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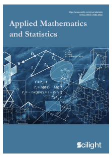

Article

Hybrid Polynomial Basis Functions for Solving Linear Partial Differential Equations

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Abstract: This study focuses on creating a numerical method that combines two polynomial basis functions to solve certain linear Partial Differential Equations (PDEs). We use a power series solution that includes Legendre and Hermite polynomials, which meet the needs of these PDEs. Then, by plugging this series solution into the PDE and using specific collocation points, we set up a system of linear equations with some unknown coefficients. We solved this using the Gauss elimination method with a computer program. We also looked at different ways to choose these collocation points to see how they affect the results. We tested our method on two cases to check its reliability, effectiveness, and accuracy. Finally, we compared our findings with established results found in other studies.

Keywords: Legendre polynomials; Hermite polynomials; linear partial differential equations; collocation points; Gauss elimination method

1. Introduction

Partial differential equations (PDEs) are important tools in math that help explain many physical, engineering, and scientific situations. They're key when investigating systems that change over time and space, as they let us model how different quantities vary with multiple factors.

PDEs have a long history, going back to great mathematicians like Leonhard Euler and Joseph-Louis Lagrange. Scientists use them in various fields like fluid dynamics, heat transfer, quantum physics, and image processing. Their widespread use in research and engineering comes from how well they capture the essential principles of changing physical processes. A key part of solving partial differential equations (PDEs) is expressing their solutions using orthogonal polynomials. These polynomials are known for being orthogonal in specific ranges when different weighting functions are applied. Partial differential equations play a crucial role in scientific computing. They've been important for modeling natural events and linking pure and applied math, making them a core part of many scientific fields. The main thing that defines any type of orthogonal polynomial is its unique relationship when different weighting functions are used in particular intervals.

According to [1], combined Chebyshev and Legendre shifted polynomial basis functions are used to obtain approximate solutions for nonlinear Partial Differential Equations, giving superior results against current studies. [2] researched an approximate calculation technique for fractional differential equations (FDEs) based on Hermite polynomials under the Caputo fractional operator. A system of algebraic equations arises from the approach, which enables Gauss elimination to solve the equations and produce the desired solution, thereby demonstrating the enhanced effectiveness of this method. [3] introduced a boundary value problem solution method through collocations that works well with high-order features. The concept of employing the Hermite-based block approach for the numerical solution of second-order nonlinear elliptic partial differential equations utilizing the interpolation and collocation procedures was examined by [4].

In [5], a new Hilbert space was constructed to satisfy the necessary initial and boundary conditions, with the goal of developing a spectral approximation method for solving nonlinear time-fractional partial integro-differential equations involving a weakly singular kernel.

A barycentric rational collocation technique was introduced in [6] to handle a set of nonlinear parabolic PDEs. Meanwhile, [7] explored a collocation strategy based on a modified form of cubic B-spline trigonometric functions to solve Fisher's reaction–diffusion equation. A meshless barycentric interpolation collocation method (MBICM) was proposed in [8] for solving a class of partial differential equations. The approximate solutions were compared with existing results in the literature and showed improvement. In [9], a numerical approach was developed for solving a class of fractional optimal control problems, relying on Hermite polynomial approximations.

In [10], a highly effective technique using shifted Chebyshev polynomials was employed to tackle PDEs. The authors selected a power series solution built from these shifted polynomials to satisfy the required conditions. In [11], Hermite polynomials were used to solve Volterra–Fredholm integral equations of the second kind numerically. An algorithm was proposed in [12] for solving variable-order fractional differential equations with boundary and initial conditions, using shifted Legendre polynomials. A Galerkin method based on shifted Legendre polynomials was developed in [13] to solve linear and nonlinear fractional delay Volterra integro-differential equations.

A wavelet collocation method using Legendre polynomials was proposed in [14] to find approximate solutions to linear and nonlinear hyperbolic telegraph equations. In [15], a numerical algorithm was developed for solving the fractional nonlinear Sobolev model by combining Legendre polynomials with a finite difference scheme, where the finite difference approach was used to discretize the time variable. An efficient method for approximating sixth-order boundary value problems was presented in [16] based on Legendre wavelets, with Legendre polynomials serving as collocation points to convert the differential equation into a system of algebraic equations. The resulting solutions were then compared with both exact solutions and existing literature.

