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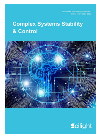

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Stability in Distribution of Stochastic Delay Differential Equations Driven by Wiener Processes versus Poisson Jumps Processes in Hilbert Spaces

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Abstract: The present paper is concerned with the distributional stability of a class of stochastic delay differential equations that are formulated within Hilbert spaces, which are driven by Wiener processes as well as Poisson jump processes. Utilizing the weak convergence method, sufficient conditions for this stability property are established. To illustrate the theoretical findings, two pertinent examples are presented.

Keywords: stability in distribution; Hilbert space; Wiener process; Poisson jump process

Mathematics Subject Classification: 60H15; 60G15; 60H05

1. Introduction

Stochastic delay differential equations (SDDEs) are extensively employed to depict diverse dynamic behaviors in a wide range of disciplines, such as financial economics, natural sciences, and engineering fields. Within the framework of stochastic dynamical systems, stability analysis constitutes a fundamental research topic, with notable forms such as mean-square stability, exponential stability, asymptotic stability, almost sure stability, and stability in probability, among others (see, e.g., [1–5]).

It should be pointed out that the aforementioned stability concepts might be excessively stringent within a stochastic setting. Accordingly, it is essential to examine whether the probability distribution of the solution converges to a limiting distribution, which need not be the Dirac delta function. This behavior demonstrates stationarity in distribution, and the corresponding limiting distribution is known as the stationary distribution. Back in 1992, a foundational definition concerning stability in distribution can be traced back to Basak and Bhattacharya [6], who introduced it for a specific category of stochastic differential equations with singular diffusion.

On the basis of this pioneering work, The analysis of stability in distribution pertaining to stochastic differential equations has reached an unprecedented level of advancement. For example, as investigated by Bao et al. [7], retarded stochastic differential equations were studied, encompassing not only neutral-type models but also those involving Lévy processes that do not necessarily possess finite second moments. It was demonstrated by Kinnally and Williams [8] that, for multidimensional stochastic delay differential equations featuring normal reflection, there exist sufficient conditions ensuring the existence and uniqueness of stationary distributions. In a follow-up investigation, Hu et al. [9] verified that the model they proposed yields a unique global positive solution, and they further confirmed that the stationary distribution associated with it is both existent and unique. Sufficient criteria for distributional stability are available for a class of highly nonlinear stochastic functional differential equations, as reported by Wang et al. [10] and Fei et al. [11]. Further contributions to this line of inquiry can be found in the works of Hu and Wang [12] and Bao et al. [13].

The growing interest in the distributional stability of Wiener-process-driven stochastic delay differential equations can be largely attributed to their capacity to naturally capture the dynamics of many systems in the real world. Owing to their ability to account for abrupt random disturbances, stochastic delay differential equations with jumps typically yield more accurate characterizations of the systems under consideration. Despite their theoretical and practical relevance, investigations into such jump-type equations remain comparatively limited. Bao et al. [14] dedicated their investigation to stochastic partial differential delay equations that feature jump processes.The authors not only proved that mild solutions exist and are unique but also derived conditions under which distributional stability can be ensured. Ref. [15] by Liu is concerned with stochastic delay differential equations in Hilbert spaces where the driving noises are given by positive semigroups and Lévy processes. In that work, sufficient criteria ensuring the existence and uniqueness of stationary distributions are established. Moreover, the Wiener-process-driven and jump-process-driven scenarios are treated separately. A separate treatment of the Wiener-process-driven and jump-process-driven cases was undertaken by Liu [16] in investigating the stationary problem for a family of Hilbert-space-based stochastic evolution systems of second order, characterized by the inclusion of memory effects. Further results on distributional stability for stochastic differential equations with memory driven by positive semigroups and Lévy processes can be found in [17]. Focusing on neutral stochastic functional differential equations, Tan et al. [18] conducted a separate analysis of the stationary distribution problem under the influence of Brownian motion and jump processes. More recently, ref. [18] extended the analysis to equations driven by fractional Brownian motion and Poisson jumps. In a different direction, ref. [19] established stability in distribution for stochastic functional differential equations driven by G-Lévy processes. Stationary distributions for stochastic functional differential equations driven by Lévy processes in Hilbert spaces were further investigated in [16].

$$\mathrm{d}(X(t)-u(X_t))=f\left(X_t\right)\mathrm{d}t+g\left(X_t\right)\mathrm{d}B(t), \quad t>0, X_0=\xi,$$

and

$$\mathrm{d}(X(t)-u(X_t))=f\left(X_t\right)\mathrm{d}t+\int_{\Gamma}\sigma\left(X_t,z\right)\tilde{N}\left(\mathrm{d}t,\mathrm{d}z\right), \quad t\geq 0, X_0=\xi.$$

It was shown by Taniguchi [20] that, for the class of non-Lipschitz stochastic functional evolution equations presented below, energy solutions exist and are uniquely determined. As discussed in Ref. [21], related findings on the controllability of multi-delay semilinear stochastic systems are available. Involving two forms of noise—Brownian motion and Poisson jumps—this equation defines u as the neutral component, $\{B(t)\}_{t\geq 0}$ as a standard Wiener process, and $\tilde{N}(\mathrm{d}t,\mathrm{d}z)$ as the compensated Poisson random measure.

$$\begin{cases}\mathrm{d}X(t)=[A(t,X(t))+f(t,X_t)]\,\mathrm{d}t+g(t,X_t)\,\mathrm{d}W(t)+\int_U k(t,X_t,y)\,q(\mathrm{d}t,\mathrm{d}y),\\ X_0=\varphi\in D([-\tau,0];H).\end{cases}$$

Building upon preliminary analysis, this paper aims to explore the distributional stability of a certain type of stochastic delay differential equations, which are considered in Hilbert spaces, accounting for two different driving mechanisms, namely Wiener processes and Poisson jump processes.

$$\mathrm{d}X(t)=[A(t,X(t))+f\left(t,X_t\right)]\,\mathrm{d}t+g\left(t,X_t\right)\mathrm{d}W(t)$$

and

$$\mathrm{d}X(t)=[A(t,X(t))+f\left(t,X_t\right)]\,\mathrm{d}t+\int_{\Gamma}\sigma\left(X_t,z\right)\tilde{N}\left(\mathrm{d}t,\mathrm{d}z\right).$$

An application of the weak convergence method yields several sufficient conditions in this paper for ensuring the distributional stability of the equations in question.

We now turn to a description of how the remainder of this investigation is structured. Section 2 is concerned with the distributional stability of stochastic delay differential equations subject to Wiener process perturbations. A generalization of the analysis to equations governed by Poisson jump processes is presented in Section 3.

