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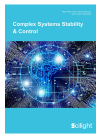

Article

Consensus of Linear Multi-Agent System with Unknown Input Time-Varying Delay by Linear Time-Varying Protocol

Jun-Hua Peng [1,2] and Kai Zhang [1,2,*]

[1] Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin 150001, China
[2] National Key Laboratory of Complex System Control and Intelligent Agent Cooperation, Harbin 150001, China
* Correspondence: kaizhang2023@hit.edu.cn

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Abstract: This paper addresses the leader-following consensus problem of linear multi-agent systems subject to unknown input time-varying delays. A linear time-varying control protocol is proposed based on the parametric Lyapunov equation (PLE), and all parameters can be computed offline. By exploiting the properties of the PLE, it is proven that the proposed method ensures asymptotic leader-following consensus without requiring any a priori information on the delay value or its upper and lower bounds. Simulation results are provided to validate the theoretical analysis and demonstrate the effect of the design parameter on the system performance under unknown input time-varying delays.

Keywords: leader-following consensus; parametric Lyapunov equation; multi-agent systems; unknown input time-varying delays

1. Introduction

The cooperative control of multi-agent systems (MASs) has attracted considerable attention in the control community over the past decades, owing to its wide applications in formation flying [1], coordinated robotics, sensor networks [2], military surveillance [3] and related areas. The fundamental objective of cooperative control is to design distributed control protocols based on local information exchange, such that all agents achieve a common global behavior, such as consensus, through interactions governed by an underlying communication topology [4,5]. In parallel, time-delay systems have also attracted sustained attention, due to the inevitable presence of delays induced by sensing, communication, computation, and actuation in practical engineering systems [6,7]. The existence of time delays, especially input delays, significantly complicates system analysis and controller design [8–10], as they may deteriorate control performance or even lead to instability. When time delays are incorporated into MASs, the challenges become more pronounced because delays interact with network couplings and distributed control structures, making stability and convergence analysis highly nontrivial. Therefore, the development of effective control strategies for MASs with input delays remains an open and challenging problem that deserves further investigation.

The consensus problem serves as a fundamental issue in cooperative control, aiming to design distributed protocols that enable all agents to agree on certain states through local interactions. In contrast to single agent systems, the consensus problem in MASs is inherently more complex due to the introduction of communication topologies, which constrain information exchange and couple individual agent dynamics with network interactions. The work in [11] laid the foundation for the consensus problem. Subsequently, convergence conditions for first-order MASs under connected undirected graphs were established in [12], and a unified analytical framework for consensus analysis was further developed in [13]. The consensus problem was later extended to directed communication graphs, where sufficient conditions based on the existence of a directed spanning tree were derived in [14]. For second-order agent dynamics, necessary and sufficient consensus conditions under directed topologies were provided in [15]. These early studies of consensus theory primarily focused on first- and second-order agent dynamics, where classical graph-theoretic tools and Lyapunov-based methods were employed to guarantee asymptotic convergence.

More recently, attention has shifted toward fully distributed consensus protocols that rely exclusively on local information without requiring global knowledge of the communication topology [16–18]. Despite these advances, research on consensus of MASs subject to input time delays, especially unknown and time-varying delays, remains relatively limited, and the fully distributed consensus problem under such delays is particularly challenging [19].

MASs with time delays constitute another vital research direction in cooperative control, since neglecting delays may significantly distort collective behaviors and even lead to instability of the overall system. In practical networked systems, time delays generally arise in two forms: communication delays caused by information transmission over networks, and input delays resulting from onboard computation, signal processing, or actuator dynamics. Owing to the intrinsic infinite-dimensional nature of time-delay systems, the analysis and control design for delayed MASs are considerably more challenging than those of delay-free systems. Existing control approaches for time-delay systems can generally be classified into infinite-dimensional and finite-dimensional methods. The former typically address delay dynamics through techniques such as predictor-based control and Lyapunov–Krasovskii functionals [20,21], while the latter aim to transform the delayed system into an equivalent delay-free or augmented finite-dimensional form for controller synthesis [22]. It should be emphasized that most of the results require precise knowledge of the time delays, or at least require the availability of their upper or lower bounds. Consequently, when the time delays are unknown and their bounds cannot be determined, these approaches may no longer be applicable. To address this limitation, a linear time-varying controller was proposed in [23] for single-agent systems with multiple input delays. In parallel, consensus problems of time-delay MASs were reported in [24–28]. However, the results in [27,28] are restricted to small delays, while the approach in [26] relies on infinite-dimensional terms, which significantly complicate implementation and analysis. A finite-dimensional control protocol based on truncated predictor feedback was proposed in [29], which is more amenable to practical implementation. Nevertheless, the consensus problem of MASs with arbitrarily large and unknown input delays remains unresolved and continues to pose significant theoretical and practical challenges.

In light of the above discussions, this paper investigates the leader-following consensus problem of MASs subject to unknown input time-varying delays. To clearly present the main results and technical developments, the remainder of this paper is organized as follows. Section II introduces necessary preliminaries, including basic concepts from graph theory, the parametric Lyapunov equation (PLE), and its key properties. In Section III, a linear time-varying state-feedback control protocol is proposed, whose parameters can be computed offline. By constructing a Lyapunov-Krasovskii-like functional, it is rigorously shown that the closed-loop MAS achieves consensus, and that the consensus error converges asymptotically to zero. Section IV provides a numerical simulation to demonstrate the effectiveness of the proposed approach. The main contributions are summarized as follows.

- A linear time-varying control protocol is developed, which avoids the nonlinear structures and infinite-dimensional representations commonly used in existing works (e.g., [26]), thereby simplifying the controller design and facilitating practical implementation.
- The proposed method accommodates completely unknown and arbitrarily large input time-varying delays. In contrast to existing results that require exact delay information or bounded delays, such as [27,28], the proposed protocol does not require any a priori knowledge on the delay values or their bounds.
- The computational burden is significantly reduced. Owing to the linear structure and the properties of the parametric Lyapunov equation (PLE), the controller parameters can be computed offline, thereby improving implementation efficiency.

Notation: Let $\mathbf{R}$ denote the set of real numbers. For a matrix $M \in \mathbf{R}^{n\times n}$, $M^{\mathrm{T}}, \|M\|, \mathrm{tr}\,(M), \lambda\,(M)$ and $\lambda_i\,(M)$ separately denotes its transpose, Euclidean norm, trace, set of eigenvalues and i-th eigenvalue. Moreover, $\bar{\lambda}\,(M)$ and $\underline{\lambda}\,(M)$ denote the maximum and minimum eigenvalue. Furthermore, $\phi\,(M) = \min_{i=1,2,\ldots n}\{\mathrm{Re}\,(\lambda_i(M))\}$ indicates the minimum real part of the eigenvalues of M. Let $\alpha\,(M)$ denotes the maximal order of the Jordan blocks associated with an eigenvalue $\lambda_i(M)$ such that $\mathrm{Re}\,(\lambda_i(M)) = \phi\,(M)$, ensuring that $\alpha\,(M) \geq 1$. If the matrix M is symmetric, $M > 0$ indicates positive definiteness. For matrices $X \in \mathbf{R}^{m\times n}$ and $Y \in \mathbf{R}^{p\times q}$, $X \otimes Y$ denotes the Kronecker product. The symbol $\mathbf{I}_p^q$, with integers $p \leq q$, is defined as set $\{p, p+1, \ldots, q\}$.

