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New Results on Global Exponential Convergence of Discontinuous HCNNs with Time-Varying Leakage Delays

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Abstract: This paper presents a class of high-order cellular neural networks (HCNNs) with mixed discontinuous activations and time-varying leakage delays. To deal with the discontinuous property, the framework of Filippov solution is invoked to solve the inexistence of the classical solutions. Then combining with the functional differential inclusions theory and inequality technique, some new verifiable algebraic criteria are given to ensure that all solutions of the proposed neural network converge exponentially to the zero vector. The results obtained in this paper not only extend earlier works on HCNNs to the discontinuous case but also complement the previous researches on discontinuous neural networks since the mixed discontinuous activations have never been touched. Consequently, the results we established are more generalized. Finally, the effectiveness of the obtained results are illustrated via numerical examples and simulations.

Keywords: high-order cellular neural network; discontinuous activations; time-varying leakage delays; exponential convergence

1. Introduction

In this paper, we consider a general class of HCNNs with mixed discontinuous activations and time-varying leakage delays as follows:

$$\begin{aligned} x_i'(t) =& -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(x_j(t-\alpha_{ijl}(t)))g_l(x_l(t-\beta_{ijl}(t))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} d_{ijl}(t)\cdot\int_0^{+\infty}\sigma_{ijl}(u)g_j(x_j(t-u))du \\ &\int_0^{+\infty}\nu_{ijl}(u)g_l(x_l(t-u))du + I_i(t),\ i=1,2,...,n, \end{aligned} \tag{1}$$

with initial conditions The initial conditions associated with system (1) are of the form

$$x_i(t)=\varphi_i(t),\quad x_i'(t)=\varphi_i'(t),\quad t\in[-\rho,0],\quad i=1,2,...,n, \tag{2}$$

where $\varphi_i(\cdot)$ and $\varphi_i'(\cdot)$ are real-valued bounded and continuous functions defined on $(-\infty,0]$, n corresponds to the number of units in a neural network; $x_i(t)$ corresponds to the state vector of the ith unit at the time t; $c_i(t)$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $a_{ij}(t)$, $b_{ijl}(t)$, and $d_{ijl}(t)$ are the first-order and second-order connection weights of

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the neural network, respectively; $\eta_i(t) > 0$ corresponds to the time-varying leakage delay; $\alpha_{ijl}(t) \geq 0$, $\beta_{ijl}(t) \geq 0$, and $\tau_{ij}(t) \geq 0$ correspond to the transmission delays; $\sigma_{ijl}(u)$ and $\nu_{ijl}(u)$ correspond to the transmission delay kernels; $I_i(t)$ denotes the external inputs at time t; f_j and g_j are the activation functions of signal transmission, which are assumed to be discontinuous. Let $\rho = \max\{\eta_i^+, \tau_{ij}^+, \alpha_{ijl}^+, \beta_{ijl}^+\}$, where $\eta_i^+ = \max\limits_{1\leq i\leq n} \sup\limits_{t\in\mathbb{R}} |\eta_i(t)|$, $\tau_{ij}^+ = \max\limits_{1\leq i,j\leq n} \sup\limits_{t\in\mathbb{R}} |\tau_{ij}(t)|$, $\alpha_{ijl}^+ = \max\limits_{1\leq i,j,l\leq n} \sup\limits_{t\in\mathbb{R}} |\alpha_{ijl}(t)|$, $\beta_{ijl}^+ = \max\limits_{1\leq i,j,l\leq n} \sup\limits_{t\in\mathbb{R}} |\beta_{ijl}(t)|$.

Generally, system (1) is able to describe the high-order cellular neural networks (HCNNs) with time-varying leakage delays. HCNNs have been taken into so many considerations due to their widely practical applications in pattern recognition, associative memory, optimization problems and so on. For more details, we refer readers to [1]. In recent several years, many researchers have discussed the convergence of solutions for the HCNNs with delays in the leakage terms, see, to name a few, [2–6]. For example, Xu [3] considered the anti-periodic solutions for HCNNs with time-varying delays in the leakage terms by using differential inequality techniques and assuming the activation functions are Lipschitz continuous as well as bounded. Relaxing the assumption imposed on the time-varying leakage delay in [3], Xiong [4] further investigated the convergence for HCNNs with time-varying leakage delays by using differential inequality techniques and assuming the activation functions are Lipschitz continuous.

From [2–6] and the related references therein, we can see that the activations are Lipschitz continuous or bounded. Actually, because of the inevitable influences from the external environment, the discontinuous activation can always exist. Discontinuous activation can be used to describe the models more practically. In recent years, since the discontinuous activation has frequently been found in some practical applications, such as mechanics, automatic control and other natural sciences fields, neural network systems with discontinuous activations have attracted more and more attentions from researchers. See, to name a few, [7–17]. For example, Forti et al. [8] discussed the global convergence of the neural networks and assumed that the activations possess jump discontinuous points. Their work was regarded as the pioneer one in the stability analysis of the neural network systems with discontinuous activations. After that, Cai et al. [7] considered the periodic dynamics of a class of time-varying delayed neural networks via differential inclusions. Taking the influence of the neutral operator into account, Kong et al. [9] further studied the dynamic behavior of a class of neutral-type neural networks with discontinuous non-monotone activations and time-varying delays.

However, to the best of our knowledge, to date, only a few investigations have been conducted for the global exponential convergence of the HCNNs with mixed discontinuous activations and time-varying leakage delays. In order to fill this gap partially, motivated by the works mentioned above, in this paper, we are concerned with a class of HCNNs with mixed discontinuous activations and time-varying leakage delays described by the differential Equation (1). Still, without imposing any additional conditions on time-varying leakage delays $\eta_i(t)$, under the concept of Filippov solution, by applying the differential inclusions and inequality technique, some sufficient conditions on the global exponential convergence of the solutions for (1) is proposed originally.

Throughout the paper, we also assumed that $c_i : \mathbb{R} \to (0, +\infty)$, η_i, τ_{ij}, α_{ijl}, $\beta_{ijl} : \mathbb{R} \to [0, +\infty)$ and I_i, a_{ij}, b_{ijl}, $d_{ijl} : \mathbb{R} \to \mathbb{R}$ are bounded and continuous on $\mathbb{R}$, $i, j, l = 1, 2, ..., n$. Moreover, we further formulate the following assumptions imposed on the activation function:

(H1) f_i and g_i is continuous except on a countable set of isolate points $\{\rho_k^i\}$ and $\{\widetilde{\rho}_k^i\}$, where there exist finite right and left limits, $f_i^+(\rho_k^i)$ and $f_i^-(\rho_k^i)$, $g_i^+(\widetilde{\rho}_k^i)$ and $g_i^-(\widetilde{\rho}_k^i)$, respectively. Moreover, f_i and g_i have a finite number of discontinuities on any compact interval of $(0, +\infty)$.

