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Exponential Stability Criteria for Fractional Order Switched System Based on Multiple Discontinuous Lyapunov Function Method

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Abstract: In this article, the exponential stability of Caputo fractional order switched system (CFOSS) that simultaneously includes unstable and stable subsystems is discussed. By combining the mode-dependent average dwell time (MDADT) technique with the multiple discontinuous Lyapunov functions (MDLF) method, the sufficient, low-conservatism conditions for such stability are obtained, and then the conditions are applied to Caputo fractional order linear switched system (CFOLSS) to derive a set of algebraic criteria for solvable linear matrix inequalities (LMIs). Next, the criteria for stability of the switched T-S fuzzy model under rapid and slow MDADT switching are determined by representing the underlying nonlinear system using the T-S fuzzy modeling approach. The findings verify that CFOSS with unstable subsystems and stable subsystems is exponentially stable when the stable subsystems stay long enough or when all unstable subsystems switch quickly enough. Ultimately, the efficacy of the result is validated via two numerical simulation examples provided.

Keywords: fractional order switched system; exponential stability; multiple discontinuous Lyapunov function; mode-dependent average dwell time

Mathematics Subject Classification (2020): 93D23; 34A08; 93C42; 37B25; 15A39

1. Introduction

A hybrid system is a one that has both continuous dynamics and sometimes exhibits discrete jumping/switched dynamics, which are the abbreviation of hybrid or dynamic systems. As a particularly important type in hybrid systems [1,2], the switched system, which is frequently used to model a range of control issues as well as certain intricate natural processes, is composed of a few discrete or continuous subsystems and switching rules that coordinate the responsibilities and relationships between the subsystems. The switched system is crucial for both theoretical and practical applications, scholars have carried out A lot of work related to it and achieved a lot of remarkable results [3–5].

Regarding the dynamical system, stability is an essential dynamic attribute. In switched system, each subsystem and switching rule have influence on the stability, so the design and stability of switching rules is the most focused and productive fields in the research of switched system. There are three main types of switched rules: time control, state control, and time state simultaneous control. The majority of research has concentrated on time-controlled switched, and appeared a variety of useful techniques. For example, the Lyapunov function method [6,7], the average dwell time method [1,8], the LaSalle invariant set principle [9,10], linear matrix inequalities [11], differential equation theory [12] and so forth. According to [13,14], it is often noted that the piecewise Lyapunov function method and the average dwell time (ADT) approach together give sufficient criteria for system stability. The same idea is also present in non-switched systems [15], which is based on the idea of impulse control theory and ADT, which discusses the sufficient conditions for global exponential stability based on Lyapunov. Liu and Shen also employed the Razumikhin methodology and the Lyapunov function method in [16] to investigate the stability of the invariant set. Regarding time-delayed hybrid systems, consistent asymptotic stability and consistent stability criteria are defined. In [17], the global exponential stability (GES) problem of a kind of nonlinear impulse switched system was studied by using impulse control theory and Lyapunov function method.

In practical engineering, local faults or sudden disturbances often exist, which will degrade the properties of subsystems and make them unstable. Therefore, the research on switched system that simultaneously includes stable and unstable subsystems has more theoretical and practical significance. In [18,19], the introduction of the mode-dependent average dwell time (MDADT) technique results in an ADT unique to each subsystem, and the ADT technology is generalized. In the following study, Zhang and Li [20] investigated the input/output state stability (IOSS) of delayed impulsive switched system via the Lyapunov-Razumikhin method and MDADT approach, deriving Razumikhin-type IOSS criteria for systems with coexisting unstable/stable impulses. Next, In [21] the strategy of fast switching is adopted for the unstable subsystem, so that the entire system stays less time in the unstable subsystem than in the stable subsystem, and the influence of the unstable subsystem on the overall system is reduced to achieve a stable state. According to [22], when the average cumulative activation duration of unstable subsystems is notably shorter than that of stable ones, then both the local exponential stability of nonlinear switched delay system and the global exponential stability of linear switched delay system can may be ensured.

Notably, switched linear system was the main focus of the aforementioned work. In recent years, switched nonlinear system have garnered increasing attention owing to their widespread applications in key fields like mobile robots, networked control system, and DC-DC converters [23,24]. However, direct analysis of nonlinear systems is challenging because of their nonlinear features. Conventional linear control approaches have limitations when it comes to handling nonlinear situations since they depend on the fundamental tenet of small-range operation to guarantee the efficacy of linear models. In order to deal with nonlinear problems, the Takagi-Sugeno (T-S) fuzzy model [25] is presented, which can arbitrarily precisely approximate smooth nonlinear functions, decompose complex nonlinear systems into multiple local linear subsystems, and effectively integrate them with fuzzy rules, providing an effective tool for solving nonlinear problems. At present, a large number of scholars have conducted in-depth research on many control problems of nonlinear systems based on system theory and fuzzy control theory, and have achieved a series of achievements [26,27]. In [28], Yu and Yan approximated discrete-time switched nonlinear system using a T-S fuzzy model, developed MLF method, and determined the stability requirements for the T-S fuzzy model switching under both slow and fast φ-dependent average dwell time switching. Ref. [29] discussed the asymptotic stability, external positivity, and positivity of large class nonuniform fractional nonlinear systems (FONSs) with bounded multitime-varying delay using the Laplace transform method and the T-S fuzzy approach. To attain H^{∞} synchronization performance of uncertain hierarchical chaotic systems in [30], Lin has suggested an adaptive approach based on an adaptive fuzzy logic controller.

The discussion above mostly concentrates on switched system of integer order. However, with the improvement of science and technology and knowledge level, scholars have found that many phenomena in nature are not integer dimension, and there are some defects in using integral calculus only to deal with problems. As a generalization of integral calculus, fractional calculus is more suitable for describing the physical change process related to historical memory, and can also better describe and reflect dynamic behavior. In real life, there are many examples of this property, including viscoelastic materials, dielectric polarization and electromagnetism, memristors, etc. [31–33]. As time goes forward, the research on the fractional order switched system has also produced many remarkable results [34,35]. For example, the stability under arbitrary switched conditions had been studied using the common Lyapunov function (CLF) [36] and the multiple Lyapunov functions (MLF) [37] techniques. Where the MLF method is more flexible than the CLF method and can reduce the conservative nature of using constraint switching to analyze and synthesize swtched system. Studies on fractional-positive switched system [36,38] and fractional-positive systems [39] have been conducted by some researchers. In [40], a continuous-time fractional order positive system's calming problem was resolved using the Lyapunov function, and a positive fractional order variable-order discrete system is proposed in [41]. In [42], Wang et al. tackled the Caputo fractional switched linear system' almost sure stability issue of that use switched signals incorporating both stochastic and deterministic.