2. Preliminaries and Problem Formulation

Let us consider a pair of separable Hilbert spaces V and H. The norm on V is denoted by $\|\cdot\|_V$, while $|\cdot|_H$ stands for the norm on H. These spaces are subject to the following.

$$V\subset H\equiv H^*\subset V^*,$$

such that V is densely embedded in H with continuous inclusion. In what follows, the symbol $\langle\cdot,\cdot\rangle$ indicates the duality pairing on $V\times V^*$, whereas $(\cdot,\cdot)$ is employed to denote the inner product in H. We begin with a probability space $(\Omega,\mathcal{F},\mathbb{P})$ carrying a filtration $\mathcal{F}tt\geq 0$, and with a separable Hilbert space K. By $\mathcal{L}(K,H)$ we mean the space consisting of all bounded linear operators taking values from K to H. We begin with a filtered stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. On this framework, we consider a K-valued Wiener-type process, denoted by $\{W_Q(t), t \geq 0\}$, which is specifically of Q-Wiener form. The operator Q, which serves as its covariance, is prescribed to be a self-adjoint, positive, trace-class operator acting on the separable Hilbert space K. Whenever the trace of Q remains bounded, the aforementioned object is conventionally designated as a K-valued Q-Wiener process that is adapted to the family $\{\mathcal{F}_t\}_{t\geq 0}$. Consider the space $L_2^0(K, H)$ drawn from $\mathcal{L}(K, H)$, which consists of those operators ζ for which $\zeta\sqrt{Q}$ is a Hilbert–Schmidt operator and the trace expression $\operatorname{tr}(\zeta Q \zeta^*)$ takes a finite value. The norm is given by

$$\|\zeta\|_{L_2^0}^2 := \|\zeta\sqrt{Q}\|_{HS}^2 = \operatorname{tr}(\zeta Q \zeta^*).$$

Introduce $\mathscr{C} = C([-\tau, 0]; H)$ as the space of continuous functions ξ mapping $[-\tau, 0]$ into H, with the norm defined by $\|\xi\| := \sup_{-\tau\leq\theta\leq 0} |\xi(\theta)|$. With $X(t)$ being a continuous H-valued stochastic process defined on $[-\tau, +\infty)$, the associated process X_t for $t \geq 0$ is given by $X_t(\theta) = X(t + \theta)$ for all $\theta \in [-\tau, 0]$. In this way, a stochastic process taking values in $\mathscr{C}$ arises.

Our study concerns the stochastic delay differential equation given below, with a Wiener process as the driving noise,

$$\begin{cases} \mathrm{d}X(t) = [A(t, X(t)) + f(t, X_t)]\,\mathrm{d}t + g(t, X_t)\,\mathrm{d}W(t), t \geq 0, \\ X_0 = \xi \in C([-\tau, 0]; H). \end{cases} \tag{1}$$

As a point of observation, the operator $A(t, \cdot) : V \to V^*$ is bounded and is allowed to cover both linear and nonlinear situations. It is assumed that f and g are measurable; here f is an H-valued function defined on $[0, \infty) \times \mathscr{C}$, and g is an $L_2^0(K, H)$-valued function over the same domain, locally bounded, and continuous. Moreover, the following assumptions is made

Assumption 1. *We assume that $A(t, \cdot) : V \to V^*$ satisfies measurability, hemicontinuity, monotonicity, and boundedness. Let $\alpha > 0$ and $\beta \geq 0$ be constants satisfying*

$$\alpha\|u - v\|_V^2 + 2\langle A(t, u) - A(t, v), u - v\rangle \leq \beta|u - v|_H^2, \quad \forall u, v \in V, \tag{2}$$

we adopt $\langle\cdot,\cdot\rangle$ as the symbol for the dual pairing that pairs V^* with V. Also, a constant $\lambda_0 > 0$ may be taken so that

$$\lambda_0|v|_H^2 \leq \|v\|_V^2, \quad \forall v \in V,$$

and we set $a := \alpha\lambda_0 - \beta > 0$.

Assumption 2. *With the constants λ_1, λ_2, λ_3 satisfying $\lambda_1 > \lambda_2 > 0$ and $\lambda_3 > 0$, in addition to a constant κ lying in $(0, 1)$ and a probability measure $\mu(\cdot)$ on $[-\tau, 0]$, we obtain for all $\phi, \psi \in \mathscr{C}$ that the following is valid:*

$$\begin{aligned} &2\langle f(t, \varphi) - f(t, \psi), \varphi(0) - \psi(0)\rangle + ||g(t, \varphi) - g(t, \psi)||_{L_2^0}^2 \\ &\leqslant -\lambda_1|\varphi(0) - \psi(0)|_H^2 + \lambda_2 \int_{-\tau}^0 |\varphi(\theta) - \psi(\theta)|_H^2 \mu(d\theta), \end{aligned} \tag{3}$$

$$||g(t, \varphi) - g(t, \psi)||_{L_2^0}^2 \leqslant \lambda_3 \left(|\varphi(0) - \psi(0)|_H^2 + \int_{-\tau}^0 \left(|\varphi(\theta) - \psi(\theta)|_H^2 \mu(d\theta)\right)\right). \tag{4}$$

We denote by $X(t; \xi)$ the unique solution to Equation (3) that starts from the initial state ξ belonging to $\mathscr{C}$, with values lying in H. Meanwhile, the mapping $X_t(\theta) = X(t + \theta)$ for θ in $[-\tau, 0]$ defines a continuous H-valued function over the time window $[-\tau, 0]$. Moreover, as shown in Mohammed [22], $\{X_t(\xi)\}_{t\geq 0}$ satisfies a strong Markov property. Let $P(\xi; t, \cdot)$ stand for the probabilistic transition rule governing $X_t(\xi)$ whenever $t \geq 0$. Furthermore, let $\mathscr{P}(\mathscr{C})$ denote the family consisting of all probability distributions defined on the measurable space $(\mathscr{C}, \mathscr{B}(\mathscr{C}))$. For any two probability measures P_1 and P_2 drawn from the space of all probability distributions on $\mathscr{C}$, we endow this space with the distance function $d_{\mathbb{L}}$ via

$$\mathrm{d}_{\mathbb{L}}(P_1, P_2) = \sup_{f\in\mathbb{L}} \left|\int_{\mathscr{C}} f(\xi)P_1(d\xi) - \int_{\mathscr{C}} f(\eta)P_2(d\eta)\right|,$$

where

$$\mathbb{L} = \{f : \mathscr{C} \to R : |f(\xi) - f(\eta)| \leq \|\xi - \eta\|_\infty \text{ and } |f(\cdot)| \leq 1 \text{ for } \xi, \eta \in \mathscr{C}\}.$$

Remark 1. *By virtue of constituting a standard dissipativity framework for stochastic evolution equations in Hilbert spaces, the monotonicity and coercivity conditions in (Assumption 1) guarantee the existence of solutions and simultaneously provide the dissipative structure upon which stability analysis relies. The parameter* $a = \alpha\lambda_0 - \beta > 0$ *quantifies the net dissipativity of the system. This condition can be relaxed in several directions: for instance,* β *may be time-dependent under suitable integrability conditions, or the inequality (2) may be replaced by a local monotonicity condition to accommodate non-globally monotone coefficients. Such generalizations indicate the robustness of the proposed approach and suggest directions for future extension.*

Following Ikeda and Watanabe [23], we now give the characterization of distributional stability.