2. Problem Statement

For MASs with single time-varying input delay, each follower agent $i \in \mathbf{I}_1^N$ is described by

$$\begin{cases} \dot{x}_i\,(t) = Ax_i\,(t) + Bu_i\,(t-\tau\,(t)) \triangleq Ax_i\,(t) + Bu_i\,(\phi\,(t)) \\ y_i\,(t) = Cx_i\,(t)\,, i \in \mathbf{I}_1^N \end{cases} \tag{1}$$

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where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{q\times n}$ are constant matrices. $u_i \in \mathbf{R}^m$ is the control input with time-varying delay $\tau(t)$, and we define $\phi(t) = t - \tau(t)$. The leader satisfies

$$\dot{x}_0(t) = Ax_0(t). \tag{2}$$

The information exchange among agents is characterized by a weighted directed graph $\mathcal{G}(\mathcal{N}, \epsilon, \mathcal{R})$, where $\mathcal{N} = \{1, 2, \ldots, N+1\}$ is the node set and $\epsilon \subseteq \mathcal{N} \times \mathcal{N}$ is the edge set. The weighted adjacency matrix is $\mathcal{R} = [r_{ij}] \in \mathbf{R}^{(N+1)\times(N+1)}$, where $r_{ij} > 0$ if $(j, i) \in \epsilon$, (i.e., agent i receives information from agent j), and $r_{ij} = 0$ otherwise. Moreover $r_{ii} = 0$. Let $\mathcal{F}_i = \{j \in \mathcal{N} | (j, i) \in \epsilon\}$ denotes the set of in-neighbors of node i. Define the associated degree matrix $\mathcal{D} = \operatorname{diag}\{\mathcal{D}_i\}_{i=0}^N \in \mathbf{R}^{(N+1)\times(N+1)}$ with $\mathcal{D}_i = \sum_{j\in\mathcal{F}_i} r_{ij}$, and the Laplacian matrix is then defined by $\mathcal{L} = [l_{ij}] = \mathcal{D} - \mathcal{R}$. In this work, the leader is assumed to have no incoming edges, so that $\mathcal{L}$ admits the block representation:

$$\mathcal{L} = \begin{bmatrix} 0 & 0_{1\times N} \\ \mathcal{L}_2 & \mathcal{L}_1 \end{bmatrix},$$

where $\mathcal{L}_1 \in \mathbf{R}^{N\times N}$ and $\mathcal{L}_2 \in \mathbf{R}^{N\times 1}$. Each follower is assumed to access solely through relative information of its neighbors and itself for feedback. Specifically, agent i has access to $\tilde{x}_i = \sum_{j=0}^N r_{ij}(x_i - x_j), i \in \mathbf{I}_1^N$. Based on the above, the consensus problem is formulated as follows.

Problem 1. *Consider the linear MAS consisting of (1) and (2) under time-varying input delay. Design a linear protocol $u_i, i \in \mathbf{I}_1^N$ such that the closed-loop system achieves leader-following consensus, i.e., $\lim_{t\uparrow\infty} \|x_i(t) - x_0(t)\|_2 = 0$ for all $i \in \mathbf{I}_1^N$.*

To address Problem 1, we impose the following assumptions and lemmas.

Assumption 1. *The graph $\mathcal{G}(\mathcal{N}, \epsilon, \mathcal{R})$ admits a directed spanning tree rooted at the leader.*

Assumption 2. *The matrix pair (A, B) is controllable, and all eigenvalues of A are zero.*

Remark 1. *The condition in Assumption 2 can be extended to the case where (A, B) is stabilizable, and the eigenvalues of A are located at the origin or in the left half-plane. Since stable modes of A do not affect the stabilizability, we impose Assumption 2 for clarity of presentation.*

Assumption 3. *There exist two unknown constants $\bar{\tau} > 0$ and $d \in [0, 1)$ such that*

$$0 \leq \tau(t) \leq \bar{\tau}, \quad \dot{\tau}(t) \leq d, \quad \forall t \geq 0, \quad i \in \mathbf{I}_1^N.$$

Lemma 1. *Refs. [14,30] Under Assumption 1, $\mathcal{L}_1$ is nonsingular and all its eigenvalues have positive real parts.*

Our subsequent analysis relies on the following PLE:

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P = -\gamma P. \tag{3}$$

Several auxiliary lemmas concerning the PLE are recalled below.

Lemma 2. *Under Assumption 2, the PLE (3) admits a unique positive definite solution $P(\gamma)$ for any $\gamma > 0$ if and only if $\gamma > -2\phi(A)$. Moreover, $P(\gamma)$ can be expressed as $P(\gamma) = W^{-1}(\gamma)$, where $W(\gamma)$ is the unique positive definite solution to the following Lyapunov equation*

$$\left(A + \frac{1}{2}\gamma I_n\right) W + W \left(A + \frac{1}{2}\gamma I_n\right)^{\mathrm{T}} = BB^{\mathrm{T}}.$$

When all eigenvalues of A are identical and purely real, then

$$\frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma} \leq \frac{\delta_{\mathrm{c}} P(\gamma)}{n\gamma}, \ \forall \gamma > -2\phi(A),$$

where δ_{c} is a constant independent of γ and can be determined by

$$\delta_{\mathrm{c}} = \sup_{\gamma > -2\phi(A)} \{\delta_{\mathrm{c}}^{\gamma}\}, \ \delta_{\mathrm{c}}^{\gamma} = \pi(\gamma)\lambda_{\max}\left(W^{-\frac{1}{2}} U W^{-\frac{1}{2}}\right),$$

where $U = U(\gamma)$ is the unique positive definite solution to the Lyapunov equation

$$\left(A + \frac{1}{2}\gamma I_n\right) U + U\left(A + \frac{1}{2}\gamma I_n\right)^{\mathrm{T}} = W.$$

In addition, $P(\gamma)$ possesses the following inequalities

$$\frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma} \geq \frac{P(\gamma)}{n\gamma} \tag{4}$$

$$P(\gamma) BB^{\mathrm{T}} P(\gamma) \leq n\gamma P(\gamma) \tag{5}$$

$$\left(\frac{\mathrm{d}P}{\mathrm{d}\gamma}\right)^{\mathrm{T}} BB^{\mathrm{T}} \frac{\mathrm{d}P}{\mathrm{d}\gamma} \leq n\frac{\mathrm{d}P}{\mathrm{d}\gamma} \tag{6}$$

$$\frac{\mathrm{d}P}{\mathrm{d}\gamma} \leq \frac{\delta_{\mathrm{c}} P(\gamma)}{n\gamma} \tag{7}$$

$$A^{\mathrm{T}} P(\gamma) A \leq 3n^2\gamma^2 P(\gamma) \tag{8}$$

Finally, for any given $\gamma_0 > 0$, let $\delta \triangleq 2\alpha(A) - 1 \geq 1$. Then, there exist constants $\mu_1 = \mu_1(A, B)$ and $\mu_2 = \mu_2(\gamma_0)$ such that

$$\mu_1 \gamma^{\delta} I_n \leq P(\gamma) \leq \mu_2 \gamma I_n, \ \gamma \in (0, \gamma_0]. \tag{9}$$

The properties mentioned above are referenced from [23,31]. For completeness, we also recall the following lemma, which will be used in the subsequent proof.

Lemma 3. *Ref. [30] Under Assumption 1, $\mathcal{L}_1$ is nonsingular. Moreover, there exists a symmetric positive definite matrix S such that $S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}} S > 0$ holds. In particular, S can be chosen as $S = \mathrm{diag}(s_1, s_2, ..., s_N)$, where $[s_1, s_2, ..., s_N]^{\mathrm{T}} = \left(\mathcal{L}_1^{\mathrm{T}}\right)^{-1} \mathbf{1}_N$.*

3. State Feedback Protocol

Motivated by the protocol in [32], we propose the following linear state-feedback protocol and present the main result.