Despite the extensive literature on polynomial collocation methods for PDEs, a systematic hybridization of Legendre and Hermite basis functions within a unified collocation framework remains underexplored. Existing methods often require many terms to achieve acceptable accuracy or fail to exploit the complementary approximation strengths of multiple polynomial families. The present work is motivated by the need for a method that achieves high accuracy with fewer terms, is straightforward to implement via Gauss elimination, and can be systematically extended to higher discretization orders.

The principal contributions of this work are: (i) the construction of a novel hybrid bivariate basis formed by combining Legendre and Hermite polynomials, which simultaneously satisfies prescribed initial and boundary conditions; (ii) the derivation of a structured collocation scheme that converts the target PDE into a linear algebraic system solvable by Gauss elimination; (iii) a systematic investigation of multiple discretization orders ($N = 2, 3, 4$) demonstrating that accuracy improves significantly with increasing N; and (iv) numerical validation on two benchmark problems confirming that the proposed method outperforms several existing techniques in the literature in terms of error norms.

2. Preliminaries

Legendre polynomials are a set of unique and orthogonal polynomials named after Adrien-Marie Legendre, who came up with them in 1782. They have various mathematical properties and find use in different areas. You can think of them as an orthogonal system when using the weight function $c(k) = 1$, with the range being from −1 to 1. In this case, $Z_p(k)$ represents a polynomial of degree n.

$$\int_{-1}^{1} Z_p(k) Z_q(k)\, dk = 0. \text{ If } p \neq q \tag{1}$$

Rodrigues' formula gives a simple way to write out the Legendre polynomials like this:

$$Z_p(k) = \frac{1}{2^p i!} \frac{d^p}{dk^p} (k^2 - 1)^p \tag{2}$$

Here are the first six Legendre polynomials:

$$Z_0(k) = 1$$
$$Z_1(k) = k$$
$$Z_2(k) = \frac{1}{2}(3k^2 - 1)$$

$$Z_3(k) = \frac{1}{2}(5k^3 - 3k)$$
$$Z_4(k) = \frac{1}{8}(35k^4 - 30k^2 + 3)$$
$$Z_5(k) = \frac{1}{8}(63k^5 - 70k^3 + 15k)$$

Hermite polynomials are a type of orthogonal polynomial that are really important in math, physics, and engineering. They're named after Charles Hermite, a French mathematician. You'll often see them in areas like probability, quantum mechanics, and numerical analysis. Basically, they are the solutions to Hermite's differential equation and come in two versions: one for physicists and one for probabilists. These polynomials are handy in math for various theories and practical uses. They are especially useful when it comes to solving differential equations and finding applications in quantum mechanics and statistical mechanics. You can write them as $A_t(m)$, and they follow a specific recurrence relation.

$$A_{t+1}(m) = 2mA_t(m) - 2tA_{t-1}(m) \tag{3}$$

For $t \geq 1$, with initial conditions $A_0(m) = 1$ and $A_1(m) = 2m$. These polynomials fit this differential equation:

$$A_t''(m) - 2mA_t'(m) + 2tA_t(m) = 0 \tag{4}$$

The Hermite polynomial can be defined using a contour integral: (where the contour is a closed curve encircling the origin, traversed counterclockwise in the complex plane)

$$A_t(r) = \frac{t!}{2\pi i}\oint e^{-u^2+2ur}u^{-t-1}du \tag{5}$$

The path goes around the starting point and moves counter-clockwise. The first few Hermite polynomials are

$$A_0(m) = 1$$
$$A_1(m) = 2m$$
$$A_2(m) = 4m^2 - 2$$
$$A_3(m) = 8m^3 - 12m$$
$$A_4(m) = 16m^4 - 48m^2 + 12$$
$$A_5(m) = 32m^5 - 160m^3 + 120m$$

