$.

j	$\mathbf{Q}_1(j\upsilon)$	$\mathbf{Q}_2(j\upsilon)$	$\mathbf{Q}_3(j\upsilon)$	$\mathbf{Q}_4(j\upsilon)$	$\cdots$	$\mathbf{Q}_{i+1}(j\upsilon)$
0	$\mathbf{I}$	$\mathcal{A}$	$\mathcal{A}^2$	$\mathcal{A}^3$	$\cdots$	$\mathcal{A}^i$
1	$\mathbf{\Theta}$	$\mathcal{B}$	$\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}$	$\mathcal{A}(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}) + \mathcal{B}\mathcal{A}^2$	$\cdots$	$\cdots$
2	$\mathbf{\Theta}$	$\mathbf{\Theta}$	$\mathcal{B}^2$	$\mathcal{A}\mathcal{B}^2 + \mathcal{B}(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A})$	$\cdots$	$\cdots$
3	$\mathbf{\Theta}$	$\mathbf{\Theta}$	$\mathbf{\Theta}$	$\mathcal{B}^3$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
i	$\mathbf{\Theta}$	$\mathbf{\Theta}$	$\mathbf{\Theta}$	$\mathbf{\Theta}$	$\cdots$	$\mathcal{B}^i$

Definition 5 (see [11]). *Delayed perturbation of two parameter Mittag-Leffler type matrix function $\mathfrak{X}^{\mathcal{A},\,\mathcal{B}}_{\upsilon,\alpha,\beta}(\cdot)$ generated by $\mathcal{A}, \mathcal{B}$ is defined by*

$$\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\beta}(\varsigma) = \begin{cases} \mathbf{\Theta}, \ \ \varsigma \in [-\upsilon, 0), \\ \mathbf{I}, \ \ \varsigma = 0, \\ \sum\limits_{i=0}^{\infty} \mathbf{Q}_{i+1}(0) \frac{\varsigma^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} + \sum\limits_{i=1}^{\infty} \mathbf{Q}_{i+1}(\upsilon) \frac{(\varsigma-\upsilon)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} \\ \ + \cdots + \sum\limits_{i=\rho}^{\infty} \mathbf{Q}_{i+1}(\rho\upsilon) \frac{(\varsigma-(\rho-1)\upsilon)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}, \ \ (\rho-1)\upsilon < \varsigma \le \rho\upsilon. \end{cases} \tag{3}$$

Lemma 1 (see [5]). *The function $\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\beta}(\cdot)$ is continuous on $[0, +\infty)$.*

Lemma 2. *The function $\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\beta}(\varsigma)$ is uniformly convergent on $[0, +\infty)$.*

Proof. When $\varsigma = 0$, $\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\beta}(\varsigma) = \mathbf{I}$, which is obviously convergent. When $(\rho - 1)\upsilon < \varsigma \le \rho\upsilon$, we have $\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\beta}(\varsigma) = \sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{\rho-1} \mathbf{Q}_{i+1}(j\upsilon) \frac{(\varsigma-j\upsilon)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}$. For real matrices $\mathcal{A}, \ \mathcal{B} \in \mathcal{R}^{n\times n}$, there exist two positive constants $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ such that $|\mathcal{A}|^i \le M^i_{\mathcal{A}}$, $|\mathcal{B}|^i \le M^i_{\mathcal{B}}$. When $j > i$, one has $\mathbf{Q}_{i+1}(j\upsilon) = \mathbf{\Theta}$. When $j \le i$, one has $|\mathbf{Q}_{i+1}(j\upsilon)| \le (M_{\mathcal{A}} + M_{\mathcal{B}})^i$. Since $(\rho - 1)\upsilon < \varsigma \le \rho\upsilon$, it follows that $0 < \varsigma - j\upsilon \le \rho\upsilon$. Thus, we have

$$\left|\sum_{i=0}^{\infty}\sum_{j=0}^{\rho-1}\mathbf{Q}_{i+1}(j\upsilon)\frac{(\varsigma-j\upsilon)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}\right|\leq\sum_{i=0}^{\infty}\frac{(M_{\mathcal{A}}+M_{\mathcal{B}})^{i}(\rho\upsilon)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}.$$

Furthermore, since $\Gamma(i\alpha+\beta)\sim(i\alpha)^{i\alpha+\beta-\frac{3}{2}}e^{-i\alpha}\sqrt{2\pi i\alpha}$ exhibits super exponential growth, we can conclude that $\sum\limits_{i=0}^{\infty}\frac{(M_{\mathcal{A}}+M_{\mathcal{B}})^{i}(\rho\upsilon)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}$ is convergent. Finally, according to the Weierstrass M-test, we can conclude that $\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\rho-1}\mathbf{Q}_{i+1}(j\upsilon)\frac{(\varsigma-j\upsilon)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}$ is uniformly convergent on $(\rho-1)\upsilon<\varsigma\leq\rho\upsilon$. In conclusion, $\mathfrak{X}_{\upsilon,\alpha,\beta}^{\mathcal{A},\mathcal{B}}(\varsigma)$ is uniformly convergent on $[0,+\infty)$. □

Definition 6. *The derivative of* $\mathfrak{X}_{\upsilon,\alpha,\beta}^{\mathcal{A},\,\mathcal{B}}(\cdot)$ *is defined as:*

$$\mathcal{Z}_{\upsilon,\alpha,\beta}^{\mathcal{A},\mathcal{B}}(\varsigma)=\begin{cases}\mathbf{\Theta},\ \ \varsigma\in[-\upsilon,0],\\ \sum\limits_{i=0}^{\infty}\mathbf{Q}_{i+1}(0)\frac{(i\alpha+\beta-1)\varsigma^{i\alpha+\beta-2}}{\Gamma(i\alpha+\beta)}+\sum\limits_{i=1}^{\infty}\mathbf{Q}_{i+1}(\upsilon)\frac{(i\alpha+\beta-1)(\varsigma-\upsilon)^{i\alpha+\beta-2}}{\Gamma(i\alpha+\beta)}\\ \ +\cdots+\sum\limits_{i=\rho}^{\infty}\mathbf{Q}_{i+1}(\rho\upsilon)\frac{(i\alpha+\beta-1)(\varsigma-(\rho-1)\upsilon)^{i\alpha+\beta-2}}{\Gamma(i\alpha+\beta)},\ \ (\rho-1)\upsilon<\varsigma\leq\rho\upsilon.\end{cases}\tag{4}$$

Lemma 3 (see [5])**.** *For any* $\varsigma\geq0,\ 0<\alpha<1,\ 0<\beta\leq1$ *and* $\alpha+\beta\geq1$, *we have*

$$\begin{aligned}|\mathfrak{X}_{\upsilon,\alpha,\beta}^{\mathcal{A},\mathcal{B}}(\varsigma)| &\leq \varsigma^{\beta-1}E_{\alpha,\beta}((|\mathcal{A}|+|\mathcal{B}|)\varsigma^{\alpha})\\ &:= \varsigma^{\beta-1}E_{\alpha,\beta}(\eta\varsigma^{\alpha}),\end{aligned}\tag{5}$$

where $\eta=|\mathcal{A}|+|\mathcal{B}|$, $E_{\alpha,\beta}(\varsigma)=\sum\limits_{j=0}^{\infty}\frac{\varsigma^{j}}{\Gamma(j\alpha+\beta)}$, $\varsigma\in\mathcal{R}$ *is the Mittag-Leffer function.*

Lemma 4 (see [29])**.** *For any* $\varsigma\geq0$, *we have*

(i) *Let* $\alpha>0$, *then* $\frac{d}{d\varsigma}E_{\alpha,1}(\varsigma)=\frac{1}{\alpha}E_{\alpha,\alpha}(\varsigma)$,

(ii) *Let* $\alpha>0,\beta\in\mathcal{R}$, *then* $\frac{d}{d\varsigma}E_{\alpha,\beta}(\varsigma)=\frac{1}{\alpha}E_{\alpha,\alpha+\beta+1}(\varsigma)+\frac{1-\beta}{\alpha}E_{\alpha,\alpha+\beta}(\varsigma)$.

