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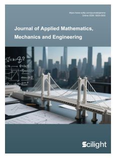

Article

k-Grahaml Sequences and Special Relations

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Abstract: In this study, we define the k-Grahaml, k-Grahaml-Lucas, and Modified k-Grahaml sequences and certain terms from these sequences are given. We then determine the generating functions, evaluate the summations, and derive the related expressions. We also relate the terms of these sequences to matrices and carry out various matrix-related computations. We express the Binet formulas of these sequences in different ways. In addition, we further study the interrelations between various forms of Grahaml-type sequences, including k-Grahaml, k-Grahaml-Lucas, and their modified versions. Finally, we discover an intriguing relationship between the k-Grahaml sequence and the Grahaml-Lucas sequence.

Keywords: Grahaml sequence; Binet formulas; recurrence relations; Lucas numbers; generating function

MSC Classification: 11B37; 11B39; 11B83; 05A15

1. Introduction

Sequences play an essential role in mathematics, offering insight into patterns and structures that arise in various conditions. A well-known and historically significant example is the Rabbit Problem, introduced by the Italian mathematician Leonardo Fibonacci in his landmark book Liber Abaci, published in the early 1200s. In this scenario, Fibonacci explored how a rabbit population would grow under ideal circumstances, assuming that each pair of rabbits begins producing another pair every month once they are two months old.

The Fibonacci and Lucas sequences are two of the most iconic and intriguing numerical sequences in mathematics. Their remarkable properties and unexpected appearances in fields such as nature, art, and architecture have fascinated mathematicians, scientists, artists, and philosophers for centuries. The Fibonacci sequence, in particular, has been applied across numerous disciplines, including engineering [1], economy [2], group theory [3], graph theory [4], liberal arts [5], cryptography [6], biomathematics [7], computer science [8], and many more.

Over the years, numerous generalizations of the Fibonacci sequence have been proposed, reflecting its profound mathematical significance and versatility. These extensions often arise by changing the sequence's recurrence relationship, initial conditions, or dimensionality, leading to entirely new families of numbers with properties that both resemble and extend those of the classical Fibonacci sequence. Some of the most well-known generalizations are the Tribonacci [9] and Tetranacci [10] sequences, in which each term is defined by the sum of three or more preceding terms rather than just two. Others involve introducing coefficients, negative indices, or using matrix representations to explore the sequence in more abstract contexts. These generalized sequences are studied not only for their theoretical interest but also for their practical applications in fields such as combinatorics, computer algorithms, coding theory, and even the modeling of real-world phenomena. Examples include the Copper Fibonacci [11], Bronze Leonardo [12], Padovan [13], Pell [14], k-Chebyshev [15], Narayana [16], k-Oresme [17], Dickson k-Fibonacci [18], Bell [19], Bigollo [20], k-Blaise [21], Copper Lucas [22], Fermat [23],

k-Edouard [24], Bernoulli-Euler polynomials [25], Hyperharmonic Fibonacci [26], Perfect square [27], k-Mersenne [28], Jacobsthal [29], and so on.

For $n \in \mathbb{N}$, Fibonacci numbers F_n, Lucas numbers $\mathcal{L}_n$, and Leonardo numbers l_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \text{ with } F_0 = 0 \text{ and } F_1 = 1,$$

$$\mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n, \text{ with } \mathcal{L}_0 = 2 \text{ and } \mathcal{L}_1 = 1,$$

$$l_{n+2} = l_{n+1} + l_n + 1, \text{ with } l_0 = 1 \text{ and } l_1 = 1.$$

Binet formulas for F_n, $\mathcal{L}_n$, and l_n are given by the following relations, respectively,

$$F_n = \frac{\varphi^n - \sigma^n}{\alpha - \beta},\ \mathcal{L}_n = \varphi^n + \sigma^n,\ l_n = 2\frac{\varphi^{n+1} - \sigma^{n+1}}{\alpha - \beta} - 1,$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\sigma = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$. In this context, the number φ represents the golden ratio, which is one of the well-known metallic ratios.

In addition, for $n \in \mathbb{N}$, Soykan [30] defined the Grahaml sequence G_n, Grahaml-Lucas sequence H_n, and Modified Grahaml sequence E_n with the following recurrence relations, respectively,

$$G_{n+3} = 2G_{n+2} + 3G_{n+1} + 5G_n, \text{ with } G_0 = 0, G_1 = 1, \text{ and } G_2 = 2,$$

$$H_{n+3} = 2H_{n+2} + 3H_{n+1} + 5H_n, \text{ with } H_0 = 3, H_1 = 2, \text{ and } H_2 = 10,$$

$$E_{n+3} = 2E_{n+2} + 3E_{n+1} + 5E_n, \text{ with } E_0 = 0, E_1 = 1, \text{ and } E_2 = 1.$$

The Binet formulas for G_n, H_n,, and E_n, are expressed by the following respective equations;

$$G_n = \frac{m^{n+1}}{(m-p)(m-r)} + \frac{p^{n+1}}{(p-m)(p-r)} + \frac{r^{n+1}}{(r-m)(r-p)},$$

$$H_n = m^n + p^n + r^n,$$

$$E_n = \frac{(m-1)m^{n+1}}{(m-p)(m-r)} + \frac{(p-1)p^{n+1}}{(p-m)(p-r)} + \frac{(r-1)r^{n+1}}{(r-m)(r-p)}.$$

Hence, $m \cong 3{,}344$, $p \cong -0{,}672 + 1{,}021i$, and $r \cong -0{,}672 - 1{,}021i$ are the roots of the characteristic equation $x^3 - 2x^2 - 3x - 5 = 0$. In [30], Soykan found many features of the Grahaml sequences.