3. Methodology

The Hybridization Collocation Method

Here, we'll talk about how to tackle partial differential equations (PDEs) using a mix of Legendre and Hermite polynomials. To show you how this works, we'll look at a PDE that has two main variables, *r* and *s*, with *j* being the variable that depends on them. Here's how it's represented:

$$T(j) = d(r,s) \tag{6}$$

Depending on the stipulated terms and circumstances:

$$j(r_0,s) = f(s) \text{ and } j(r,s_0) = u(r) \tag{7}$$

Let *j* be a function we don't know yet, while d, f, and u are functions we do know, and the other values stay constant. First, we assume our solution is a two-variable power series that uses Legendre and Hermite polynomials. This solution has to fit the requirements shown in (8) below.

$$j(r,s) = \mu(r,s) + \gamma(r,s)\sum_{a=0}^{M}\sum_{b=0}^{N} w_{ab}\, Z_a^{*}(r)\overset{\sim\sim}{A}_b(s) \tag{8}$$

It's important to recognize that $\mu(r,s)$ and $\gamma(r,s)$ are traits of both r and s, and we need to choose them in a way that makes sure Equation (8) meets the rules outlined in Condition (7). Next, we plug our guessed solution from (8) into (6), which gives us a residual equation we call $G(r,s) = 0$, containing coefficients we label as w_{ab}. By using the right points of collocation $[0,1]$ by $[0,1]$; we end up with a system of equations that includes these unknown coefficients w_{ab}.

$$Tp = e \tag{9}$$

In this situation, T is the known value of the matrix, p is the column vector with the Legendre and Hermite coefficients, and e is the set value on the right side. What's key to remember is that both T and e are made up of real numbers that are figured out using the collocation points in the residuals equation $B(r, s) = 0$.

Now, ${Q_l}^2$ as collocation points become a requisite when we consider Q_l polynomial where $Q_l = N + 1$. Next, we set up the equations to find the unknown coefficients and plug them into (8) to get an approximate solution for (6). Discretizing the domain is important and needs attention. We'll look at different discretization patterns before picking the best one to get good results. The collocation points in the [0, 1] interval are selected as equally-spaced nodes of the form $x^i = i/(N+1)$, $i = 1, 2,\ldots, N+1$, where N is the discretization order. This choice ensures that the residual equation is enforced at interior points of the domain, avoiding endpoint singularities and providing a well-conditioned system of linear equations. Equal spacing is adopted here for simplicity and reproducibility; Chebyshev-Gauss-Lobatto nodes are noted as an alternative that can further improve conditioning for higher N.

For a discretization order N, the proposed method generates a dense $(N+1)^2 \times (N+1)^2$ linear system, which Gauss elimination solves in $O((N+1)^6)$ operations. In contrast, finite difference methods on an $n \times n$ grid produce an $O(n^2)$ sparse system solved in $O(n^3)$ operations via direct solvers. The hybrid collocation method is therefore most advantageous for problems where high accuracy is required with small N (e.g., smooth solutions on regular domains), since the dense system remains tractable. Finite element or finite difference methods are preferred when N is large or the domain is irregular. The present results confirm that $N = 4$ yields errors several orders of magnitude smaller than competing methods, demonstrating the efficiency of the approach for the benchmark problems considered.

4. Numerical Examples and Their Results

Example 1. *First we consider a linear partial differential equation of the form.*

$$9\frac{\partial^2 F}{\partial j^2} - \frac{\partial^2 F}{\partial k^2} + e^j - e^{-j} = 0 \tag{10}$$

Taking the conditions into consideration:

$$F(j, 0) = j \text{ and} F_k(j, 0) = sin\, j \tag{11}$$

where the analytical solution of the given equation is;

$$F(j, k) = j + \frac{1}{3} sin\, j\, sin\, 3\, k - \frac{2}{9} sinh\, j + \frac{2}{9} sinh\, j\, cosh\, 3\, k \tag{12}$$