Based on the above discussions and the conclusions of the existing literatures, the exponential stability of Caputo fractional order switched system (CFOSS) that simultaneously include unstable and stable subsystems is the main topic of this work, and mainly uses MDADT slow switching and fast switching techniques combined with the multiple discontinuous Lyapunov functions (MDLF) method to give sufficient conditions for system stability. Then it is divided into two parts. The first part gives a solvable linear matrix inequalities (LMIs) conditions when the switched linear system is stable. In the second part, T-S fuzzy model is used to solve the stability problem of switched nonlinear system based on fuzzy rules of switched system under local linear model.

The following categories highlight this paper's primary innovations and contributions:

1. In most papers such as [43,44], the authors studied some stabilities, but did not consider unstable subsystem. In contrast, this paper considers switched system comprising both stable and unstable subsystems, which may more realistically reflect the objective process of change.
2. Considering the diversity of switching rules, we give two switching schemes when dealing with two types of subsystems. The first is to use the hybrid MDADT approach that applies fast/slow switching mechanisms separately to deal with two subsystem types. The second is to make the two subsystems follow the uniform MDADT slow switching technology. The common denominator of the two classes of methods is that the stable subsystem is activated over a certain threshold, thereby compensating state divergence induced by unstable subsystem.
3. In order to obtain a tighter MDADT bound, we construct an independent MLF in each subinterval, and obtain a novel approach to the MDLF. It significantly lessens the conclusion's conservative nature by just requiring every Lyapunov function to be sectionally continuous in the activated system mode's dwell time.
4. This paper presents the LMIs conditions of exponential stability of switched linear system, approximates the nonlinear subsystems of switched systems employing the T-S fuzzy model, extends the conclusion to switched nonlinear system, and derives the stability conditions of switched T-S fuzzy models under fast and slow MDADT switching.

What follows is the remainder of this paper: Section 2 gives the relevant definitions, lemmas and system descriptions. The exponential stability of a switched system with various subsystem scenarios is examined in Section 3. Two useful numerical simulation examples are presented in Section 4 to confirm the usefulness of the suggested approach.

2. Preliminary

This section mainly introduces the relevant definitions and lemmas of fractional calculus, system models, and system stability, laying the foundation for further research on the stability of fractional switching systems in the future.

2.1. Calculus of Fractional Order

Definition 1. *Ref.* [45]. $y:(0,\infty)\to\mathbb{R}$ *has a* $\alpha>0$ *fractional order Riemann-Liouville integral, which is defined as*

$$ {}_{u_0}J_u^{\alpha}y(u)=\frac{1}{\Gamma(\alpha)}\int_{u_0}^{u}\frac{y(\tau)}{(u-\tau)^{1-\alpha}}\,\mathrm{d}\tau, $$

where $\Gamma(s)=\int_0^{\infty}t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t$ *is the Gamma function.*

Definition 2. *Ref.* [46]. *For a function* y*:* $(0,\infty)\to\mathbb{R}$*,* $n\in\mathbb{Z}^{+}$ *it's Caputo fractional derivative of order* $\alpha(\alpha>0)$ *is defined as*

$$ {}_{u_0}^{C}D_u^{\alpha}y(u)=\frac{1}{\Gamma(n-\alpha)}\int_{u_0}^{u}\frac{y^{(n)}(\tau)}{(u-\tau)^{1+\alpha-n}}\,\mathrm{d}\tau,\ \ n-1<\alpha\le n, $$

Notably, for the case where $n=1$*, the Caputo fractional derivative is simplified to*

$$ {}_{u_0}^{C}D_u^{\alpha}y(u)=\frac{1}{\Gamma(1-\alpha)}\int_{u_0}^{u}\frac{y'(\tau)}{(u-\tau)^{\alpha}}\,\mathrm{d}\tau,\ \ 0<\alpha<1. $$

The Mittag-Leffler function assumes a pivotal role in theoretical explorations of fractional calculus, acting as a necessary tool for the stability analysis of fractional differential equations.

Definition 3. *Ref.* [8]. *The Mittag-Leffler function with a single parameter is defined as follows:*

$$ E_{\alpha}(z)=E_{\alpha}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+1)},\quad \alpha>0. $$

Definition 4. *Ref.* [8]. *The Mittag-Leffler function with two parameters is defined as follows:*

$$ E_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)},\quad \alpha,\beta>0. $$

Remark 1. *For a complex number $z \in \mathbb{C}$*

- *When $\beta = 1$, the two-parameter Mittag-Leffler function reduces to its single-parameter counterpart;*
- *When $\alpha = \beta = 1$, it further reduces to the exponential function $E_{1,1}(z) = \mathrm{e}^z$, as defined.*

Evidently, the exponential function is a special case of the single-parameter function, which in turn is a subset of the two-parameter Mittag-Leffler function.

2.2. Some Inequalities

Lemma 1. *Ref.* [47]. *Let $\alpha \in (0,1)$, $x(u) \in \mathbb{R}^n$ be continuously differentiable functions. For any moment when $u \geq u_0$, the following inequality holds*

$$\frac{1}{2} {}_{u_0}^{c}D_u^{\alpha}\left[x^T(u)Px(u)\right] \leq x^T(u)P {}_{u_0}^{c}D_u^{\alpha} x(u),$$

where P is a positive definite matrix.