Definition 1. *We say that the segment process* $X_t(\xi)$ *enjoys distributional stability whenever there is a measure* π *within* $\mathscr{P}(\mathscr{C})$ *such that, for an arbitrary choice of* $\xi \in \mathscr{C}$*, the transition law* $P(\xi; t, \cdot)$ *approaches* π *in the weak sense as* t *becomes arbitrarily large. Stated differently,*

$$\lim_{t\to\infty} d_{\mathbb{L}}(P(\xi;t,\cdot),\pi(\cdot)) = 0, \quad \forall \xi \in \mathscr{C}.$$

Lemma 1. *On the condition that Assumptions 1 and 2 are satisfied,*

$$\sup_{t\geq 0}\sup_{\xi\in M} \mathbb{E}\|X_t(\xi)\|_\infty^2 < \infty, \tag{5}$$

in which M *denotes a bounded subset of* $\mathscr{C}$*.*

Proof. Making use of the classical Itô's formula (see [24]), for any $\varepsilon > 0$, we obtain

$$\begin{aligned} e^{\varepsilon t}|X(t;\xi)|_H^2 = |\xi(0)|_H^2 &+ \varepsilon \int_0^t e^{\varepsilon s}|X(s;\xi)|_H^2 \mathrm{d}s \\ &+ 2\int_0^t e^{\varepsilon s}\left\langle A(s,X(s;\xi)) + f\left(s,X_s(\xi)\right), X(s;\xi)\right\rangle \mathrm{d}s \\ &+ \int_0^t e^{\varepsilon s}\left|g\left(s,X_s(\xi)\right)\right|_{L_2^0}^2 \mathrm{d}s + 2\int_0^t e^{\varepsilon s}\left\langle g\left(s,X_s(\xi)\right), X(s;\xi)\right\rangle \mathrm{d}W(s). \end{aligned}$$

Applying expectation to both sides yields

$$\begin{aligned} e^{\varepsilon t}\mathbb{E}|X(t;\xi)|_H^2 = |\xi(0)|_H^2 &+ \varepsilon\mathbb{E}\int_0^t e^{\varepsilon s}|X(s;\xi)|_H^2 \mathrm{d}s \\ &+ \mathbb{E}\int_0^t 2e^{\varepsilon s}\left\langle A(s,X(s;\xi)), X(s;\xi)\right\rangle \mathrm{d}s \\ &+ \mathbb{E}\int_0^t e^{\varepsilon s}\left[2\langle f\left(s,X_s(\xi)\right), X(s;\xi)\rangle + |g(s,(X_s(\xi))|_{L_2^0}^2\right] \mathrm{d}s. \end{aligned} \tag{6}$$

Using Equation (2), we get

$$\begin{aligned} 2\langle A(s,X(s;\xi)), x(s;\xi)\rangle &\leqslant -\alpha\|X(s;\xi)\|_V^2 + \beta|X(s;\xi)|_H^2 \\ &\leqslant \beta|X(s;\xi)|_H^2 - \alpha\lambda_0|X(s;\xi)|_H^2 \\ &= -a|X(s;\xi)|_H^2. \end{aligned} \tag{7}$$

Substituting (3), (7) into (6), we have

$$\begin{aligned} e^{\varepsilon t}\mathbb{E}|X(t;\xi)|_H^2 \leq |\xi(0)|_H^2 &+ \varepsilon\mathbb{E}\int_0^t e^{\varepsilon s}|X(s;\xi)|_H^2 \mathrm{d}s \\ &- a\mathbb{E}\int_0^t e^{\varepsilon s}|X(s;\xi)|_H^2 \mathrm{d}s - \lambda_1\mathbb{E}\int_0^t e^{\varepsilon s}|X(s;\xi)|_H^2 \mathrm{d}s \\ &+ \lambda_2\mathbb{E}\int_0^t\int_{-\tau}^0 e^{\varepsilon s}|X(s+\theta;\xi)|_H^2 \mu(\mathrm{d}\theta)\mathrm{d}s. \end{aligned} \tag{8}$$

Now we compute

$$\begin{aligned}\int_0^t \int_{-\tau}^0 e^{\varepsilon s}|X(s+\theta;\xi)|_H^2\mu(\mathrm{d}\theta)\mathrm{d}s &= \int_{-\tau}^0 \left(\int_0^t e^{\varepsilon s}|X(s+\theta;\xi)|_H^2\mathrm{d}s\right)\mu(\mathrm{d}\theta)\\ &\le \int_{-\tau}^0 \mu(\mathrm{d}\theta)e^{\varepsilon\tau}\int_{-\tau}^t e^{\varepsilon s}\left|X(s;\xi)\right|_H^2\mathrm{d}s\\ &\le e^{\varepsilon\tau}\int_{-\tau}^0 e^{\varepsilon s}\left|X(s;\xi)\right|_H^2\mathrm{d}s + e^{\varepsilon\tau}\int_0^t e^{\varepsilon s}\left|X(s;\xi)\right|_H^2\mathrm{d}s\\ &\le \frac{e^{\varepsilon\tau}}{\varepsilon}\|\xi\|_\infty^2 + e^{\varepsilon\tau}\int_0^t e^{\varepsilon s}\left|X(s;\xi)\right|_H^2\mathrm{d}s. \end{aligned} \tag{9}$$

Substituting (9) into (8), it follows that

$$e^{\varepsilon t}\mathbb{E}|X(t;\xi)|_H^2 \le (1+\frac{\lambda_2 e^{\varepsilon\tau}}{\varepsilon})\|\xi\|_\infty^2 + (\varepsilon - a - \lambda_1 + \lambda_2 e^{\varepsilon\tau})\mathbb{E}\int_0^t e^{\varepsilon s}\left|X(s;\xi)\right|_H^2\mathrm{d}s.$$

Let $\varepsilon_1 = a+\lambda_1-\lambda_2 e^{\varepsilon\tau}$ and $c_1 = 1+\frac{\lambda_2 e^{\varepsilon\tau}}{\varepsilon}$. Given that $\lambda_1 > \lambda_2 > 0$, we can select $\varepsilon \in (0,1)$ small enough for $\varepsilon_1 > 0$ to hold. Applying the conventional Gronwall lemma, we finally arrive at

$$E|X(t;\xi)|_H^2 \le c_1 e^{-\varepsilon_1 t}\|\xi\|_\infty^2 < \infty, \quad \text{for all } t \ge -\tau. \tag{10}$$