Lemma 5. *For any* $\varsigma\geq0,\ 0<\alpha<1,\ \beta>0$, *and* $\alpha+\beta>1$ *then the following holds true:*

$$|\mathcal{Z}_{\upsilon,\alpha,\beta}^{\mathcal{A},\mathcal{B}}(\varsigma)|\leq\varsigma^{\beta-2}\left[\varsigma\frac{d}{d\varsigma}E_{\alpha,\beta}(\eta\varsigma^{\alpha})+\beta E_{\alpha,\beta}(\eta\varsigma^{\alpha})\right].\tag{6}$$

Proof. By (4), without loss of generality, for $(\rho-1)\upsilon<\varsigma<\rho\upsilon,\rho=1,2,\ldots$ we have

$$\begin{aligned}|\mathcal{Z}_{\upsilon,\alpha,\beta}^{\mathcal{A},\mathcal{B}}(\varsigma)| &\leq \left|\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbf{Q}_{i+1}(j\upsilon)\frac{(i\alpha+\beta-1)(\varsigma-j\upsilon)^{i\alpha+\beta-2}}{\Gamma(i\alpha+\beta)}\right|\\ &\leq \sum_{i=0}^{\infty}|\mathbf{Q}_{i+1}(0)|\frac{(i\alpha+\beta-1)\varsigma^{i\alpha+\beta-2}}{\Gamma(i\alpha+\beta)}+\sum_{i=1}^{\infty}|\mathbf{Q}_{i+1}(\upsilon)|\frac{(i\alpha+\beta-1)(\varsigma-\upsilon)^{i\alpha+\beta-2}}{\Gamma(i\alpha+\beta)}\\ &\quad+\cdots+\sum_{i=\rho}^{\infty}|\mathbf{Q}_{i+1}(\rho\upsilon)|\frac{(i\alpha+\beta-1)(\varsigma-(\rho-1)\upsilon)^{i\alpha+\beta-2}}{\Gamma(i\alpha+\beta)}\\ &= \sum_{i=0}^{\infty}(|\mathcal{A}|+|\mathcal{B}|)^{i}\frac{(i\alpha+\beta)\varsigma^{i\alpha+\beta-2}}{\Gamma(i\alpha+\beta)}\\ &= \varsigma^{\beta-2}\left[\sum_{i=0}^{\infty}(|\mathcal{A}|+|\mathcal{B}|)^{i}\frac{i\alpha\varsigma^{i\alpha}}{\Gamma(i\alpha+\beta)}+\sum_{i=0}^{\infty}(|\mathcal{A}|+|\mathcal{B}|)^{i}\frac{\beta\varsigma^{i\alpha}}{\Gamma(i\alpha+\beta)}\right]\\ &= \varsigma^{\beta-2}\left[\varsigma\times\frac{d}{d\varsigma}E((|\mathcal{A}|+|\mathcal{B}|)\varsigma^{\alpha})+\beta E_{\alpha,\beta}((|\mathcal{A}|+|\mathcal{B}|)\varsigma^{\alpha})\right]\\ &:= \varsigma^{\beta-2}\left[\varsigma\times\frac{d}{d\varsigma}E(\eta\varsigma^{\alpha})+\beta E_{\alpha,\beta}(\eta\varsigma^{\alpha})\right].\end{aligned}$$

The proof is complete. □

Remark 2. *For $0 < \alpha < 1$ and $\beta = 1$, we have* $|\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(\varsigma)| \leq \varsigma^{-1}\Big[\varsigma \times \frac{1}{\alpha}E_{\alpha,\alpha}(\eta\varsigma^{\alpha}) + E_{\alpha,1}(\eta\varsigma^{\alpha})\Big]$. *For* $0 < \alpha = \beta < 1$, *we have* $|\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma)| \leq \varsigma^{\alpha-2}\Big[\frac{\varsigma}{\alpha}E_{\alpha,2\alpha-1}(\eta\varsigma^{\alpha}) + \frac{\varsigma(1-\alpha)}{\alpha}E_{\alpha,2\alpha}(\eta\varsigma^{\alpha}) + \alpha E_{\alpha,\alpha}(\eta\varsigma^{\alpha})\Big]$.

Lemma 6. *The function* $\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\beta}(\varsigma)$ *is uniformly convergent on* $[0, +\infty)$.

Proof. Its proof process is the same as that of Lemma 2, so we will not elaborate on it in detail. □

Lemma 7. *A* $\mathcal{R}^n$*-value stochastic process* $\{\Upsilon(\varsigma) : -\upsilon \leq \varsigma \leq \mathcal{T}\}$ *is called a solution of* (2) *if* $\Upsilon(\varsigma)$ *satisfies the integral equation of the following form*

$$\Upsilon(\varsigma) = \begin{cases} \mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(\varsigma+\upsilon)\Psi(-\upsilon) + \int_{-\upsilon}^{0} \mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)[({}^{C}D^{\alpha}_{-\upsilon^{+}}\Psi)(s) - \mathcal{A}\Psi(s)]ds \\ +\int_{0}^{\varsigma} \mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)\varphi(s,\Upsilon(s))ds + \int_{0}^{\varsigma} \mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)\psi(s,\Upsilon(s))dw(s), \ \varsigma \in [0,\mathcal{T}], \\ \Psi(\varsigma), \ \varsigma \in [-\upsilon, 0], \end{cases} \tag{7}$$

where $\Upsilon(\varsigma)$ *is* $\mathcal{F}_{\varsigma}$*-adapted and* $\mathbb{E}(\int_{-\upsilon}^{\mathcal{T}} |\Upsilon(\varsigma)|^2 d\varsigma) < \infty$.

Definition 7 (see [26])**.** *The system* (2) *is deemed to have an exponentially stable trivial solution in the* p*-th moment if two positive constants* λ *and* C *exist, satisfying the inequality:*

$$\mathbb{E}(|\Upsilon(\varsigma)|^p) \leq C\mathbb{E}|\Upsilon(0)|e^{-\lambda\varsigma}.$$

3. Itô's Formula of Caputo Version

In this section, we present the Caputo fractional Itô's formula, which specifies the differentiation rules for functions of Caputo fractional stochastic processes. We first derive the one-dimensional form of this formula and subsequently extend it to the multi-dimensional scenario.

Let $V \in \mathcal{C}^{1,2}(\mathcal{R}^n \times \mathcal{R}^+, \mathcal{R})$ denotes the set of all real-valued functions $V(\varsigma, \Upsilon(\varsigma))$ defined on $\mathcal{R}^n \times \mathcal{R}^+$ such that they are continuously twice differentiable in ς and once in Υ. We defined

$$V_{\varsigma} = \frac{\partial V}{\partial \varsigma},\ V_{\Upsilon} = \left(\frac{\partial V}{\partial \Upsilon_1}, \cdots, \frac{\partial V}{\partial \Upsilon_n} \right),\ V_{\Upsilon\Upsilon} = \begin{pmatrix} \frac{\partial^2 V}{\partial \Upsilon_1 \partial \Upsilon_1}, \cdots, \frac{\partial^2 V}{\partial \Upsilon_1 \partial \Upsilon_n} \\ \vdots \qquad\qquad \vdots \\ \frac{\partial^2 V}{\partial \Upsilon_n \partial \Upsilon_1}, \cdots, \frac{\partial^2 V}{\partial \Upsilon_n \partial \Upsilon_n} \end{pmatrix},$$

Clearly, when $V \in \mathcal{C}^{1,2}(\mathcal{R} \times \mathcal{R}^+, \mathcal{R})$, we have $V_{\Upsilon} = \frac{\partial V}{\partial \Upsilon}$ and $V_{\Upsilon\Upsilon} = \frac{\partial^2 V}{\partial \Upsilon^2}$.

Theorem 1. *Let* $\Upsilon(\varsigma)$ *be an Itô process on* $\varsigma > 0$ *with* (2), *let* $n = 1$ *and* $V \in \mathcal{C}^{1,2}(\mathcal{R} \times \mathcal{R}^+, \mathcal{R})$, *then* $V(\varsigma, \Upsilon(\varsigma))$ *is again an Itô process with* (2) *given by*

$$\begin{aligned} V(\varsigma,\Upsilon(\varsigma)) \quad = \quad & V(0,\Upsilon(0)) + \int_0^{\varsigma} \Big[V_s(s,\Upsilon(s)) + \frac{1}{2}V_{\Upsilon\Upsilon}(s,\Upsilon(s))\psi^2(s,\Upsilon(s))\Big]ds \\ & + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))\Big\{ \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(s+\upsilon)\Psi(-\upsilon) \\ & + \int_{-\upsilon}^{0} \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)[({}^{C}D^{\alpha}_{-\upsilon^{+}}\Psi)(u) - \mathcal{A}\Psi(u)]du \\ & + \varphi(s,\Upsilon(s)) + \int_0^{s} \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))du \\ & + \int_0^{s} \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)\Big\}ds \\ & + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))\psi(s,\Upsilon(s))dw(s). \end{aligned}$$

Proof. Let $0 = \varsigma_0 < \varsigma_1 < \cdots < \varsigma_m = \varsigma,\ \triangle\varsigma_k = \varsigma_{k+1} - \varsigma_k,\ \triangle\Upsilon_k = \Upsilon(\varsigma_{k+1}) - \Upsilon(\varsigma_k),\ T_1 = \max\limits_{0\leq k\leq m-1} \triangle\varsigma_k,\ T_2 = \max\limits_{0\leq k\leq m-1} \triangle\Upsilon(\varsigma_k)$. According to Taylor's theorem, we have

$$\begin{aligned}V(\varsigma,\Upsilon(\varsigma)) - V(0,\Upsilon(0)) &= \sum_{k=0}^{m-1}\Big[V(\varsigma_{k+1},\Upsilon(\varsigma_{k+1})) - V(\varsigma_k,\Upsilon(\varsigma_{k+1}))\Big] \\ &\quad + \sum_{k=0}^{m-1}\Big[V(\varsigma_k,\Upsilon(\varsigma_{k+1})) - V(\varsigma_k,\Upsilon(\varsigma_k))\Big] \\ &:= V_1 + V_2. \end{aligned} \tag{8}$$