Lemma 2. *Ref.* [48]. *Given the conditions where any $t \in \mathbb{R}$ and $t > 0$ with $|arg(t)| < \frac{\pi}{2}$, there exists some satisfying the condition that*

$$E_{\alpha}(t) \leq \frac{1}{\alpha}\exp(t^{\frac{1}{\alpha}}), \quad 0 < \alpha < 1,\ t > t_0.$$

Lemma 3. *Ref.* [43]. *Suppose $V(t)$ is a continuously differentiable non-negative function on $[t_0, +\infty)$, then there exists $\lambda \in \mathbb{R}$ such that*

$${}_{t_0}^{C}D_t^{\alpha} V(x(t)) \leq \lambda V(x(t)), \quad 0 < \alpha < 1,$$

then, it follows that

$$V(x(t)) \leq V\left(x\left(t_0\right)\right) E_{\alpha}\left(\lambda\left(t - t_0\right)^{\alpha}\right).$$

2.3. System Descriptions

The following fractional order switched system (CFOSS) model is formulated:

$${}_{t_0}^{C}D_t^{\alpha} x(t) = f_{\sigma(t)}(x(t)), \quad x\left(t_0\right) = x_0, \tag{1}$$

where x_0 is the system's initial state, t_0 is initial moment, $x(t) \in \mathbb{R}^n$, ${}_{t_0}^{C}D_t^{\alpha}$ is a fractional derivative operator, and $\alpha \in (0,1)$. The switching signal σ: $[t_0, +\infty) \to \mathbf{M} = \{1, \ldots, M\}$ is a right-continuous piecewise constant continuous function. Let $\sigma(t) = i \in \mathbf{M}$, and f_i is globally Lipschitz continuous, where $t \in [t_s, t_{s+1})$ $(s = 0, 1, 2, \cdots)$. Considering the stability of subsystems, this chapter divides the set into $\mathbf{M} = \mathcal{U} \cup \mathcal{S}$, where the set of unstable subsystems is denoted by $\mathcal{U}$, while $\mathcal{S}$ denotes the set of stable subsystems.

Definition 5. *Ref.* [43]. *For any initial circumstances $x\left(t_0\right)$, if there exist positive constants $K > 0$ and $\gamma > 0$ such that the system solution satisfies $x(t)$*

$$\|x(t)\| \leq K \left\|x\left(t_0\right)\right\| \mathrm{e}^{-\gamma(t - t_0)},$$

then system (1) *has a marginal γ under switched signal $\sigma(t)$ and is globally uniformly exponentially stable (GUES).*

Definition 6. *Ref.* [18]. *For constants $\tau_{ai} > 0$, $N_{0i} \geq 0$, if the inequality*

$$\frac{T_i\left(t_2, t_1\right)}{\tau_{ai}} - N_{0i} \leq N_{\sigma_i}\left(t_2, t_1\right) \leq N_{0i} + \frac{T_i\left(t_2, t_1\right)}{\tau_{ai}},\ t_2 > t_1 \geq 0$$

holds, then τ_{ai} as the MDADT of the ith subsystem, where $T_i\left(t_2, t_1\right)$ and $N_{\sigma_i}\left(t_2, t_1\right)$ represent the switched numbers and the total running time of the ith subsystem over the time interval $[t_1, t_2]$, respectively.

Remark 2. *An effective method for designing switching rules is proposed in Definition 6, i.e., MDADT technology, which is less conservative than traditional MDT technology. Because it allows each subsystem to have its own ADT, correspondingly, the runtime of the entire system $[t_1, t_2]$ is also subdivided into the total runtime of each subsystem $T_i\left(t_2, t_1\right)$.*

Definition 7. *Ref.* [21]. *Given constants* $\tau_a^c > 0$, $N_0^c \geq 0$, *the following*

$$N^c(t_2, t_1) \geq N_0^c + \frac{T^c(t_2, t_1)}{\tau_a^c}, \quad t_2 > t_1 \geq 0$$

holds, then τ_a^c *is the fast switching MDADT. For the fast switched subsystem, the switched numbers and total running time across the interval* $[t_1, t_2]$ *are represented by* $N^c(t_2, t_1)$ *and* $T^c(t_2, t_1)$, *respectively.*

Remark 3. *In the above two definitions, we build two types of MDADT based on various strategies to address the impacts of unstable subsystems. The former is still adapted to the stable subsystem, and the latter can be applied to the unstable subsystem construction switched rules.*

Remark 4. *In general, we usually assume that the mode-dependent chatter bounds* $N_{0i} = 0$ *and* $N_0^c = 0$.

Assumption 1. *There exist positive constant* $0 < \theta < H_{\sigma(t_s)}^j$ *for* $j = 1, 2, \cdots, G_{\sigma(t_s)}$ *such that for* $\sigma(t_s)$ *is a stable subsystem sequence.*

Remark 5. *To avoid the dwell time of the stable subsystem being too short, in addition to adopting the fast/slow dual switching mechanism, we have also set the lower bound of the dwell time of the stable subsystem.*

3. Main Result

This section introduces the exponential stability for CFOSS (1) with stable and unstable subsystems in linear and nonlinear forms, analyzed via two methods:

- S1: Stable subsystems employ MDADT slow switching, while all unstable subsystems adopt MDADT fast switching, ensuring unstable subsystems have far shorter activation durations than stable ones.
- S2: Both subsystem types use uniform MDADT slow switching, with the total activation time ratio between stable and unstable subsystems bounded by specific thresholds.

Additionally, by investigating stability and constructing a novel MDLF, we address the stability analysis of switching systems composed solely of unstable subsystems, thereby enhancing existing results.

With the foundation of the multiple Lyapunov function, we construct an independent MLF between every two consecutive switched moments $[t_m, t_{m+1})$. As can be seen from Figure 1, during the operation of a certain subsystem, the MLF is also discontinuous, and called multiple discontinuous Lyapunov function (MDLF):

$$V(x(t)) = V_{\sigma(t)}^{g(t)}(x(t)) \in \mathcal{C}^1,$$

where $\sigma(t) = i \in \mathcal{M}$ and $g(t) \in \mathcal{G}_{\sigma(t)} = \left\{0, 1, \cdots, G_{\sigma(t)} - 1\right\}$, any time interval $[t_m, t_{m+1})$ is divided into $G_{\sigma(t_m)}$ segments, i.e., $G_{\sigma(t_m)}$ also represents the number of segments per Lyapunov function. The interval of each fragment is represented as $L_{\sigma(t_m)}^i = \left[t_m + J_{\sigma(t_m)}^i, t_m + J_{\sigma(t_m)}^{i+1}\right), \quad i \in \mathcal{G}_{\sigma(t)}$, and the interval length is denoted as $H_{\sigma(t_m)}^i, \forall i \in \left\{1, 2, \cdots, G_{\sigma(t_m)}\right\}$, so $J_{\sigma(t_m)}^i = \sum_{j=1}^i H_{\sigma(t_m)}^j$ and $[t_m, t_{m+1}) = \cup_j L_{\sigma(t_m)}^i$, $i \in \mathcal{G}_{\sigma(t)}$.