The segment process $X_t(\xi)$ will now be shown to be bounded in the appropriate sense. Exploiting Itô's formula alongside the identities (5) and (13), we obtain, for arbitrary $t \ge 0$ and for all θ in the range $[-\tau, 0]$, that

$$\begin{aligned}|X(t+\theta;\xi)|_H^2 &= |X(t-\tau;\xi)|_H^2 + M(t,\theta) + \int_{t-\tau}^{t+\theta} 2\left\langle A(s,X(s;\xi)),X(s;\xi)\right\rangle\mathrm{d}s\\ &\quad + \int_{t-\tau}^{t+\theta}[2\langle f\left(s,X_s(\xi)\right),X(s;\xi)\rangle + |g\left(s,X_s(\xi)\right)|_{L_2^0}^2]\mathrm{d}s\\ &\le |X(t-\tau;\xi)|_H^2 + M(t,\theta) - (a+\lambda_1)\int_{t-\tau}^{t+\theta}|X(s;\xi)|_H^2\mathrm{d}s\\ &\quad + \lambda_2\int_{t-\tau}^{t+\theta}\int_{-\tau}^0 |X(s+\theta;\xi)|_H^2\mu(\mathrm{d}\theta)\mathrm{d}s\\ &\le |X(t-\tau;\xi)|_H^2 + M(t,\theta)\\ &\quad + \lambda_2\int_{t-\tau}^{t+\theta}\int_{-\tau}^0 |X(s+\theta;\xi)|_H^2\mu(\mathrm{d}\theta)\mathrm{d}s,\end{aligned} \tag{11}$$

where

$$M(t,\theta) = \int_{t-\tau}^{t+\theta}\langle g\left(s,X_s(\xi)\right),X(s;\xi)\rangle\mathrm{d}W(s).$$

Exploiting the Burkholder-Davis-Gundy inequality in combination with (4) yields a certain positive real number c_2 for which the following holds

$$\begin{aligned}\mathbb{E}\left(\sup_{-\tau\le\theta\le 0}|M(t,\theta)|_H\right) &\le c_2\mathbb{E}\left(\int_{t-\tau}^t \left|\langle g\left(s,X_s(\xi)\right),X(s;\xi)\rangle\right|_H^2\mathrm{d}s\right)^{\frac{1}{2}}\\ &\le c_2\mathbb{E}\left(\sup_{t-\tau\le s\le t}|X(s;\xi)|_H^2\int_{t-\tau}^t \left|g\left(s,X_s(\xi)\right)\right|_{L_2^0}^2\mathrm{d}s\right)^{\frac{1}{2}}\\ &\le \frac{1}{2}\mathbb{E}\left(\sup_{-\tau\le\theta\le 0}|X(t+\theta;\xi)|_H^2\right) + \frac{c_2^2}{2}\mathbb{E}\int_{t-\tau}^t \left|g\left(s,X_s(\xi)\right)\right|_{L_2^0}^2\mathrm{d}s\\ &\le \frac{1}{2}\mathbb{E}\left(\sup_{-\tau\le\theta\le 0}|X(t+\theta;\xi)|_H^2\right) + \frac{\lambda_3 c_2^2}{2}\mathbb{E}\int_{t-\tau}^t \left|X(s;\xi)\right|_H^2\mathrm{d}s\\ &\quad + \frac{\lambda_3 c_2^2}{2}\mathbb{E}\int_{t-\tau}^t\int_{-\tau}^0 \left|X(s+\theta;\xi)\right|_H^2\mu(\mathrm{d}\theta)\mathrm{d}s.\end{aligned}$$

Substituting this into (11), it can be seen

$$\begin{aligned}\mathbb{E}\left(\sup_{-\tau\le\theta\le 0}|X(t+\theta;\xi)|_H^2\right) &\le \mathbb{E}|X(t-\tau;\xi)|_H^2 + \tfrac{1}{2}\mathbb{E}\left(\sup_{-\tau\le\theta\le 0}|X(t+\theta;\xi)|_H^2\right)\\ &\quad + \tfrac{\lambda_3 c_2^2}{2}\mathbb{E}\int_{t-\tau}^t |X(s;\xi)|_H^2\mathrm{d}s + \left(\lambda_2 + \tfrac{\lambda_3 c_2^2}{2}\right)\mathbb{E}\int_{t-\tau}^t\int_{-\tau}^0 |X(s+\theta;\xi)|_H^2\mu(\mathrm{d}\theta)\mathrm{d}s.\end{aligned}$$

Thus,

$$\mathbb{E}\left(\sup_{-\tau\le\theta\le 0}|X(t+\theta;\xi)|_H^2\right)\le 2\mathbb{E}|X(t-\tau;\xi)|_H^2+\lambda_3c_2^2\mathbb{E}\int_{t-\tau}^{t}|X(s;\xi)|_H^2\mathrm{d}s$$
$$+(2\lambda_2+\lambda_3c_2^2)\mathbb{E}\int_{t-2\tau}^{t}|X(s;\xi)|_H^2\mathrm{d}s.$$

Thus, for all t not smaller than τ, in view of (10), one can obtain a positive constant $c_3>0$ with the property that

$$\mathbb{E}\|X_t(\xi)\|_\infty^2=\mathbb{E}\sup_{-\tau\le\theta\le 0}|X(t+\theta;\xi)|_H^2\le c_3e^{-\varepsilon_1 t}\|\xi\|_\infty^2<\infty.$$

On the time domain defined by $[0,\tau]$, one readily arrives at the conclusion that

$$\begin{aligned}\mathbb{E}\|X_t(\xi)\|_\infty^2&=\mathbb{E}\sup_{-\tau\le\theta\le 0}|X(t+\theta;\xi)|_H^2\\&\le\mathbb{E}\left[\|\xi\|_\infty^2+\sup_{-\tau\le\theta\le 0}|X(\tau+\theta;\xi)|_H^2\right]\\&\le\|\xi\|_\infty^2+c_3e^{-\varepsilon_1\tau}\|\xi\|_\infty^2\\&\le(1+c_3e^{-\varepsilon_1\tau})e^{\varepsilon_1\tau-\varepsilon_1 t}\|\xi\|_\infty^2\\&=c_4e^{-\varepsilon_1 t}\|\xi\|_\infty^2<\infty,\end{aligned}\tag{12}$$

where $c_4=(1+c_3e^{-\varepsilon_1\tau})e^{\varepsilon_1\tau}$.

In conclusion, (5) is verified. □

Lemma 2. *Assuming that Assumptions 1 and 2 holds, for $\forall\,\xi,\eta\in M$, we have*

$$\lim_{t\to\infty}\sup_{\xi,\eta\in M}\mathbb{E}\|X_t(\xi)-X_t(\eta)\|_\infty^2=0,\tag{13}$$

in which $X_t(\xi)$ and $X_t(\eta)$ represent the solutions of (1) subject to the initial conditions $X_0=\xi$ and $X_0=\eta$, respectively.