First, perform a univariate Taylor expansion of V_1 with respect to ς. Set $\Upsilon = \Upsilon(\varsigma_{k+1})$ and expand it at $\varsigma = \varsigma_k$, we have

$$V(\varsigma_{k+1},\Upsilon(\varsigma_{k+1})) = V(\varsigma_k,\Upsilon(\varsigma_{k+1})) + (\varsigma_{k+1} - \varsigma_k)V_\varsigma(\varsigma_k,\Upsilon(\varsigma_{k+1})).$$

We can obtain

$$\begin{aligned}\sum_{k=0}^{m-1}\Big[V(\varsigma_{k+1},\Upsilon(\varsigma_{k+1})) - V(\varsigma_k,\Upsilon(t_{k+1}))\Big] &= \sum_{k=0}^{m-1}(\varsigma_{k+1} - \varsigma_k)V_t(\varsigma_k,\Upsilon(\varsigma_{k+1})) \\ &= \sum_{k=0}^{m-1} V_\varsigma(\varsigma_k,\Upsilon(\varsigma_{k+1}))\triangle\varsigma_k.\end{aligned}$$

When $T_1 \to 0$, one can get

$$\sum_{k=0}^{m-1} V_\varsigma(\varsigma_k,\Upsilon(\varsigma_{k+1}))\triangle\varsigma_k \to \int_0^\varsigma V_s(s,\Upsilon(s))ds. \tag{9}$$

Then, conduct a univariate Taylor expansion of V_2 regarding Υ. Set $\varsigma = \varsigma_k$ constant and perform the expansion at $\Upsilon = \Upsilon(\varsigma_k)$, and thus we get

$$\begin{aligned}V(\varsigma_k,\Upsilon(\varsigma_{k+1})) &= V(\varsigma_k,\Upsilon(\varsigma_k)) + [\Upsilon(\varsigma_{k+1}) - \Upsilon(\varsigma_k)]V_\Upsilon(\varsigma_k,\Upsilon(\varsigma_k)) \\ &\quad + \frac{1}{2}[\Upsilon(\varsigma_{k+1}) - \Upsilon(\varsigma_k)]^2 V_\Upsilon(\varsigma_k,\Upsilon(\varsigma_k)).\end{aligned}$$

We can obtain

$$\begin{aligned}\sum_{k=0}^{m-1}\Big[V(\varsigma_k,\Upsilon(\varsigma_{k+1})) - V(\varsigma_k,\Upsilon(\varsigma_k))\Big] &= \sum_{k=0}^{m-1}[\Upsilon(\varsigma_{k+1}) - \Upsilon(\varsigma_k)]V_\Upsilon(\varsigma_k,\Upsilon(\varsigma_k)) \\ &\quad + \sum_{k=0}^{m-1}\frac{1}{2}[\Upsilon(\varsigma_{k+1}) - \Upsilon(\varsigma_k)]^2 V_\Upsilon(\varsigma_k,\Upsilon(\varsigma_k)) \\ &= \sum_{k=0}^{m-1} V_\Upsilon(\varsigma_k,\Upsilon(\varsigma_k))\triangle\Upsilon(\varsigma_k) \\ &\quad + \sum_{k=0}^{m-1} V_\Upsilon(\varsigma_k,\Upsilon(\varsigma_k))\triangle\Upsilon^2(\varsigma_k).\end{aligned}$$

When $T_2 \to 0$, one can get

$$\sum_{k=0}^{m-1} V_\Upsilon(\varsigma_k,\Upsilon(\varsigma_{k+1}))\triangle\Upsilon(\varsigma_k) \to \int_0^\varsigma V_\Upsilon(s,\Upsilon(s))d\Upsilon(s). \tag{10}$$

By (7), we has an equivalent form

$$\begin{aligned}
d\Upsilon(\varsigma) &= \Upsilon'(\varsigma)d\varsigma \\
&= \mathcal{Z}_{\upsilon,\alpha,1}^{\mathcal{A},\mathcal{B}}(\varsigma+\upsilon)\Psi(-\upsilon)d\varsigma + \left(\int_{-\upsilon}^{0} \mathcal{Z}_{\upsilon,\alpha,\alpha}^{\mathcal{A},\mathcal{B}}(\varsigma-s)[({}^{C}D_{-\upsilon^+}^{\alpha}\Psi)(s) - A\Psi(s))]ds\right)d\varsigma \\
&\quad +\varphi(\varsigma,\Upsilon(\varsigma))d\varsigma + \left(\int_0^{\varsigma} \mathcal{Z}_{\upsilon,\alpha,\alpha}^{\mathcal{A},\mathcal{B}}(\varsigma-s)\varphi(s,\Upsilon(s))ds\right)d\varsigma \\
&\quad +\psi(\varsigma,\Upsilon(\varsigma))dw(t) + \left(\int_0^{\varsigma} \mathcal{Z}_{\upsilon,\alpha,\alpha}^{\mathcal{A},\mathcal{B}}(\varsigma-s)\psi(s,\Upsilon(s))dw(s)\right)d\varsigma.
\end{aligned}$$

Therefore

$$\begin{aligned}
\triangle\Upsilon(\varsigma_k) &= \mathcal{Z}_{\upsilon,\alpha,1}^{\mathcal{A},\mathcal{B}}(\varsigma_k+\upsilon)\Psi(-\upsilon)\triangle\varsigma_k + \int_{-\upsilon}^{0} \mathcal{Z}_{\upsilon,\alpha,\alpha}^{\mathcal{A},\mathcal{B}}(\varsigma_k-s)[({}^{C}D_{-\upsilon^+}^{\alpha}\Psi)(s) - \mathcal{A}\Psi(s))]ds\triangle\varsigma_k \\
&\quad +\varphi(\varsigma_k,\Upsilon(\varsigma_k))\triangle\varsigma_k + \int_0^{\varsigma_k} \mathcal{Z}_{h,\alpha,\alpha}^{\mathcal{A},\mathcal{B}}(\varsigma_k-s)\varphi(s,\Upsilon(s))ds\triangle\varsigma_k \\
&\quad +g(\varsigma_k,\Upsilon(\varsigma_k))\triangle w(\varsigma_k) + \int_0^{\varsigma_k} \mathcal{Z}_{\upsilon,\alpha,\alpha}^{\mathcal{A},B}(\varsigma_k-s)\psi(s,\Upsilon(s))dw(s)\triangle\varsigma_k.
\end{aligned}$$

Based on $d\varsigma d\varsigma = 0$, $dw(\varsigma)d\varsigma = 0$, $dw(\varsigma)dw(\varsigma) = \varsigma$, one can get

$$(\triangle\Upsilon(\varsigma_k))^2 = \psi^2(\varsigma_k,\Upsilon(\varsigma_k))\triangle\varsigma_k.$$

When $T_2 \to 0$, one can get

$$\begin{aligned}
\sum_{k=0}^{m-1} V_{\Upsilon}(\varsigma_k,\Upsilon(\varsigma_{k+1}))\triangle\Upsilon^2(\varsigma_k) &= \sum_{k=0}^{m-1} V_{\Upsilon}(\varsigma_k,\Upsilon(\varsigma_{k+1}))\psi^2(\varsigma_k,\Upsilon(\varsigma_k))\triangle\varsigma_k \\
&\to \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))\psi^2(\varsigma_k,\Upsilon(\varsigma_k))ds.
\end{aligned} \tag{11}$$

Substituting (9)–(11) into (8), we obtain that

$$\begin{aligned}
V(\varsigma,\Upsilon(\varsigma)) &= V(0,\Upsilon(0)) + \int_0^{\varsigma} V_s(s,\Upsilon(s))ds + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))d\Upsilon(s) \\
&\quad + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))g^2(\varsigma_k,\Upsilon(\varsigma_k))ds \\
&= V(0,\Upsilon_0) + \int_0^{\varsigma} V_s(s,\Upsilon(s))ds + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))\mathcal{Z}_{\upsilon,\alpha,1}^{\mathcal{A},\mathcal{B}}(s+\upsilon)\Psi(-\upsilon)ds \\
&\quad + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))\int_{-\upsilon}^{0} \mathcal{Z}_{\upsilon,\alpha,\alpha}^{\mathcal{A},\mathcal{B}}(s-u)[({}^{C}D_{-\upsilon^+}^{\alpha}\Psi)(u) - \mathcal{A}\Psi(u))]duds \\
&\quad + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))\varphi(s,\Upsilon(s))ds \\
&\quad + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))\int_0^{s} \mathcal{Z}_{\upsilon,\alpha,\alpha}^{\mathcal{A},\mathcal{B}}(s-u)\varphi(u,\Upsilon(u))duds \\
&\quad + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))\psi(s,\Upsilon(s))dw(s) \\
&\quad + \int_0^{\varsigma} V_{\Upsilon}(s,\Upsilon(s))\int_0^{s} \mathcal{Z}_{\varsigma,\alpha,\alpha}^{\mathcal{A},\mathcal{B}}(s-u)\psi(u,\Upsilon(u))dw(u)ds \\
&\quad + \frac{1}{2}\int_0^{\varsigma} V_{\Upsilon\Upsilon}(s,\Upsilon(s))\psi^2(s,\mathcal{Y}(s))ds.
\end{aligned}$$

The proof is now complete. □

We proceed to extend the 1-dimensional Itô's formula to the multi-dimensional setting. Let $w(\varsigma) = (w_1(\varsigma), w_2(\varsigma), \cdots, w_n(\varsigma))$ be an n-dimensional Brown motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$, which is adapted to the filtration $\{\mathcal{F}_\varsigma\}_{\varsigma \geq 0}$. Given that the proof follows a similar line of reasoning as the 1-dimensional case, it is omitted herein.