Assume that the bivariate series solution for the linear PDE (10) as;

$$F(j, k) = j + k\, sin\, j + (jk^2) \sum_{q=0}^{4} \sum_{s=0}^{4} a_{qs} T_q^*(j) P_s^*(k) \tag{13}$$

where $\omega(j, k) = j + k\, sin\, j$ and $\lambda(j, k) = jk^2$. While $T_q^*(j)$ and $P_s^*(k)$ represent the polynomial basis functions: Legendre and Hermite. Initial conditions (Equation (11) are satisfied by the anticipated solution Equation (13)). Now, differentiate Equation (13) partially with relation to j and k to acquire the derivatives as;

$$\begin{aligned}
&\frac{\partial^2 F}{\partial k^2} = 2ja_{00} + \frac{3}{2} jk(35j^4 - 30j^2 + 3)a_{41} + 6jk(3j^2 - 1)a_{21} + 2j^2(8k^3 - 12k)a_{13} + \\
&20jk^2(3j^2 - 1)a_{22} + 20jk^2(5j^3 - 3j)a_{32} + 5jk^2(35j^4 - 30j^2 + 3)a_{42} + 4jk \\
&(24k^2 - 12)a_{03} + 4jk(64k^3 - 96k)a_{04} + 4j^2k(64k^3 - 96k)a_{14} + 4j^2k(24k^2 - 12)a_{13} \\
&+ \frac{1}{2} jk^2(35j^4 - 30j^2 + 3)(48k^2 - 24)a_{44} + jk^2(3j^2 - 1)(96k^2 - 48)a_{24} + jk^2(5j^3 - 3j) \\
&(96k^2 - 48)a_{34} + j(35j^4 - 30j^2 + 3)(4k^4 - 12k^2 + 3)a_{44} + j(35j^4 - 30j^2 + 3) \\
&(2k^3 - 3k)a_{43} + 2j(3j^2 - 1)(2k^2 - 1)a_{22} + 2j(3j^2 - 1)(4k^3 - 6k)a_{23} + 2j(3j^2 - 1) \\
&(8k^4 - 24k^2 + 6)a_{24} + 2j(5j^3 - 3j)(2k^2 - 1)a_{32} + 2j(5j^3 - 3j)(4k^3 - 6k)a_{33} + 2j \\
&(5j^3 - 3j)(8k^4 - 24k^2 + 6)a_{34} + \frac{1}{2} j(35j^4 - 30j^2 + 3)(2k^2 - 1)a_{42} + 2jk(35j^4 - 30j^2 + 3)
\end{aligned}$$

$$(16k^3-24k)a_{44}+2jk(35j^4-30j^2+3)(6k^2-3)a_{43}+4jk(3j^2-1)(12k^2-6)a_{23}+$$

$$4jk(3j^2-1)(32k^3-48k)a_{24}+4jk(5j^3-3j)(12k^2-6)a_{33}+4jk(5j^3-3j)$$

$$(32k^3-48k)a_{34}+jk^2(192k^2-96)a_{04}+j(3j^2-1)a_{20}+j(5j^3-3j)a_{30}+\frac{1}{4}j$$

$$(35j^4-30j^2+3)a_{40}+j^2k^2(192k^2-96)a_{14}+6jk(5j^3-3j)a_{31}+6jk^3(35j^4-30j^2+3)$$

$$a_{43}+24jk^3(3j^2-1)a_{23}+2j(4k^2-2)a_{02}+2j(8k^3-12k)a_{03}++2j(16k^4-48k^2+12)a_{04}$$

$$+24jk^3(5j^3-3j)a_{33}+2j^2(4k^2-2)a_{12}+2j^2(16k^4-48k^2+12)a_{14}+2j^2a_{10}$$

$$+48jk^3a_{03}+12jka_{01}+40jk^2a_{02}+12j^2ka_{11}+40j^2k^2a_{12}+48j^2k^3a_{13}$$

and

$$\frac{\partial^2 F}{\partial j^2}=\frac{1}{4}jk^3(420j^2-60)a_{41}+\frac{1}{8}jk^2(420j^2-60)a_{40}+2k^2(4k^2-2)a_{12}+2k^2$$