Proof. Let us consider the separation between two solution branches of (1) emerging from distinct initial points, denoted by

$$X(t;\xi)-X(t;\eta)=\xi(0)-\eta(0)+\int_0^t[A(s,X(s;\xi))-A(s,X(s;\eta))]\,\mathrm{d}s$$
$$+\int_0^t[f(s,X_s(\xi))-f(s,X_s(\eta))]\,\mathrm{d}s+\int_0^t[g(s,X_s(\xi))-g(s,X_s(\eta))]\,\mathrm{d}W(s).$$

Through the use of Itô's formula together with the conditions (2) and (3), one arrives at for an arbitrary $\varepsilon>0$,

$$\begin{aligned}e^{\varepsilon t}\mathbb{E}|X(t;\xi)-X(t;\eta)|_H^2&=\mathbb{E}|\xi(0)-\eta(0)|_H^2+\varepsilon\mathbb{E}\int_0^t e^{\varepsilon s}|X(s;\xi)-X(s;\eta)|_H^2\mathrm{d}s\\&\quad+2\mathbb{E}\int_0^t e^{\varepsilon s}\langle A(s,X(s;\xi))-A(s,X(s;\eta)),X(s;\xi)-X(s;\eta)\rangle\,\mathrm{d}s\\&\quad+2\mathbb{E}\int_0^t e^{\varepsilon s}\langle f(s,X_s(\xi))-f(s,X_s(\eta)),X(s;\xi)-X(s;\eta)\rangle\,\mathrm{d}s\\&\quad+\mathbb{E}\int_0^t e^{\varepsilon s}|g(s,X_s(\xi))-g(s,X_s(\eta))|_{L_2^0}^2\mathrm{d}s\\&\le\mathbb{E}|\xi(0)-\eta(0)|_H^2+\varepsilon\mathbb{E}\int_0^t e^{\varepsilon s}|X(s;\xi)-X(s;\eta)|_H^2\mathrm{d}s\\&\quad-(a+\lambda_1)\mathbb{E}\int_0^t e^{\varepsilon s}|X(s;\xi)-X(s;\eta)|_H^2\mathrm{d}s\\&\quad+\lambda_2\mathbb{E}\int_0^t\int_{-\tau}^0 e^{\varepsilon s}|X(s+\theta;\xi)-X(s+\theta;\eta)|_H^2\mu(\mathrm{d}\theta)\mathrm{d}s.\end{aligned}$$

Using a similar argument to that in (10), we can select a pair of strictly positive parameters, ε_2 and c_5, with the property that

$$E|X(t;\xi) - X(t;\eta)|_H^2 \le c_5 e^{-\varepsilon_2 t}\|\xi - \eta\|_\infty^2, \quad \text{for all } t \ge -\tau.$$

Furthermore, for $t \ge 0$, it follows that,

$$\mathbb{E}\|X_t(\xi) - X_t(\eta)\|_\infty^2 \le c_5 e^{-\varepsilon_2 t}\|\xi - \eta\|_\infty^2. \tag{14}$$

Leting $t \to \infty$ in (14), we get (13). Thus the proof is finished. □

Theorem 1. *Provided that Assumptions 1 and 2 are fulfilled, the segment process* $\{X_t(\xi)\}_{t\ge 0}$ *of (1) possesses distributional stability.*

Proof. As a first step, A central part of our argument is to verify that $P(\xi;t,\cdot)_{t\ge 0}$ behaves as a Cauchy sequence under the topology of $\mathscr{P}(\mathscr{C})$. Utilizing the attributes of conditional expectation in combination with the Markov structure possessed by $\{X_t(\xi)\}_{t\ge 0}$, the following holds for each $f \in \mathbb{L}$ and all $t, s > 0$:

$$\begin{aligned}
|\mathbb{E}f(X_{t+s}(\xi)) - \mathbb{E}f(X_t(\eta))| &= |\mathbb{E}\left[\mathbb{E}(f(X_{t+s}(\xi))|\mathscr{F}_s)\right] - \mathbb{E}f(X_t(\eta))| \\
&= \left|\int_{\mathscr{C}} \mathbb{E}f(X_t(\zeta))P(\xi;s,\mathrm{d}\zeta) - \mathbb{E}f(X_t(\eta))\right| \\
&\le \int_{\mathscr{C}} |\mathbb{E}f(X_t(\zeta)) - \mathbb{E}f(X_t(\eta))|\, P(\xi;s,\mathrm{d}\zeta) \\
&\le 2P(\xi;s,\bar{B}_r) + \int_{B_r} |\mathbb{E}f(X_t(\zeta)) - \mathbb{E}f(X_t(\eta))|\, P(\xi;s,\mathrm{d}\zeta),
\end{aligned} \tag{15}$$

where $B_r := \{\xi \in \mathscr{C} : \|\xi\|_\infty \le r\}$ and $\bar{B}_r := \mathscr{C} - B_r$. According to Lemma 1, for any arbitrary $\varepsilon \in (0,1)$, We can take $r > 0$ to be sufficiently great so as to guarantee

$$P(\xi;s,\bar{B}_r) < \frac{\varepsilon}{4}, \quad \forall s > 0. \tag{16}$$

Moreover, an application of Lemma 2 yields a time $T > 0$ for which

$$\begin{aligned}
\sup_{f\in\mathbb{L}} |\mathbb{E}f(X_t(\zeta)) - \mathbb{E}f(X_t(\eta))| &\le \mathbb{E}\|X_t(\zeta) - X_t(\eta)\|_\infty \\
&\le \frac{\varepsilon}{2}, \quad \zeta, \eta \in \mathscr{C}, t \ge T.
\end{aligned} \tag{17}$$

Upon substituting (16) and (17) into (15), one obtains

$$|\mathbb{E}f(X_{t+s}(\xi)) - \mathbb{E}f(X_t(\eta))| \le \varepsilon, \quad t \ge T, s > 0.$$

In light of the fact that f is selected without loss of generality from the class $\mathbb{L}$, we can get

$$\mathrm{d}_{\mathbb{L}}\left(P(\xi;t+s,\cdot) - P(\eta;t,\cdot)\right) = \sup_{f\in\mathbb{L}} |\mathbb{E}f(X_{t+s}(\xi)) - \mathbb{E}f(X_t(\eta))| \le \varepsilon,$$

when both t is at least T and s is strictly positive. From this, we deduce that the one-parameter family $\{P(\xi;t,\cdot)\}$ with $t \ge 0$ is a Cauchy sequence in the function space $\mathscr{P}(\mathscr{C})$. Therefore, we can ensure that precisely one member $\pi(\cdot)$ of the collection $\mathscr{P}(\mathscr{C})$ exists, which satisfies

$$\lim_{t\to\infty} \mathrm{d}_{\mathbb{L}}\left(P(\xi;t,\cdot), \pi(\cdot)\right) = 0. \tag{18}$$

Which means that the segment process $\{X_t(\xi)\}_{t\ge 0}$ is distributionally stable. We have thus proved the theorem. □

Remark 2. *Compared with the existing results on distributional stability for stochastic evolution equations in [16], Theorem 1 extends the stability criteria to stochastic delay differential equations driven by Wiener processes in Hilbert spaces. The monotonicity and dissipativity conditions adopted here are more general, and our result covers the case of time-dependent delay systems that were not fully addressed in [16].*

3. Stability in Distribution: Poisson Jump Case

The focus of the remainder of this paper lies on stochastic functional differential equations that are subject to Poisson jump process perturbations. Consider a measurable space $(\mathbb{Y}, \mathscr{B}(\mathbb{Y}), \lambda(\cdot))$, a countable set $D_p \subseteq \mathbb{R}_+$, and an adapted process $p : D_p \to \mathbb{Y}$ with values in $\mathbb{Y}$. According to Kinnally and Williams [8], Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space equipped with the Poisson random measure $N(\cdot,\cdot) : \mathscr{B}(\mathbb{R}_+ \times \mathbb{Y}) \times \Omega \to \mathbb{Y} \cup \{0\}$. Then, this measure can be represented as

$$N((0,t] \times \Gamma) = \sum_{s \in D_p, s \leq t} \mathbb{I}_\Gamma (p(s)), \quad \Gamma \in \mathscr{B}(\mathbb{Y}).$$

With a Poisson point process p serving as the fundamental object, the accompanying Poisson random measure is denoted by N, acting on the same measurable space. We adopt the notation $\lambda(\cdot) := \mathbb{E}N((0,t] \times \cdot)$ for the intensity. It then follows that the compensated Poisson random measure given by $\tilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \lambda(\mathrm{d}z)\mathrm{d}t$ is a martingale. Suppose that $\mathscr{D} = D([-\tau, 0]; H)$ represents the space of càdlàg functions ζ mapping the interval $[-\tau, 0]$ into H, where the associated norm is given by $\|\zeta\|_\infty = \sup_{-\tau \leq \theta \leq 0} |\zeta(\theta)|$.