Theorem 2. *Let $\Upsilon(\varsigma)$ be an Itô process on $\varsigma > 0$ with (2), let $V \in \mathcal{C}^{1,2}(\mathcal{R}^n \times \mathcal{R}^+, \mathcal{R})$, then $V(\varsigma, \mathcal{Y}(\varsigma))$ is again an Itô process with (2) given by*

$$
\begin{aligned}
V(\varsigma, \Upsilon(\varsigma)) \quad = \quad & V(0, \Upsilon_0) + \int_0^\varsigma V_s(s, \Upsilon(s))ds + \int_0^\varsigma V_\Upsilon(s, \Upsilon(s))\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(s+\upsilon)\Psi(-\upsilon)ds \\
& + \int_0^\varsigma V_\Upsilon(s, \Upsilon(s))\varphi(s, \Upsilon(s))ds \\
& + \int_0^\varsigma V_\Upsilon(s, \Upsilon(s)) \int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u, \Upsilon(u))duds \\
& + \int_0^\varsigma V_\Upsilon(s, \Upsilon(s))\psi(s, \Upsilon(s))dw(s) \\
& + \int_0^\varsigma V_\Upsilon(s, \Upsilon(s)) \int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u, \Upsilon(u))dw(u)ds \\
& + \frac{1}{2}\int_0^\varsigma trace(\psi^\top(s, \Upsilon(s))V_{\Upsilon\Upsilon}(s, \Upsilon(s))\psi(s, \Upsilon(s)))ds.
\end{aligned}
$$

4. L^p-Estimstes and p-th Exponential Stability

The p-moment estimation of the solutions of stochastic differential equations is of great significance for the study of their stability. In this section, we will present two p-th moment estimation formulas for system (2) and investigate its exponential stability.

For the convenience of subsequent stability analysis, we denote $y_1(\varsigma) = [({}^C D^\alpha_{-\upsilon^+}\Psi)(\varsigma) - \mathcal{A}\Psi(\varsigma)]$. We introduce the following assumptions:

Assumption 1. $M_1 = [\frac{1}{\alpha^2}(E_{\alpha,2\alpha-1}(\eta\mathcal{T}^\alpha))^2 + \frac{1}{\alpha^2}(E_{\alpha,2\alpha}(\eta\mathcal{T}^\alpha))^2 + \alpha^2(E_{\alpha,\alpha}(\eta\mathcal{T}^\alpha))^2]$.

Assumption 2. *Exists two positive constants K_1, K_2, such that $|\varphi(\varsigma, \Upsilon(\varsigma))| \leq K_1$ and $|\psi(\varsigma, \Upsilon(\varsigma))| \leq K_2$.*

4.1. L^p-Estimstes

In this section, we will derive two p-th moment estimation formulas through Itô's formula.

Theorem 3. *Let $p \geq 2$, and assume that Assumptions 1 and 2 hold, there exists positive constants K_3, K_4 such that*

$$
\Upsilon^\top(\varsigma)\left[\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(\varsigma+\upsilon)\Psi(-\upsilon) + \int_{-\upsilon}^0 \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)y_1(s)ds\right] \leq K_3(1+|\Upsilon(\varsigma)|^2), \tag{12}
$$

$$
\Upsilon^\top(\varsigma)\varphi(\varsigma, \Upsilon(\varsigma)) + \frac{p-1}{2}|\psi(\varsigma, \Upsilon(\varsigma))|^2 \leq K_4(1+|\Upsilon(\varsigma)|^2). \tag{13}
$$

Then we obtain the p-th moment estimation formula

$$
\mathbb{E}|\Upsilon(\varsigma)|^p \leq 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon(0)|^p)e^{\tilde{C}\varsigma},
$$

where

$$
\tilde{C} = \left[p(K_3+K_4) + p\sqrt{3M_1\frac{\mathcal{T}^{2\alpha-1}}{2\alpha-1}}(K_1+K_2)\right].
$$

Proof. By Itô's formula of Caputo version and Equation (12), we can derive that

$$\begin{aligned}
(1+|\Upsilon(\varsigma)|^2)^{\frac{p}{2}} &= (1+|\Upsilon(0)|^2)^{\frac{p}{2}} + p\int_0^\varsigma (1+|\Upsilon|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(s+\upsilon)\Psi(-\upsilon)ds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\int_{-\upsilon}^0 \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)[({}^C D^\alpha_{-\upsilon^+}\Psi)(u)-\mathcal{A}\Psi(u))]duds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))duds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\varphi(s,\Upsilon(s))ds \\
&\quad + \frac{p}{2}\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\left(\frac{(p-2)|\Upsilon^\top(s)\psi(s,\Upsilon(s))|^2}{1+|\Upsilon(s)|^2}+|\psi(s,\Upsilon(s))|^2\right)ds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)ds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\psi(s,\Upsilon(s))dw(s) \\
&\le (1+|\Upsilon(0)|^2)^{\frac{p}{2}} + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\Big[\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(s+\upsilon)\Psi(-\upsilon) \\
&\quad + \int_{-\upsilon}^0 \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)y_1(u)du\Big]ds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\left[\Upsilon^\top(s)\varphi(s,\Upsilon(s))+\frac{p-1}{2}|\psi(s,\Upsilon(s))|^2\right]ds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))duds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)ds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\psi(s,\Upsilon(s))dw(s) \\
&\le (1+|\Upsilon(0)|^2)^{\frac{p}{2}} + pK_3\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds + pK_4\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))duds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)ds \\
&\quad + p\int_0^\varsigma (1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\psi(s,\Upsilon(s))dw(s).
\end{aligned}$$

Set $G(s) = \int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))du$, $\tilde{G}(s) = \int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)$. According to Assumptions 1 and 2 and Equation (6), it can be obtained that

$$\begin{aligned}
\mathbb{E}|G(s)|^2 &\le \mathbb{E}\left|\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))du\right|^2 \le K_1^2 M_1\int_0^s (s-u)^{2\alpha-2}du \\
&:= 3K_1^2 M_1\frac{s^{2\alpha-1}}{2\alpha-1}. \\
\mathbb{E}|\tilde{G}(s)|^2 &\le \mathbb{E}\left|\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)\right|^2 \le K_2^2 M_1\int_0^s (s-u)^{2\alpha-2}du \\
&:= 3K_2^2 M_1\frac{s^{2\alpha-1}}{2\alpha-1}.
\end{aligned}$$

Therefore, we can get

$$
\begin{aligned}
\mathbb{E}((1+|\Upsilon(\varsigma)|^2)^{\frac{p}{2}}) \leq\ & 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon(0)|^p)+p(K_3+K_4)\int_0^{\varsigma}\mathbb{E}(1+|\Upsilon|^2)^{\frac{p}{2}}ds \\
& +\mathbb{E}\left(p\int_0^{\varsigma}(1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))duds\right) \\
& +\mathbb{E}\left(p\int_0^{\varsigma}(1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)ds\right) \\
& +\mathbb{E}\left(p\int_0^{\varsigma}(1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\psi(s,\Upsilon(s))dw(s)\right) \\
\leq\ & 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon(0)|^p)+p(K_3+K_4)\int_0^{\varsigma}\mathbb{E}(1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds \\
& +\mathbb{E}\left(p\int_0^{\varsigma}(1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)G(s)ds\right) \\
& +\mathbb{E}\left(p\int_0^{\varsigma}(1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}\Upsilon^\top(s)\tilde{G}(s)ds\right) \\
\leq\ & 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon(0)|^p)+p(K_3+K_4)\int_0^{\varsigma}\mathbb{E}(1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds \\
& +p\int_0^{\varsigma}\sqrt{\mathbb{E}|(1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}|\Upsilon^\top(s)|^2E|G(s)|^2}ds \\
& +p\int_0^{\varsigma}\sqrt{\mathbb{E}|(1+|\Upsilon(s)|^2)^{\frac{p}{2}-1}|\Upsilon^\top(s)|^2E|\tilde{G}(s)|^2}ds \\
\leq\ & 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon(0)|^p)+p(K_3+K_4)\int_0^{\varsigma}\mathbb{E}(1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds \\
& +p\int_0^{\varsigma}\sqrt{\mathbb{E}((1+|\Upsilon(s)|^2)^{p-2}|\Upsilon^\top(s)|^2)3K_1^2M_1\frac{s^{2\alpha-1}}{2\alpha-1}}ds \\
& +p\int_0^{\varsigma}\sqrt{\mathbb{E}((1+|\Upsilon(s)|^2)^{p-2}|\Upsilon^\top(s)|^2)3K_2^2M_1\frac{s^{2\alpha-1}}{2\alpha-1}}ds \\
\leq\ & 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon|^p)+p(K_3+K_4)\int_0^{\varsigma}\mathbb{E}(1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds \\
& +pK_1\sqrt{3M_1\frac{\mathcal{T}^{2\alpha-1}}{2\alpha-1}}\int_0^{\varsigma}\mathbb{E}(1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds \\
& +pK_2\sqrt{3M_1\frac{\mathcal{T}^{2\alpha-1}}{2\alpha-1}}\int_0^{\varsigma}\mathbb{E}(1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds \\
\leq\ & 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon(0)|^p)+\Bigg[p(K_3+K_4) \\
& +p\sqrt{3M_1\frac{s^{2\alpha-1}}{2\alpha-1}}(K_1+K_2)\Bigg]\int_0^{\varsigma}\mathbb{E}(1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds \\
\leq\ & 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon(0)|^p)+\tilde{C}\int_0^{\varsigma}\mathbb{E}(1+|\Upsilon(s)|^2)^{\frac{p}{2}}ds.
\end{aligned}
$$