Using the recurrence relation of the classical Fibonacci sequence as a foundation, researchers have introduced the concept of k-Fibonacci sequences, which hold significant relevance in number theory. In particular, Falcon and Plaza, in their work [31], developed the k-Fibonacci sequence and explored numerous properties associated with it. Later, Falcon extended this line of study by defining the k-Lucas sequences [32]. Furthermore, in another contribution, Falcon applied the Hankel transform to the k-Fibonacci sequence, leading to a novel way of deriving terms of the Fibonacci sequence [33].

As demonstrated by these developments, a wide variety of generalizations of the Fibonacci and Lucas sequences have been proposed to date. Building upon this rich mathematical tradition, the present study introduces new extensions that draw inspiration from both the k-Fibonacci and Grahaml sequences. Specifically, we define and examine three new families of sequences, which we name the k-Grahaml, k-Grahaml-Lucas, and Modified k-Grahaml sequences. These are denoted by the symbols $R_{k,n}$, $L_{k,n}$, and $M_{k,n}$, respectively.

This article is structured in three distinct sections.

In chapter 2, we introduce the k-Grahaml, k-Grahaml-Lucas, and Modified k-Grahaml sequences. We explore their characteristic equations, derive Binet formulas, and present their generating functions and summation identities. Several properties of these sequences are also discussed in detail. Furthermore, we demonstrate how to derive Binet formulas using their generating functions. The chapter concludes by establishing matrix representations that relate to the terms of these sequences.

In chapter 3, we focus on identifying connections between the k-Grahaml, k-Grahaml-Lucas, Modified k-Grahaml, and their classical counterparts: Grahaml, Grahaml-Lucas, and Modified Grahaml sequences. By examining these relationships, we reveal significant links among their terms. The chapter concludes with the discovery of a noteworthy relationship between the k-Grahaml and Grahaml-Lucas sequences.

2. k-Grahaml, k-Grahaml-Lucas, and Modified k-Grahaml Sequences

In this section, a new generalization of Grahaml sequences is studied. Then we obtain many properties of this generalization such as special summation formulas and generating functions. In addition, we associate the terms of these sequences with matrices.

Definition 1. *For $k \in \mathbb{R}$, and $n \in \mathbb{N}$, the k-Grahaml $R_{k,n}$, k-Grahaml-Lucas $L_{k,n}$, and Modified k-Grahaml $M_{k,n}$ are defined by, respectively;*

$$R_{k,n+3} = 2R_{k,n+2} + 3R_{k,n+1} + 5R_{k,n} + k, \text{ with } R_{k,0} = 0,\ R_{k,1} = 1, \text{ and } R_{k,2} = 2,$$

$$L_{k,n+3} = 2L_{k,n+2} + 3L_{k,n+1} + 5L_{k,n} + k, \text{ with } L_{k,0} = 3,\ L_{k,1} = 2, \text{ and } L_{k,2} = 10,$$

$$M_{k,n+3} = 2M_{k,n+2} + 3M_{k,n+1} + 5M_{k,n} + k, \text{ with } M_{k,0} = 0,\ M_{k,1} = 1, \text{ and } M_{k,2} = 1.$$

Also, the four-order recurrence relations of the sequences $R_{k,n}$, $L_{k,n}$, and $M_{k,n}$ are as follows,

$$R_{k,n+4} = 3R_{k,n+3} + R_{k,n+2} + 2R_{k,n+1} - 5R_{k,n},$$

$$\text{with } R_{k,0} = 0,\ R_{k,1} = 1, R_{k,2} = 2,\ R_{k,3} = k + 7,$$

$$L_{k,n+4} = 3L_{k,n+3} + L_{k,n+2} + 2L_{k,n+1} - 5L_{k,n},$$

$$\text{with } L_{k,0} = 3,\ L_{k,1} = 2, L_{k,2} = 10,\ L_{k,3} = k + 41,$$

$$M_{k,n+4} = 3M_{k,n+3} + M_{k,n+2} + 2M_{k,n+1} - 5M_{k,n},$$

$$\text{with } M_{k,0} = 0,\ M_{k,1} = 1,\ M_{k,2} = 1,\ M_{k,3} = k + 5,$$

Then let's give some information about the equations of these sequences.

The characteristic equation of the k-Grahaml, k-Grahaml-Lucas, and Modified k-Grahaml sequences is

$$t^4 - 3t^3 - t^2 - 2t + 5 = 0 = (t^3 - 2t^2 - 3t - 5)(t - 1) = 0.$$

The roots of this equation are;

$$m \cong 3.344,\ p \cong -0.672 + 1.021i,\ r \cong -0.672 - 1.021i, \text{ and } s = 1.$$

The relationship between these roots is given below;

$$mp + mr + ms + pr + ps + rs = -1,\ m + p + r + s = 3,\ mprs = 5, \text{ and}$$

$$mpr + mps + mrs + prs = 2.$$

The $R_{k,n}$, $L_{k,n}$, and $M_{k,n}$ values for the six ten n natural numbers are given below;

- $R_{k,0} = 0,\ L_{k,0} = 3,\ M_{k,0} = 0,$
- $R_{k,1} = 1,\ L_{k,1} = 2,\ M_{k,1} = 1,$
- $R_{k,2} = 2,\ L_{k,2} = 10,\ M_{k,2} = 1,$
- $R_{k,3} = k + 7,\ L_{k,3} = k + 41,\ M_{k,3} = k + 5,$
- $R_{k,4} = 3k + 25,\ L_{k,4} = 3k + 122,\ M_{k,4} = 3k + 18,$
- $R_{k,5} = 10k + 81,\ L_{k,5} = 10k + 417,\ M_{k,5} = 10k + 56,$
- $R_{k,6} = 35k + 272,\ L_{k,6} = 35k + 1405,\ M_{k,6} = 35k + 191.$

In the following theorem, we establish the Binet formulas of the k-Grahaml, k-Grahaml-Lucas, and Modified k-Grahaml sequences.