Our subsequent analysis focuses on stochastic functional differential equations that incorporate a Poisson jump process as the driving noise.

$$\begin{cases} \mathrm{d}X(t) = [A(t, X(t)) + f(t, X_t)]\,\mathrm{d}t + \int_\Gamma \sigma(X_t, z)\,\tilde{N}(\mathrm{d}t, \mathrm{d}z), t \geq 0, \\ X_0 = \xi \in \mathscr{D}([-\tau, 0]; H), \end{cases} \tag{19}$$

where $f : [0, \infty) \times \mathscr{D} \to H$, $\sigma : \mathscr{D} \times \Gamma \to H$ are measurable, locally bounded and continuous.

The following set of assumptions is required for the analysis.

Assumption 3. *Suppose that $\mu_1 > \mu_2 > 0$ and $\mu_3 > 0$ are constants, and take a probability measure $\rho(\cdot)$ defined on the finite interval $[-\tau, 0]$. Then, given any $\phi, \varphi \in \mathscr{D}$, the following conditions hold:*

$$\begin{aligned} &2\langle f(t,\varphi) - f(t,\psi), \varphi(0) - \psi(0)\rangle + \int_\Gamma |\sigma(\varphi, z) - \sigma(\phi, z)|_H^2 \lambda(dz) \\ &\leqslant -\mu_1 |\varphi(0) - \psi(0)|_H^2 + \mu_2 \int_{-\tau}^0 |\varphi(\theta) - \psi(\theta)|_H^2 \rho(d\theta), \end{aligned} \tag{20}$$

$$\int_\Gamma |\sigma(\varphi, z) - \sigma(\phi, z)|_H^2 \lambda(dz) \leqslant \mu_3 \left(|\varphi(0) - \psi(0)|_H^2 + \int_{-\tau}^0 |\varphi(\theta) - \psi(\theta)|_H^2 \rho(d\theta) \right) \tag{21}$$

Theorem 2. *Under the Assumptions 1 and 3, the segment process $\{X_t(\xi)\}_{t \geq 0}$ of (19) is distributionally stable.*

Proof. Repeating the steps carried out in the demonstration of Theorem 1, we merely need to verify that for every bounded set $M \subseteq \mathscr{D}$,

$$\sup_{t \geq 0} \sup_{\xi \in M} \mathbb{E}\|X_t(\xi)\|_\infty^2 < \infty, \tag{22}$$

and

$$\lim_{t \to \infty} \sup_{\xi, \eta \in M} \mathbb{E}\|X_t(\xi) - X_t(\eta)\|_\infty^2 = 0. \tag{23}$$

Next, let us prove (22) first. By virtue of Itô's formula, for an arbitrary $\varepsilon > 0$, one has

$$\begin{aligned} e^{\varepsilon t}|X(t;\xi)|_H^2 = |\xi(0)|_H^2 &+ \varepsilon \int_0^t e^{\varepsilon s}|X(s;\xi)|_H^2 \mathrm{d}s \\ &+ 2\int_0^t e^{\varepsilon s} \langle A(s, X(s;\xi)), X(s;\xi)\rangle\,\mathrm{d}s \\ &+ \int_0^t e^{\varepsilon s} \left[2\langle f(s, X_s(\xi)), X(s;\xi)\rangle + \int_\Gamma |\sigma(X_s(\xi), z)|_H^2 \lambda(\mathrm{d}z) \right] \mathrm{d}s \\ &+ \int_0^t \int_\Gamma e^{\varepsilon s} \left[2\langle \sigma(X_s(\xi), z), X(s;\xi)\rangle + |\sigma(X_s(\xi), z)|_H^2 \right] \tilde{N}(\mathrm{d}s, \mathrm{d}z). \end{aligned}$$

Taking the expectation of the above equation and using (2), (22), we can obtain

$$e^{\varepsilon t} E\left|X(t;\xi)\right|_H^2 \leq |\xi(0)|_H^2 + \varepsilon E\int_0^t e^{\varepsilon s}|X(s;\xi)|_H^2 \mathrm{d}s - (a+\mu_1)\, E\int_0^t e^{\varepsilon s}|X(s;\xi)|_H^2 \mathrm{d}s + \mu_2 E\int_0^t\int_{-\tau}^0 e^{\varepsilon s}|X(s+\theta;\xi)|_H^2 \mu(\mathrm{d}\theta)\mathrm{d}s.$$

As in the argument leading to (10), one obtains that

$$E|X(t;\xi)|_H^2 \leq c_6 e^{-\varepsilon_3 t}\|\xi\|_\infty^2 < \infty, \quad \text{for all } t \geq -\tau. \tag{24}$$

Using Itô's formula along with (2) and (20) yields, provided that $t \geq \tau$ holds and θ is chosen from the interval $[-\tau, 0]$

$$\begin{aligned}
|X(t+\theta;\xi)|_H^2 &= |X(t-\tau;\xi)|_H^2 + G(t,\theta) + \int_{t-\tau}^{t+\theta} 2\left\langle A(s, X(s;\xi)), X(s;\xi)\right\rangle \mathrm{d}s \\
&\quad + \int_{t-\tau}^{t+\theta}\left[2\langle f\left(s, X_s(\xi)\right), X(s;\xi)\rangle + \int_\Gamma \left|\sigma\left(X_s(\xi), z\right)\right|_H^2 \lambda(\mathrm{d}z)\right]\mathrm{d}s \\
&\leq X(t-\tau;\xi)|_H^2 + N(t,\theta) + \mu_2\int_{t-\tau}^{t+\theta}\int_{-\tau}^0 |X(s+\theta;\xi)|_H^2\mu(\mathrm{d}\theta)\mathrm{d}s,
\end{aligned} \tag{25}$$

where

$$G(t,\theta) = \int_{t-\theta}^{t+\theta}\int_\Gamma \left[2\left\langle \sigma\left(X_s(\xi), z\right), X(s;\xi)\right\rangle + \left|\sigma\left(X_s(\xi), z\right)\right|_H^2\right]\tilde{N}\left(\mathrm{d}s, \mathrm{d}z\right).$$