Theorem 1. *Let Assumptions 1–3 hold. Then under the distributed control law*

$$u_i(t) = -k_0 B^T P(\gamma(t)) \tilde{x}_i(t), \tag{10}$$

$$\gamma(t) = \frac{\gamma_0}{(t+1)^{\mu}}, \tag{11}$$

the MAS consisting of (1) and (2) achieves leader-following consensus (equivalently, the consensus error dynamics is asymptotically stable), where $P(\gamma)$ is the unique positive definite solution to the PLE (3), $k_0 > 0$ is a constant to be designed, $\gamma_0 > 0$ and $\mu \in (0, 1)$ are constants.

Moreover, differentiating $\gamma(t)$ yields $\dot{\gamma}(t) = -\mu\gamma_0/(t+1)^{\mu+1} = -\mu\gamma^{1/\mu+1}/\gamma_0^{1/\mu} < 0$.

Proof. Define the vectors $x = \left[x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, ..., x_N^{\mathrm{T}}\right]^{\mathrm{T}}$ and $u = \left[u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, ..., u_N^{\mathrm{T}}\right]^{\mathrm{T}}$. Substituting the protocol (10) into the agent dynamics (1) yields the closed-loop system

$$\dot{x}_i(t) = Ax_i(t) - k_0 BB^T P(\gamma(\phi(t))) \tilde{x}_i(\phi(t)). \tag{12}$$

In compact form, (12) can be written as

$$\begin{aligned}\dot{x}(t) &= (I_N \otimes A) x(t) - k_0\left(\mathcal{L}_1 \otimes BB^T P(\gamma(\phi(t)))\right) x(\phi(t)) \\ &= (I_N \otimes A) x(t) - k_0\left(\mathcal{L}_1 \otimes BB^T P(\gamma(t))\right) x(t) + \Delta(t),\end{aligned} \tag{13}$$

where $\Delta(t)$ is defined by

$$\Delta(t) = k_0\left(\mathcal{L}_1 \otimes BB^{\mathrm{T}}P(\gamma(t))\right)x(t) - k_0\left(\mathcal{L}_1 \otimes BB^{\mathrm{T}}P(\gamma(\phi(t)))\right)x(\phi(t)). \tag{14}$$

For notational convenience, we suppress the explicit time dependence when no confusion arises; for example, we write P for $P(\gamma(t))$ and $P(\phi)$ for $P(\gamma(\phi(t)))$.

Consider the following Lyapunov function

$$V_x = x^{\mathrm{T}}(S \otimes P(\gamma))x, \tag{15}$$

where S is defined in Lemma 3. Differentiating $V_x(t)$ along the trajectories of (13) yields

$$\begin{aligned}\dot{V}_x =& x^{\mathrm{T}}\left(S \otimes \frac{\mathrm{d}P}{\mathrm{d}t}\right)x + \dot{x}^{\mathrm{T}}(S \otimes P)x + x^{\mathrm{T}}(S \otimes P)\dot{x}\\ =&\dot{\gamma}x^{\mathrm{T}}\left(S \otimes \frac{\mathrm{d}P}{\mathrm{d}\gamma}\right)x + x^{\mathrm{T}}\left(S \otimes \left(A^{\mathrm{T}}P + PA\right)\right)x\\ &- k_0x^{\mathrm{T}}\left(\left(S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}}S\right) \otimes PBB^{\mathrm{T}}P\right)x + 2\Delta^{\mathrm{T}}(S \otimes P)x.\end{aligned} \tag{16}$$

Since S is positive definite, there holds $\underline{\lambda}(S)I_N \leq S \leq \bar{\lambda}(S)I_N$, and Lemma 3 ensures $S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}}S > 0$. Therefore it follows

$$k_0x^{\mathrm{T}}\left(\left(S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}}S\right) \otimes PBB^{\mathrm{T}}P\right)x \geq k_0\underline{\lambda}\left(S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}}S\right)\underline{\lambda}\left(S^{-1}\right)x^{\mathrm{T}}\left(S \otimes PBB^{\mathrm{T}}P\right)x. \tag{17}$$

Substituting (17) into (16) leads to

$$\begin{aligned}\dot{V}_x \leq&\dot{\gamma}x^{\mathrm{T}}\left(S \otimes \frac{\mathrm{d}P}{\mathrm{d}\gamma}\right)x + x^{\mathrm{T}}\left(S \otimes \left(A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P\right)\right)x\\ &+ \left(1 - k_0\underline{\lambda}\left(S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}}S\right)\underline{\lambda}\left(S^{-1}\right)\right)x^{\mathrm{T}}\left(S \otimes PBB^{\mathrm{T}}P\right)x + 2\Delta^{\mathrm{T}}(S \otimes P)x\\ \leq&\frac{\dot{\gamma}}{n\gamma}x^{\mathrm{T}}(S \otimes P)x - \gamma x^{\mathrm{T}}(S \otimes P)x + 2\Delta^{\mathrm{T}}(S \otimes P)x\\ &+ \left(1 - k_0\underline{\lambda}\left(S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}}S\right)\underline{\lambda}\left(S^{-1}\right)\right)x^{\mathrm{T}}\left(S \otimes PBB^{\mathrm{T}}P\right)x,\end{aligned} \tag{18}$$

where the second inequality holds due to the PLE (3) and property (4). By selecting appropriate k_0 such that $k_0\underline{\lambda}\left(S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}}S\right)\underline{\lambda}\left(S^{-1}\right) \geq 1$, Equation (18) can be continued as

$$\begin{aligned}\dot{V}_x \leq& \frac{\dot{\gamma}}{n\gamma}x^{\mathrm{T}}(S \otimes P)x - \gamma x^{\mathrm{T}}(S \otimes P)x + 2\Delta^{\mathrm{T}}(S \otimes P)x\\ \leq& \frac{\dot{\gamma}}{n\gamma}x^{\mathrm{T}}(S \otimes P)x - \gamma x^{\mathrm{T}}(S \otimes P)x + \frac{2}{\gamma}\Delta^{\mathrm{T}}(S \otimes P)\Delta + \frac{\gamma}{2}x^{\mathrm{T}}(S \otimes P)x\\ =& -\left(\gamma - \frac{\dot{\gamma}}{n\gamma} - \frac{\gamma}{2}\right)x^{\mathrm{T}}(S \otimes P)x + \frac{2}{\gamma}\Delta^{\mathrm{T}}(S \otimes P)\Delta\\ \leq& -\alpha x^{\mathrm{T}}(S \otimes P)x + \frac{2}{\gamma}\Delta^{\mathrm{T}}(S \otimes P)\Delta,\end{aligned} \tag{19}$$

where we used Young's inequality in the second inequality and we defined $\alpha \triangleq \gamma - \frac{\dot{\gamma}}{n\gamma} - \frac{\gamma}{2}$ in the third inequality.

Then we address the term $\Delta(t)$ in (19), which was defined in (14). Define $\psi(t) \triangleq k_0\left(\mathcal{L}_1 \otimes BB^{\mathrm{T}}P(\gamma(t))\right)x(t)$. Differentiating $\psi(t)$ and substituting (13) yields

$$\begin{aligned}\dot{\psi}(t) =&k_0\left(\mathcal{L}_1 \otimes BB^{\mathrm{T}}\frac{\mathrm{d}P}{\mathrm{d}t}\right)x(t) + k_0\left(\mathcal{L}_1 \otimes BB^{\mathrm{T}}P(t)\right)\dot{x}(t)\\ =&k_0\dot{\gamma}\left(\mathcal{L}_1 \otimes BB^{\mathrm{T}}\frac{\mathrm{d}P}{\mathrm{d}\gamma}\right)x(t) + k_0\left(\mathcal{L}_1 \otimes BB^{\mathrm{T}}P(t)\right)\\ &\times\left((I_N \otimes A)x(t) - k_0\left(\mathcal{L}_1 \otimes BB^{T}P(\phi)\right)x(\phi)\right)\\ =&k_0\dot{\gamma}\left(\mathcal{L}_1 \otimes BB^{\mathrm{T}}\frac{\mathrm{d}P}{\mathrm{d}\gamma}\right)x(t) + k_0\left(\mathcal{L}_1 \otimes BB^{\mathrm{T}}P(t)A\right)x(t)\\ &- k_0^2\left(\mathcal{L}_1^2 \otimes BB^{\mathrm{T}}P(t)BB^{\mathrm{T}}P(\phi)\right)x(\phi).\end{aligned} \tag{20}$$