Theorem 1. *Let $k \in \mathbb{R}$ and $n \in \mathbb{N}$. We obtain*

i. $R_{k,n} = \frac{(m^2-m+k)m^n}{(m-p)(m-1)(m-r)} + \frac{(p^2-p+k)p^n}{(p-m)(p-r)(p-1)} + \frac{(r^2-r+k)r^n}{(r-m)(r-p)(r-1)} + \frac{k}{(1-m)(1-p)(1-r)},$

ii. $L_{k,n} = \frac{(2m^3+4m^2+(9+k)m-15)m^{n-1}}{(m-p)(m-1)(m-r)} + \frac{(2p^3+4p^2+(9+k)p-15)p^{n-1}}{(p-m)(p-r)(p-1)} + \frac{(2r^3+4r^2+(9+k)r-15)r^{n-1}}{(r-m)(r-p)(r-1)} + \frac{k}{(1-m)(1-p)(1-r)},$

iii. $M_{k,n} = \frac{(m^2-2m+k+1)m^n}{(m-p)(m-1)(m-r)} + \frac{(p^2-2p+k+1)p^n}{(p-m)(p-r)(p-1)} + \frac{(r^2-2r+k+1)r^n}{(r-m)(r-p)(r-1)} + \frac{k}{(1-m)(1-p)(1-r)}.$

Proof. i. The Binet form of a sequence is as follows

$$R_{k,n} = Am^n + Bp^n + Cr^n + Ds^n.$$

For these n values, we obtain;

$$R_{k,0} = A + B + C + D,$$

$$R_{k,1} = Am + Bp + Cr + Ds,$$

$$R_{k,2} = Am^2 + Bp^2 + Cr^2 + Ds^2,$$

$$R_{k,3} = Am^3 + Bp^3 + Cr^3 + Ds^3.$$

We find

$$A = \frac{(m^2-m+k)}{(m-p)(m-1)(m-r)},\ B = \frac{(p^2-p+k)}{(p-m)(p-r)(p-1)},\ C = \frac{(r^2-r+k)}{(r-m)(r-p)(r-1)},\text{ and } D = \frac{k}{(1-m)(1-p)(1-r)}.$$

The proofs of the others are shown similarly. □

In the following theorems, we give the generating functions of the k-Grahaml $R_{k,n}$, k-Grahaml -Lucas $L_{k,n}$, and Modified k-Grahaml $M_{k,n}$ sequences. In addition, we obtain Binet formulas of $R_{k,n}$, $L_{k,n}$, and $M_{k,n}$ sequences with the help of generating functions.

Theorem 2. *The generating functions for k-Grahaml $R_{k,n}$, k-Grahaml-Lucas $L_{k,n}$, and Modified k-Grahaml $M_{k,n}$ sequences are given as follows, respectively,*

i. $r(t) = \sum_{n=0}^{\infty} R_{k,n}t^n = \frac{t-t^2+kt^3}{5t^4-2t^3-t^2-3t+1}$,

ii. $l(t) = \sum_{n=0}^{\infty} L_{k,n}t^n = \frac{3-7t+t^2+(k+3)t^3}{5t^4-2t^3-t^2-3t+1}$,

iii. $m(t) = \sum_{n=0}^{\infty} M_{k,n}t^n = \frac{t-2t^2+(k+1)t^3}{5t^4-2t^3-t^2-3t+1}$.

Proof. i. For the k-Grahaml sequence, we have

$r(t) = \sum_{n=0}^{\infty} R_{k,n}t^n = t + 2t^2 + (k+7)t^3 + \sum_{n=4}^{\infty} R_{k,n}t^n$

$= t + 2t^2 + (k+7)t^3 + 3\sum_{n=4}^{\infty} R_{k,n-1}t^n + \sum_{n=4}^{\infty} R_{k,n-2}t^n + 2\sum_{n=4}^{\infty} R_{k,n-3}t^n - 5\sum_{n=4}^{\infty} R_{k,n-4}t^n$,

$= t + 2t^2 + (k+7)t^3 + 3t\sum_{n=3}^{\infty} R_{k,n}t^n + t^2\sum_{n=4}^{\infty} R_{k,n}t^n + 2t^3\sum_{n=1}^{\infty} R_{k,n}t^n - 5t^4\sum_{n=4}^{\infty} R_{k,n}t^n$.

Thus, we have

$$r(t) = \sum_{n=0}^{\infty} R_{k,n}t^n = \frac{t-t^2+kt^3}{5t^4-2t^3-t^2-3t+1}.$$

The proofs of the others are shown similarly. □

Theorem 3. *For $R_{k,n}$, $L_{k,n}$, and $M_{k,n}$ sequences, the Binet formulas can be obtained with the help of the generating functions.*

Proof. With the help of the roots of the characteristic equation of these sequences, the roots of the $5t^4 - 2t^3 - t^2 - 3t + 1 = 0$ equation become $\frac{1}{m}$, $\frac{1}{p}$, $\frac{1}{r}$, and $\frac{1}{s}$. For $R_{k,n}$, we have

$$\frac{t-t^2+kt^3}{5t^4-2t^3-t^2-3t+1} = \frac{(m^2-m+k)}{(m-p)(m-1)(m-r)}\frac{1}{1-mt} + \frac{(p^2-p+k)}{(p-m)(p-r)(p-1)}\frac{1}{1-pt} + \frac{(r^2-r+k)}{(r-m)(r-p)(r-1)}\frac{1}{1-rt} + \frac{k}{(1-m)(1-p)(1-r)}\frac{1}{1-st}$$

$$= \sum_{n=0}^{\infty}\left(\frac{(m^2-m+k)m^n}{(m-p)(m-1)(m-r)} + \frac{(p^2-p+k)p^n}{(p-m)(p-r)(p-1)} + \frac{(r^2-r+k)r^n}{(r-m)(r-p)(r-1)} + \frac{k}{(1-m)(1-p)(1-r)}\right)t^n.$$

$= \sum_{n=0}^{\infty} R_{k,n}t^n$.