Thanks to the Burkholder-Davis-Gundy inequality combined with (23), a positive constant c_7 can be found satisfying the condition

$$\begin{aligned}
\mathbb{E}\left(\sup_{-\tau\leq\theta\leq 0}|N(t,\theta)|\right) &\leq c_7\mathbb{E}\left(\sum_{s\in D_p, t-\tau\leq s\leq t}\left[2\left\langle X(s;\xi), \sigma\left(X_s(\xi), p(s)\right)\right\rangle + \left|\sigma\left(X_s(\xi), p(s)\right)\right|_H^2\right]^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}c_7\mathbb{E}\left(\sum_{s\in D_p, t-\tau\leq s\leq t}\left|\sigma\left(X_s(\xi), p(s)\right)\right|_H^4\right)^{\frac{1}{2}} \\
&\quad + 2\sqrt{2}c_7\mathbb{E}\left(\sum_{s\in D_p, t-\tau\leq s\leq t}|X(s;\xi)|_H^2\left|\sigma\left(X_s(\xi), p(s)\right)\right|_H^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}c_7\mathrm{E}\left(\sum_{s\in D_p, t-\tau\leq s\leq t}\left|\sigma\left(X_s(\xi), p(s)\right)\right|_H^2\right) \\
&\quad + 2\sqrt{2}c_7\mathbb{E}\left(\sup_{-\tau\leq\theta\leq 0}|X(t+\theta;\xi)|_H^2\sum_{s\in D_p, t-\tau\leq s\leq t}\left|\sigma\left(X_s(\xi), p(s)\right)\right|_H^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\mathbb{E}\left(\sup_{-\tau\leq\theta\leq 0}|X(t+\theta;\xi)|_H^2\right) + (4c_7^2 + \sqrt{2}c_7)\mathbb{E}\left(\sum_{s\in D_p, t-\tau\leq s\leq t}\left|\sigma\left(X_s(\xi), p(s)\right)\right|_H^2\right) \\
&\leq \frac{1}{2}\mathbb{E}\left(\sup_{-\tau\leq\theta\leq 0}|X(t+\theta;\xi)|_H^2\right) + (4c_7^2 + \sqrt{2}c_7)\mathbb{E}\left(\int_{t-\tau}^t\int_\Gamma\left|\sigma\left(X_s(\xi), z\right)\right|_H^2\lambda(\mathrm{d}z)\mathrm{d}s\right) \\
&\leq \frac{1}{2}\mathbb{E}\left(\sup_{-\tau\leq\theta\leq 0}|X(t+\theta;\xi)|_H^2\right) + (4c_7^2 + \sqrt{2}c_7)\mu_3\mathbb{E}\int_{t-\tau}^t |X(s;\xi)|_H^2 ds \\
&\quad + (4c_7^2 + \sqrt{2}c_7)\mu_3\mathbb{E}\int_{t-\tau}^t\int_\Gamma |X(s+\theta;\xi)|_H^2\rho(\mathrm{d}\theta)\mathrm{d}s.
\end{aligned} \tag{26}$$

For $t \geq \tau$, substituting (26) into (25), and using (24), there exist constant $c_8 > 0$ satisfying

$$\mathbb{E}\|X_t(\xi)\|_\infty^2 = \mathbb{E}\sup_{-\tau\leq\theta\leq 0}|X(t+\theta;\xi)|_H^2 \leq c_8 e^{-\varepsilon_3 t}\|\xi\|_\infty^2 < \infty.$$

Furthermore, similar to (12), we get

$$\mathbb{E}\|X_t(\xi)\|_\infty^2 = \mathbb{E} \sup_{-\tau\leq\theta\leq 0} |X(t+\theta;\xi)|_H^2 \leq c_9 e^{-\varepsilon_3 t}\|\xi\|_\infty^2 < \infty, \quad t\in[0,\tau].$$

It follows that (22) is satisfied for any $t \geq 0$.

Next, let us examine the difference between two solutions arising from Equation (19) corresponding to distinct initial values $X_0 = \xi, X_0 = \eta \in M$, namely

$$\begin{aligned} X(t;\xi) - X(t;\eta) = \xi(0) - \eta(0) &+ \int_0^t [A(s,X(s;\xi)) - A(s,X(s;\eta))]\,\mathrm{d}s \\ &+ \int_0^t [f(s,X_s(\xi)) - f(s,X_s(\eta))]\,\mathrm{d}s \\ &+ \int_0^t\int_\Gamma [\sigma(X_s(\xi),z) - \sigma(X_s(\eta),z)]\,\tilde{N}(\mathrm{d}t,\mathrm{d}z). \end{aligned}$$

By an argument analogous to that of Lemma 2, we obtain (23). This completes the proof. □

Remark 3. *In contrast to the stability results for jump–diffusion systems in [22], Theorem 2 establishes distributional stability for stochastic delay differential equations driven by Poisson jump processes under weaker dissipativity assumptions. Our framework allows for non–Lipschitz coefficients and infinite-dimensional state spaces, which significantly extends the applicable scope of [22].*

4. Examples

We now turn to two examples that serve to illustrate the results from the preceding sections. To this end, let $H := L^2(0,\pi)$ and let $\mathbb{R} := (-\infty,\infty)$. Let $U \in B_\sigma(\mathbb{R}\setminus\{0\})$ where $0 \notin \overline{U}$. Introduce the operator $A = \partial^2/\partial x^2$, which acts as a second-order differential operator on the separable Hilbert space H. The domain of A is given by

$$D(A) = \left\{u \in H : \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in H,\ u(0) = u(\pi) = 0\right\}.$$

Let $e_n := \sqrt{2/\pi}\sin(nx)$ $(n = 1,2,3,\dots)$. There exists a complete orthogonal basis $\{e_n\}$ in H such that each element e_n is an eigenfunction of the operator A. Let the eigenvalues σ_n be defined by $Ae_n = \sigma_n e_n$ for $n = 1,2,3,\dots$. The operator A satisfies the well-known estimate $\langle Av, v\rangle \leq -\|v\|_V^2$ for any $v \in V$, which in turn implies that the inequality in Assumption 1 holds.