Using the definition of $\phi(t) = t - \tau(t)$ and integrating both sides of (20), $\Delta(t)$ can be expressed as

$$\Delta(t) = \psi(t) - \psi(\phi(t)) = \int_{\phi(t)}^{t} \dot{\psi}(s)\,\mathrm{d}s = \int_{t-\tau}^{t} \theta_1(s) + \theta_2(s)\,\mathrm{d}s, \tag{21}$$

where

$$\theta_1(s) = k_0 \dot{\gamma} \left(\mathcal{L}_1 \otimes BB^{\mathrm{T}} \frac{\mathrm{d}P}{\mathrm{d}\gamma} \right) x(s) + k_0 \left(\mathcal{L}_1 \otimes BB^{\mathrm{T}} P(s) A \right) x(s),$$

and

$$\theta_2(s) = -k_0^2 \left(\mathcal{L}_1^2 \otimes BB^{\mathrm{T}} P(s) BB^{\mathrm{T}} P(\phi(s)) \right) x(\phi(s)).$$

Applying Young's inequality to separate the contributions of θ_1 and θ_2, and then using Jensen's inequality over the interval $[\phi(t), t]$, we obtain the estimate

$$\begin{aligned}
\Delta^{\mathrm{T}}\Delta =& \left(\int_{t-\tau}^{t} \theta_1(s) + \theta_2(s)\,\mathrm{d}s \right)^{\mathrm{T}} \left(\int_{t-\tau}^{t} \theta_1(s) + \theta_2(s)\,\mathrm{d}s \right) \\
\leq& \left(1 + \frac{1}{k_1}\right) \left(\int_{t-\tau}^{t} \theta_1(s)\,\mathrm{d}s \right)^{\mathrm{T}} \left(\int_{t-\tau}^{t} \theta_1(s)\,\mathrm{d}s \right) \\
&+ (1 + k_1) \left(\int_{t-\tau}^{t} \theta_2(s)\,\mathrm{d}s \right)^{\mathrm{T}} \left(\int_{t-\tau}^{t} \theta_2(s)\,\mathrm{d}s \right) \\
\leq& \tau \left(1 + \frac{1}{k_1}\right) \int_{t-\tau}^{t} \theta_1^{\mathrm{T}}(s)\,\theta_1(s)\,\mathrm{d}s \\
&+ \tau (1 + k_1) \int_{t-\tau}^{t} \theta_2^{\mathrm{T}}(s)\,\theta_2(s)\,\mathrm{d}s,
\end{aligned} \tag{22}$$

On the one hand, we first bound θ_1 as

$$\begin{aligned}
&\theta_1^{\mathrm{T}}(s)\,\theta_1(s) \\
=& k_0^2 \dot{\gamma}^2 x^{\mathrm{T}}(s) \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \otimes \left(\frac{\mathrm{d}P}{\mathrm{d}\gamma} \right)^{\mathrm{T}} BB^{\mathrm{T}} BB^{\mathrm{T}} \frac{\mathrm{d}P}{\mathrm{d}\gamma} \right) x(s) \\
&+ 2k_0^2 \dot{\gamma} x^{\mathrm{T}}(s) \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \otimes \left(\frac{\mathrm{d}P}{\mathrm{d}\gamma} \right)^{\mathrm{T}} BB^{\mathrm{T}} BB^{\mathrm{T}} P(s) A \right) x(s) \\
&+ k_0^2 x^{\mathrm{T}}(s) \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \otimes A^{\mathrm{T}} P(s) BB^{\mathrm{T}} BB^{\mathrm{T}} P(s) A \right) x(s) \\
\leq& 2k_0^2 \dot{\gamma}^2 x^{\mathrm{T}}(s) \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \otimes \left(\frac{\mathrm{d}P}{\mathrm{d}\gamma} \right)^{\mathrm{T}} BB^{\mathrm{T}} BB^{\mathrm{T}} \frac{\mathrm{d}P}{\mathrm{d}\gamma} \right) x(s) \\
&+ 2k_0^2 x^{\mathrm{T}}(s) \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \otimes A^{\mathrm{T}} P(s) BB^{\mathrm{T}} BB^{\mathrm{T}} P(s) A \right) x(s) \\
\leq& 2k_0^2 \delta_c \bar{\lambda} \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \right) \bar{\lambda} \left(S^{-1} \right) \|B\|^2 \frac{\dot{\gamma}^2}{\gamma} x^{\mathrm{T}}(s) (S \otimes P) x(s) \\
&+ 6k_0^2 n^3 \bar{\lambda} \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \right) \bar{\lambda} \left(S^{-1} \right) \|B\|^2 \gamma^3 x^{\mathrm{T}}(s) (S \otimes P(s)) x(s) \\
=& 2k_0^2 \delta_c \mu^2 \gamma_0^{-\frac{2}{\mu}} \bar{\lambda} \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \right) \bar{\lambda} \left(S^{-1} \right) \|B\|^2 \gamma^{1+\frac{2}{\mu}} x^{\mathrm{T}}(s) (S \otimes P) x(s) \\
&+ 6k_0^2 n^3 \bar{\lambda} \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \right) \bar{\lambda} \left(S^{-1} \right) \|B\|^2 \gamma^3 x^{\mathrm{T}}(s) (S \otimes P(s)) x(s),
\end{aligned} \tag{23}$$

where we have used Young's inequality in the first inequality and the PLE properties (4)–(8) in the second inequality. Noting that $\dot{\gamma}(t) = -\mu \gamma^{1/\mu+1} / \gamma_0^{1/\mu}$, Equation (23) can be further simplified to

$$\begin{aligned}
&\theta_1^{\mathrm{T}}(s)\,\theta_1(s) \\
=& 2k_0^2 \delta_c \mu^2 \gamma_0^{-\frac{2}{\mu}} \bar{\lambda} \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \right) \bar{\lambda} \left(S^{-1} \right) \|B\|^2 \gamma^{1+\frac{2}{\mu}} x^{\mathrm{T}}(s) (S \otimes P) x(s) \\
&+ 6k_0^2 n^3 \bar{\lambda} \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \right) \bar{\lambda} \left(S^{-1} \right) \|B\|^2 \gamma^3 x^{\mathrm{T}}(s) (S \otimes P(s)) x(s) \\
=& 2k_0^2 \delta_c \mu^2 \gamma_0^{-\frac{2}{\mu}} \bar{\lambda} \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \right) \bar{\lambda} \left(S^{-1} \right) \|B\|^2 \gamma^{1+\frac{2}{\mu}} V_x(s) \\
&+ 6k_0^2 n^3 \bar{\lambda} \left(\mathcal{L}_1^{\mathrm{T}} \mathcal{L}_1 \right) \bar{\lambda} \left(S^{-1} \right) \|B\|^2 \gamma^3 V_x(s).
\end{aligned} \tag{24}$$