$$(16k^4-48k^2+12)a_{14}+2k^2(8k^3-12k)a_{13}+k^2(140j^3-60j)(4k^4-12k^2+3)a_{44}$$

$$+k^2(140j^3-60j)(2k^3-3k)a_{43}+18k^2j(2k^2-1)a_{22}+18k^2j(4k^3-6k)a_{23}$$

$$+18k^2j(8k^4-24k^2+6)a_{24}+2k^2(15j^2-3)(2k^2-1)a_{32}+30k^2j^2(2k^2-1)a_{32}+$$

$$2k^2(15j^2-3)(4k^3-6k)a_{33}+30k^2j^2(4k^3-6k)a_{33}+2k^2(15j^2-3)(8k^4-24k^2+6)a_{34}$$

$$30k^2j^2(8k^4-24k^2+6)a_{34}+\frac{1}{2}k^2(140j^3-60j)(2k^2-1)a_{42}+k\left(-j+\frac{1}{6}j^3-\frac{1}{120}j^5\right)$$

$$+\frac{1}{2}jk^2(420j-60)(4k^4-12k^2+3)a_{44}+\frac{1}{2}jk^2(420j-60)(4k^4-12k^2+3)a_{44}+$$

$$+\frac{1}{2}jk^2(420j-60)(2k^3-3k)a_{43}+\frac{1}{4}jk^2(420j^2-60)(2k^2-1)a_{42}+\frac{1}{2}k^3(140j^3-60j)a_{41}$$

$$+2k^3(15j^2-3)a_{31}+30k^3j^2a_{31}+15k^2j^2a_{30}+4k^3a_{11}+18k^3ja_{21}+9k^2ja_{20}+\frac{1}{4}k^2$$

$$(140j^3-60j)a_{40}+k^2(15j^2-3)a_{30}+2k^2a_{10}$$

Now, substitute the above partial derivatives into Equation (13) to have the residual form of equation $H(j,k)=0$. Then, collocation at $\left(\frac{1}{5},1\right),\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{1}{5},\frac{3}{5}\right),\left(\frac{1}{5},\frac{2}{5}\right),\left(\frac{1}{5},\frac{1}{5}\right)\left(\frac{1}{5},0\right),\left(\frac{2}{5},1\right),\left(\frac{2}{5},\frac{4}{5}\right)$, $\left(\frac{2}{5},\frac{3}{5}\right),\left(\frac{2}{5},\frac{2}{5}\right),\left(\frac{2}{5},\frac{1}{5}\right),\left(\frac{2}{5},0\right),\left(\frac{3}{5},1\right),\left(\frac{3}{5},\frac{4}{5}\right),\left(\frac{3}{5},\frac{3}{5}\right),\left(\frac{3}{5},\frac{2}{5}\right),\left(\frac{3}{5},\frac{1}{5}\right),\left(\frac{3}{5},0\right),\left(\frac{4}{5},1\right),\left(\frac{4}{5},\frac{4}{5}\right),\left(\frac{4}{5},\frac{3}{5}\right),\left(\frac{4}{5},\frac{2}{5}\right),\left(\frac{4}{5},\frac{1}{5}\right)$ $\left(\frac{4}{5},0\right)$, $\left(1,\frac{3}{5}\right)$, since our $N = 4$, which in turns gives a system of twenty-five linear algebraic equations. Having solved the system by method of Gauss elimination or computational software, we have the unknowns as;

$a_{00}=1.115835657, a_{01}=0.1908856637, a_{02}=-0.08615993146, a_{03}=0.1477219593,$
$a_{04}=-0.01914247005, a_{10}=-0.1006960191, a_{11}=0.1202854170, a_{12}=-0.06519389975,$
$a_{13}=0.01621041354, a_{14}=-0.002701731953, a_{20}=0.3856576552, a_{21}=-0.2613153517,$ $a_{22}=$
$0.1850732302, a_{23}=-0.05152110529,$ $a_{24}=0.008586847784, a_{30}=-0.05114833842,$
$a_{31}=0.06108438939, a_{32}=-0.03310054622, a_{33}=0.008228979479, a_{34}=-0.001371495174$
$a_{40}=0.002757690329, a_{41}=-0.001750967685, a_{42}=-0.00001570375646,$
$a_{43}=0.0002092820231, a_{44}=-0.00003488061030$