By virtue of Gronwall's inequality, it can be obtained that

$$\mathbb{E}((1+|\Upsilon(\varsigma)|^2)^{\frac{p}{2}}) \leq 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon(0)|^p)e^{\tilde{C}\varsigma}.$$

Thus, we obtain the p-th moment estimation formula

$$\mathbb{E}|\Upsilon(\varsigma)|^p \leq 2^{\frac{p}{2}-1}(1+\mathbb{E}|\Upsilon(0)|^p)e^{\tilde{C}\varsigma}.$$

The proof is now complete. □

Theorem 4. *Let $p \geq 2$, if the following conditions hold*

$$|\varphi(\varsigma, \Upsilon(\varsigma))|^p \vee \psi(\varsigma, \Upsilon(\varsigma))|^p \leq K_5(1 + |\Upsilon(\varsigma)|^p). \tag{14}$$

where K_5 a positive constant. Then there is a p-th moment estimation formula

$$\mathbb{E}(\sup_{0\leq\varsigma\leq\mathcal{T}} |\Upsilon(\varsigma)|^p) \leq (1 + C_1)e^{C_2\varsigma},$$

where

$$\begin{aligned}
C_1 &= 4^{p-1}\left[|\Psi(-\upsilon)|^p\Big(E_{\alpha,1}(\eta(\mathcal{T}+\upsilon)^\alpha)\Big)^p + \frac{W_1(\mathcal{T}+\upsilon)^{p\alpha-1}}{p\alpha-1}\Big(E_{\alpha,\alpha}(\eta(\mathcal{T}+h)^\alpha)\Big)^p\right],\\
C_2 &= 4^{p-1}K_5\Big(E_{\alpha,\alpha}(\eta\mathcal{T}^\alpha)\Big)^p\left[\left(\frac{\mathcal{T}^{\frac{p\alpha}{p-1}+1}}{\frac{p\alpha}{p-1}+1}\right)^{p-1} + C_p\left(\frac{\mathcal{T}^{\frac{2\alpha p}{p-2}+1}}{\frac{2\alpha p}{p-2}+1}\right)^{\frac{p-2}{2}}\right],\\
W_1 &= \mathbb{E}\left(\int_{-\upsilon}^0 |y_1(u)|^{\frac{p}{p-1}}ds\right)^{p-1} < \infty,\\
C_p &= \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}}.
\end{aligned}$$

Proof. By using the inequality $|a+b+c+d|^p \leq 4^{p-1}|a|^p + 4^{p-1}|b|^p + 4^{p-1}|c|^p + 4^{p-1}|d|^p$ for any $p > 1$, we can obtain the following result:

$$\begin{aligned}
\mathbb{E}(\sup_{0\leq t\leq\mathcal{T}}|\Upsilon(\varsigma)|^p) \leq\ & 4^{p-1}\mathbb{E}|\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(\varsigma+\upsilon)\Psi(-\upsilon)|^p\\
&+4^{p-1}\mathbb{E}\left|\int_{-\upsilon}^0 \mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)[({}^CD^\alpha_{-\upsilon^+}\Psi)(s) - \mathcal{A}\Psi(s)]ds\right|^p\\
&+4^{p-1}\mathbb{E}\left|\int_0^\varsigma \mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)\varphi(s,\Upsilon(s))ds\right|^p\\
&+4^{p-1}\mathbb{E}\left(\sup_{0\leq\varsigma\leq\mathcal{T}}\left|\int_0^\varsigma \mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\upsilon-s)\psi(s,\Upsilon(s))dw(s)\right|^p\right)\\
:=\ & \Omega_1+\Omega_2+\Omega_3+\Omega_4.
\end{aligned}$$

By (5) and Hölder's inequality, we have

$$\begin{aligned}
\Omega_1 &= 4^{p-1}\mathbb{E}|\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(\varsigma+\upsilon)\Psi(-\upsilon)|^p\\
&\leq 4^{p-1}|\Psi(-\upsilon)|^p\Big(E_{\alpha,1}(\eta(\mathcal{T}+\upsilon)^\alpha)\Big)^p,\\
\Omega_2 &= 4^{p-1}\mathbb{E}\left|\int_{-\upsilon}^0 \mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)[({}^CD^\alpha_{-\upsilon^+}\Psi)(s)-\mathcal{A}\Psi(s)]ds\right|^p\\
&\leq 4^{p-1}\mathbb{E}\left(\int_{-\upsilon}^0\left|\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)[({}^CD^\alpha_{-\upsilon^+}\Psi)(s)-\mathcal{A}\Psi(s)]\right|ds\right)^p\\
&\leq 4^{p-1}\Big(E_{\alpha,\alpha}(\eta(\mathcal{T}+\upsilon)^\alpha)\Big)^p\mathbb{E}\left(\int_{-\upsilon}^0|(\varsigma-s)^{\alpha-1}y_1(s)|ds\right)^p\\
&\leq 4^{p-1}\Big(E_{\alpha,\alpha}(\eta(\mathcal{T}+\upsilon)^\alpha)\Big)^p\left(\int_{-\upsilon}^0|\varsigma-s|^{p(\alpha-1)}ds\right)^{\frac{1}{p}\times p}\mathbb{E}\left(\int_{-\upsilon}^0|y_1(u)|^{\frac{p}{p-1}}ds\right)^{\frac{p-1}{p}\times p}\\
&\leq 4^{p-1}W_1\frac{(\mathcal{T}+\upsilon)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\Big(E_{\alpha,\alpha}(\eta(\mathcal{T}+\upsilon)^\alpha)\Big)^p,
\end{aligned}$$

and

$$\begin{aligned}
\Omega_3 &= 4^{p-1}\mathbb{E}\bigg|\int_0^{\varsigma}\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)\varphi(s,\Upsilon(s))ds\bigg|^p\\
&\leq 4^{p-1}\mathbb{E}\bigg(\int_0^{\varsigma}\bigg|\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)\varphi(s,\Upsilon(s))\bigg|ds\bigg)^p\\
&\leq 4^{p-1}\bigg(E_{\alpha,\alpha}(\eta\mathcal{T}^{\alpha})\bigg)^p\bigg(\int_0^{\varsigma}(\varsigma-s)^{\frac{p\alpha}{p-1}}ds\bigg)^{p-1}\mathbb{E}\bigg(\int_0^{\varsigma}|\varphi(s,\Upsilon(s))|^p ds\bigg)\\
&\leq 4^{p-1}K_5\bigg(E_{\alpha,\alpha}(\eta\mathcal{T}^{\alpha})\bigg)^p\bigg(\frac{\mathcal{T}^{\frac{p\alpha}{p-1}+1}}{\frac{p\alpha}{p-1}+1}\bigg)^{p-1}\int_0^{\varsigma}(1+\mathbb{E}(\sup_{0\leq s\leq\varsigma}|\Upsilon(s)|^p)ds.
\end{aligned}$$