On the other hand, following the same method we have the estimate of θ_2 as

$$\begin{aligned}
&\theta_2^{\mathrm{T}}(s)\,\theta_2(s)\\
=&k_0^4 x^{\mathrm{T}}(\phi)\left(\left(\mathcal{L}_1^2\right)^{\mathrm{T}}\mathcal{L}_1^2\otimes P(\phi)\,BB^{\mathrm{T}}P(s)\,BB^{\mathrm{T}}BB^{\mathrm{T}}P(s)\,BB^{\mathrm{T}}P(\phi)\right)x(\phi)\\
\leq&k_0^4 n^2\mu_2\bar{\lambda}\left(\left(\mathcal{L}_1^2\right)^{\mathrm{T}}\mathcal{L}_1^2\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^4\gamma^2(s)\,\gamma(\phi)\,x^{\mathrm{T}}(\phi)\left(S\otimes P(\phi)\right)x(\phi)\\
\leq&k_0^4 n^2\mu_2\bar{\lambda}\left(\left(\mathcal{L}_1^2\right)^{\mathrm{T}}\mathcal{L}_1^2\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^4\gamma^3(\phi)\,x^{\mathrm{T}}(\phi)\left(S\otimes P(\phi)\right)x(\phi)\\
=&k_0^4 n^2\mu_2\bar{\lambda}\left(\left(\mathcal{L}_1^2\right)^{\mathrm{T}}\mathcal{L}_1^2\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^4\gamma^3(\phi)\,V_x(\phi),
\end{aligned}\tag{25}$$

where we used (5) and (9) in the first inequality and $\gamma(s)<\gamma(\phi)$ in the second inequality. To handle the delayed argument $\phi(s)$ in the integral term, we define $\vartheta=s-\tau(s)=\phi(s)$. Then we have the fact that $\mathrm{d}\vartheta=1-\dot{\tau}(s)\,\mathrm{d}s$ and $\tau(s)<\bar{\tau}$, which yields

$$\begin{aligned}
&\int_{t-\tau}^{t}\gamma^3(\phi(s))\,V_x(\phi(s))\,\mathrm{d}s\\
=&\int_{t-\tau-\tau(t-\tau)}^{t-\tau}\frac{1}{1-\dot{\tau}(s)}\gamma^3(\vartheta)\,V_x(\vartheta)\,\mathrm{d}\vartheta\\
\leq&\frac{1}{1-d}\int_{t-2\bar{\tau}}^{t}\gamma^3(\vartheta)\,V_x(\vartheta)\,\mathrm{d}\vartheta\\
=&\frac{1}{1-d}\int_{t-2\bar{\tau}}^{t}\gamma^3(s)\,V_x(s)\,\mathrm{d}s.
\end{aligned}\tag{26}$$

Then together with (26), substituting (24) and (25) into (22) yields

$$\begin{aligned}
\Delta^{\mathrm{T}}\Delta\leq&\tau\left(1+\frac{1}{k_1}\right)\int_{t-\tau}^{t}\theta_1^{\mathrm{T}}(s)\,\theta_1(s)\,\mathrm{d}s+\tau(1+k_1)\int_{t-\tau}^{t}\theta_2^{\mathrm{T}}(s)\,\theta_2(s)\,\mathrm{d}s\\
\leq&2k_0^2\delta_c\mu^2\gamma_0^{-\frac{2}{\mu}}\tau\left(1+\frac{1}{k_1}\right)\bar{\lambda}\left(\mathcal{L}_1^{\mathrm{T}}\mathcal{L}_1\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^2\int_{t-\tau}^{t}\gamma^{1+\frac{2}{\mu}}V_x(s)\,\mathrm{d}s\\
&+6k_0^2n^3\tau\left(1+\frac{1}{k_1}\right)\bar{\lambda}\left(\mathcal{L}_1^{\mathrm{T}}\mathcal{L}_1\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^2\int_{t-\tau}^{t}\gamma^3V_x(s)\,\mathrm{d}s\\
&+k_0^4n^2\mu_2\tau(1+k_1)\,\bar{\lambda}\left(\left(\mathcal{L}_1^2\right)^{\mathrm{T}}\mathcal{L}_1^2\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^4\int_{t-\tau}^{t}\gamma^3(\phi)\,V_x(\phi)\,\mathrm{d}s\\
\leq&2k_0^2\delta_c\mu^2\gamma_0^{-\frac{2}{\mu}}\tau\left(1+\frac{1}{k_1}\right)\bar{\lambda}\left(\mathcal{L}_1^{\mathrm{T}}\mathcal{L}_1\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^2\int_{t-\tau}^{t}\gamma^{1+\frac{2}{\mu}}V_x(s)\,\mathrm{d}s\\
&+6k_0^2n^3\tau\left(1+\frac{1}{k_1}\right)\bar{\lambda}\left(\mathcal{L}_1^{\mathrm{T}}\mathcal{L}_1\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^2\int_{t-\tau}^{t}\gamma^3V_x(s)\,\mathrm{d}s\\
&+k_0^4n^2\mu_2\tau(1+k_1)\,\frac{1}{1-d}\bar{\lambda}\left(\left(\mathcal{L}_1^2\right)^{\mathrm{T}}\mathcal{L}_1^2\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^4\int_{t-2\bar{\tau}}^{t}\gamma^3(s)\,V_x(s)\,\mathrm{d}s.
\end{aligned}\tag{27}$$

For any $k_1>0$, we can upper-bound the coefficients in (27) by a constant $k_2>0$. Specifically, there exists a $k_1>0$ such that

$$\begin{aligned}
k_2\triangleq&\max\left\{\delta_c\mu^2\gamma_0^{-\frac{2}{\mu}},3n^3\right\}\times2k_0^2\tau\left(1+\frac{1}{k_1}\right)\bar{\lambda}\left(\mathcal{L}_1^{\mathrm{T}}\mathcal{L}_1\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^2\\
=&\frac{1}{1-d}k_0^4n^2\mu_2\tau(1+k_1)\,\bar{\lambda}\left(\left(\mathcal{L}_1^2\right)^{\mathrm{T}}\mathcal{L}_1^2\right)\bar{\lambda}\left(S^{-1}\right)\|B\|^4.
\end{aligned}$$

Then (27) can be written as

$$\Delta^{\mathrm{T}}\Delta\leq3k_2\int_{t-2\bar{\tau}}^{t}\gamma^3(s)\,V_x(s)\,\mathrm{d}s.\tag{28}$$

Substituting (28) into (19) yields

$$\begin{aligned}\dot{V}_x &\leq -\alpha x^{\mathrm{T}} (S \otimes P) x + \frac{2}{\gamma} \Delta^{\mathrm{T}} (S \otimes P) \Delta \\ &\leq -\alpha V_x(t) + 6k_2\mu_2\bar{\lambda}\left(S^{-1}\right) \int_{t-2\bar{\tau}}^{t} \gamma^3(s) V_x(s)\,\mathrm{d}s, \end{aligned} \tag{29}$$

where we used (5) in the second inequality. Then we introduce the following Lyapunov functional

$$V(t) = V_x(t) + V_0(t),$$

where

$$V_0(t) = k_3 \int_{t-2\bar{\tau}}^{t} \left(3 + \frac{s-t}{\bar{\tau}}\right) \gamma^3(s) V_x(s)\,\mathrm{d}s, \tag{30}$$

with $k_3 > 0$ being a constant to be designed. Note that the weighting function $3 + (s-t)/\bar{\tau} \in [1, 3]$ for all $s \in [t - 2\bar{\tau}, t]$, which implies

$$k_3 \int_{t-2\bar{\tau}}^{t} \gamma^3(s) V_x(s)\,\mathrm{d}s \leq V_0(t) \leq 3k_3 \int_{t-2\bar{\tau}}^{t} \gamma^3(s) V_x(s)\,\mathrm{d}s. \tag{31}$$

By Leibniz's rule, differentiating $V_0(t)$ yields

$$\dot{V}_0(t) = 3k_3\gamma^3(t) V_x(t) - k_3\gamma^3(t-2\bar{\tau}) V_x(t-2\bar{\tau}) - \frac{k_3}{\bar{\tau}} \int_{t-2\bar{\tau}}^{t} \gamma^3(s) V_x(s)\,\mathrm{d}s. \tag{32}$$