Similarly, we find the Binet formulas of the $L_{k,n}$ and $M_{k,n}$ sequences. □

Next, we give special sum formulas of the k-Grahaml $R_{k,n}$, k-Grahaml-Lucas $L_{k,n}$, and Modified k-Grahaml $M_{k,n}$ sequences.

Theorem 4. *Let* $k \in \mathbb{R}$ *and* $n \in \mathbb{N}$. *We obtain*

i. $\sum_{s=0}^{n} R_{k,s} = \frac{1}{9}(10R_{k,n} + 8R_{k,n-1} + 5R_{k,n-2} - (n-2)k - 1)$,

ii. $\sum_{s=0}^{n} L_{k,s} = \frac{1}{9}(10L_{k,n} + 8L_{k,n-1} + 5L_{k,n-2} - (n-2)k + 1)$,

iii. $\sum_{s=0}^{n} M_{k,s} = \frac{1}{9}(10M_{k,n} + 8M_{k,n-1} + 5M_{k,n-2} - (n-2)k)$.

Proof. ii. From the definition of the k-Grahaml-Lucas sequence, we obtain

$$R_{k,3} = 2R_{k,2} + 3R_{k,1} + 5R_{k,0} + k,$$
$$R_{k,4} = 2R_{k,3} + 3R_{k,2} + 5R_{k,1} + k,$$
$$\vdots$$
$$R_{k,n} = 2R_{k,n-1} + 3R_{k,n-2} + 5R_{k,n-3} + k.$$

So, we have

$$-3 + \sum_{s=0}^{n} R_{k,s} = 2\sum_{s=2}^{n-1} R_{k,s} + 3\sum_{s=1}^{n-2} R_{k,s} + 5\sum_{s=0}^{n-3} R_{k,s} + (n-2)k,$$

$$-3 + \sum_{s=0}^{n} R_{k,s} = 2\left(-R_{k,n} - R_{k,0} - R_{k,1} + \sum_{s=0}^{n} R_{k,s}\right) + 3\left(-R_{k,n} - R_{k,n-1} + \sum_{s=0}^{n} R_{k,s}\right)$$
$$+5\left(-R_{k,n} - R_{k,n-1} - R_{k,n-2} + \sum_{s=0}^{n} R_{k,s}\right) + (n-2)k.$$

Thus, we obtain

$$\sum_{s=0}^{n} R_{k,s} = \frac{1}{9}(10R_{k,n} + 8R_{k,n-1} + 5R_{k,n-2} - (n-2)k - 1).$$

The proofs of the others are shown similarly. □

In the following theorem, we associate the terms of the k-Grahaml $R_{k,n}$, k-Grahaml-Lucas $L_{k,n}$, and Modified k-Grahaml $M_{k,n}$ sequences with matrices.

Theorem 5. *Let* $k \in \mathbb{R}$, *and* $n \in \mathbb{N}$. *The following equations are true:*

i. *For the* k*-Grahaml sequence,*

1. $\begin{bmatrix} R_{k,n+3} \\ R_{k,n+2} \\ R_{k,n+1} \\ R_{k,n} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} k+7 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, **2.** $\begin{bmatrix} R_{k,n} \\ R_{k,n+1} \\ R_{k,n+2} \\ R_{k,n+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 2 & -5 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ 2 \\ k+7 \end{bmatrix}$,

ii. *For the* k*-Grahaml-Lucas sequence,*

1. $\begin{bmatrix} L_{k,n+3} \\ L_{k,n+2} \\ L_{k,n+1} \\ L_{k,n} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} k+41 \\ 10 \\ 2 \\ 3 \end{bmatrix}$, **2.** $\begin{bmatrix} L_{k,n} \\ L_{k,n+1} \\ L_{k,n+2} \\ L_{k,n+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 2 & -5 \end{bmatrix}^n \begin{bmatrix} 3 \\ 2 \\ 10 \\ k+41 \end{bmatrix}$,

iii. *For the Modified* k*-Grahaml sequence,*

1. $\begin{bmatrix} M_{k,n+3} \\ M_{k,n+2} \\ M_{k,n+1} \\ M_{k,n} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} k+5 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, **2.** $\begin{bmatrix} M_{k,n} \\ M_{k,n+1} \\ M_{k,n+2} \\ M_{k,n+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 2 & -5 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ 1 \\ k+5 \end{bmatrix}$,

Proof. i. 1. Let show the proof by induction over n. For $n = 1$, the equality is true. For $n - 1$, assume the equality is true. We obtain

$$\begin{bmatrix} 3 & 1 & 2 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} k+7 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} k+7 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 2 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R_{k,n+2} \\ R_{k,n+1} \\ R_{k,n} \\ R_{k,n-1} \end{bmatrix}$$

$$= \begin{bmatrix} R_{k,n+3} \\ R_{k,n+2} \\ R_{k,n+1} \\ R_{k,n} \end{bmatrix}.$$

From the last equation, for n, it can be seen that the equality is true.