By applying the Burkholder-Davis-Gundy (BDG) inequality and Hölder's inequality, we can obtain

$$\begin{aligned}
\Omega_4 &= 4^{p-1}\mathbb{E}\bigg(\sup_{0\leq\varsigma\leq\mathcal{T}}\bigg|\int_0^{\varsigma}\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)\psi(s,\Upsilon(s))dw(s)\bigg|^p\bigg)\\
&\leq 4^{p-1}C_p\mathbb{E}\bigg(\int_0^{\varsigma}\bigg|\mathfrak{X}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-s)\psi(s,\Upsilon(s))\bigg|^2ds\bigg)^{\frac{p}{2}}\\
&\leq 4^{p-1}C_p\bigg(E_{\alpha,\alpha}(\eta\mathcal{T}^{\alpha})\bigg)^p\bigg(\int_0^{\varsigma}|(\varsigma-s)^{\alpha-1}\psi(s,\Upsilon(s))|^2ds\bigg)^{\frac{p}{2}}\\
&\leq 4^{p-1}C_p\bigg(E_{\alpha,\alpha}(\eta\mathcal{T}^{\alpha})\bigg)^p\bigg(\int_0^{\varsigma}|\tau-s|^{2\alpha\frac{p}{p-2}}ds\bigg)^{\frac{p}{2}\times\frac{p-2}{p}}\mathbb{E}\bigg(\int_0^{\varsigma}|\psi(s,\Upsilon(s))|^{2\frac{p}{2}}ds\bigg)^{\frac{p}{2}\times\frac{2}{p}}\\
&\leq 4^{p-1}C_p\bigg(E_{\alpha,\alpha}(\eta\mathcal{T}^{\alpha})\bigg)^p\bigg(\int_0^{\varsigma}|\varsigma-s|^{\frac{2\alpha p}{p-2}}ds\bigg)^{\frac{p-2}{2}}\mathbb{E}\bigg(\int_0^{\varsigma}|\psi(s,\Upsilon(s))|^p ds\bigg)\\
&\leq 4^{p-1}K_5C_p\bigg(E_{\alpha,\alpha}(\eta\mathcal{T}^{\alpha})\bigg)^p\bigg(\frac{\mathcal{T}^{\frac{2\alpha p}{p-2}+1}}{\frac{2\alpha p}{p-2}+1}\bigg)^{\frac{p-2}{2}}\int_0^{\varsigma}(1+\mathbb{E}(\sup_{0\leq s\leq\varsigma}|\Upsilon(s)|^p))ds,
\end{aligned}$$

then

$$\begin{aligned}
&\mathbb{E}(\sup_{0\leq t\leq b}|\Upsilon(t)|^p)\\
\leq\;& 4^{p-1}|\Psi(-\upsilon)|^p\bigg(E_{\alpha,1}(\eta(\mathcal{T}+\upsilon)^{\alpha})\bigg)^p+4^{p-1}W_1\frac{(\mathcal{T}+\upsilon)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\bigg(E_{\alpha,\alpha}(\eta(\mathcal{T}+\upsilon)^{\alpha})\bigg)^p\\
&+4^{p-1}K_5\bigg(E_{\alpha,\alpha}(\eta\mathcal{T}^{\alpha})\bigg)^p\bigg(\frac{\mathcal{T}^{\frac{p\alpha}{p-1}+1}}{\frac{p\alpha}{p-1}+1}\bigg)^{p-1}\int_0^{\varsigma}(1+\mathbb{E}(\sup_{0\leq s\leq\varsigma}|\Upsilon(s)|^p))ds\\
&+4^{p-1}K_5C_p\bigg(E_{\alpha,\alpha}(\eta\mathcal{T}^{\alpha})\bigg)^p\bigg(\frac{\mathcal{T}^{\frac{2\alpha p}{p-2}+1}}{\frac{2\alpha p}{p-2}+1}\bigg)^{\frac{p-2}{2}}\int_0^{\varsigma}(1+\mathbb{E}(\sup_{0\leq s\leq\varsigma}|\Upsilon(s)|^p))ds\\
:=\;& 4^{p-1}\bigg[|\Psi(-\upsilon)|^p\bigg(E_{\alpha,1}(\eta(\mathcal{T}+\upsilon)^{\alpha})\bigg)^p+W_1\frac{(\mathcal{T}+\upsilon)^{p\alpha-1}}{p\alpha-1}\bigg(E_{\alpha,\alpha}(\eta(\mathcal{T}+\upsilon)^{\alpha})\bigg)^p\bigg]\\
&+4^{p-1}K_5\bigg(E_{\alpha,\alpha}(\eta\mathcal{T}^{\alpha})\bigg)^p\bigg[\bigg(\frac{\mathcal{T}^{\frac{p\alpha}{p-1}+1}}{\frac{p\alpha}{p-1}+1}\bigg)^{p-1}+C_p\bigg(\frac{\mathcal{T}^{\frac{2\alpha p}{p-2}+1}}{\frac{2\alpha p}{p-2}+1}\bigg)^{\frac{p-2}{2}}\bigg]\int_0^{\varsigma}(1+\mathbb{E}(\sup_{0\leq s\leq\varsigma}|\Upsilon(s)|^p))ds.
\end{aligned}$$

Based on

$$\begin{aligned}
1+\mathbb{E}(\sup_{0\leq\varsigma\leq\mathcal{T}}|\Upsilon(\varsigma)|^p) &\leq 1+C_1+C_2\int_0^{\varsigma}(1+\mathbb{E}(\sup_{0\leq s\leq\varsigma}|\Upsilon(s)|^p)ds\\
&\leq (1+C_1)e^{C_2\varsigma}.
\end{aligned}$$

Finally, we obtain the p-th moment estimation formula

$$\mathbb{E}(\sup_{0\leq\varsigma\leq\mathcal{T}}|\Upsilon(\varsigma)|^p)\leq(1+C_1)e^{C_2\varsigma}.$$

The proof is now complete. □

4.2. p-th Exponential Stability

In this section, we will establish the exponential stability theorem for the solutions of system (2). We define an operator $\mathfrak{L}V$ from $\mathcal{R}^n \times \mathcal{R}^+$ to $\mathcal{R}$ by

$$\begin{aligned} \mathfrak{L}V &= V_\varsigma(\varsigma,\Upsilon(\varsigma))(\varsigma,\Upsilon(\varsigma)) + V_\Upsilon(\varsigma,\Upsilon(\varsigma))\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(\varsigma+\upsilon)\Psi(-\upsilon) \\ &\quad + V_\Upsilon(\varsigma,\Upsilon(\varsigma))\int_{-\upsilon}^{0}\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-\upsilon)[({}^C D^\alpha_{-\upsilon^+}\Psi)(s)-\mathcal{A}\Psi(s))]ds \\ &\quad + V_\Upsilon(\varsigma,\Upsilon(\varsigma))\varphi(\varsigma,\Upsilon(\varsigma)) \\ &\quad + \frac{1}{2}\mathrm{trace}[\psi^\top(s,\Upsilon(s))V_{\Upsilon\Upsilon}(\varsigma,\Upsilon(\varsigma))\psi(s,\Upsilon(s))]. \end{aligned} \tag{15}$$

Theorem 5. *Assume Assumptions 1 and 2 hold and fot the function $V \in \mathcal{C}^{1,2}(\mathcal{R}^n \times \mathcal{R}^+, \mathcal{R})$ exists four positive constants $r_i (i = 1, 2, 3)$ and K_6, for all $\Upsilon \in \mathcal{R}^n$ and $\varsigma \in \mathcal{R}$ such that*

(i) $r_1|\Upsilon(\varsigma)|^p \le V(\varsigma,\Upsilon(\varsigma)) \le r_2|\Upsilon(\varsigma)|^p$,

(ii) $\mathfrak{L}V(\varsigma,\Upsilon(\varsigma)) \le -r_3 V(\varsigma,\Upsilon(\varsigma))$,

(iii) $|V_\Upsilon(\varsigma,\Upsilon(\varsigma))| \le K_6(1+|\Upsilon(\varsigma)|^p)$,

(iv) $2K_6\sqrt{3M_1\frac{\mathcal{T}^{2\alpha-1}}{2\alpha-1}}(K_1+K_2) < r_3$.

Then, the trivial solution of system (2) is p-th moment exponentially stable.

Proof. For each sufficiently large integer $n > |\Psi(0)|$, define the stopping time sequence $\tau_n = \inf\{\varsigma > 0 : |\Upsilon(\varsigma)| \ge n\}$. This localization procedure is introduced to ensure that the stochastic integrals involved in the subsequent Itô's formula are true martingales with zero expectation. Then, for the stopped process at $\varsigma \wedge \tau_n$, it follows from the Itô's formula of Caputo version that

$$\begin{aligned} & e^{r_3(\varsigma \wedge \tau_n)}V(\varsigma\wedge\tau_n,\Upsilon(\varsigma\wedge\tau_n)) \\ = \; & V(0,\Upsilon(0)) + \int_0^{\varsigma\wedge\tau_n} r_3 e^{r_3 s}V(s,\Upsilon(s))ds + \int_0^{\varsigma\wedge\tau_n} e^{r_3 s}V_s(s,\Upsilon(s))ds \\ & + \int_0^{\varsigma\wedge\tau_n} e^{r_3 s}V_\Upsilon(s,\Upsilon(s))\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(s+\upsilon)\Psi(-\upsilon)ds \\ & + \int_0^{\varsigma\wedge\tau_n} e^{r_3 s}V_\Upsilon(s,\Upsilon(s))\int_{-\upsilon}^{0}\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)[({}^C D^\alpha_{-\upsilon^+}\Psi)(u)-\mathcal{A}\Psi(u))]duds \\ & \quad + \int_0^{\varsigma\wedge\tau_n} e^{r_3 s}V_\Upsilon(s,\Upsilon(s))\varphi(s,\Upsilon(s))ds \\ & \quad + \int_0^{\varsigma\wedge\tau_n} e^{r_3 s}V_\Upsilon(s,\Upsilon(s))\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))duds \\ & \quad + \int_0^{\varsigma\wedge\tau_n} e^{r_3 s}V_\Upsilon(s,\Upsilon(s))\psi(s,\Upsilon(s))dw(s) \\ & \quad + \int_0^{\varsigma\wedge\tau_n} e^{r_3 s}V_\Upsilon(s,\Upsilon(s))\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)ds \\ & \quad + \frac{1}{2}\int_0^{\varsigma\wedge\tau_n} e^{r_3 s}\mathrm{trace}[\psi^\top(s,\Upsilon(s))V_{\Upsilon\Upsilon}(s,\Upsilon(s))\psi(s,\Upsilon(s))]ds. \end{aligned}$$