Combining (29) and (32) gives

$$\begin{aligned}\dot{V}(t) =& \dot{V}_x(t) + \dot{V}_0(t) \\ \leq& -\alpha V_x(t) + 6k_2\mu_2\bar{\lambda}\left(S^{-1}\right) \int_{t-2\bar{\tau}}^{t} \gamma^3(s) V_x(s)\,\mathrm{d}s \\ &+ 3k_3\gamma^3 V_x(t) - k_3\gamma^3(t-2\bar{\tau}) V_x(t-2\bar{\tau}) \\ &- \frac{k_3}{\bar{\tau}} \int_{t-2\bar{\tau}}^{t} \gamma^3(s) V_x(s)\,\mathrm{d}s \\ \leq& -\left(\alpha - 3k_3\gamma^3\right) V_x(t) \\ &+ \left(6k_2\mu_2\bar{\lambda}\left(S^{-1}\right) - \frac{k_3}{\bar{\tau}}\right) \int_{t-2\bar{\tau}}^{t} \gamma^3(s) V_x(s)\,\mathrm{d}s, \end{aligned} \tag{33}$$

where we chose $k_3 > 6k_2\mu_2\bar{\lambda}\left(S^{-1}\right)\bar{\tau}$ in the last inequality. Then by using (31), we obtain

$$\dot{V}(t) \leq -k_4 V_x(t) - k_5 V_0(t), \tag{34}$$

where we defined $k_4 = \alpha - 3k_3\gamma^3$ and $k_5 = -(6k_2\mu_2\bar{\lambda}\left(S^{-1}\right) - k_3/\bar{\tau})/3k_3$. Recalling the definition of α and γ, we have

$$\begin{aligned}k_4 &= \gamma - \frac{\dot{\gamma}}{n\gamma} - \frac{\gamma}{2} - 3k_3\gamma^3 \\ &= \left(\frac{1}{2} + \frac{1}{n\gamma_0}\right)\gamma - 3k_3\gamma^3. \end{aligned}$$

Since $\gamma(t)$ is strictly decreasing and tends toward zero along time, there exist a finite time $t_0 > 0$ such that for all $t > t_0$, the inequalities $k_4 \geq \gamma/4$ and $k_5 \geq \gamma/4$ hold, namely

$$\gamma(t) \leq \min\left\{\sqrt{\frac{n\gamma_0 + 4}{12nk_3\gamma_0}}, \frac{4}{3\bar{\tau}} - \frac{8k_2\mu_2\bar{\lambda}\left(S^{-1}\right)}{k_3}\right\}.$$

Then (34) can be continued as

$$\dot{V}(t) \leq -\frac{\gamma}{4} V(t),$$

which, by using comparison principle, can be further continued as

$$V(t) \leq \exp\left(-\frac{1}{4}\int_{t_0}^{t}\gamma(s)\mathrm{d}s\right)V(t_0). \tag{35}$$

Substituting $\gamma(t) = \gamma_0/\,(t+1)^{\mu}$ with $\mu \in (0,1)$ into (35) yields

$$\begin{aligned} V(t) &\leq \exp\left(-\frac{\gamma_0\left((t+1)^{1-\mu} - (t_0+1)^{1-\mu}\right)}{4(1-\mu)}\right)V(t_0) \\ &= \frac{f\,(t_0)}{f\,(t)}V(t_0),\ \forall t \geq t_0, \end{aligned} \tag{36}$$

where $f\,(t) = \exp(\gamma_0(t+1)^{1-\mu}/4(1-\mu))$. In addition, since $V(t) \geq V_x(t) = x^{\mathrm{T}}(t)\,(S \otimes P\,(\gamma))\,x(t)$, it follows from (15) and (30) that

$$\begin{aligned} V(t) &\geq \mu_1\gamma^{\delta}\underline{\lambda}(S)\,\|x(t)\|^2 \\ &= \frac{\mu_1\gamma_0^{\delta}\underline{\lambda}(S)}{(t+1)^{\delta\mu}}\,\|x(t)\|^2, \end{aligned} \tag{37}$$

where the inequality holds due to (9). By using (36) and (37), we deduce that

$$\|x(t)\|^2 \leq \frac{f\,(t_1)}{\mu_1\gamma_0^{\delta}}\frac{(t+1)^{\delta\mu}}{f\,(t)}V(t_1),\ \forall t \geq t_1. \tag{38}$$

In addition, since the closed-loop system is a linear time-delay system with uniformly bounded coefficients, then its solutions depend continuously on the initial history. In particular, for any $T \geq 0$, there exists a constant $c_1 = c_1(T) > 0$ such that (see pp. 141–143 in [33])

$$\|x(t)\| \leq c_1\left(\|x_0\| + \sup_{\theta\in[-\bar{\tau},0)}\|u_0(\theta)\|\right), \forall t \in [0,T]. \tag{39}$$

Thus, it follows from $V(t) \leq c_2\,\|x(t)\|^2$ and (36)–(38), where $c_2 > 0$ is a constant, that $\lim_{t\to\infty}\|x(t)\| = 0$. Therefore, the origin of the closed-loop system (13) is attractive. Also, it follows from (38) and (39) that, for any given $\varepsilon > 0$, there exists a $\delta_0 > 0$ such that $(\|x_0\| + \sup_{\theta\in[-\bar{\tau},0)}\|u_0(\theta)\|) \leq \delta_0 \Rightarrow \|x(t)\| \leq \varepsilon, \forall t \geq 0$, namely, the closed-loop system (13) is also stable in the sense of Lyapunov. Since the origin is Lyapunov stable and attractive, the closed-loop system (13) is asymptotically stable, which establishes the fact that the consensus is achieved, and the proof is finished. □

Remark 2. *In Theorem 1, the condition $\dot{\tau}(t) \leq d < 1$ is imposed to guarantee that $1 - \dot{\tau}(t) \geq 1 - d > 0$. This property is essential for the change of variables $\vartheta = \phi(s)$ used in (26), which allows us to bound the integral terms involving the delayed argument. It is worth noting that the Lyapunov candidate $V_x = x^{\mathrm{T}}(S \otimes P(\gamma))x$ is not uniformly positive definite with respect to x, since $P(\gamma)$ vanishes as $\gamma(t) \to 0$. Hence, there does not exist a time-independent constant $d_1 > 0$ such that $V_x(t) \geq d_1\,\|x\|^2$ for all $t \geq 0$. This lack of a uniform lower bound prevents a direct application of standard Razumikhin-type arguments based on time-invariant comparison functions [34]. The augmented functional $V\,(t) = V_x\,(t) + V_0\,(t)$ is therefore introduced to overcome this issue and to establish asymptotic stability. Finally, although $\tau(t)$ (and d) appear in the analysis through the proof, they are not needed in the implementation of the protocol (10) and (11), and the controller parameters can be chosen without any a priori knowledge of the delay magnitude or its bounds.*

Remark 3. *To cope with unknown time-varying input delays, we introduce the time-varying parameter $\gamma\,(t)$ in (11). Consequently, the closed-loop dynamics becomes time-varying. The stability result established in Theorem 1 is therefore asymptotic rather than uniform exponential. This can be seen more explicitly from (38). Noting that the time-varying terms consist of a polynomial factor $(t+1)^{\delta\mu}$ multiplied by an exponentially decaying factor $f^{-1}(t) = \exp(-\gamma_0(t+1)^{1-\mu}/4(1-\mu))$, the convergence rate is sub-exponential and is governed by $\int_{t_1}^{t}\gamma\,(s)\,\mathrm{d}s$, which can also be seen from $\dot{V}\,(t) \leq -\gamma(t)V(t)/4$. In addition, the parameter μ directly controls the convergence rate.*