(H2) For each $j = 1, 2, ..., n$, there exist two nonnegative constants $\mathcal{A}_j$ and $\mathcal{B}_j$ such that

$$\sup_{\gamma_j \in \overline{co}[f_j(x_j)]} |\gamma_j| \leq \mathcal{A}_j |x_j|, \quad \text{and}$$
$$\sup_{\omega_j \in \overline{co}[g_j(x_j)]} |\omega_j| \leq \mathcal{B}_j |x_j|^{\frac{1}{2}}, \quad \forall x_j \in \mathbb{R},$$

where

$$\overline{co}[f_j(x_j)] = \big[\min\{f_j^-(x_j), f_j^+(x_j)\}, \max\{f_j^-(x_j), f_j^+(x_j)\}\big],$$
$$\overline{co}[g_j(x_j)] = \big[\min\{g_j^-(x_j), g_j^+(x_j)\}, \max\{g_j^-(x_j), g_j^+(x_j)\}\big].$$

Remark 1. *One can see the activations in* (1) *are discontinuous, unbounded and non-monotonic, that means the activations are not continuous, Lipschitz continuous or smooth, which are different from the related references in the literature, such as [2–6]. The results established in the present paper extend the previous work about HCNNs to the discontinuous cases.*

Remark 2. *The activation functions* $f_j(x)(j=1,2)$ *and* $g_j(x)(j=1,2)$ *are both linearly increasing, but they possess different growth rates. So, this paper presents the mixed discontinuous activations, which are different from those in [7,9,10,15–18] and the references related therein.*

The structure of this paper is as follows. In Section 2, some basic definitions and preliminary lemmas are introduced. In Section 3, the main results of the global exponential convergence of the solution is presented. In Section 4, we provide two numerical examples to demonstrate the theoretical results and methods. Moreover, some remarks are also given to make some comparisons and illustrate the advantages of the established results.

2. Preliminaries

Given a bounded and continuous function f, we denote

$$f^{+}=\sup_{t\in\mathbb{R}}|f(t)|,\quad f^{-}=\inf_{t\in\mathbb{R}}|f(t)|.$$

Let $\mathbb{R}^n$ be n-dimensional vector space. For any $x=(x_1,x_2,...,x_n)^{\top}\in\mathbb{R}^n$, its norm is defined by $\|x\|=\sup_{t\in\mathbb{R}}\max_{1\le i\le n}|x_i(t)|$.

Since the neural network model considered in this paper is discontinuous, the classic solution is not suitable for neural networks with discontinuous activations. Hence, the definition in sense of Filippov solutions is introduced in the following, see [19,20].

Consider the non-autonomous delayed differential equation of the vector form:

$$\dot{x}(t)=f(t,x(t),x(t-\tau)),\quad a.e.\ t>t_0, \tag{3}$$

where t denotes time; $x(t)=(x_1(t),x_2(t),...,x_n(t))^{\top}$ denotes state vector; $x(t-\tau(t))=(x_1(t-\tau(t)),x_2(t-\tau(t)),...,x_n(t-\tau(t)))^{\top}$ represents time-varying delayed state vector and the time delay $\tau(t)$ is a continuous function; dx/dt denotes the time derivative of x and $f:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is measurable and essentially locally bounded. In this case, delayed differential Equation (3) is allowed to possess discontinuous right-hand side.

Construct the set-valued map $K[F(t,x(t),x(t-\tau(t)))]:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to 2^{\mathbb{R}^n}$:

$$K[F(t,x(t),x(t-\tau(t)))]=\bigcap_{\rho_1>0,\rho_2>0}\ \bigcap_{meas(\mathbb{N})=0,meas(\mathbb{M})=0} f_x^c,$$

where $f_x^c=\overline{co}[f(t,B(x,\rho_1)\setminus\mathbb{N},B(x(t-\tau(t)),\rho_2)\setminus\mathbb{M})]$;
$meas(\mathbb{N})(meas(\mathbb{M}))$ is the Lebesgue measure of set $\mathbb{N}(\mathbb{M})$; intersection is taken over all sets $\mathbb{N}(\mathbb{M})$ of measure zero and over all $\rho_1>0(\rho_2>0)$; $B(x,\rho_1)$ is the ball of center x and radius ρ_1; $B(x(t-\tau(t)),\rho_2)$ is the ball of center $x(t-\tau(t))$ and radius ρ_2; $\overline{co}[\mathbb{E}]$ is the closure of the convex hull of some set $\mathbb{E}$.

Definition 1. *The function* $x(t)$ *defined on a non-degenerate interval* $\mathbb{I}\in\mathbb{R}$ *is called a Filippov solution for delayed differential Equation* (3)*, if it is absolutely continuous on any compact subinterval* $[t_1,t_2]$ *of* $\mathbb{I}$*, and for a.e.* $t\in\mathbb{I}$*,* $x(t)$ *satisfies the following functional differential inclusion*

$$\frac{dx}{dt}\in F(t,x(t),x(t-\tau(t))).$$

By Definition 1, $x(t)=(x_1(t),x_2(t),...,x_n(t))^{\top}$ is the solution of initial value problems (1) on $[0,b)$, $b\in[0,+\infty)$, if $x_i(t)(i=1,2,...,n)$ is absolutely continuous on any compact subinterval of $[0,b)$ and satisfy the following inclusion:

$$\frac{dx_i(t)}{dt} \in -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\overline{co}\big[f_j(x_j(t-\tau_{ij}(t)))\big]$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\overline{co}\big[g_j(x_j(t-\alpha_{ijl}(t)))\big]\overline{co}\big[g_l(x_l(t-\beta_{ijl}(t)))\big]$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n} d_{ijl}(t)\int_0^{+\infty}\sigma_{ijl}(u)\overline{co}\big[g_j(x_j(t-u))\big]du$$
$$\cdot\int_0^{+\infty}\nu_{ijl}(u)\overline{co}\big[g_l(x_l(t-u))\big]du + I_i(t),$$
$$\text{for } a.e.\ t\in[0,b),\ i=1,2,...,n.$$

Then, for $i = 1, 2, ..., n$, it is obvious that the set-valued maps:

$$\frac{dx_i(t)}{dt} \hookrightarrow -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\overline{co}\big[f_j(x_j(t-\tau_{ij}(t)))\big]$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\overline{co}\big[g_j(x_j(t-\alpha_{ijl}(t)))\big]\overline{co}\big[g_l(x_l(t-\beta_{ijl}(t)))\big]$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n} d_{ijl}(t)\int_0^{+\infty}\sigma_{ijl}(u)\overline{co}\big[g_j(x_j(t-u))\big]du$$
$$\cdot\int_0^{+\infty}\nu_{ijl}(u)\overline{co}\big[g_l(x_l(t-u))\big]du + I_i(t)$$

have nonempty compact convex values. Thus, they are upper semi-continuous and measurable. By the measurable selection theorem, if $x_i(t)$ is the solution of systems (1), there exist two measurable functions $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)^\top : [-\rho, b) \to \mathbb{R}^n$ and $\omega = (\omega_1, \omega_2, ..., \omega_n)^\top : [-\rho, b) \to \mathbb{R}^n$ such that $\gamma_j(t) \in \overline{co}[f_j(x_j(t))]$ and $\omega_j(t) \in \overline{co}[g_j(x_j(t))]$ for a.e. $t \in [-\rho, b)$, and

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t-\tau_{ij}(t))\\ &+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\omega_j(t-\alpha_{ijl}(t))\omega_l(t-\beta_{ijl}(t))\\ &+\sum_{j=1}^{n}\sum_{l=1}^{n} d_{ijl}(t)\int_0^{+\infty}\sigma_{ijl}(u)\omega_j(t-u)du\\ &\cdot\int_0^{+\infty}\nu_{ijl}(u)\omega_l(t-u)du + I_i(t),\ \text{for } a.e.\ t\geq 0,\ i=1,2,...,n,\end{aligned} \tag{4}$$

where $\rho = \max\{\eta_i^+, \tau_{ij}^+, \alpha_{ijl}^+, \beta_{ijl}^+\}$.