Obviously $\int_0^{\varsigma\wedge\tau_n} e^{r_3 s}V_\Upsilon(s,\Upsilon(s))\psi(s,\Upsilon(s))dw(s)$ is martingale. According to (15) and Conditions (*i*), we get

$$
\begin{aligned}
&\mathbb{E}\bigg(e^{r_3(\varsigma\wedge\tau_n)}V(\varsigma\wedge\tau_n,\Upsilon(\varsigma\wedge\tau_n))\bigg)\\
= \;&\mathbb{E}V(0,\Upsilon(0))+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} r_3e^{r_3s}V(s,\Upsilon(s))ds+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_s(s,\Upsilon(s))ds\\
&+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_\Upsilon(s,\Upsilon(s))\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(s+\upsilon)\Psi(-\upsilon)ds\\
&+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_\Upsilon(s,\Upsilon(s))\int_{-\upsilon}^0 \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-\upsilon)[({}^CD^\alpha_{-\upsilon^+}\Psi)(u)-\mathcal{A}\Psi(u))]duds\\
&+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_\Upsilon(s,\Upsilon(s))\varphi(s,\Upsilon(s))ds\\
&+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_\Upsilon(s,\Upsilon(s))\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))duds\\
&+\frac{1}{2}\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}\text{trace}[\psi^\top(s,\Upsilon(s))V_{\Upsilon\Upsilon}(s,\Upsilon(s))\psi(s,\Upsilon(s))]ds\\
&+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_\Upsilon(s,\Upsilon(s))\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)ds\\
= \;&\mathbb{E}V(0,\Upsilon(0))+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}\bigg[r_3V(s,\Upsilon(s))+\mathfrak{L}V(s,\Upsilon(s))\bigg]ds\\
&+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_\Upsilon(s,\Upsilon(s))\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))duds\\
&+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_\Upsilon(s,\Upsilon(s))\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)ds\\
\leq \;&r_2\mathbb{E}|Y(0)|^p+\mathbb{E}\int_0^{t\wedge\tau_n} e^{r_3s}[r_3V(s,\Upsilon(s))-r_3V(s,\Upsilon(s))]ds\\
&+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_\Upsilon(s,\Upsilon(s))\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))duds\\
&+\mathbb{E}\int_0^{\varsigma\wedge\tau_n} e^{r_3s}V_\Upsilon(s,\Upsilon(s))\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)ds.
\end{aligned}
$$

Set $H(s)=\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))du,\ \tilde{H}(s)=\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)$. According to Assumptions 1 and 2, it can be obtained that

$$
\begin{aligned}
\mathbb{E}|H(s)|^2 \;&\leq\; \mathbb{E}\bigg|\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\varphi(u,\Upsilon(u))du\bigg|^2\\
&\leq\; K_1^2M_1\int_0^s (s-u)^{2\alpha-2}du\leq 3K_1^2M_1\frac{s^{2\alpha-1}}{2\alpha-1}.\\
\mathbb{E}|\tilde{H}(s)|^2 \;&\leq\; \mathbb{E}\bigg|\int_0^s \mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(s-u)\psi(u,\Upsilon(u))dw(u)\bigg|^2\\
&\leq\; K_2^2M_1\int_0^s (s-u)^{2\alpha-2}du\leq 3K_2^2M_1\frac{s^{2\alpha-1}}{2\alpha-1}.
\end{aligned}
$$

Next, based on the above formula we can get

$$
\begin{aligned}
&\mathbb{E}\Big(e^{r_3(\varsigma \wedge \tau_n)}V(\varsigma\wedge\tau_n,\Upsilon(\varsigma\wedge\tau_n))\Big)\\
\leq\;& r_2\mathbb{E}|\Upsilon(0)|^p+\int_0^{\varsigma\wedge\tau_n}e^{r_3 s}\mathbb{E}(V_\Upsilon(s,\Upsilon(s))H(s))ds\\
&+\int_0^{\varsigma\wedge\tau_n}e^{r_3 s}\mathbb{E}(V_\Upsilon(s,\Upsilon(s))\tilde{H}(s))ds\\
\leq\;& r_2\mathbb{E}|\Upsilon(0)|^p+\int_0^{\varsigma\wedge\tau_n}e^{r_3 s}\sqrt{\mathbb{E}|V_\Upsilon(s,\Upsilon(s))|^2\mathbb{E}|H(s)|^2}ds\\
&+\int_0^{\varsigma\wedge\tau_n}e^{r_3 s}\sqrt{\mathbb{E}|V_\Upsilon(s,\Upsilon(s))|^2\mathbb{E}|\tilde{H}(s)|^2}ds\\
\leq\;& r_2\mathbb{E}|\Upsilon(0)|^p+\int_0^{\varsigma\wedge\tau_n}e^{r_3 s}\sqrt{2K_6^2(1+\mathbb{E}|\Upsilon(s)|^{2p})3K_1^2M_1\frac{s^{2\alpha-1}}{2\alpha-1}}ds\\
&+\int_0^{\varsigma\wedge\tau_n}e^{r_3 s}\sqrt{2K_6^2(1+\mathbb{E}|\Upsilon(s)|^{2p})3K_2^2M_1\frac{s^{2\alpha-1}}{2\alpha-1}}ds\\
\leq\;& r_2\mathbb{E}|\Upsilon(0)|^p+2K_6(K_1+K_2)\sqrt{3M_1\frac{\mathcal{T}^{2\alpha-1}}{2\alpha-1}}\int_0^{\varsigma\wedge\tau_n}e^{r_3 s}\mathbb{E}|\Upsilon(s)|^p ds.
\end{aligned}
$$

According to the a priori estimation established in Theorem 3, the solution of the system does not explode in finite time, which guarantees that $\tau_n\to\infty$ almost surely as $n\to\infty$. Thus, $\varsigma\wedge\tau_n\to\varsigma$ almost surely. Therefore, we have

$$
\liminf_{n\to\infty}\mathbb{E}\Big[e^{r_3(\varsigma\wedge\tau_n)}V(\varsigma\wedge\tau_n,\Upsilon(\varsigma\wedge\tau_n))\Big]\geq\mathbb{E}\Big(e^{r_3\varsigma}V(\varsigma,\Upsilon(\varsigma))\Big).
$$

Simultaneously, using the condition $r_1|\Upsilon(\varsigma)|^p\leq V(\varsigma,\Upsilon(\varsigma))$, we can obtain

$$
\begin{aligned}
r_1e^{r_3\varsigma}\mathbb{E}|\Upsilon(\varsigma)|^p\;\leq\;& \mathbb{E}(e^{r_3\varsigma}V(\varsigma,\Upsilon(\varsigma)))\\
\leq\;& r_2\mathbb{E}|\Upsilon(0)|^p+2K_6(K_1+K_2)\sqrt{3M_1\frac{\mathcal{T}2\alpha-1}{2\alpha-1}}\int_0^{\varsigma}e^{r_3 s}\mathbb{E}|\Upsilon(s)|^p ds\\
\leq\;& r_2\mathbb{E}|\Upsilon(0)|^p e^{2K_6(K_1+K_2)\sqrt{3M_1\frac{\mathcal{T}^{2\alpha-1}}{2\alpha-1}}\varsigma}.
\end{aligned}
$$

Then

$$
\mathbb{E}|\Upsilon(\varsigma)|^p\;\leq\;\frac{r_2}{r_1}\mathbb{E}|\Upsilon(0)|^p e^{[2K_6(K_1+K_2)\sqrt{3M_1\frac{\mathcal{T}^{2\alpha-1}}{2\alpha-1}}-r_3]\varsigma}.
$$

□

5. Examples

Example 1. *Set $\alpha=0.6,\ \upsilon=0.5$ and $\mathcal{T}=1$. We consider*

$$
\begin{cases}
({}^CD_{0^+}^{0.6}\Upsilon)(\varsigma)=\mathcal{A}\Upsilon(\varsigma)+\mathcal{B}\Upsilon(\varsigma-\upsilon)+\varphi(\varsigma,\Upsilon(\varsigma))+\psi(\varsigma,\Upsilon(\varsigma))\dfrac{dw(\varsigma)}{d\varsigma},\ \varsigma\in[0,1]\\
\Upsilon(\varsigma)=\Psi(\varsigma),\ \varsigma\in[-0.5,0].
\end{cases}
\tag{16}
$$