4. Simulation Results

In this section, numerical simulations are conducted to verify the effectiveness of the proposed control strategy. We consider a MAS described by (1) and (2), consisting of one leader and six followers. The system satisfies Assumption 2, and each follower agent is modeled as a double-integrator system, which is a representative model widely adopted in MASs and consensus studies. The system matrices are given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The communication topology among the agents is depicted as

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix}.$$

It can be verified that the above Laplacian matrix satisfies the conditions stated in Assumption 1. To further reflect realistic delayed environments, a time-varying delay is introduced and selected as

$$\tau(t) = 0.3 + 0.3\sin(5t),$$

which satisfies Assumption 3. By solving PLE (3), we obtain

$$P(\gamma(t)) = \begin{bmatrix} \gamma^3(t) & \gamma^2(t) \\ \gamma^2(t) & 2\gamma(t) \end{bmatrix},$$

which is subsequently used to construct the control protocol (10). Specifically, after fixing the scheduling law $\gamma(t) = \gamma_0/(t+1)^{\mu}$, the matrix $P(\gamma(t))$ is obtained by direct substitution into the above analytical expression, and then the control input is computed from protocol (10) at each time instant. The initial states of the leader and the followers are randomly selected as $x_0(0) = [10, -5]^{\mathrm{T}}$, $x_1(0) = [0, -15]^{\mathrm{T}}$, $x_2(0) = [4, -13]^{\mathrm{T}}$, $x_3(0) = [-4, -11]^{\mathrm{T}}$, $x_4(0) = [12, -9]^{\mathrm{T}}$, $x_5(0) = [-8, -6]^{\mathrm{T}}$, $x_6(0) = [15, -3]^{\mathrm{T}}$.

Following the theoretical design procedure, we first compute S from Lemma 3, and then use $S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}}S$ to derive a feasible lower bound of the control gain k_0 from the stability condition. A more detailed discussion of this parameter selection procedure is given in Remark 4. Under the above settings, the control parameter is required to satisfy

$$k_0 > \frac{1}{\underline{\lambda}\left(S\mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}}S\right)\underline{\lambda}\left(S^{-1}\right)} = \frac{1}{1.2228 * 0.1667} \approx 0.1363.$$

After this feasibility condition is satisfied, k_0, γ_0 and μ are further adjusted to illustrate the trade-off between convergence speed and control effort. In the simulations, we select $k_0 = 2$ to ensure sufficient convergence performance. For the time-varying parameter $\gamma(t)$, the design parameters are chosen as $\gamma_0 = 0.5$ and $\mu = 0.8$. Under these conditions, the simulation results are presented in Figures 1 and 2.

Figure 1 illustrates the state trajectories of the leader and the followers, from which it can be observed that all follower agents asymptotically converge to the leader's state, thereby achieving leader–following consensus. Figure 2 depicts the control input trajectories of all followers. It can be seen that the control inputs are influenced by the introduced time-varying input delay, while the proposed control strategy still guarantees the convergence of the system, demonstrating the effectiveness and robustness of the proposed consensus control scheme.

To further investigate the influence of the control gain k_0 on the system performance, a parametric study is conducted by selecting $k_0 = 1.5$, 2, 2.5, and 3, respectively. Since follower 6 exhibits the most pronounced state and control variations in Figures 1 and 2, we further examine the norm of its state $||x_6(t)||$ and its control input $u_6(t)$ under different values of k_0, and the results are shown in Figures 3 and 4.

Figure 3 illustrates the evolution of the state norm of follower 6 while Figure 4 presents the control input trajectories of follower 6, both for different values of k_0 with γ_0 and μ fixed. It can be observed that increasing k_0 leads to a faster convergence rate, but also results in larger control variations, more pronounced oscillations, and higher peak values.

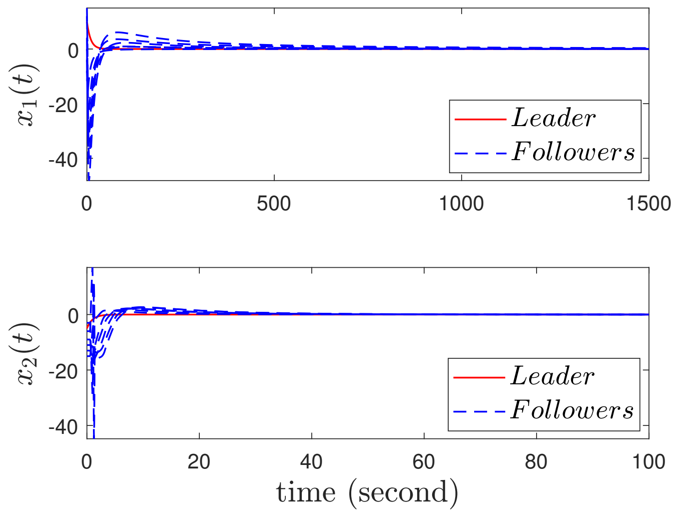

Figure 1. State trajectories of agents.

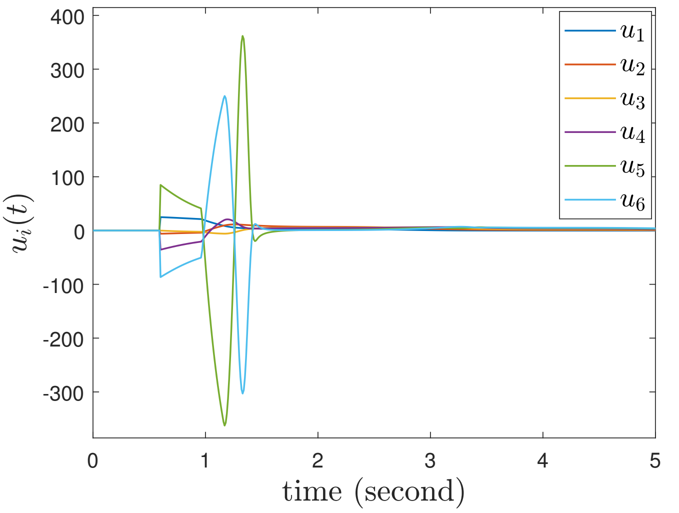

Figure 2. Control inputs of agents.

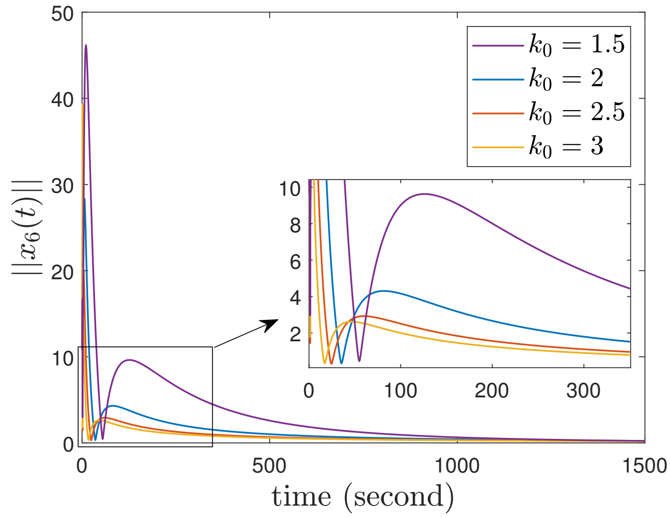

Figure 3. The state's norm of follower 6 under different k_0 ($\gamma_0 = 0.5$, $\mu = 0.8$).

Similarly, Figures 5 and 6 show the influence of the parameter μ under fixed γ_0 and k_0, where the plotted quantities are again $||x_6(t)||$ and $u_6(t)$. A smaller μ results in a faster decay of the state norm, indicating an improved convergence rate, while the control inputs exhibit larger magnitudes and more rapid variations, leading to higher peaks.