The proofs of the others may be found similarly. □

Theorem 6. *Let* $k \in \mathbb{R}$ *and* $t \in \mathbb{N}$. *The following equations are satisfied:*

i. $$det \begin{bmatrix} R_{k,t+3} & R_{k,t+2} & R_{k,t+1} & R_{k,t} \\ R_{k,t+2} & R_{k,t+1} & R_{k,t} & R_{k,t-1} \\ R_{k,t+1} & R_{k,t} & R_{k,t-1} & R_{k,t-2} \\ R_{k,t} & R_{k,t-1} & R_{k,t-2} & R_{k,t-3} \end{bmatrix} = 5^{t-3}(k^4 + 8k^3 + 7k^2 + 45k),$$

ii. $$det \begin{bmatrix} L_{k,t+3} & L_{k,t+2} & L_{k,t+1} & L_{k,t} \\ L_{k,t+2} & L_{k,t+1} & L_{k,t} & L_{k,t-1} \\ L_{k,t+1} & L_{k,t} & L_{k,t-1} & L_{k,t-2} \\ L_{k,t} & L_{k,t-1} & L_{k,t-2} & L_{k,t-3} \end{bmatrix} = 5^{t-3}(k^4 + 64k^3 + 1231k^2 + 11079k),$$

iii. $$det \begin{bmatrix} M_{k,t+3} & M_{k,t+2} & M_{k,t+1} & M_{k,t} \\ M_{k,t+2} & M_{k,t+1} & M_{k,t} & M_{k,t-1} \\ M_{k,t+1} & M_{k,t} & M_{k,t-1} & M_{k,t-2} \\ M_{k,t} & M_{k,t-1} & M_{k,t-2} & M_{k,t-3} \end{bmatrix} = 5^{t-3}(k^4 + 9k^3 + 34k^2 + 81k).$$

Proof. The proofs are shown by the induction method using definitions and determinant properties. □

Theorem 7. *For* $R_{k,n}$, $L_{k,n}$, *and* $M_{k,n}$ *sequences, the Binet formulas can be obtained with the help of the matrices.*

Proof. The following relation is used for proof (see for details Corollary 3.1 in [34]).

$$t_n = \frac{1}{\det(\Lambda)} \sum_{j=1}^{i} r_{c+1-j} \det(\Lambda_j).$$

Thus,

$$R_{k,n} = \frac{1}{\det(\Lambda)} \sum_{j=1}^{i} R_{k,c+1-j} \det(\Lambda_j).$$

Let $c = i = 4$, $\Lambda = \begin{bmatrix} m^3 & m^2 & m & 1 \\ p^3 & p^2 & p & 1 \\ r^3 & r^2 & r & 1 \\ s^3 & s^2 & s & 1 \end{bmatrix}$, $\Lambda_1 = \begin{bmatrix} m^{n-1} & m^2 & m & 1 \\ p^{n-1} & p^2 & p & 1 \\ r^{n-1} & r^2 & r & 1 \\ s^{n-1} & s^2 & s & 1 \end{bmatrix}$, $\Lambda_2 = \begin{bmatrix} m^3 & m^{n-1} & m & 1 \\ p^3 & p^{n-1} & p & 1 \\ r^3 & r^{n-1} & r & 1 \\ s^3 & s^{n-1} & s & 1 \end{bmatrix}$, $\Lambda_3 = \begin{bmatrix} m^3 & m^2 & m^{n-1} & 1 \\ p^3 & p^2 & p^{n-1} & 1 \\ r^3 & r^2 & r^{n-1} & 1 \\ s^3 & s^2 & s^{n-1} & 1 \end{bmatrix}$, and $\Lambda_4 = \begin{bmatrix} m^3 & m^2 & m & m^{n-1} \\ p^3 & p^2 & p & p^{n-1} \\ r^3 & r^2 & r & r^{n-1} \\ s^3 & s^2 & s & s^{n-1} \end{bmatrix}$. So, we obtain

$$R_{k,n} = \frac{1}{\det(\Lambda)} \sum_{j=1}^{4} R_{k,5-j} \det(\Lambda_j)$$

$$= \frac{1}{\det(\Lambda)} \left(R_{k,4} \det(\Lambda_1) + R_{k,3} \det(\Lambda_2) + R_{k,2} \det(\Lambda_3) + R_{k,1} \det(\Lambda_4)\right)$$

$$= \frac{(m^2-m+k)m^n}{(m-p)(m-1)(m-r)} + \frac{(p^2-p+k)p^n}{(p-m)(p-r)(p-1)} + \frac{(r^2-r+k)r^n}{(r-m)(r-p)(r-1)} + \frac{k}{(1-m)(1-p)(1-r)}.$$

Similarly, the Binet formula of the $L_{k,n}$, and $M_{k,n}$ sequences are found. □

3. Special Relations

In this chapter, we examine the relationships among the k-Grahaml, k-Grahaml-Lucas, and Modified k-Grahaml sequences.

In the following theorems, we examine the relations among the k-Grahaml $R_{k,n}$, k-Grahaml-Lucas $L_{k,n}$, and Modified k-Grahaml $M_{k,n}$ sequences. The chapter concludes with the discovery of a noteworthy relationship between the k-Grahaml and Grahaml-Lucas sequences.

Theorem 8. *Let $k \in \mathbb{R}$, and $n \in \mathbb{N}$. The following equations are true:*

i. $R_{k,n} = -\frac{3k^2+136k+1370}{k^3+64k^2+1231k+11079} L_{k,n+3} + \frac{8k^2+408k+3199}{k^3+64k^2+1231k+11079} L_{k,n+2} + \frac{-2k^2+11k+3570}{k^3+64k^2+1231k+11079} L_{k,n+1} + \frac{k^3+61k^2+948k+5680}{k^3+64k^2+1231k+11079} L_{k,n}$,

ii. $R_{k,n} = -\frac{k+4}{k^3+9k^2+34k+81} M_{k,n+3} + \frac{3k+17}{k^3+9k^2+34k+81} M_{k,n+2} + \frac{k^2+6k+3}{k^3+9k^2+34k+81} M_{k,n+1} + \frac{k^3+8k^2+26k+65}{k^3+9k^2+34k+81} M_{k,n}$,

iii. $L_{k,n} = \frac{3k^2+17k+2}{k^3+8k^2+6k+50} R_{k,n+3} - \frac{8k^2+69k+31}{k^3+8k^2+6k+50} R_{k,n+2} + \frac{2k^2+38k+183}{k^3+8k^2+6k+50} R_{k,n+1} + \frac{k^3+11k^2+21k-109}{k^3+8k^2+6k+50} R_{k,n}$,