3. Global Exponential Convergence

Theorem 1. *Suppose that the assumptions (H1) and (H2) hold, and the following assumptions are satisfied*

(H3) *For i, j, $l = 1, 2, ..., n$, the delay kernels σ_{ijl}, $\nu_{ijl} : [0, +\infty) \to \mathbb{R}$ are continuous, $|\sigma_{ijl}|e^{\kappa t}$ and $|\nu_{ijl}|e^{\kappa t}$ are integrable on $[0, +\infty)$ for a certain positive constant κ.*

(H4) *For each $i = 1, 2, ..., n$, $I_i(t)$ is bounded and continuous on $[0, +\infty)$, and there exist positive constants λ_0 and ξ_i such that*

$$I_i(t) = O(e^{-\lambda_0 t}),\ \text{ as }\ t \to +\infty,$$

and

$$\sup_{t\geq 0}\max_{1\leq i\leq n}\Big\{F_i(t,\lambda_0)\Big(\frac{c_i^+}{c_i^- -\lambda_0}+1\Big)\Big\}<1,$$
$$\sup_{t\geq 0}\max_{1\leq i\leq n}\{-c_i^-+\lambda_0+F_i(t,\lambda_0)\}<0,$$

where

$$\begin{aligned}F_i(t,\lambda_0)=&|c_i(t)||\eta_i(t)|e^{\lambda_0\eta_i^+}+\xi_i^{-1}\sum_{j=1}^n|a_{ij}(t)|\mathcal{A}_j\xi_je^{\lambda_0\tau_{ij}^+}\\&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n|b_{ijl}(t)|\mathcal{B}_j\xi_j^{\frac{1}{2}}\mathcal{B}_l\xi_l^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0(\alpha_{ijl}^++\beta_{ijl}^+)}\\&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n|d_{ijl}(t)|\int_0^{+\infty}|\sigma_{ijl}(u)|\mathcal{B}_j\xi_j^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0u}du\\&\cdot\int_0^{+\infty}|\nu_{ijl}(u)|\mathcal{B}_l\xi_l^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0u}du+\lambda_0.\end{aligned}$$

Then, for every solution $x(t)$ of system (1) with any initial value conditions (2), there exists a positive constant λ such that

$$x_i(t)=O(e^{-\lambda t}),\ \text{as}\ t\to+\infty, i=1,2,...,n.$$

Proof. Suppose that $x(t)=(x_1(t),x_2(t),...,x_n(t))^\top$ is a solution of neural network system (1) with initial conditions (2). Let

$$\begin{aligned}y(t)=&(y_1(t),y_2(t),...,y_n(t))^\top\\=&(\xi_1^{-1}x_1(t),\xi_2^{-1}x_2(t),...,\xi_n^{-1}x_n(t))^\top.\end{aligned}$$

Then, from (1), it follows that

$$\begin{aligned}y_i'(t)=&-c_i(t)y_i(t-\eta_i(t))+\xi_i^{-1}\sum_{j=1}^n a_{ij}(t)f_j(x_j(t-\tau_{ij}(t)))\\&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)g_j(x_j(t-\alpha_{ijl}(t)))g_l(x_l(t-\beta_{ijl}(t)))\\&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n d_{ijl}(t)\int_0^{+\infty}\sigma_{ijl}(u)g_j(x_j(t-u))du\\&\cdot\int_0^{+\infty}\nu_{ijl}(u)g_l(x_l(t-u))du+\xi_i^{-1}I_i(t).\end{aligned}$$

By Definition 1, we can further have that there exists $\gamma(t)=(\gamma_1(t),\gamma_2(t),...,\gamma_n(t))^\top$ and $\omega(t)=(\omega_1(t),\omega_2(t),...,\omega_n(t))^\top$, where $\gamma_j(t)\in\overline{co}[f_j(x_j(t))]$ and $\omega_j(t)\in\overline{co}[g_j(x_j(t))]$ such that

$$\begin{aligned}y_i'(t)=&-c_i(t)y_i(t-\eta_i(t))+\xi_i^{-1}\sum_{j=1}^n a_{ij}(t)\gamma_j(t-\tau_{ij}(t))\\&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)\omega_j(t-\alpha_{ijl}(t))\omega_l(t-\beta_{ijl}(t))\\&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n d_{ijl}(t)\int_0^{+\infty}\sigma_{ijl}(u)\omega_j(t-u)du\\&\cdot\int_0^{+\infty}\nu_{ijl}(u)\omega_l(t-u)du+\xi_i^{-1}I_i(t),\\&\text{for } a.e.\ t\geq 0,\ i=1,2,...,n.\end{aligned}\tag{5}$$

In view of (H4), we can choose a constant $\lambda \in (0, \min\{\lambda_0, \kappa, c_i^-\})$ such that

$$\sup_{t\in\mathbb{R}} \max_{1\le i\le n} \Big\{\Big(1+\frac{c_i^+}{c_i^- - \lambda}\Big)F_i(t,\lambda)\Big\} < 1,$$
$$\sup_{t\in\mathbb{R}} \max_{1\le i\le n} \{-c_i^- + \lambda + F_i(t,\lambda)\} < 0,$$

namely,

$$\begin{aligned}
&\sup_{t\ge 0} \max_{1\le i\le n} \Big\{\Big(1+\frac{c_i^+}{c_i^- - \lambda}\Big)\cdot\Big[|c_i(t)||\eta_i(t)|e^{\lambda\eta_i^+}\\
&+\xi_i^{-1}\sum_{j=1}^n |a_{ij}(t)|\mathcal{A}_j\xi_j e^{\lambda\tau_{ij}^+}\\
&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n |b_{ijl}(t)|\mathcal{B}_j\xi_j^{\frac12}\mathcal{B}_l\xi_l^{\frac12}e^{\frac12\lambda(\alpha_{ijl}^+ + \beta_{ijl}^+)}\\
&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n |d_{ijl}(t)|\int_0^{+\infty}|\sigma_{ijl}(u)|\mathcal{B}_j\xi_j^{\frac12}e^{\frac12\lambda u}du\\
&\cdot\int_0^{+\infty}|\nu_{ijl}(u)|\mathcal{B}_l\xi_l^{\frac12}e^{\frac12\lambda u}du + \lambda\Big]\Big\} < 1,
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
&\sup_{t\ge 0} \max_{1\le i\le n} \Big\{(c_i^- - \lambda)\Big(\frac{F_i(t,\lambda)}{c_i^- - \lambda} - 1\Big)\Big\}\\
&=\sup_{t\ge 0} \max_{1\le i\le n} \Big\{-c_i^- + \lambda + |c_i(t)||\eta_i(t)|e^{\lambda\eta_i^+}\\
&+\xi_i^{-1}\sum_{j=1}^n |a_{ij}(t)|\mathcal{A}_j\xi_j e^{\lambda\tau_{ij}^+}\\
&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n |b_{ijl}(t)|\mathcal{B}_j\xi_j^{\frac12}\mathcal{B}_l\xi_l^{\frac12}e^{\frac12\lambda(\alpha_{ijl}^+ + \beta_{ijl}^+)}\\
&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n |d_{ijl}(t)|\int_0^{+\infty}|\sigma_{ijl}(u)|\mathcal{B}_j\xi_j^{\frac12}e^{\frac12\lambda u}du\\
&\cdot\int_0^{+\infty}|\nu_{ijl}(u)|\mathcal{B}_l\xi_l^{\frac12}e^{\frac12\lambda u}du\Big\} < 0.
\end{aligned} \tag{7}$$

Let

$$\|y(t)\|_1 = \max_{1\le i\le n}\{|y_i(t)|, |y_i'(t)|\},$$
$$\|\varphi\|_\xi = \max\Big\{\sup_{t\le 0}\max_{1\le i\le n}\xi_i^{-1}|\varphi_i(t)|, \sup_{t\le 0}\max_{1\le i\le n}\xi_i^{-1}|\varphi_i'(t)|\Big\}.$$