Substituting all a'_{qs} in Equation (13), we obtain an approximate form of solution for the differential equation as thus;

$$
\begin{aligned}
&F(j,k) = 0.3817713274jk^3 + 1.115835657jk^2 + j + jk - \frac{1}{6}j^3k + \frac{1}{120}j^5k - \frac{1}{5040}j^7k + \\
&0.2405708340j^2k^3 - 0.06519389975j^2k^2(4k^2-2) - 0.01914247005jk^2(16k^4 - 48k^2 + 12) \\
&+0.01621041354j^2k^2(8k^3 - 12k) - 0.002701731953j^2k^2(16k^4 - 48k^2 + 12) + \\
&0.1477219593jk^2(8k^3 - 12k) - 0.08615993146jk^2(4k^2 - 2) - 0.00001744030515jk^2 \\
&(35j^4 - 30j^2 + 3)(4k^4 - 12k^2 + 3) + 0.0001046410116jk^2(35j^4 - 30j^2 + 3)(2k^3 - 3k) \\
&+0.1850732302jk^2(3j^2 - 1)(2k^2 - 1) - 0.05152110529jk^2(3j^2 - 1)(4k^3 - 6k) + \\
&0.008586847784jk^2(3j^2 - 1)(8k^4 - 24k^2 + 6) - 0.03310054622jk^2(5j^3 - 3j)(2k^2 - 1) \\
&+0.008228979479jk^2(5j^3 - 3j)(4k^3 - 6k) - 0.001371495174jk^2(5j^3 - 3j) \\
&(8k^4 - 24k^2 + 6) - 0.000003925939115jk^2(35j^4 - 30j^2 + 3)(2k^2 - 1) - 0.2613153517 \\
&jk^2(3j^2 - 1) - 0.0004377419212jk^2(35j^4 - 30j^2 + 3) + 0.06108438939jk^3(5j^3 - 3j) \\
&-0.1006960191j^2k^2 + 0.0003447112911jk^2(35j^4 - 30j^2 + 3) - 0.02557416921jk^2 \\
&(5j^3 - 3j) + 0.1928288276jk^2(3j^2 - 1)
\end{aligned}
$$

But the exact solution of (10) is

$$
F(j,k) = j + \tfrac{1}{3} sin\, j\, sin\, 3\, k - \tfrac{2}{9} sin\, h\, j + \tfrac{2}{9} sin\, h\, j\, cosh\, 3\, k.
$$

Discussion of Results

Looking at Tables 1 and 2, the approximate results obtained with the current approach align well with the theoretical solution for Example 1, showing strong agreement. The collocation points selected within the domain boundaries prove effective based on the outcomes achieved. To improve the approximate solution for the PDEs, this method also calls for testing different discretization patterns. As the results show, when different schemes ($N = 2$, $N = 3$, and $N = 4$) were examined, the approach performed better than existing ones, with finer discretization patterns yielding more accurate results.

Table 1. Show the computations of the results when $k = 0.5$ at different points of discretization.

k	j	Exact Solution	Present Scheme at $N = 2$	Present Scheme at $N = 3$	Present Scheme at $N = 4$
0.5	0.001	0.001633033794	0.001685081449	0.001632410590	0.0016330941220
0.5	0.003	0.004899101252	0.005055245384	0.004897231568	0.0048992825440
0.5	0.005	0.008165168327	0.008425412446	0.008162051942	0.0081654710820
0.5	0.007	0.011431234770	0.011795584700	0.011426871310	0.0114316592600
0.5	0.009	0.01469730030	0.015165764230	0.014691689260	0.0146978469200
0.5	0.01	0.01633033266	0.016850857380	0.016324097600	0.0163309404800

Table 2. Show the error norms from Table 1 results, also with the existing literature.