iv. $L_{k,n} = \frac{3k^2-17k+25}{k^3+9k^2+34k+81} M_{k,n+3} - \frac{8k^2+61k+86}{k^3+9k^2+34k+81} M_{k,n+2} + \frac{3k^2+53k+204}{k^3+9k^2+34k+81} M_{k,n+1} + \frac{k^3+11k^2+25k-62}{k^3+9k^2+34k+81} M_{k,n}$,

v. $M_{k,n} = \frac{k+4}{k^3+8k^2+6k+50} R_{k,n+3} - \frac{2k+17}{k^3+8k^2+6k+50} R_{k,n+2} - \frac{k^2+7k-6}{k^3+8k^2+6k+50} R_{k,n+1} + \frac{k^3+9k^2+15k+52}{k^3+8k^2+6k+50} R_{k,n}$,

vi. $M_{k,n} = -\frac{3k^2+137k+1465}{k^3+64k^2+1231k+11079} L_{k,n+3} + \frac{8k^2+419k+3623}{k^3+64k^2+1231k+11079} L_{k,n+2} - \frac{3k^2+45k-2928}{k^3+64k^2+1231k+11079} L_{k,n+1} + \frac{k^3+62k^2+994k+5993}{k^3+64k^2+1231k+11079} L_{k,n}$,

Proof. i. The following relation is used for proofs;

$$R_{k,n} = a \times L_{k,n+3} + b \times L_{k,n+2} + c \times L_{k,n+1} + d \times L_{k,n}.$$

For these n values, we obtain;

$$R_{k,0} = a \times L_{k,3} + b \times L_{k,2} + c \times L_{k,1} + d \times L_{k,0},$$

$$R_{k,1} = a \times L_{k,4} + b \times L_{k,3} + c \times L_{k,2} + d \times L_{k,1},$$

$$R_{k,2} = a \times L_{k,5} + b \times L_{k,4} + c \times L_{k,3} + d \times L_{k,2},$$

$$R_{k,3} = a \times L_{k,6} + b \times L_{k,5} + c \times L_{k,4} + d \times L_{k,3}.$$

If the information is written instead of, we obtain

$$0 = a \times (k+41) + b \times 10 + c \times 2 + d \times 3,$$

$$1 = a \times (3k+122) + b \times (k+41) + c \times 10 + d \times 2,$$

$$2 = a \times (10k+417) + b \times (3k+122) + c \times (k+41) + d \times 10,$$

$$k+7 = a \times (35k+1405) + b \times (10k+417) + c \times (3k+122) + d \times (k+41).$$

when this equation is solved, we find

$$a = -\frac{3k^2+136k+1370}{k^3+64k^2+1231k+11079},\ b = \frac{8k^2+408k+3199}{k^3+64k^2+1231k+11079},$$

$$c = \frac{-2k^2+11k+3570}{k^3+64k^2+1231k+11079}, \text{ and } d = \frac{k^3+61k^2+948k+5680}{k^3+64k^2+1231k+11079}.$$

The proofs of the others are shown similarly. □

Theorem 9. *Let $k \in \mathbb{R}$, and $n \in \mathbb{N}$. The following equations are true:*

For Grahaml sequence

i. $G_n = \frac{-k+5}{k^3+8k^2+6k+50} R_{k,n+3} + \frac{k^2+6k-10}{k^3+8k^2+6k+50} R_{k,n+2} - \frac{k^2+10k+15}{k^3+8k^2+6k+50} R_{k,n+1} + \frac{5k+20}{k^3+8k^2+6k+50} R_{k,n}$,

ii. $G_n = -\frac{8k+139}{k^3+64k^2+1231k+11079} L_{k,n+3} + \frac{k^2+57k+737}{k^3+64k^2+1231k+11079} L_{k,n+2} - \frac{k^2+44k+123}{k^3+64k^2+1231k+11079} L_{k,n+1} - \frac{5k+475}{k^3+8k^2+6k+50} L_{k,n}$,

iii. $G_n = \frac{5}{k^3+9k^2+34k+81} M_{k,n+3} + \frac{k^2+5k-1}{k^3+9k^2+34k+81} M_{k,n+2} - \frac{k^2+10k+24}{k^3+9k^2+34k+81} M_{k,n+1} + \frac{5k+20}{k^3+9k^2+34k+81} M_{k,n}$.

For Grahaml-Lucas sequence

i. $H_n = \frac{3k^2+16k+7}{k^3+8k^2+6k+50} R_{k,n+3} - \frac{7k^2+63k+41}{k^3+8k^2+6k+50} R_{k,n+2} + \frac{k^2+28k+168}{k^3+8k^2+6k+50} R_{k,n+1} + \frac{3k^2+19k-134}{k^3+8k^2+6k+50} R_{k,n}$,

ii. $H_n = \frac{3k^2+128k+1231}{k^3+64k^2+1231k+11079} L_{k,n+3} - \frac{7k^2+351k+2462}{k^3+64k^2+1231k+11079} L_{k,n+2} + \frac{k^2-55k-3693}{k^3+64k^2+1231k+11079} L_{k,n+1} + \frac{3k^2+278k+4924}{k^3+8k^2+6k+50} L_{k,n}$

iii. $H_n = \frac{3k^2+18k+34}{k^3+9k^2+34k+81} M_{k,n+3} - \frac{7k^2+59k+104}{k^3+9k^2+34k+81} M_{k,n+2} + \frac{k^2+37k+177}{k^3+9k^2+34k+81} M_{k,n+1} + \frac{3k^2+4k-107}{k^3+9k^2+34k+81} M_{k,n}$.