For any $\varepsilon > 0$, we can have that

$$\|y(t)\|_1 \le (\|\varphi\|_\xi + \varepsilon)e^{-\lambda t} < M(\|\varphi\|_\xi + \varepsilon)e^{-\lambda t},\ \ t\in(-\infty, 0],$$

where $M > \sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big\{\frac{c_i^- - \lambda}{F_i(t,\lambda)}\Big\}$ is a sufficiently large constant such that

$$|\xi_i^{-1}I_i(t)| < \lambda M(\|\varphi\|_\xi + \varepsilon)e^{-\lambda t},\ \ t\in\mathbb{R},\ i = 1, 2, ..., n,$$

In addition, from (H4) and (7), it follows that

$$\frac{F_i(t,\lambda)}{c_i(t)-\lambda} < 1,\ M > \frac{c_i^- - \lambda}{F_i(t,\lambda)} > 1,\ \frac{1}{M} < \frac{F_i(t,\lambda)}{c_i^- - \lambda},\ t\in\mathbb{R},\ i = 1, 2, ..., n. \tag{8}$$

In the following, we will show

$$\|y(t)\|_1 < M(\|\varphi\|_\xi + \varepsilon)e^{-\lambda t},\ \ t > 0.$$

Otherwise, there must exist $i = 1, 2, ..., n$ and $\theta > 0$ such that

$$\begin{cases} \|y(t)\|_1 = \max\limits_{1\le i\le n}\{|y_i(t)|, |y_i'(t)|\} = M(\|\varphi\|_\xi + \varepsilon)e^{-\lambda\theta}, \\ \|y(t)\|_1 < M(\|\varphi\|_\xi + \varepsilon)e^{-\lambda t}, \ \ t \in (-\infty, \theta). \end{cases}$$

Note that

$$\begin{aligned} y_i'(s) + c_i(s)y_i(s) = c_i(s)\int_{s-\eta_i(s)}^{s} y'(u)du + \xi_i^{-1}\sum_{j=1}^{n} a_{ij}(s)\gamma_j(s-\tau_{ij}(s)) \\ + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\omega_j(s-\alpha_{ijl}(s))\omega_l(s-\beta_{ijl}(s)) \\ + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} d_{ijl}(s)\int_0^{+\infty}\sigma_{ijl}(u)\omega_j(s-u)du\int_0^{+\infty}\nu_{ijl}(u)\omega_l(s-u)du \\ + \xi_i^{-1}I_i(s), \ \ s\in[0,t], \ \ t\in[0,\theta]. \end{aligned}$$

Multiplying both sides of the above equation by $e^{\int_0^s c_i(u)du}$, and integrating it on $[0, t]$, we get

$$\begin{aligned} y_i(t) = & y_i(0)e^{-\int_0^t c_i(u)du} + \int_0^t e^{-\int_s^t c_i(u)du}\Big[c_i(s)\int_{s-\eta_i(s)}^{s} y'(u)du \\ & + \xi_i^{-1}\sum_{j=1}^{n} a_{ij}(s)\gamma_j(s-\tau_{ij}(s)) \\ & + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\omega_j(s-\alpha_{ijl}(s))\omega_l(s-\beta_{ijl}(s)) \\ & + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} d_{ijl}(s)\int_0^{+\infty}\sigma_{ijl}(u)\omega_j(s-u)du \\ & \cdot\int_0^{+\infty}\nu_{ijl}(u)\omega_l(s-u)du + \xi_i^{-1}I_i(s)\Big]ds, \ \ t\in[0,\theta]. \end{aligned}$$

Thus, with the help of (H4), we have

$$\begin{aligned} |y_i(\theta)| = & \Big|y_i(0)e^{-\int_0^\theta c_i(u)du} + \int_0^\theta e^{-\int_s^\theta c_i(u)du}\Big[c_i(s)\int_{s-\eta_i(s)}^{s} y'(u)du \\ & + \xi_i^{-1}\sum_{j=1}^{n} a_{ij}(s)\gamma_j(s-\tau_{ij}(s)) \\ & + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\omega_j(s-\alpha_{ijl}(s))\omega_l(s-\beta_{ijl}(s)) \\ & + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} d_{ijl}(s)\int_0^{+\infty}\sigma_{ijl}(u)\omega_j(s-u)du \\ & \cdot\int_0^{+\infty}\nu_{ijl}(u)\omega_l(s-u)du + \xi_i^{-1}I_i(s)\Big]ds\Big| \\ \le & (\|\varphi\|_\xi + \varepsilon)e^{-c_i^-\theta} + \int_0^\theta e^{-\int_s^\theta c_i(u)du}\Big[c_i(s)\int_{s-\eta_i(s)}^{s} |y'(u)|du \\ & + \xi_i^{-1}\sum_{j=1}^{n} |a_{ij}(t)|\mathcal{B}_j|\xi_j y_j(s-\tau_{ij}(s))| \\ & + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} |b_{ijl}(t)|\mathcal{A}_j\xi_j^{\frac{1}{2}}|y_j(s-\alpha_{ijl}(s))|^{\frac{1}{2}} \\ & \cdot\mathcal{A}_l\xi_l^{\frac{1}{2}}|y_l(s-\beta_{ijl}(s))|^{\frac{1}{2}} \\ & + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n} |d_{ijl}(t)|\int_0^{+\infty}|\sigma_{ijl}(u)|\mathcal{A}_j\xi_j^{\frac{1}{2}}|y_j(s-u)|^{\frac{1}{2}}du \\ & \cdot\int_0^{+\infty}|\nu_{ijl}(u)|\mathcal{A}_l\xi_l^{\frac{1}{2}}|\xi_l y_l(s-u)|^{\frac{1}{2}}du + \xi_i^{-1}|I_i(s)|\Big]ds, \end{aligned}$$

then we can further obtain

$$
\begin{aligned}
&|y_i(\theta)| \le (\|\varphi\|_\xi+\varepsilon)e^{-c_i^-\theta}+\int_0^\theta e^{-\int_s^\theta c_i(u)du}\Big\{\\
&c_i(s)\eta_i(s)M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda(s-\eta_i(s))}\\
&+\xi_i^{-1}\sum_{j=1}^n|a_{ij}(t)|\mathcal{A}_j\xi_jM(\|\varphi\|_\xi+\varepsilon)e^{-\lambda(s-\tau_{ij}(s))}\\
&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n|b_{ijl}(t)|\mathcal{B}_j\xi_j^{\frac12}\mathcal{B}_l\xi_l^{\frac12}\Big[M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda(s-\alpha_{ijl}(s))}\Big]^{\frac12}\\
&\cdot\Big[M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda(s-\beta_{ijl}(s))}\Big]^{\frac12}\\
&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n|d_{ijl}(t)|\int_0^{+\infty}|\sigma_{ijl}(u)|\mathcal{B}_j\xi_j^{\frac12}\Big[M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda(s-u)}\Big]^{\frac12}du\\
&\cdot\int_0^{+\infty}|\nu_{ijl}(u)|\mathcal{B}_l\xi_l^{\frac12}\Big[M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda(s-u)}\Big]^{\frac12}du\\
&+\lambda M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda s}\Big\}ds\\
&\le(\|\varphi\|_\xi+\varepsilon)e^{-c_i^-\theta}+\int_0^\theta e^{-\int_s^\theta c_i(u)du}e^{-\lambda s}ds\cdot M(\|\varphi\|_\xi+\varepsilon)F_i(t,\lambda)\\
&\le(\|\varphi\|_\xi+\varepsilon)e^{-c_i^-\theta}+e^{-c_i^-\theta}M(\|\varphi\|_\xi+\varepsilon)F_i(t,\lambda)\int_0^\theta e^{(c_i^--\lambda)s}ds\\
&=M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda\theta}\Big[\frac{1}{M}e^{(\lambda-c_i^-)\theta}+\frac{F_i(t,\lambda)}{c_i^--\lambda}(1-e^{(\lambda-c_i^-)\theta})\Big]\\
&=M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda\theta}\Big[\Big(\frac{1}{M}-\frac{F_i(t,\lambda)}{c_i^--\lambda}\Big)e^{(\lambda-c_i^-)\theta}+\frac{F_i(t,\lambda)}{c_i^--\lambda}\Big],
\end{aligned}
\tag{9}
$$

which together with (8) leads to

$$|y_i(\theta)| < M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda\theta},$$

thus,

$$
\begin{aligned}
\|y(\theta)\|_1&=\max_{1\le i\le n}\{|y_i(\theta)|,|y_i'(\theta)|\}\\
&=|y_i'(\theta)|=M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda\theta}.
\end{aligned}
\tag{10}
$$