k	j	Karunakar, et al (2019) [10] at $N = 3$	Proposed method at $N = 2$	Proposed method at $N = 3$	Proposed method at $N = 4$
0.5	0.001	4.99371×10^{-4}	5.20477×10^{-5}	6.23204×10^{-7}	6.03280×10^{-8}
0.5	0.003	1.49811×10^{-3}	1.56144×10^{-4}	1.86968×10^{-6}	1.81292×10^{-7}
0.5	0.005	2.49684×10^{-3}	2.60244×10^{-4}	3.11639×10^{-6}	3.02755×10^{-7}
0.5	0.007	3.49557×10^{-3}	3.64350×10^{-4}	4.36345×10^{-6}	4.24490×10^{-7}
0.5	0.009	4.49428×10^{-3}	4.68464×10^{-4}	5.61104×10^{-6}	5.46620×10^{-7}
0.5	0.01	4.99362×10^{-3}	5.20525×10^{-4}	6.23506×10^{-6}	6.07820×10^{-7}

Example 2. *Turning to the second example, this time involving a linear partial differential equation.*

$$
\frac{1}{2}h^2\frac{\delta^2 B}{\delta h^2} - \frac{\delta^2 B}{\delta w^2} = 0 \tag{14}
$$

with the conditions that

$$
0 \le h \le 1, \qquad 0 \le w \le 1 \text{ and } B(h,0) = h, B_w(h,0) = h^2 \tag{15}
$$

The exact solution for the differential equation is provided as follows:

$$
B(h,w) = h + h^2\, sin\, h\,(w) \tag{16}
$$

Assume that the bivariate series solution for the PDE (14) as;

$$B(h,w) = (h + h^2 w) + w^2 \sum_{x=0}^{4} \sum_{y=0}^{4} a_{xy} L_x^*(h) H_y^*(w) \tag{17}$$

where $\beta(h,w) = h + h^2 w and \alpha(h,w) = w^2$. Initial conditions (Equation (15) are satisfied by the anticipated solution Equation (17)). Next, Equation (17) is partially differentiated with respect to h and w to obtain the corresponding derivatives needed in Equation (14). By following the same steps used in Example 1, the unknown coefficients come out as follows:

$a_{00} = 0.01721267320, a_{01} = -0.005030094250, a_{02} = 0.01113285868, a_{03} = 0.002433821275,$
$a_{04} = -0.0004150336775, a_{10} = -0.03987542587, a_{11} = 0.07888238309, a_{12} = -0.02594580700,$
$a_{13} = -0.004789002723, a_{14} = 0.0008960504372, a_{20} = 0.03791188723, a_{21} = -0.01874371321$
$a_{22} = 0.02518221861, a_{23} = 0.004480479027, a_{24} = -0.0007467465046 a_{30} = -0.02132907818$
$a_{31} = 0.04272992498, a_{32} = -0.01450838674, a_{33} = -0.001848350697, a_{34} = 0.0003080584496$
$a_{40} = 0.006666464297, a_{41} = -0.01536363182, a_{42} = 0.004959574433, a_{43} = a_{44} = 0$

Substituting all a'_{xy} in Equation (17), we obtain an approximate form of solution for the differential equations as thus;