For Modified Grahaml sequence

i. $E_n = \frac{9}{k^3+8k^2+6k+50} R_{k,n+3} + \frac{k^2+4k-27}{k^3+8k^2+6k+50} R_{k,n+2} - \frac{2k^2+17k+9}{k^3+8k^2+6k+50} R_{k,n+1} + \frac{k^2+13k+27}{k^3+8k^2+6k+50} R_{k,n}$,

ii. $E_n = -\frac{9k+234}{k^3+64k^2+1231k+11079} L_{k,n+3} + \frac{k^2+68k+1161}{k^3+64k^2+1231k+11079} L_{k,n+2} - \frac{2k^2+100k+765}{k^3+64k^2+1231k+11079} L_{k,n+1} + \frac{k^2+41k-162}{k^3+8k^2+6k+50} L_{k,n}$

iii. $E_n = \frac{k+9}{k^3+9k^2+34k+81} M_{k,n+3} + \frac{k^2+2k-18}{k^3+9k^2+34k+81} M_{k,n+2} - \frac{2k^2+16k+27}{k^3+9k^2+34k+81} M_{k,n+1} + \frac{k^2+13k+36}{k^3+9k^2+34k+81} M_{k,n}$.

Proof. The structure of the proofs resembles that of Theorem 8. □

The following theorem provides a fascinating relation between the k-Grahaml sequence and the Grahaml-Lucas sequence.

Theorem 10. *Let* $k \in \mathbb{R}$, *and* $n \in \mathbb{N}$. *The following equation is true:*

$$R_{k,3n} = R_{k,2n} H_n + R_{k,2n} - 5^n R_{k,-n} - \frac{1}{2} R_{k,n}(H_n^2 + 2H_n - H_{2n}).$$

Proof. Let $R_{k,n} = \frac{(m^2-m+k)m^n}{(m-p)(m-1)(m-r)} + \frac{(p^2-p+k)p^n}{(p-m)(p-r)(p-1)} + \frac{(r^2-r+k)r^n}{(r-m)(r-p)(r-1)} + \frac{k}{(1-m)(1-p)(1-r)} = Am^n + Bp^n + Cr^n + Ds^n$.

We obtain

$5^n R_{k,-n} = m^n p^n r^n s^n (Am^{-n} + Bp^{-n} + Cr^{-n} + Ds^{-n})$

$= Ap^n r^n s^n + Bm^n r^n s^n + Cm^n p^n s^n + Dm^n p^n r^n$

$= p^n r^n s^n + m^n r^n s^n + m^n p^n s^n + m^n p^n r^n (A + B + C + D) - m^n p^n r^n (A + B + C)$

$-m^n p^n s^n (A + B + D) - m^n r^n s^n (A + C + D) - p^n r^n s^n (B + C + D)$.

Then we get

$R_{k,2n}(H_n+1)=(Am^{2n}+Bp^{2n}+Cr^{2n}+Ds^{2n})(m^n+p^n+r^n+s^n)$

$=Am^{3n}+Bp^{3n}+Cr^{3n}+Ds^{3n}$

$-(p^nr^ns^n+m^nr^ns^n+m^np^ns^n+m^np^nr^n)(A+B+C+D)$

$+(p^nr^ns^n+m^nr^ns^n+m^np^ns^n+m^np^nr^n)(A+B+C+D)$

$-m^np^nr^n(A+B+C)-m^np^ns^n(A+B+D)-m^nr^ns^n(A+C+D)$

$-p^nr^ns^n(B+C+D)+\frac{1}{2}(Am^n+Bp^n+Cr^n+Ds^n)(m^n+p^n+r^n+s^n)^2-(m^{2n}+p^{2n}+r^{2n}+s^{2n})$

$=R_{k,3n}-(p^nr^ns^n+m^nr^ns^n+m^np^ns^n+m^np^nr^n)(A+B+C+D)+5^nR_{k,-n}+\frac{1}{2}R_{k,n}((H_n+1)^2-(H_{2n}+1))$

In addition, $A+B+C+D=R_{k,0}=0$. Thus, we get

$$R_{k,3n}=R_{k,2n}H_n+R_{k,2n}-5^nR_{k,-n}-\frac{1}{2}R_{k,n}(H_n^2+2H_n-H_{2n}).$$

Similar methods are used to prove the others. □

4. Conclusions

In this study, we present and conduct a comprehensive exploration of three new recursive sequence generalizations: the k-Grahaml sequence, k-Grahaml-Lucas sequence, and Modified k-Grahaml sequence. We initiate our investigation by formally defining each of these sequences and illustrating their foundational characteristics through selected terms, which help demonstrate their unique structure and progression.

Following the introduction of these sequences, we derive their corresponding generating functions. These functions act as compact and elegant mathematical tools that encapsulate the sequences' entire behavior, making it possible to analyze their properties systematically and efficiently.

We then turn our attention to evaluating several summation identities related to these sequences. A substantial portion of our analysis is devoted to exploring how these sequences can be represented using matrices. By constructing and manipulating matrix forms of the sequences, we reveal deeper structural insights and perform computations that enrich our understanding of their recurrence behavior.

Additionally, we derive multiple variants of Binet-type formulas for each of the sequences. These explicit formulas allow for the direct computation of sequence terms without reliance on recursive methods, offering both theoretical elegance and practical utility. Beyond individual analysis, we examine how these different forms of the Grahaml-inspired sequences relate to one another. Specifically, we investigate the mathematical connections between the k-Grahaml and k-Grahaml-Lucas sequences, as well as their modified counterparts, focusing on both recursive definitions and underlying algebraic features.

Finally, one of the most significant findings of our research is the discovery of a noteworthy and somewhat unexpected relationship between the k-Grahaml sequence and the Grahaml-Lucas sequence. This connection not only deepens the theoretical framework surrounding these generalized sequences but also paves the way for further mathematical exploration and potential interdisciplinary applications.

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No AI tools were utilized for this paper.