On the other hand, by (9), we can have

$$
\begin{aligned}
|y_i'(\theta)|\le&|c_i(\theta)y_i(\theta)|+c_i(\theta)\int_{\theta-\eta_i(\theta)}^\theta|y'(s)|ds\\
&+\xi_i^{-1}\sum_{j=1}^n|a_{ij}(\theta)||\gamma_j(\theta-\tau_{ij}(\theta))|\\
&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n|b_{ijl}(\theta)||\omega_j(\theta-\alpha_{ijl}(\theta))||\omega_l(\theta-\beta_{ijl}(\theta))|\\
&+\xi_i^{-1}\sum_{j=1}^n\sum_{l=1}^n|d_{ijl}(\theta)|\Big|\int_0^{+\infty}\sigma_{ijl}(u)\omega_j(\theta-u)du\\
&\int_0^{+\infty}\nu_{ijl}(u)\omega_l(\theta-u)du\Big|+|\xi_i^{-1}I_i(\theta)|,
\end{aligned}
$$

then we further have

$$
\begin{aligned}
|y_i'(\theta)| \leq & c_i^+|y_i(\theta)| + |c_i(\theta)|\int_{\theta-\eta_i(\theta)}^{\theta}|y_i'(s)|ds \\
& + \xi_i^{-1}\sum_{j=1}^{n}|a_{ij}(\theta)|\mathcal{A}_j\xi_j|y_j(\theta-\tau_{ij}(\theta))| \\
& + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|b_{ijl}(\theta)|\mathcal{B}_j\xi_j^{\frac{1}{2}}|y_j(\theta-\alpha_{ijl}(\theta))|^{\frac{1}{2}} \\
& \cdot \mathcal{B}_l\xi_l^{\frac{1}{2}}|y_l(\theta-\beta_{ijl}(\theta))|^{\frac{1}{2}} \\
& + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|d_{ijl}(\theta)|\int_0^{+\infty}|\sigma_{ijl}(u)|\mathcal{B}_j\xi_j^{\frac{1}{2}}|y_j(\theta-u)|^{\frac{1}{2}}du \\
& \cdot \int_0^{+\infty}|\nu_{ijl}(u)|\mathcal{B}_l\xi_l^{\frac{1}{2}}|y_l(\theta-u)|^{\frac{1}{2}}du + |\xi_i^{-1}I_i(\theta)| \\
< & c_i^+M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda\theta}\Big[\Big(\frac{1}{M}-\frac{F_i(\theta,\lambda)}{c_i^- -\lambda}\Big)e^{(\lambda-c_i^-)\theta} \\
& + \frac{F_i(\theta,\lambda)}{c_i^- -\lambda}\Big] + M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda\theta}\Big[|c_i(\theta)||\eta_i(\theta)|e^{\lambda\eta_i^+} \\
& + \xi_i^{-1}\sum_{j=1}^{n}|a_{ij}(\theta)|\mathcal{A}_j\xi_je^{\lambda\tau_{ij}^+} \\
& + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|b_{ijl}(\theta)|\mathcal{B}_j\xi_j^{\frac{1}{2}}\mathcal{B}_l\xi_l^{\frac{1}{2}}e^{\frac{1}{2}\lambda(\alpha_{ijl}^++\beta_{ijl}^+)} \\
& + \xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|d_{ijl}(\theta)|\int_0^{+\infty}|\sigma_{ijl}(u)|\mathcal{B}_j\xi_j^{\frac{1}{2}}e^{\frac{1}{2}\lambda u}du \\
& \cdot \int_0^{+\infty}|\nu_{ijl}(u)|\mathcal{B}_l\xi_l^{\frac{1}{2}}e^{\frac{1}{2}\lambda u}du + \lambda\Big] \\
= & M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda\theta}\Big\{c_i^+\Big[\Big(\frac{1}{M}-\frac{F_i(\theta,\lambda)}{c_i^- -\lambda}\Big)e^{(\lambda-c_i^-)\theta} \\
& + \frac{F_i(\theta,\lambda)}{c_i^- -\lambda}\Big] + F_i(\theta,\lambda)\Big\} \\
= & M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda\theta}\Big\{c_i^+\Big(\frac{1}{M}-\frac{F_i(\theta,\lambda)}{c_i^- -\lambda}\Big)e^{(\lambda-c_i^-)\theta} \\
& + F_i(\theta,\lambda)\Big(\frac{c_i^+}{c_i^- -\lambda}+1\Big)\Big\},
\end{aligned}
$$

by (6) and (8), we have

$$|y_i'(\theta)| < M(\|\varphi\|_\xi+\varepsilon)e^{-\lambda\theta}.$$

Obviously, we can see that there exists a contradiction. Letting $\varepsilon \to 0^+$, we eventually get that

$$\|y(t)\|_1 \leq M\|\varphi\|_\xi e^{-\lambda t}, \ \ t > 0.$$

Up to now, the proof is complete. □

4. Numerical Examples and Remarks

In this section, we present two topical examples to demonstrate the results obtained in previous sections.

Example 1. *Consider the following HCNNs with mixed discontinuous activations and time-varying leakage delays:*

$$
\begin{aligned}
x_1'(t) = & - c_1(t)x_1(t-\eta_1(t)) \\
& + a_{11}(t)f_1(x_1(t-\tau_{11}(t))) \\
& + a_{12}(t)f_2(x_2(t-\tau_{12}(t))) \\
& + b_{111}(t)g_1(x_1(t-\alpha_{111}(t)))g_1(x_1(t-\beta_{111}(t))) \\
& + b_{121}(t)g_2(x_2(t-\alpha_{121}(t)))g_1(x_1(t-\beta_{121}(t))) \\
& + b_{112}(t)g_1(x_1(t-\alpha_{112}(t)))g_2(x_2(t-\beta_{112}(t))) \\
& + b_{122}(t)g_2(x_2(t-\alpha_{122}(t)))g_2(x_2(t-\beta_{122}(t))) \\
& + d_{112}(t)\int_0^{+\infty}\sigma_{112}(u)g_1(x_1(t-u))du \\
& \cdot \int_0^{+\infty}\nu_{112}(u)g_2(x_2(t-u))du + I_1(t),
\end{aligned}
$$

$$
\begin{aligned}
x_2'(t) = & - c_2(t)x_2(t-\eta_2(t)) \\
& + a_{21}(t)f_1(x_1(t-\tau_{21}(t))) + a_{22}(t)f_2(x_2(t-\tau_{22}(t))) \\
& + b_{211}(t)g_1(x_1(t-\alpha_{211}(t)))g_1(x_1(t-\beta_{211}(t))) \\
& + b_{221}(t)g_2(x_2(t-\alpha_{221}(t)))g_1(x_1(t-\beta_{221}(t))) \\
& + b_{212}(t)g_1(x_1(t-\alpha_{212}(t)))g_2(x_2(t-\beta_{212}(t))) \\
& + b_{222}(t)g_2(x_2(t-\alpha_{222}(t)))g_2(x_2(t-\beta_{222}(t))) \\
& + d_{212}(t)\int_0^{+\infty}\sigma_{212}(u)g_1(x_1(t-u))du \\
& \cdot \int_0^{+\infty}\nu_{212}(u)g_2(x_2(t-u))du + I_2(t),
\end{aligned}
\tag{11}
$$