The MDLF framework offers distinct advantages for switched system stability analysis in comparison to the traditional MLF method:

- At switching instants, MDLF loosens MLF's stringent continuity and non-increasing requirements. Restrictions on dwell duration and switching frequency are greatly relaxed because it simply needs an average decay state throughout each switching cycle.
- MDLF allows piecewise continuous functions with deliberate jumps at switching locations, which significantly simplifies function design for complicated systems, in contrast to MLF, which depends on continuously differentiable Lyapunov functions.
- MDLF permits transient Lyapunov function expansion in unstable phases of systems with unstable subsystems, which is offset by decay in stable phases. In many real-world situations, MLF is unable to achieve steady functioning under looser switching rules.

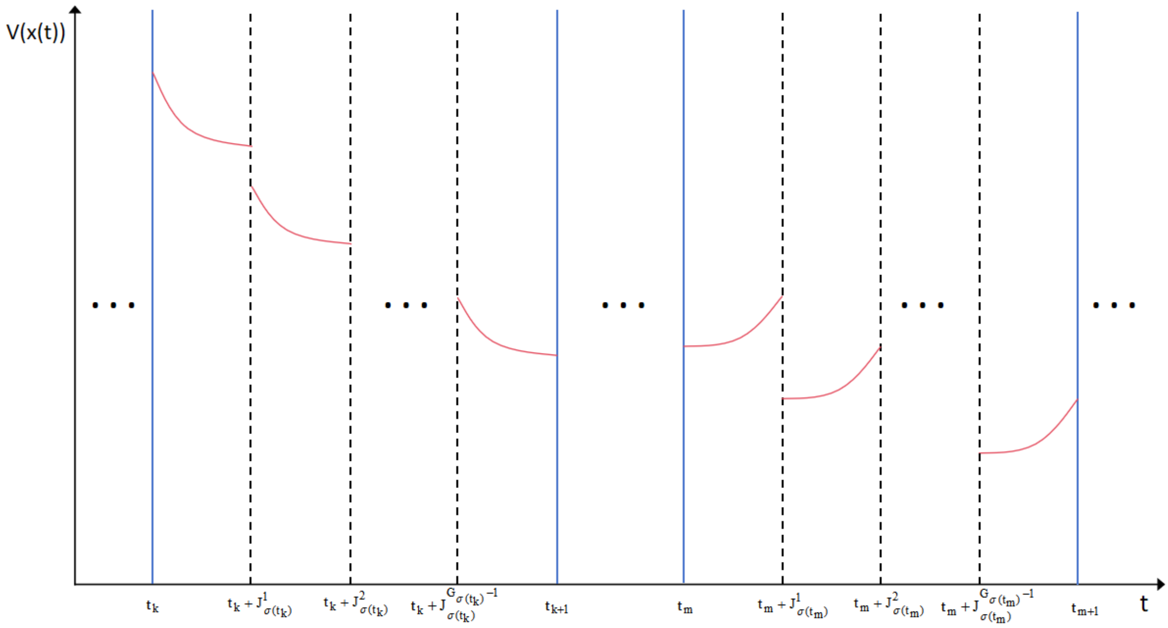

Figure 1. Multiple discontinuous Lyapunov function.

The Switching Signal of Fast/Slow Switching

In (S1), the stable subsystems implement MDADT slow switching, and has its own Lyapunov function V_i^j, and all unstable subsystems perform the same fast switching, and have a common Lyapunov function V_c^j.

Lemma 4. *For system* (1)*, consider constants* $\lambda_i < 0$*,* $\mu_i > 1$*,* $0 < \eta_i < 1$*,* $\forall i \in \mathcal{S}$ *and* $\lambda_c > 0$*,* $0 < \mu_c < 1$*,* $0 < \eta_c < 1$*,* $\forall i \in \mathcal{U}$*, two postive constants* α_1*,* α_2*, and let* $\forall \sigma(t_k) = p$*,* $\sigma\left(t_k^-\right) = q$*. Assume that there exist a set of* C^1 *non-negative functions* $V_i^j(x(t))$*:* $\mathbb{R}^n \to \mathbb{R}$*,* $\forall i \in \mathcal{S}$*,* $j \in \mathcal{G}_{\sigma(t)}$*, and* $V_c^j(x(t))$*:* $\mathbb{R}^n \to \mathbb{R}$*,* $\forall i \in \mathcal{U}$*,* $j \in \mathcal{G}_{\sigma(t)}$*, such that*

(A1) $$\begin{cases} \alpha_1\|x(t)\|^2 \le V_i^j(x(t)) \le \alpha_2\|x(t)\|^2, & \forall i \in \mathcal{S}, \\ \alpha_1\|x(t)\|^2 \le V_c^j(x(t)) \le \alpha_2\|x(t)\|^2, & \forall i \in \mathcal{U}, \end{cases}$$

(A2) $$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} V_i^j(x(t)) \le \lambda_i V_i^j(x(t)), & \forall i \in \mathcal{S}, \\ {}_{t_0}^{C}D_t^{\alpha} V_c^j(x(t)) \le \lambda_c V_c^j(x(t)), & \forall i \in \mathcal{U}, \end{cases}$$

(A3) $$\begin{cases} V_i^j(x(t_k + J_i^j)) \le \eta_i V_i^{j-1}(x(t_k + J_i^j)), & \forall i \in \mathcal{S}, j \neq 0, \\ V_c^j(x(t_k + J_c^j)) \le \eta_c V_c^{j-1}(x(t_k + J_c^j)), & \forall i \in \mathcal{U}, j \neq 0, \end{cases}$$