Where

$$
\mathcal{A}=\begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix},\ \mathcal{B}=\begin{pmatrix}0.03 & 0\\ 0 & 0.03\end{pmatrix},\ \Psi(\varsigma)=\begin{pmatrix}\varsigma+1\\ 0.3\varsigma+1\end{pmatrix},
$$

$$
\varphi(\varsigma,\Upsilon(\varsigma))=\begin{pmatrix}-0.25\varsigma\tanh(\Upsilon_1(\varsigma))\\ -0.25\varsigma\tanh(\Upsilon_2(\varsigma))\end{pmatrix},\ \psi(\varsigma,\Upsilon(\varsigma))=\begin{pmatrix}0.05\varsigma\arctan(\Upsilon_1(\varsigma))\\ 0.05\varsigma\arctan(\Upsilon_2(\varsigma))\end{pmatrix}.
$$

According to Assumption 2, we can obtain $\|\varphi(\varsigma,\Upsilon(\varsigma))\| \leq 0.354$, $\|\psi(\varsigma,\Upsilon(\varsigma))\| \leq 0.111$. From (12), we can derive

$$\begin{aligned}
& \Upsilon^\top(\varsigma)\left[\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{h,\alpha,a}(\varsigma+h)\Psi(-h)+\int_{-h}^{0}\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{h,\alpha,\alpha}(\varsigma-s)y_1(s)ds\right] \\
\leq\ & \Upsilon^\top(\varsigma)\left[\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{h,\alpha,a}(\mathcal{T}+h)\Psi(-h)+\int_{-h}^{0}\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{h,\alpha,\alpha}(\mathcal{T}-s)y_1(s)ds\right] \\
=\ & 0.415\Upsilon_1(\varsigma)+0.460\Upsilon_2(\varsigma) \\
\leq\ & 0.44(1+\|\Upsilon(\varsigma)\|^2),
\end{aligned}$$

and

$$\begin{aligned}
& \Upsilon^\top(\varsigma)\varphi(\varsigma,\Upsilon(\varsigma))+\frac{p-1}{2}\|\psi(\varsigma,\Upsilon(\varsigma))\|^2 \\
\leq\ & -0.25[\Upsilon_1(\varsigma)\tanh(\Upsilon_1(\varsigma))+\Upsilon_2(\varsigma)\tanh(\Upsilon_2(\varsigma))] \\
& +0.003[\arctan^2(\Upsilon_1(\varsigma))+\arctan^2(\Upsilon_2(\varsigma))] \\
\leq\ & 0.183(1+\|\Upsilon(\varsigma)\|^2).
\end{aligned}$$

Thus far, all the assumptions of Theorem 3 are satisfied, hence the solution of system (17) satisfies the p-th moment estimation formula.

Example 2. *Set $\alpha = 0.74,\ \upsilon = 0.4$ and $\mathcal{T} = 1$. We consider*

$$\begin{cases}({}^C D^{0.74}_{0^+}\Upsilon)(\varsigma) = \mathcal{A}\Upsilon(\varsigma)+\mathcal{B}\Upsilon(\varsigma-\upsilon)+\varphi(\varsigma,\Upsilon(\varsigma))+\psi(\varsigma,\Upsilon(\varsigma))\dfrac{dw(\varsigma)}{d\varsigma},\ \varsigma\in[0,1] \\ \Upsilon(\varsigma)=\Psi(\varsigma),\ \varsigma\in[-0.4,0].\end{cases} \tag{17}$$

Where

$$\mathcal{A}=\begin{pmatrix}-0.6 & 0\\ 0 & -0.6\end{pmatrix},\ \mathcal{B}=\begin{pmatrix}0.02 & 0\\ 0 & 0.02\end{pmatrix},\ \Psi(\varsigma)=\begin{pmatrix}0.5\varsigma-1\\ 0.3\varsigma-0.5\end{pmatrix},$$

$$\varphi(\varsigma,\Upsilon(\varsigma))=\begin{pmatrix}-0.2e^{-\varsigma}\sin(\Upsilon_1(\varsigma))\\ -0.2e^{-\varsigma}\sin(\Upsilon_2(\varsigma))\end{pmatrix},\ \psi(\varsigma,\Upsilon(\varsigma))=\begin{pmatrix}-0.04\varsigma\sin(\Upsilon_1(\varsigma))\\ -0.04\varsigma\sin(\Upsilon_2(\varsigma))\end{pmatrix}.$$

Take $V(\varsigma,\Upsilon(\varsigma))=(1+e^{-\varsigma^2})|\Upsilon(\varsigma)|^2$, where $|\Upsilon(\varsigma)|^2=\Upsilon_1^2(\varsigma)+\Upsilon_2^2(\varsigma)$ It is clear that $|\Upsilon(\varsigma)|^2 \leq V(\varsigma,\Upsilon(\varsigma)) \leq 2|\Upsilon(\varsigma)|^2$, and $r_1=1,\ r_2=2,\ p=2$. $V_\Upsilon(\varsigma,\Upsilon(\varsigma))=2(1+e^{-\varsigma^2})(\Upsilon_1(\varsigma),\Upsilon_2(\varsigma))$, $|V_\Upsilon(\varsigma,\Upsilon(\varsigma))|\leq 2(1+|\Upsilon(\varsigma)|^2)$, $V_{\Upsilon\Upsilon}(\varsigma,\Upsilon(\varsigma))=2(1+e^{-\varsigma^2})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

From (15), we can derive

$$\begin{aligned}
\mathfrak{L}V =\ & V_\varsigma(\varsigma,\Upsilon(\varsigma))(\varsigma,\Upsilon(\varsigma))+V_\Upsilon(\varsigma,\Upsilon(\varsigma))\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,1}(\varsigma+\upsilon)\Psi(-\upsilon) \\
& +V_\Upsilon(\varsigma,\Upsilon(\varsigma))\int_{-\upsilon}^{0}\mathcal{Z}^{\mathcal{A},\mathcal{B}}_{\upsilon,\alpha,\alpha}(\varsigma-\upsilon)[({}^C D^{\alpha}_{-\upsilon^+}\Psi)(s)-\mathcal{A}\Psi(s))]ds \\
& +V_\Upsilon(\varsigma,\Upsilon(\varsigma))\varphi(\varsigma,\Upsilon(\varsigma))+\frac{1}{2}trace[\psi^\top(s,\Upsilon(s))V_{\Upsilon\Upsilon}(\varsigma,\Upsilon(\varsigma))\psi(s,\Upsilon(s))] \\
\leq\ & -0.8705\Upsilon_1(\varsigma)-0.4499\Upsilon_2(\varsigma)-0.2487\Upsilon_1(\varsigma) \\
& -0.1348\Upsilon_2(\varsigma)-0.2006|\Upsilon(\varsigma)|^2+0.0021|\Upsilon(\varsigma)|^2 \\
\leq\ & -1.1192\Upsilon_1(\varsigma)-0.3151\Upsilon_2(\varsigma)-0.1985|\Upsilon(\varsigma)|^2 \\
\leq\ & -0.71(1+e^{\varsigma^2})|\Upsilon(\varsigma)|^2 \\
\leq\ & -0.71V(\varsigma,\Upsilon(\varsigma)).
\end{aligned}$$

Let $r_3=0.5$, then $\mathfrak{L}V(\varsigma,\Upsilon(\varsigma))\leq -r_3V(\varsigma,\Upsilon(\varsigma))$. Moreover, due to $|\varphi(\varsigma,\Upsilon(\varsigma))|\leq 0.0108,\ |\psi(\varsigma,\Upsilon(\varsigma))|\leq 0.0032$. Thus $2K_6(K_1+K_2)\sqrt{3M_1\frac{\mathcal{T}^{2\alpha-1}}{2\alpha-1}}-r_3 = 2\times 2\times(0.0108+0.0032)\times 6.33-0.71=-0.3555<0$. Hence, all the conditions of Theorem 5 are true. Therefore, we see that the trivial solution of system (2) is 2th moment exponentially stable.

6. Conclusions

In this paper, we investigate the p-th moment estimation and p-th moment exponential stability for a class of Caputo fractional stochastic delay differential equations. By constructing a novel Itô's formula adapted to the Mittag-Leffler type matrix polynomial function, we successfully overcome the mathematical difficulties arising from the fractional memory effect and the pure time delay. Utilizing stochastic analysis methods such as Hölder's inequality and the BDG inequality, we derive the p-th moment estimation formula for the system's solution. Furthermore, several sufficient conditions are established to verify the p-th moment exponential stability of the system's solution. This work provides new insights for stability research at the intersection of fractional calculus and stochastic delay systems. For future research directions, the current results could be extended to fractional stochastic systems with neutral-type delays or systems driven by more complex noise processes.

Author Contributions

Y.Z.: Designed the research plan and procedures, conducted the key content derivation, and drafted the manuscript. M. L.: Provided technical support and supervision, revised the manuscript, and finalized the paper. All authors have read and agreed to the published version of the manuscript.

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Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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