where $n=2$, and

$$
\begin{gathered}
c_1(t) = 2.4 + 0.3\sin t, c_2(t) = 2.4 + 0.4\cos t; \\
(a_{ij}(t))_{2\times 2} = \begin{pmatrix} \frac{(1+|t|)\sin t}{1+40|t|} & \frac{(1+|t|)\sin t}{1+36|t|} \\ \frac{(2+|t|)\sin t}{1+40|t|} & \frac{(2+|t|)\sin t}{1+36|t|} \end{pmatrix}; \\
(b_{1jl}(t))_{2\times 2} = \begin{pmatrix} 0.002\sin(2t)e^{-t} & 0.03\cos(2t)e^{-t} \\ 0.06\cos(2t)e^{-t} & 0.07\sin(2t)e^{-t} \end{pmatrix}, \\
(b_{2jl}(t))_{2\times 2} = \begin{pmatrix} 0.005\cos(2t)e^{-t} & 0.02\sin(2t)e^{-t} \\ 0.005\sin(2t)e^{-t} & 0.02\cos(2t)e^{-t} \end{pmatrix}; \\
d_{112}(t) = \frac{1}{10}\sin(2t)e^{-t}, d_{212}(t) = \frac{1}{10}\cos(2t)e^{-t}, \\
d_{ijl}(t) = 0, ijl \neq 112, ijl \neq 212; \alpha_{ijl} = e^{-\sin t}, \\
\beta_{ijl} = e^{-\cos t}, \sigma_{ijl}(u) = \nu_{ijl}(u) = e^{-u},\ i,j,l = 1,2; \\
I_1(t) = e^{-2|t|}\sin^4 t,\ \ I_2(t) = e^{-2|t|}\sin^5 t; \\
\eta_i(t) = 0.0008,\ i = 1,2,\ \ (\tau_{ij})_{2\times 2} = \begin{pmatrix} 0.1\cos^2 t & 0.1\sin^2 t \\ 0.1\sin^2 t & 0.1\cos^2 t \end{pmatrix}.
\end{gathered}
$$

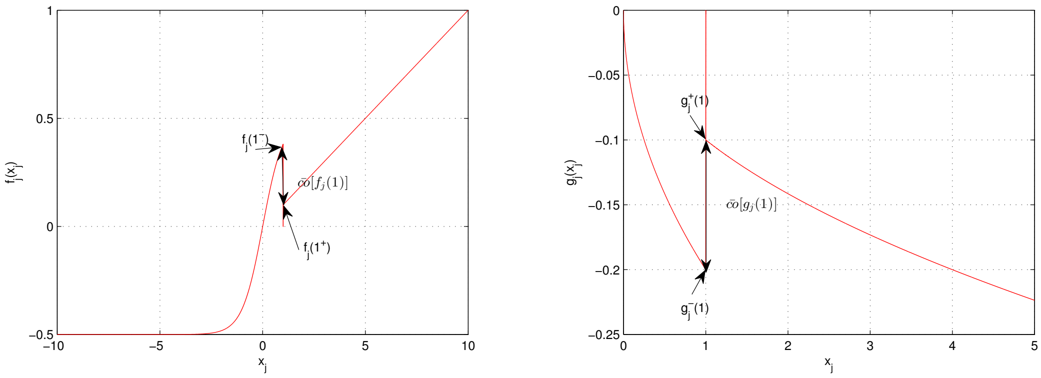
Moreover, let

$$
f_1(x) = f_2(x) = \begin{cases} 0.5\tanh(x), & x \le 1; \\ 0.1x, & x > 1, \end{cases}
$$

$$
g_1(x) = g_2(x) = \begin{cases} -0.2|x|^{\frac{1}{2}}, & 0 < x \le 1; \\ -0.1|x|^{\frac{1}{2}}, & x > 1. \end{cases}
$$

It is easy to see that the activation functions $f_1(x)$ and $f_2(x)$ are discontinuous, non-decreasing. The activation function $f_j(x)$ has a discontinuous point $x=1$ and $\overline{co}[f_j(1)] = [f_j^+(1), f_j^-(1)] = [0.1, 0.5\tanh(1)]$, $j = 1,2$.

And, the activation functions $g_1(x)$ and $g_2(x)$ are discontinuous, non-increasing. The activation function $g_j(x)$ has a discontinuous point $x = 1$ and $\overline{co}[g_j(1)] = [g_j^-(1), g_j^+(1)] = [-0.2, -0.1]$, $j = 1, 2$. Thus, (H1) and (H2) are satisfied. This fact can be seen in Figure 1.



(a) (b)

Figure 1. (**a**) Discontinuous activation functions $f_j (j = 1, 2)$ for system (11); (**b**) Discontinuous activation functions $g_j (j = 1, 2)$ for system (11).

Moreover, we can have $\mathcal{A}_1 = \mathcal{A}_2 = 0.5$, $\mathcal{B}_1 = \mathcal{B}_2 = 0.2$, and $\lambda_0 = 2$.
Let $\xi_1 = 63$ and $\xi_2 = 47$, we can have

$$\sup_{t\geq 0}\left\{F_1(t,\lambda_0)\left(\frac{c_1^+}{c_1^- - \lambda_0}+1\right)\right\} \approx 0.6553 < 1,$$
$$\sup_{t\geq 0}\left\{F_2(t,\lambda_0)\left(\frac{c_2^+}{c_2^- - \lambda_0}+1\right)\right\} \approx 0.4103 < 1,$$
$$\sup_{t\geq 0}\{-c_1^- + \lambda_0 + F_1(t,\lambda_0)\} \approx -0.3855 < 0,$$
$$\sup_{t\geq 0}\{-c_2^- + \lambda_0 + F_2(t,\lambda_0)\} \approx -0.4102 < 0,$$