(A4) $$\begin{cases} V_p^0\left(x\left(t_k\right)\right) \le \mu_p V_q^{G_q-1}\left(x\left(t_k^-\right)\right), & \forall (p,q) \in \mathcal{S} \times \mathcal{S}, \\ V_p^0\left(x\left(t_k\right)\right) \le \mu_p V_c^{G_c-1}\left(x\left(t_k^-\right)\right), & \forall (p,q) \in \mathcal{S} \times \mathcal{U}, \\ V_c^0\left(x\left(t_k\right)\right) \le \mu_c V_q^{G_q-1}\left(x\left(t_k^-\right)\right), & \forall (p,q) \in \mathcal{U} \times \mathcal{S}, \end{cases}$$

then for the switching signal that meets the following conditions

$$\begin{cases} a_i = \dfrac{\ln\left(\mu_i E_\alpha\left(\lambda_i \theta^\alpha\right)\right) + \bar{G}_i \ln\left(\eta_i E_\alpha\left(\lambda_i \theta^\alpha\right)\right)}{\tau_{ai}} < 0, & i \in \mathcal{S}, \\ a_i = a_c = \dfrac{\ln\left(\frac{\mu_c}{\alpha}\right) + \bar{G}_c \ln\left(\frac{\eta_c}{\alpha}\right)}{\tau_a^c} + \lambda_c^{\frac{1}{\alpha}} < 0, & i \in \mathcal{U}, \end{cases} \tag{2}$$

where τ_{ai} *and* τ_a^c *represent the MDADT of* $V_i(x(t))$ *and* $V_c(x(t))$*, respectively, the system* (1) *is GUES with marginal*

$$\gamma = -\frac{1}{2}\max\left\{a_c, a_1, a_2, \cdots, a_r\right\}. \tag{3}$$

Proof. For arbitrary $t \in [t_m + J^{g(t)}_{\sigma(t_m)}, t_m + J^{g(t)+1}_{\sigma(t_m)}) \subseteq [t_m, t_{m+1})$, from Lemma 3 and condition (**A2**), one has

$$V^{g(t)}_{\sigma_{(t)}}(x(t)) \leq V^{g(t)}_{\sigma_{(t_m)}}(x(t_m + J^{g(t)}_{\sigma(t_m)}))E_\alpha\left[\lambda_{\sigma\left(t_m^{g(t)}\right)}\left(t - t_m - J^{g(t)}_{\sigma(t_m)}\right)^\alpha\right]. \tag{4}$$

It follows from (4) and conditon (**A3 and A4**) that

$$\begin{aligned}
&V^{g(t)}_{\sigma_{(t)}}(x(t)) \\
&\leq \eta_{\sigma(t_m)}V^{g(t)-1}_{\sigma_{t_m}}\left(x\left((t_m + J^{g(t)}_{\sigma(t_m)})^-\right)\right)E_\alpha\left[\lambda_{\sigma(t_m)}\left(t - t_m - J^{g(t)}_{\sigma(t_m)}\right)^\alpha\right] \\
&\leq \eta_{\sigma(t_m)}V^{g(t)-1}_{\sigma_{t_m}}\left(x\left((t_m + J^{g(t)-1}_{\sigma(t_m)})^-\right)\right)E_\alpha\left[\lambda_{\sigma(t_m)}\left(t - t_m - J^{g(t)}_{\sigma(t_m)}\right)^\alpha\right] \\
&\quad\times E_\alpha\left[\lambda_{\sigma(t_m)}\left(H^{g(t)}_{\sigma(t_m)}\right)^\alpha\right] \\
&\leq \cdots \\
&\leq \eta^{g(t)}_{\sigma(t_m)}E_\alpha\left[\lambda_{\sigma(t_m)}\left(t - t_m - J^{g(t)}_{\sigma(t_m)}\right)^\alpha\right]\prod_{j=1}^{g(t)}E_\alpha\left[\lambda_{\sigma(t_m)}\left(H^{j}_{\sigma(t_m)}\right)^\alpha\right]V^0_{\sigma_{t_m}}\left(x\left((t_m)\right)\right) \\
&\leq \eta^{g(t)}_{\sigma(t_m)}\mu_{\sigma(t_m)}V^{G_{\sigma(t_m)}-1}_{\sigma_{t_m-1}}\left(x\left(t_m^-\right)\right)E_\alpha\left[\lambda_{\sigma(t_m)}\left(t - t_m - J^{g(t)}_{\sigma(t_m)}\right)^\alpha\right] \\
&\quad\times\prod_{j=1}^{g(t)}E_\alpha\left[\lambda_{\sigma(t_m)}\left(H^{j}_{\sigma(t_m)}\right)^\alpha\right] \\
&\leq \eta^{g(t)}_{\sigma(t_m)}\eta^{G_{\sigma(t_{m-1})}-1}_{\sigma(t_i)}\mu_{\sigma(t_m)}V^0_{\sigma(t_{m-1})}\left(x\left(t_{m-1}\right)\right)E_\alpha\left[\lambda_{\sigma(t_m)}\left(t - t_m - J^{g(t)}_{\sigma(t_m)}\right)^\alpha\right] \\
&\quad\times\prod_{j=1}^{g(t)}E_\alpha\left[\lambda_{\sigma(t_m)}\left(H^{j}_{\sigma(t_m)}\right)^\alpha\right]\prod_{j=1}^{G_{\sigma(t_{m-1})}}E_\alpha\left[\lambda_{\sigma(t_{m-1})}\left(H^{j}_{\sigma(t_{m-1})}\right)^\alpha\right] \\
&\leq \cdots \\
&\leq \eta^{g(t)}_{\sigma(t_m)}\prod_{i=0}^{m-1}\eta^{G_{\sigma(t_i)}-1}_{\sigma(t_i)}\prod_{i=1}^{m}\mu_{\sigma(t_i)}E_\alpha\left[\lambda_{\sigma(t_m)}\left(t - t_m - J^{g(t)}_{\sigma(t_m)}\right)^\alpha\right] \\
&\quad\times\prod_{j=1}^{g(t)}E_\alpha\left[\lambda_{\sigma(t_m)}\left(H^{j}_{\sigma(t_m)}\right)^\alpha\right]V^0_{\sigma(t_0)}\left(x\left(0\right)\right)\prod_{i=0}^{m-1}\prod_{j=1}^{G_{\sigma(t_i)}}E_\alpha\left[\lambda_{\sigma(t_i)}\left(H^{j}_{\sigma(t_i)}\right)^\alpha\right].
\end{aligned} \tag{5}$$