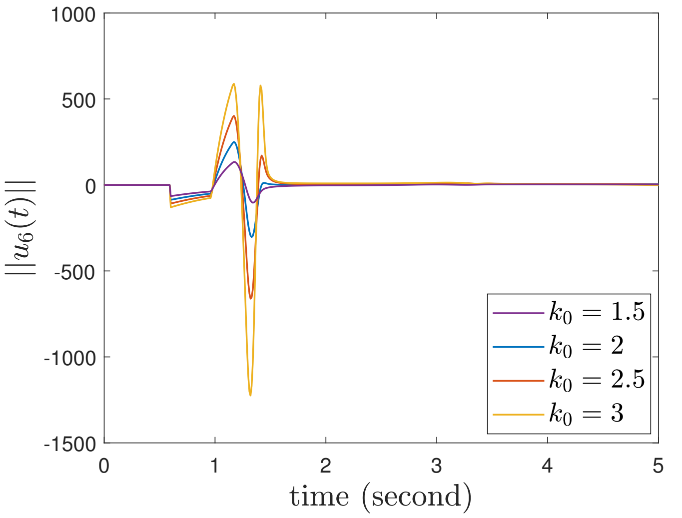

Figure 4. The control input of follower 6 under different k_0 ($\gamma_0 = 0.5$, $\mu = 0.8$).

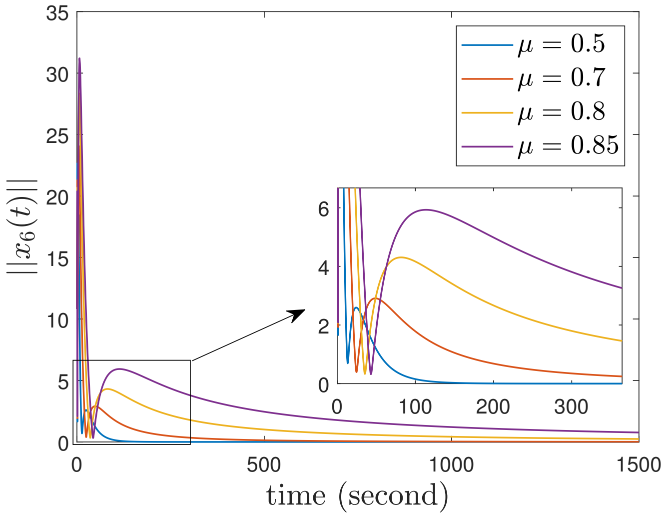

Figure 5. The state's norm of follower 6 under different μ ($\gamma_0 = 0.5$, $k_0 = 2$).

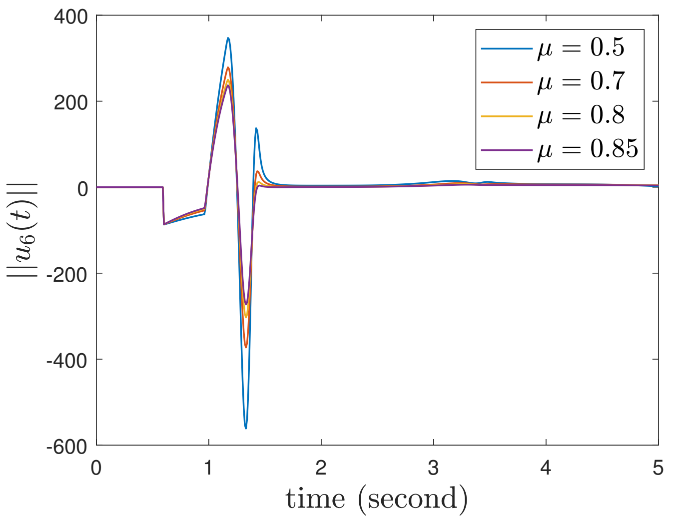

Figure 6. The control input of follower 6 under different μ ($\gamma_0 = 0.5$, $k_0 = 2$).

Overall, the results in Figures 3–6 reveal a clear trade-off between convergence speed and control effort. A smaller k_0 or a larger μ yields smoother control inputs at the expense of slower convergence, while a larger k_0 or a smaller μ accelerates convergence but introduces stronger input fluctuations and higher control peaks. These observations are consistent with the theoretical analysis. Therefore, in practical applications, the parameters k_0 and μ should be selected by jointly considering the transient performance requirements and the allowable control input magnitude/variation.

Remark 4. *The controller parameters can be selected in a structured manner.*

First, the feedback gain k_0 should be chosen to satisfy

$$k_0 \lambda\big(SL_1 + L_1^T S\big)\, \lambda\big(S^{-1}\big) \geq 1,$$

which is the sufficient condition required in the stability analysis. Therefore, k_0 is not chosen heuristically, but according to an explicit theoretical lower bound. In practice, k_0 can be selected slightly above this bound to guarantee stability, and then increased if faster convergence is desired.

Next, the parameters γ_0 and μ in

$$\gamma(t) = \frac{\gamma_0}{(t+1)^{\mu}}, \gamma_0 > 0,\ \mu \in (0,1),$$

are used to shape the transient performance of the closed-loop system. Specifically, γ_0 determines the initial scale of the time-varying gain matrix $P(\gamma(t))$, while μ governs the decay rate of $\gamma(t)$. A larger k_0 or a smaller μ generally leads to faster convergence, but also causes larger control peaks and stronger input variations. By contrast, a smaller k_0 or a larger μ yields smoother control inputs at the expense of slower convergence. Therefore, after satisfying the theoretical condition on k_0, the parameters γ_0 and μ can be tuned according to the desired trade-off between convergence speed and control effort.

Although the above procedure provides a theoretically guided rule, the final choice of parameters within the admissible range still depends on practical performance requirements. This also motivates future research on adaptive or self-tuning controller design to simplify parameter tuning.

Remark 5. *Compared with [26–29], and [23], the difference of the present work can be understood more directly from three aspects. First, unlike [27,28], which study delayed consensus under bounded-delay setting, this paper considers leader-following consensus of linear MASs with completely unknown input time-varying delays, and the implementation does not require the actual delay value or its bounds. Second, unlike the predictor-based designs in [26,29], the proposed controller is not constructed through predictor compensation, but is generated directly from the solution of the PLE, which preserves a finite-dimensional linear time-varying structure and is easier to compute. Third, unlike [23], which focuses on stabilization of a single delayed system, this paper addresses the distributed consensus problem of MASs under directed communication graphs. In this sense, the novelty of the present work lies in extending the PLE-based linear time-varying design from single-system stabilization to multi-agent consensus under unknown time-varying input delays.*

5. Conclusions

This paper has investigated the leader-following consensus for a class of multi-agent systems (MASs) with unknown time-varying delays. To address the challenges introduced by such delays, a linear time-varying control framework based on a parametric Lyapunov equation (PLE) has been developed. By introducing the properties of the PLE, a time-varying parameter $\gamma(t)$ is incorporated into the control design, and the solution of the PLE enables the construction of the proposed protocol. Sufficient conditions are derived to guarantee the asymptotic convergence of all follower states. In contrast to existing methods, the proposed approach does not require any information about the time delay, including its actual value or its upper and lower bounds. Numerical simulations further verified the effectiveness of the proposed control strategy. Future work will focus on developing fully distributed protocols for MASs with unknown input time-varying delays, as well as investigating adaptive controller design. Another interesting direction is to extend the present framework to fractional-order MASs, particularly in the presence of stochastic impulsive effects [35], switching [36] and delay [37], as well as in more general stochastic delay settings [38–41].

Author Contributions

K.Z.: conceptualization, supervision, writing—review and editing. J.-H.P.: methodology, software, investigation, visualization, writing—original draft preparation, validation. All authors have read and agreed to the published version of the manuscript.

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Not applicable.

Data Availability Statement

The simulation data and related materials supporting the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

During the preparation of this work, the authors used ChatGPT to assist with language polishing. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

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