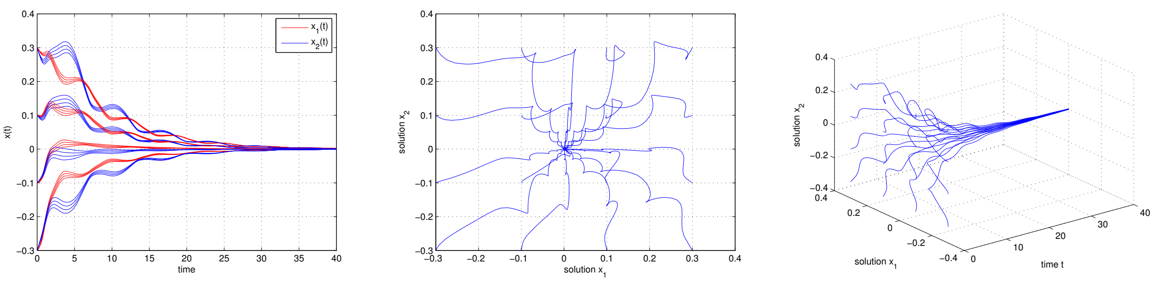
where

$$\begin{aligned}
F_1(t,\lambda_0) =& |c_1(t)||\eta_1(t)|e^{\lambda_0\eta_1^+} \\
&+ \xi_1^{-1}|a_{11}(t)|\mathcal{A}_1\xi_1 e^{\lambda_0\tau_{11}^+} + \xi_1^{-1}|a_{12}(t)|\mathcal{A}_2\xi_2 e^{\lambda_0\tau_{12}^+} \\
&+ \xi_1^{-1}\Big[|b_{111}(t)|\mathcal{B}_1\xi_1^{\frac{1}{2}}\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0(\alpha_{111}^+ + \beta_{111}^+)} \\
&+ |b_{112}(t)|\mathcal{B}_1\xi_1^{\frac{1}{2}}\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0(\alpha_{112}^+ + \beta_{112}^+)}\Big] \\
&+ \xi_1^{-1}\Big[|b_{121}(t)|\mathcal{B}_2\xi_2^{\frac{1}{2}}\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0(\alpha_{121}^+ + \beta_{121}^+)} \\
&+ |b_{122}(t)|\mathcal{B}_2\xi_2^{\frac{1}{2}}\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0(\alpha_{122}^+ + \beta_{122}^+)}\Big] \\
&+ \xi_1^{-1}\Big[|d_{111}(t)|\int_0^{+\infty}|\sigma_{111}(u)|\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&\cdot\int_0^{+\infty}|\nu_{111}(u)|\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&+ |d_{112}(t)|\int_0^{+\infty}|\sigma_{112}(u)|\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&\cdot\int_0^{+\infty}|\nu_{112}(u)|\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du\Big] \\
&+ \xi_1^{-1}\Big[|d_{121}(t)|\int_0^{+\infty}|\sigma_{121}(u)|\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&\cdot\int_0^{+\infty}|\nu_{121}(u)|\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&+ |d_{122}(t)|\int_0^{+\infty}|\sigma_{122}(u)|\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&\cdot\int_0^{+\infty}|\nu_{122}(u)|\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du\Big] + \lambda_0,
\end{aligned} \tag{12}$$

and

$$
\begin{aligned}
F_2(t,\lambda_0) =&|c_2(t)||\eta_2(t)|e^{\lambda_0\eta_2^+} \\
&+\xi_2^{-1}|a_{21}(t)|\mathcal{A}_1\xi_1 e^{\lambda_0\tau_{21}^+}+\xi_2^{-1}|a_{22}(t)|\mathcal{A}_2\xi_2 e^{\lambda_0\tau_{22}^+} \\
&+\xi_2^{-1}\Big[|b_{211}(t)|\mathcal{B}_1\xi_1^{\frac{1}{2}}\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0(\alpha_{211}^+ +\beta_{211}^+)} \\
&+|b_{212}(t)|\mathcal{B}_1\xi_1^{\frac{1}{2}}\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0(\alpha_{212}^+ +\beta_{212}^+)}\Big] \\
&+\xi_2^{-1}\Big[|b_{221}(t)|\mathcal{B}_2\xi_2^{\frac{1}{2}}\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0(\alpha_{221}^+ +\beta_{221}^+)} \\
&+|b_{222}(t)|\mathcal{B}_2\xi_2^{\frac{1}{2}}\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0(\alpha_{222}^+ +\beta_{222}^+)}\Big] \\
&+\xi_2^{-1}\Big[|d_{211}(t)|\int_0^{+\infty}|\sigma_{211}(u)|\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&\cdot\int_0^{+\infty}|\nu_{211}(u)|\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&+|d_{212}(t)|\int_0^{+\infty}|\sigma_{212}(u)|\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&\cdot\int_0^{+\infty}|\nu_{212}(u)|\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du\Big] \\
&+\xi_2^{-1}\Big[|d_{221}(t)|\int_0^{+\infty}|\sigma_{221}(u)|\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&\cdot\int_0^{+\infty}|\nu_{221}(u)|\mathcal{B}_1\xi_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&+|d_{222}(t)|\int_0^{+\infty}|\sigma_{222}(u)|\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du \\
&\cdot\int_0^{+\infty}|\nu_{222}(u)|\mathcal{B}_2\xi_2^{\frac{1}{2}}e^{\frac{1}{2}\lambda_0 u}du\Big]+\lambda_0.
\end{aligned}
\tag{13}
$$

As a result, the coefficients of neural system (11) *satisfy all the conditions in Theorem 1. Hence, we can conclude that all solutions of the neural system* (11) *converge exponentially to the zero vector. This fact can be presented in the following Figure 2 by MATLAB software.*



(**a**) (**b**) (**c**)

Figure 2. (**a**) Time response of the state variables x_1 and x_2 for discontinuous HCNNs (11) with random initial conditions; (**b**) Phase plane behavior of the state variables x_1 and x_2 for discontinuous HCNNs (11); (**c**) Three-dimensional trajectory of state variables x_1 and x_2 for discontinuous HCNNs (11).

Remark 3. *For all we know, there is no research on the global exponential convergence of the HCNNs with mixed discontinuous activations and time-varying leakage delays. We also mention that all results in the references cited in the present paper cannot be directly applied to establish the results on the global exponential convergence of the solutions for the discontinuous HCNNs* (11)*. This implies that the results of this paper are essentially new.*

Example 2. *Consider the following HCNNs with mixed discontinuous activations and time-varying leakage delays:*

$$\begin{cases} x_1'(t) = & -c_1(t)x_1(t-\eta_1(t)) + a_{11}(t)f_1(x_1(t-\tau_{11}(t))) \\ & +a_{12}(t)f_2(x_2(t-\tau_{12}(t))) \\ & +b_{112}(t)g_1(x_1(t-\alpha_{112}(t)))g_2(x_2(t-\beta_{112}(t))) \\ & +d_{112}(t)\int_0^{+\infty}\sigma_{112}(u)g_1(x_1(t-u))du \\ & \cdot\int_0^{+\infty}\nu_{112}(u)g_2(x_2(t-u))du + I_1(t), \\ x_2'(t) = & -c_2(t)x_2(t-\eta_2(t)) + a_{21}(t)f_1(x_1(t-\tau_{21}(t))) \\ & +a_{22}(t)f_2(x_2(t-\tau_{22}(t))) \\ & +b_{212}(t)g_1(x_1(t-\alpha_{212}(t)))g_2(x_2(t-\beta_{212}(t))) \\ & +d_{212}(t)\int_0^{+\infty}\sigma_{212}(u)g_1(x_1(t-u))du \\ & \int_0^{+\infty}\nu_{212}(u)g_2(x_2(t-u))du + I_2(t), \end{cases} \tag{14}$$

where $n = 2$*, and*

$$c_1(t) = 2.1 + 0.2\sin t, c_2(t) = 2.6 + 0.3\cos t;$$
$$(a_{ij}(t))_{2\times 2} = \begin{pmatrix} \frac{(1+|t|)\sin t}{1+40|t|} & \frac{(1+|t|)\sin t}{1+36|t|} \\ \frac{(2+|t|)\sin t}{1+40|t|} & \frac{(2+|t|)\sin t}{1+36|t|} \end{pmatrix};$$
$$b_{112}(t) = d_{112}(t) = 0.1\sin(2t)e^{-t},$$
$$b_{212}(t) = d_{212}(t) = 0.1\cos(2t)e^{-t},$$
$$d_{ijl}(t) = 0, ijl \neq 112, ijl \neq 212;$$
$$I_1(t) = e^{-2|t|}\sin^4 t,\ \ I_2(t) = e^{-2|t|}\sin^5 t;$$
$$\alpha_{ijl} = e^{-\sin t}, \beta_{ijl} = e^{-\cos t}, \sigma_{ijl}(u) = \nu_{ijl}(u) = e^{-u},\ i,j,l = 1,2;$$
$$\eta_i(t) = 0.0008,\ i = 1,2,\ \ (\tau_{ij})_{2\times 2} = \begin{pmatrix} 0.1\cos^2 t & 0.1\sin^2 t \\ 0.1\sin^2 t & 0.1\cos^2 t \end{pmatrix}.$$