If we set

$$\bar{G}_{\sigma(t_i)} = \begin{cases} g(t),\ i = m, \\ G_{\sigma(t_i)} - 1,\ i = 0, 1, 2, \cdots, m-1, \end{cases}$$

,

$$\bar{H}^j_{\sigma(t_i)} = \begin{cases} t - t_m - J^{g(t_m)+1}_{\sigma(t_m)},\ i = m, j = \bar{G}_{\sigma(t_m)} + 1, \\ H^j_{\sigma(t_i)},\ other, \end{cases}$$

and $\mu_{\sigma(t_0)} = 1$, we can rewrite (5) as

$$V^{g(t)}_{\sigma_{(t)}}(x(t)) \leq \prod_{i=0}^{m}\left\{\prod_{j=1}^{\bar{G}_{\sigma(t_i)}+1}E_\alpha\left[\lambda_{\sigma(t_i)}\left(H^j_{\sigma(t_i)}\right)^\alpha\right]\eta^{\bar{G}_{\sigma(t_i)}}_{\sigma(t_i)}\mu_{\sigma(t_i)}\right\}V^0_{\sigma(t_0)}\left(x\left(0\right)\right). \tag{6}$$

Because the stable subsystem executes slow switching and the unstable subsystem executes fast switching, we can obtain

$$\begin{aligned}
V^{g(t)}_{\sigma_{(t)}}(x(t)) \leq &\prod_{\sigma(t_i)\in\mathcal{S}}\left\{\eta^{\bar{G}_{\sigma(t_i)}}_{\sigma(t_i)}\mu_{\sigma(t_i)}\prod_{j=1}^{\bar{G}_{\sigma(t_i)}+1}E_\alpha\left[\lambda_{\sigma(t_i)}\left(H^j_{\sigma(t_i)}\right)^\alpha\right]\right\} \\
&\times\prod_{\sigma_{(t_i)}\in\mathcal{U}}\left\{\eta_c^{\bar{G}_c}\mu_c\prod_{j=1}^{\bar{G}_c+1}E_\alpha\left[\lambda_c\left(H^j_c\right)^\alpha\right]\right\}V^0_{\sigma(t_0)}\left(x\left(0\right)\right).
\end{aligned} \tag{7}$$

It follows from Lemma 2 and Assumption 1, then

$$
\begin{aligned}
V_{\sigma(t)}^{g(t)}(x(t)) \leq & \prod_{\sigma(t_i)\in\mathcal{S}} \left[\eta_{\sigma(t_i)}^{\bar{G}_{\sigma(t_i)}} \mu_{\sigma(t_i)} \prod_{j=1}^{\bar{G}_{\sigma(t_i)}+1} E_\alpha\left(\lambda_{\sigma(t_i)}\theta^\alpha\right) \right] \\
& \times \prod_{\sigma_{(t_i)}\in\mathcal{U}} \left\{ \eta_c^{\bar{G}_c}\mu_c \prod_{j=1}^{\bar{G}_c+1} \frac{1}{\alpha} \exp\left\{ \lambda_c^{\frac{1}{\alpha}} H_c^j \right\} \right\} V_{\sigma(t_0)}^0\left(x\left(0\right)\right) \\
= & \prod_{\sigma(t_i)\in\mathcal{S}} \left\{ \eta_{\sigma(t_i)}^{\bar{G}_{\sigma(t_i)}} \mu_{\sigma(t_i)} \left[E_\alpha\left(\lambda_{\sigma(t_i)}\theta^\alpha\right)\right]^{\bar{G}_{\sigma(t_i)}+1} \right\} V_{\sigma(t_0)}^0\left(x\left(0\right)\right) \\
& \times \prod_{\sigma_{(t_i)}\in\mathcal{U}} \left\{ \eta_c^{\bar{G}_c}\mu_c \left(\frac{1}{\alpha}\right)^{\bar{G}_c+1} \exp\left\{ \lambda_c^{\frac{1}{\alpha}}\left(t_{i+1}-t_i\right) \right\} \right\} \\
= & \prod_{i\in\mathcal{S}} \left\{ \eta_i^{\bar{G}_i}\mu_i \left[E_\alpha\left(\lambda_i\theta^\alpha\right)\right]^{\bar{G}_i+1} \right\}^{N_{\sigma i}(t,t_0)} \left\{ \eta_c^{\bar{G}_c}\mu_c \left(\frac{1}{\alpha}\right)^{\bar{G}_c+1} \right\}^{N^c(t,t_0)} \\
& \times \exp\left\{ \sum_{\sigma(t_i)\in\mathcal{U}} \lambda_c^{\frac{1}{\alpha}}\left(t_{i+1}-t_i\right) \right\} V_{\sigma(t_0)}^0\left(x\left(0\right)\right) \\
= & \prod_{i\in\mathcal{S}} \left\{ \left[\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right] \left[\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right]^{\bar{G}_i} \right\}^{N_{\sigma i}(t,t_0)} \\
& \times \left[\frac{\mu_c}{\alpha} \left(\frac{\eta_c}{\alpha}\right)^{\bar{G}_c} \right]^{N^c(t,t_0)} \exp\left\{ \lambda_c^{\frac{1}{\alpha}} T^c(t_0,t) \right\} V_{\sigma(t_0)}^0\left(x\left(0\right)\right) \\
= & \exp\left\{ \sum_{i\in\mathcal{S}} N_{\sigma i}(t,t_0) \left[\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) + \bar{G}_i \ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)\right] \right. \\
& \left. + N^c(t,t_0) \left[\ln\left(\frac{\mu_c}{\alpha}\right) + \bar{G}_c \ln\left(\frac{\eta_c}{\alpha}\right)\right] + \lambda_c^{\frac{1}{\alpha}} T^c(t_0,t) \right\} V_{\sigma(t_0)}^0\left(x\left(0\right)\right) \\
\leq & M_1 \exp\left\{ \sum_{i\in\mathcal{S}} \frac{\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) + \bar{G}_i \ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)}{\tau_{ai}} T_i(t,t_0) \right. \\
& \left. + \left[\frac{\ln\left(\frac{\mu_c}{\alpha}\right) + \bar{G}_c \ln\left(\frac{\eta_c}{\alpha}\right)}{\tau_a^c} + \lambda_c^{\frac{1}{\alpha}} \right] T^c(t,t_0) \right\} V_{\sigma(t_0)}^0\left(x\left(0\right)\right),
\end{aligned}
$$