$$\begin{aligned}
&B(h,w) = -0.004789002723 w^2 h(8w^3 - 12w) + 0.001239893608 w^2 (35h^4 - 30h^2 + 3)\\
&(2w^2 - 1) - 0.02594580700 w^2 h(4w^2 - 2) - 0.001848350697 w^2 (5h^3 - 3h)(4w^3 - 6w)\\
&-0.01450838674 w^2 (5h^3 - 3h)(2w^2 - 1) - 0.0007467465046 w^2 (3h^2 - 1)(8w^4 - 24w^2 + 6)\\
&+0.0003080584496 w^2 (5h^3 - 3h)(8w^4 - 24w^2 + 6) + 0.02518221861 w^2 (3h^2 - 1)(2w^2 - 1)\\
&+0.004480479027 w^2 (3h^2 - 1)(4w^3 - 6w) + 0.0008960504372 w^2 h(16w^4 - 48w^2 + 12)\\
&+0.04272992498 w^3 (5h^3 - 3h) + h^2 w - 0.01006018850 w^3 + 0.01721267320 w^2 + h\\
&-0.01874371321 w^3 (3h^2 - 1) + 0.01113285868 w^2 (4w^2 - 2) + 0.002433821275 w^2 (8w^3 - 12w)\\
&-0.0004150336775 w^2 (16w^4 - 48w^2 + 12) - 0.03987542587 w^2 h + 0.01895594362 w^2 (3h^2 - 1)\\
&-0.01066453909 w^2 (5h^3 - 3h) + 0.0008333080121 w^2 (35h^4 - 30h^2 + 3) + 0.1577647662 w^3 h\\
&-0.003840907955 w^3 (35h^4 - 30h^2 + 3)
\end{aligned}$$

The results obtained through the present scheme agree favorably with the theoretical solution of example two and demonstrate good performance according to the table above. The collation points selected inside the boundary area demonstrate satisfactory results through achieved outcomes. To get better PDE approximate results various discretization patterns must be selected as part of this method. The solution showed us different schemes from *N* = 2 to *N* = 3 and *N* = 4, where the scheme performed effectively compared to the existing scheme (as shown Tables 3 and 4).

Table 3. Show the computations of the results when *h* = 0.5 at different points of discretization.

h	*w*	Exact solution	Present scheme at *N* = 2	Present scheme at *N* = 3	Present scheme at *N* = 4
0.5	0.001	0.5002500000	0.5002500000	0.5002500000	0.5002500000
0.5	0.003	0.5007500011	0.5007500010	0.5007500011	0. 5007500010
0.5	0.005	0.5012500052	0.5012500048	0.5012500053	0.5012500049
0.5	0.007	0.5017500143	0.5017500131	0.5017500144	0.5017500136
0.5	0.009	0.5022500304	0.5022500280	0.5022500304	0.5022500293
0.5	0.01	0.5025000417	0.5025000384	0.5025000416	0.5025000402

Table 4. Show the error norms from Table 3 results, also with the existing literature.

h	*w*	Karunakar, et al (2019) [10] at *N* = 3	Proposed method at *N* = 2	Proposed method at *N* = 3	Proposed method at *N* = 4
0.5	0.001	1.562×10^{-7}	0.00000000	0.00000000	0.00000000
0.5	0.003	1.4028×10^{-6}	1.0000×10^{-10}	0.00000000	1.0000×10^{-10}
0.5	0.005	3.8893×10^{-6}	4.0000×10^{-10}	1.0000×10^{-10}	3.0000×10^{-10}
0.5	0.007	7.6087×10^{-6}	1.2000×10^{-9}	1.0000×10^{-10}	7.0000×10^{-10}
0.5	0.009	1.2554×10^{-5}	2.4000×10^{-9}	0.00000000	1.1000×10^{-9}
0.5	0.01	1.5484×10^{-5}	3.3000×10^{-9}	1.0000×10^{-10}	1.5000×10^{-9}

5. Conclusion and Remarks

A numerical strategy blending collocation with a mix of Legendre and Hermite polynomial bases was put into practice and proved effective for solving a set of linear PDEs. In this approach, a trial solution built from these hybrid polynomials was put forward, and it met the necessary conditions of the equations. When held up against earlier methods found in the literature, tweaking the initial guess paid off—yielding an approximate solution that required fewer terms. The numerical work relied on collocation techniques, and convergence was confirmed by setting a tolerance level close to the exact solution. Two test cases were examined, and the results were compared with those in existing studies. The findings showed that the proposed method works well and is capable of reaching convergence using fewer terms. That said, it would be worth exploring two-dimensional nonlinear PDEs, especially those involving complex boundary conditions or irregular domains.

Author Contributions

A.E.A.: conceptualization, methodology, software; A.S.O.: data curation, writing—original draft preparation; J.O.A.: visualization, investigation; M.O.O.: supervision; M.E.A.: software, validation; writing—reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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