References

1. de Oliveira, R.R.; Alves, F.R.V. An investigation of the bivariate complex fibonacci polynomials supported in didactic engineering: An application of Theory of Didactics Situations (TSD). *Acta Sci.* **2019**, *21*, 170–195.
2. Guo, J.; Li, J.; Sato, Y.; et al. A gated recurrent unit model with fibonacci attenuation particle swarm optimization for carbon emission prediction. *Processes* **2024**, *12*, 1063.
3. Şanlı, Z.; Köroğlu, T. Some group actions and Fibonacci numbers. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2022**, *71*, 273–284.
4. Akgüneş, N.; Bardakçı, G.B.; Nacaroğlu, Y. On k-Fibonacci graphs. *Montes Taurus J. Pure Appl. Math.* **2024**, *6*, 146–159.
5. Sekido, H.; Iwasaki, M. Integer representations in numeration system with Fibonacci squares. *Res. Math.* **2022**, *9*, 2069647.
6. Aydınyüz, S.; Aşcı, M. The Moore-Penrose Inverse of the Rectangular Fibonacci Matrix and Applications to the Cryptology. *Adv. Appl. Discret. Math.* **2023**, *40*, 195–211.
7. Avazzadeh, Z.; Hassani, H.; Agarwal, P.; et al. Optimal study on fractional fascioliasis disease model based on generalized Fibonacci polynomials. *Math. Methods Appl. Sci.* **2023**, *46*, 9332–9350.
8. Stojmenovic, I. Recursive algorithms in computer science courses: Fibonacci numbers and binomial coefficients. *IEEE Trans. Educ.* **2000**, *43*, 273–276.
9. Taher, H.; Dash, S.K. Common Terms of k-Pell and Tribonacci Numbers. *Eur. J. Pure Appl. Math.* **2024**, *17*, 135–146.
10. E. Waddill, M. The Tetranacci sequence and generalizations. *Fibonacci Q.* **1992**, *30*, 9–19.
11. Özkan, E.; Akkuş, H. Copper ratio obtained by generalizing the Fibonacci sequence. *AIP Adv.* **2024**, *14*, 075207.
12. Özkan, E.; Akkuş, H. Generalized Bronze Leonardo sequence. *Notes Number Theory Discret. Math.* **2024**, *30*, 811–824.
13. Bravo, J.J.; Herrera, J.L. Generalized Padovan sequences. *Commun. Korean Math. Soc.* **2022**, *37*, 977–988.
14. Bravo, J.J.; Herrera, J.L.; Luca, F. On a generalization of the Pell sequence. *Math. Bohem.* **2021**, *146*, 199–213.
15. Özkan, E.; Akkus, H. On K-Chebsyhev Sequence. *Wseas Trans. Math.* **2023**, *22*, 503–507.
16. Sivaraman, R. Generalized narayana sequences and figurate numbers. *Adv. Math. Sci. J.* **2020**, *9*, 7977–7984.
17. Özkan, E.; Akkuş, H. A New Approach to k-Oresme and k-Oresme-Lucas Sequences. *Symmetry* **2024**, *16*, 1407.
18. Özkan, E.; Akkus, H. On Generalizations of Dickson k-Fibonacci Polynomials. *WSEAS Trans. Math.* **2024**, *23*, 856–862.
19. Aigner, M. A characterization of the Bell numbers. *Discret. Math.* **1999**, *205*, 207–210.
20. Soykan, Y.; Okumus, I.; Bilgin, N.G. On generalized bigollo numbers. *Asian Res. J. Math.* **2023**, *19*, 72–88.
21. Akkuş, H.; Özkan, E. A note on Blaise sequences and their applications. *Discret. Math. Algorithms Appl.* **2025**, *18*, *2550050*.
22. Akkuş, H.; Özkan, E. Copper Fibonacci, Copper Lucas polynomials and their some special transformations and hyperbolic quaternions. *Proc. Indian Natl. Sci. Acad. USA* **2025**, *92*, 213–226.
23. Normenyo, B.; Rihane, S.; Togbe, A. Fermat and Mersenne numbers in k-Pell sequence. *Mat. Stud.* **2021**, *56*, 115–123.
24. Akkuş, H.; Özkan, E. A Note on Edouard Sequence. *J. Sci. Arts* **2025**, *25*, 337–348.
25. Ramírez, W.; Kızılateş, C.; Bedoya, D.; et al. On certain properties of three parametric kinds of Apostol-type unified Bernoulli-Euler polynomials. *Aims Math.* **2025**, *10*, 137–158.
26. Tuglu, N.; Kızılateş, C.; Kesim, S. On the harmonic and hyperharmonic Fibonacci numbers. *Adv. Differ. Equ.* **2015**, *1*, 297.
27. Altıparmak, İ.; Akkuş, H.; Özkan, E. On a Generalization of the Perfect Square Sequence and its Polynomial. *Math. Montisnigri* **2025**, *62*, 64–77.
28. Özkan, E.; Şen, B.; Akkuş, H.; et al. A Study on the k-Mersenne and k-Mersenne-Lucas Sequences. *Univers. J. Math. Appl.* **2025**, *8*, 1–7.
29. Dikmen, C.M.; Altınsoy, M. On third order hyperbolic jacobsthal numbers. *Konuralp J. Math.* **2022**, *10*, 118–126.
30. Soykan, Y. On generalized Grahaml numbers. *J. Adv. Math. Comput. Sci.* **2020**, *35*, 42–57.
31. Falcon, S.; Plaza, A. On the Fibonacci k-numbers, *Chaos Solitons Fractals* **2007**, *32*, 1615–1624.
32. Falcon, S. On the k-Lucas numbers. *Int. J. Contemp. Math. Sci.* **2011**, *6*, 1039–1050.
33. Falcon, S. Catalan Transform of the k-Fibonacci sequence. *Commun. Korean Math. Soc.* **2013**, *28*, 827–832.
34. Kiliç, E.; Stanica, P. A matrix approach for general higher order linear recurrences. *Bull. Malays. Math. Sci. Soc.* **2011**, *34*, 51–67.