where the running time of the ith stable subsystem is indicated by $T_i(t_0,t)$, the total running time of all unstable subsystems is shown by $T^c(t_0,t)$, and

$$
M_1 = \begin{cases}
\exp\left\{ \sum_{i\in\mathcal{S}} \left[N_{0i}(t,t_0)\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) - N_{0i}(t,t_0)\bar{G}_i \ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) \right] \right. \\
\left. \quad + N_0^c(t,t_0) \left[\ln\left(\frac{\mu_c}{\alpha}\right) + \bar{G}_c \ln\left(\frac{\eta_c}{\alpha}\right) \right] \right\}, \quad \ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) > 0, \\
\exp\left\{ \sum_{i\in\mathcal{S}} \left[-N_{0i}(t,t_0)\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) - N_{0i}(t,t_0)\bar{G}_i \ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) \right] \right. \\
\left. \quad + N_0^c(t,t_0) \left[\ln\left(\frac{\mu_c}{\alpha}\right) + \bar{G}_c \ln\left(\frac{\eta_c}{\alpha}\right) \right] \right\}, \quad \ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) < 0.
\end{cases}
$$

Next, we set

$$
\begin{cases}
a_i = \dfrac{\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) + \bar{G}_i \ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)}{\tau_{ai}}, \ i\in\mathcal{S} \\
a_c = \dfrac{\ln\left(\frac{\mu_c}{\alpha}\right) + \bar{G}_c \ln\left(\frac{\eta_c}{\alpha}\right)}{\tau_a^c} + \lambda_c^{\frac{1}{\alpha}}, \ i\in\mathcal{U},
\end{cases} \tag{8}
$$

then we have

$$V_{\sigma(t)}^{g(t)}(x(t)) \leq M_1 \exp\left\{\sum_{i\in\mathcal{S}} a_i T_i(t_0,t) + a_c T^c(t_0,t)\right\} V_{\sigma(t_0)}^{0}\left(x\left(0\right)\right).$$

Set $\gamma = -\frac{1}{2}\max\left\{a_c, a_1, a_2, \cdots, a_r\right\}$, $K = \sqrt{\frac{\alpha_2}{\alpha_1}M_1}$. Based on condition (**A1**) and Equation (8), it can be inferred that

$$\alpha_1 \left\|x(t)\right\|^2 \leq M_1 \exp\left\{\sum_{i=1}^{r} a_i T_i(t_0,t) + a_c T^c(t_0,t)\right\} \alpha_2 \left\|x(t_0)\right\|^2.$$

Therefore

$$\begin{aligned}\left\|x(t)\right\| &\leq \sqrt{\frac{\alpha_2}{\alpha_1}M_1}\left\|x(t_0)\right\| \exp\left\{\frac{1}{2}\left(\sum_{i=1}^{r} a_i T_i(t_0,t) + a_c T^c(t_0,t)\right)\right\}\\ &\leq K\mathrm{e}^{-\gamma(t-t_0)}\left\|x\left(t_0\right)\right\|.\end{aligned}$$

It is obvious that $\lim\limits_{t\to+\infty} x(t) = 0$, and for any switched signal that satisfies (2), system (1) is GUES with marginal γ. □

Remark 6. *When choosing the Lyapunov function, the overall system energy and conservatism are reduced, and the energy function is no longer required to decrease but not increase. Lemma 4 allows the V function to jump at the moment of switched, allowing $V_i(x(t_k)) > V_j(x(t_{k-1}))$, but this increase is bounded. At the same time, for unstable subsystems, the increase in the energy function can be compensated by the time of system stays in the stable subsystem.*

Remark 7. *In Lemma 4, a fast/slow switching strategy is proposed considering the features of unstable systems and stable subsystems. This adaptive switching rule achieves dynamic coordination of the two subsystems' residence times. Notably, its construction logic differs fundamentally from traditional time-dependent switching methods (e.g., DT, ADT), as the latter use global uniform parameters independent of system modes, failing to realize the mode-specific switching control proposed here.*

When the function $f_{\sigma(t)}(x(t))$ of system (1) is linear, then the Caputo fractional order switched linear system (CFOSLS) is obtained by simplifying system (1) before introducing the key theorems:

$$ {}_{t_0}^{C}D_t^{\alpha} x(t) = A_{\sigma(t)}(x(t)), \quad x\left(t_0\right) = x_0. \tag{9}$$

Theorem 1. *For CFOSLS (9), assume there exist constants $\lambda_i < 0$, $\mu_i > 1$, $0 < \eta_i < 1$, $\forall i \in \mathcal{S}$ and $\lambda_c > 0$, $0 < \mu_c < 1$, $0 < \eta_c < 1$, $\forall i \in \mathcal{U}$, and let $\forall \sigma\left(t_k\right) = p$, $\sigma\left(t_k^-\right) = q$. If there exists a collection of matrices $P_i^j > 0$, $i \in \mathcal{M}, j \in \mathcal{G}_{\sigma(t)}$, such that*

(A21) $\begin{cases} A_i^T P_i^j + P_i^j A_i \leq \lambda_i P_i^j, & \forall i \in \mathcal{S}, \\ A_c^T P_c^j + P_c^j A_c \leq \lambda_c P_c^j, & \forall i \in \mathcal{U}, \end{cases}$

(A31) $\begin{cases} P_i^j \leq \eta_i P_i^{j-1}, & \forall i \in \mathcal{S}, j \neq 0, \\ P_c^j \leq \eta_c P_c^{j-1}, & \forall i \in \mathcal{U}, j \neq 0, \end{cases}$

(A41) $\begin{cases} P_p^0 \leq \mu_p P_q^{G_q - 1}, & \forall\left(p,\ q\right) \in \mathcal{S}\times\mathcal{S}, \\ P_p^0 \leq \mu_p P_c^{G_c - 1}, & \forall\left(p,\ q\right) \in \mathcal{S}\times\mathcal{U}, \\ P_c^0 \leq \mu_c P_q^{G_q - 1}, & \forall\left(p,\ q\right) \in \mathcal{U}\times\mathcal{S}, \end{cases}$

then for any switched signal σ that satisfies (2), the system (9) is GUES with marginal γ (3).