Moreover, let

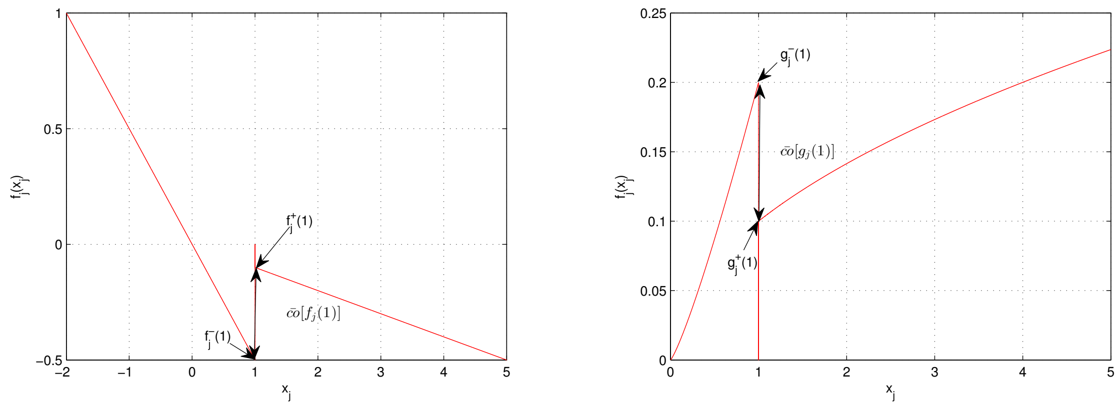
$$f_1(x) = f_2(x) = \begin{cases} -0.5x, & x \leq 1; \\ -0.1x, & x > 1, \end{cases}$$
$$g_1(x) = g_2(x) = \begin{cases} 0.2|x|^{\frac{1}{2}}, & 0 < x \leq 1; \\ 0.1|x|^{\frac{1}{2}}, & x > 1. \end{cases}$$

It is easy to see that the activation functions $f_1(x)$ *and* $f_2(x)$ *are discontinuous, non-increasing. The activation function* $f_j(x)$ *has a discontinuous point* $x = 1$ *and* $\overline{co}[f_j(1)] = [f_j^-(1), f_j^+(1)] = [-0.5, -0.1]$*,* $j = 1,2$*. And, the activation functions* $g_1(x)$ *and* $g_2(x)$ *are discontinuous, non-decreasing. The activation function* $g_j(x)$ *has a discontinuous point* $x = 1$ *and* $\overline{co}[g_j(1)] = [g_j^+(1), g_j^-(1)] = [0.1, 0.2]$*,* $j = 1,2$*. Thus, (H1) and (H2) are satisfied. This fact can be seen in Figure 3.*

Moreover, we can have $\mathcal{A}_1 = \mathcal{A}_2 = 0.5$*,* $\mathcal{B}_1 = \mathcal{B}_2 = 0.2$*, and* $\lambda_0 = 2$*. Let* $\xi_1 = 75$ *and* $\xi_2 = 51$*, we can have*

$$\sup_{t\geq 0}\Big\{F_1(t,\lambda_0)\Big(\frac{c_1^+}{c_1^- - \lambda_0} + 1\Big)\Big\} \approx 0.4967 < 1,$$
$$\sup_{t\geq 0}\Big\{F_2(t,\lambda_0)\Big(\frac{c_2^+}{c_2^- - \lambda_0} + 1\Big)\Big\} \approx 0.5836 < 1,$$
$$\sup_{t\geq 0}\{-c_1^- + \lambda_0 + F_1(t,\lambda_0)\} \approx -0.1501 < 0,$$
$$\sup_{t\geq 0}\{-c_2^- + \lambda_0 + F_2(t,\lambda_0)\} \approx -0.5853 < 0,$$

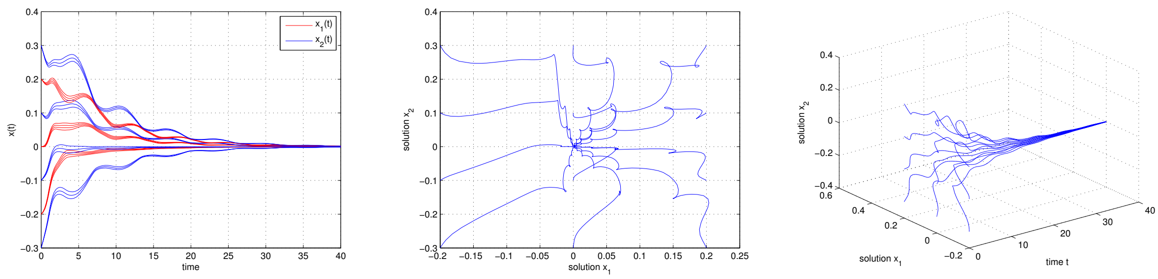
where $F_1(t,\lambda_0)$ *and* $F_2(t,\lambda_0)$ *are defined in (12) and (13).*



(**a**) (**b**)

Figure 3. (**a**) Discontinuous activation functions $f_j(j = 1, 2)$ for system (14); (**b**) Discontinuous activation functions $g_j(j = 1, 2)$ for system (14).

As a result, the coefficients of neural system (14) *satisfy all the conditions in Theorem 1. Hence, we can conclude that all solutions of the neural system* (14) *converge exponentially to the zero vector. This fact can be presented in the following Figure 4 by MATLAB software.*



(**a**) (**b**) (**c**)

Figure 4. (**a**) Time response of the state variables x_1 and x_2 for discontinuous HCNNs (14) with random initial conditions; (**b**) Phase plane behavior of the state variables x_1 and x_2 for discontinuous HCNNs (14); (**c**) Three-dimensional trajectory of state variables x_1 and x_2 for discontinuous HCNNs (14).

Remark 4. *From Examples 1 and 2, one can see the activations are discontinuous, unbounded and non-monotonic, that means the activations are not continuous, Lipschitz continuous or smooth, which are different from the related references in the literature, such as [2–6]. The results established in the present paper extend the previous work about HCNNs to the discontinuous cases.*

Remark 5. *In Example 1, the activation functions* $f_j(x)(j = 1, 2)$ *are discontinuous, non-decreasing, and* $g_j(x)(j = 1, 2)$ *are discontinuous, non-increasing. While in Example 2, the activation functions* $f_j(x)(j = 1, 2)$ *are discontinuous, non-increasing, and* $g_j(x)(j = 1, 2)$ *are discontinuous, non-decreasing. So, this paper presents a class of discontinuous neural networks with mixed discontinuous activations, which are different from those in [7,9,15–18] and the references related therein.*

5. Conclusions

In this paper, we are concerned with a class of HCNNs with mixed discontinuous activations and time-varying leakage delays. By applying differential inclusions theory and inequality technique, some sufficient conditions have been provided to guarantee the global exponential convergence of the solutions. In addition, two typical numerical examples and simulations are given to illustrate the effectiveness of the proposed criterion. The present paper can be regarded as the first time to study the dynamic behaviors of the discontinuous HCNNs and investigate the global exponential convergence of the HCNNs with mixed discontinuous activations and time-varying leakage delays. Consequently, some previous results are enriched and complemented.

As pointed out in [21,22], compared with periodic effects, almost periodic effects are more frequent in many real world applications. In fact, by a recent work [23,24], to some extent and in the sense of category, the "amount" of almost periodic functions (not periodic) is far more than the "amount" of continuous periodic functions. That is to say, almost periodic oscillatory behavior is considered to be more accordant with reality. The almost periodic systems are as a natural extension of the periodic ones. However, there is few results on the almost periodic solutions for the time-delayed HCNNs. These will be our further researches.

Author Contributions

Y.X.: data curation, writing–original draft preparation; N.Z.: writing–reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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