Example 1. *The following equation serves as the object of our study: a stochastic delay differential equation involving a finite delay $\tau > 0$:*

$$\begin{cases} \mathrm{d}X(t) = \left(\frac{\partial^2}{\partial^2 x}X(t) - hX_t + e^t\right)\,\mathrm{d}t + \left(X_t + \int_{-\tau}^0 X_t(\theta)\rho(\mathrm{d}\theta) + \cos t\right)\mathrm{d}W(t), t \geq 0, \\ X_0 = \xi \in \mathscr{C}([-\tau,0];H), \end{cases} \tag{27}$$

For a certain positive constant denoted by h, Let $f(t,\phi) = -h\phi + e^t, g(t,\phi) = \phi + \int_{-\tau}^0 \phi(\theta)\rho(\mathrm{d}\theta) + \cos t$. By Hölder inequality, it is to show that

$$\begin{aligned} |g(t,\phi) - g(t,\varphi)|^2 &= \left|\phi(0) - \varphi(0) + \int_{-\tau}^0 \phi(\theta) - \varphi(\theta)\rho(\mathrm{d}\theta)\right|^2 \\ &\leq 2|\phi(0) - \varphi(0)|^2 + 2\left|\int_{-\tau}^0 (\phi(\theta) - \varphi(\theta))\rho(\mathrm{d}\theta)\right|^2 \\ &\leq 2|\phi(0) - \varphi(0)|^2 + 2\left[\left(\int_{-\tau}^0 |\phi(\theta) - \varphi(\theta)|^2\rho(\mathrm{d}\theta)\right)^{\frac{1}{2}}\left(\int_{-\tau}^0 \rho(\mathrm{d}\theta)\right)^{\frac{1}{2}}\right]^2 \\ &\leq 2\left(|\phi(0) - \varphi(0)|^2 + \int_{-\tau}^0 |\phi(\theta) - \varphi(\theta)|^2\rho(\mathrm{d}\theta)\right), \end{aligned} \tag{28}$$

and

$$\begin{aligned} &2\langle\phi(0)-\varphi(0), f(t,\phi)-f(t,\varphi)\rangle+|g(t,\phi)-g(t,\varphi)|^2 \\ &=2\langle\phi(0)-\varphi(0), -h\left(\phi(0)-\varphi(0)\right)\rangle+|g(t,\phi)-g(t,\varphi)|^2 \\ &\leq -2h\left|\phi(0)-\varphi(0)\right|^2+|g(t,\phi)-g(t,\varphi)|^2 \\ &\leq -(2h-2)\left|\phi(0)-\varphi(0)\right|^2+2\int_{-\tau}^{0}\left|\phi(\theta)-\varphi(\theta)\right|^2\rho(\mathrm{d}\theta) \end{aligned} \tag{29}$$

If $h>2$, (28) and (29) satisfy the Assumption 2. Thanks to Theorem 1, it can be deduced that the solution process of (27) *enjoys stability in distribution whenever the parameter h fulfills $h>2$.*

To illustrate the theoretical result, Figure 1 shows one sample path of Equation (27) with $h=3$. While a single trajectory does not directly represent the stationary distribution, its tendency to remain bounded and oscillate near zero after a short transient period strongly suggests that the sequence $\{X(t)\}_{t\geq 0}$ is said to converge in distribution to a limiting distribution. This is consistent with the stability in distribution proved in Theorem 1.

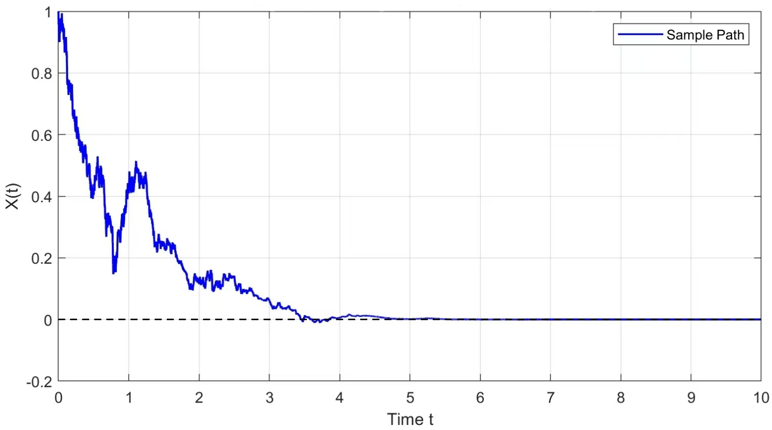

Figure 1. A sample path of Equation (27) with $h=3$.

Example 2. *Suppose we have a one-dimensional stochastic delay differential equation subject to jumps, which can be described as:*

$$\begin{cases} \mathrm{d}X(t)=\left(\frac{\partial^2}{\partial^2 x}X(t)-kX_t+e^t\right)\,\mathrm{d}t+\int_1^\infty zX_t\tilde{N}(\,\mathrm{d}z,\,\mathrm{d}t), \quad t\geq 0, \\ X_0=\xi\in\mathscr{D}([-\tau,0];H), \end{cases} \tag{30}$$

with k taken as a positive constant.. Denote by $f(t,\phi)=-k\phi+e^t$ and $\sigma(\phi,z)=z\phi$. Suppose that $\lambda(\cdot)$ is the intensity measure associated with the Poisson measure N, satisfying $\int_1^\infty z^2\lambda(\mathrm{d}z)<\infty$. Conduct the similar procedure as Example 1, It may be proved that

$$\begin{aligned} &2\langle\phi(0)-\varphi(0), f(t,\phi)-f(t,\varphi)\rangle+\int_1^\infty|\sigma(\phi,z)-\sigma(\varphi,z)|^2\lambda(\mathrm{d}z) \\ &\leq -\left(2k-\int_1^\infty z^2\lambda(\mathrm{d}z)\right)|\phi(0)-\varphi(0)|^2. \end{aligned} \tag{31}$$

If $k>\frac{1}{2}\int_1^\infty z^2\lambda(\mathrm{d}z)$, (31) satisfies the Assumption 3. Hence, it follows from Theorem 2 that the solution process to (30) exhibits distributional stability provided that $k>\dfrac{1}{2}\displaystyle\int_1^\infty z^2\lambda(\mathrm{d}z)$.

Figure 2 displays a sample path of Equation (29) with $k=1.5$. The trajectory is subject to both continuous Wiener perturbations and discrete Poisson jumps. Although individual jumps cause sudden deviations, the path consistently returns to a bounded region near zero. This recovery behavior is not a proof of stability by itself, but it provides visual evidence that the underlying probability distribution does not diverge — a key feature of stability in distribution as established in Theorem 2.

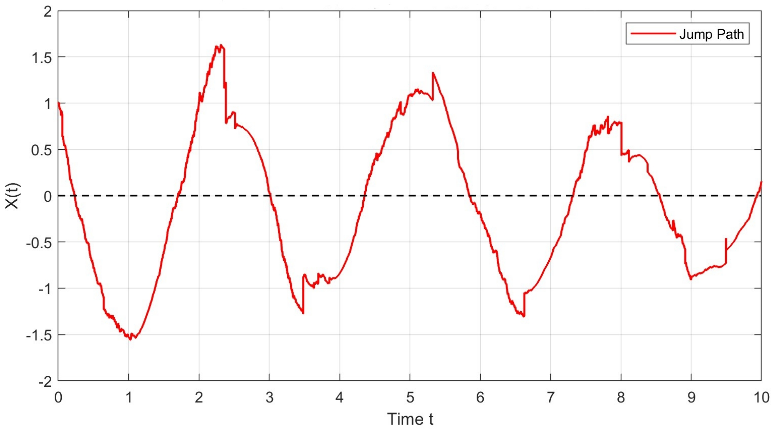

Figure 2. A sample path of Equation (30) with $k = 1.5$.

Author Contributions

J.Z.: conceptualization, methodology, data curation, writing—original draft preparation, validation; W.W.: supervision, writing—reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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