Proof. As for any subsystems, the MDLF are then constructed as follows:

$$V_\xi^j\left(x\left(t\right)\right) = x^T\left(t\right) P_\xi^j x\left(t\right), \tag{10}$$

where $\sigma(t) = \xi = \begin{cases} i, & i \in \mathcal{S}, \\ c, & i \in \mathcal{U}. \end{cases}$, $g(t) = j \in \mathcal{G}_{\sigma(t)}$.

Taking $\alpha_1 = \min_{\xi\in\mathbf{M}}(\lambda_{\min}(P_\xi^j))$, $\alpha_2 = \max_{\xi\in\mathbf{M}}(\lambda_{\max}(P_\xi^j))$, and it is obvious to get condition (**A1**). Based on Lemma 1 and the condition (**A21**), then

$$\begin{aligned}{}_{t_0}^{C}D_t^{\alpha}V_\xi^j\left(x\left(t\right)\right) &\leq {}_{t_0}^{C}D_t^{\alpha}[x^T\left(t\right)P_\xi^j x\left(t\right)]\\ &\leq 2x^T\left(t\right)P_\xi^j {}_{t_0}^{C}D_t^{\alpha}x\left(t\right)\\ &= x^T\left(t\right)\left(A_i^T P_\xi^j + P_\xi^j A_i\right)x\left(t\right)\\ &\leq x^T\left(t\right)\lambda_\xi^j P_\xi^j x\left(t\right)\\ &= \lambda_\xi^j V_\xi^j\left(x\left(t\right)\right).\end{aligned}$$

Using the condition (**A31**) and let $\sigma(t_k) = \xi$, then we can obtain

$$\begin{aligned}&V_\xi^j\left(x\left(t_k + J_\xi^j\right)\right) - \eta_\xi V_\xi^{j-1}\left(x\left(t_k + J_\xi^j\right)\right)\\ &\leq x^T\left(t_k+J_\xi^j\right)P_\xi^j x\left(t_k+J_\xi^j\right) - \eta_\xi x^T\left(t_k+J_\xi^j\right)P_\xi^{j-1}x\left(t_k+J_\xi^j\right)\\ &\leq x^T\left(t_k + J_\xi^j\right)\left(P_\xi^j - \eta_\xi P_\xi^{j-1}\right)x\left(t_k + J_\xi^j\right)\\ &\leq 0.\end{aligned}$$

According to the condition (**A41**) and we let $\sigma(t_k) = i$, $\sigma(t_k^-) = j$ also have

$$\begin{aligned}&V_i^0(x(t_k)) - \mu_i V_j^{G_j-1}(x(t_k^-))\\ &= x^T\left(t_k\right)P_i^0 x\left(t_k\right) - \mu_i x^T\left(t_k^-\right)P_j^{G_j-1}x\left(t_k^-\right)\\ &= x^T\left(t_k\right)\left(P_i^0 - \mu_i P_j^{G_j-1}\right)x\left(t_k\right)\\ &\leq 0.\end{aligned}$$

Similarly, we can say that $\begin{cases}V_i^0(x(t_k)) - \mu_i V_c^{G_c-1}(x(t_k^-)) < 0,\\ V_c^0(x(t_k)) - \mu_c V_j G_j - 1(x(t_k^-)) < 0,\end{cases}$ so the condition (**A4**) is proven. System (9) is GUES for any initial condition if the switching signal satisfies (2), as per Lemma 4. □

When the function $f_{\sigma(t)}(x(t))$ of system (1) is nonlinear, then the system (1) is termed the Caputo fractional order switched nonlinear system (CFOSNS).

Before presenting the key theorem, each nonlinear subsystem is described using the following T-S fuzzy model: The fuzzy rule m for the pth subsystem is: IF z_{p1}^m is F_{p1}^m and ... and z_{pu}^m is F_{pu}^m, THEN:

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = A_{pm}x(t), \quad x\left(t_0\right) = x_0, \tag{11}$$

where $p \in \mathcal{M}$, $m \in \{1, 2, \ldots, \gamma_p\}$, the number of fuzzy rules and sets is denoted by γ_p and u , respectively. $z_p(t) = [z_{p1}(t), z_{p2}(t), ..., z_{pu}(t)]$ are the premise variables, $F_{p1}^m, ..., F_{pu}^m$ denote the sets of fuzzy, and A_{pm} is a constant matrix of known appropriate dimensions. Consequently, the global system model can be acquired as follows using fuzzy blending:

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = \sum_{m=1}^{\gamma_p} h_{pm}(t)A_{pm}x(t), \quad x\left(t_0\right) = x_0, \tag{12}$$

where $h_{pm}(t),\ p \in \mathcal{M},\ m \in \{1, 2, \cdots, \gamma_p\}$, are normalized membership functions which satisfy:

$$h_{pm}(t) = \frac{\prod_{j=1}^{u} F_{pj}^m(z_{pj}(t))}{\sum_{m=1}^{\gamma_p}\prod_{j=1}^{u} F_{pj}^m(z_{pj}(t))} \geq 0,$$

where $F_{pj}^m(z_{pj}(t))$ is the grade of the membership function of $z_{pj}(t)$ in F_{pj}^m, and $h_{pm}(t) \geq 0$, $\sum_{m=1}^{\gamma_p} h_{pm}(t) = 1$.

Remark 8. *In particular, the T-S fuzzy model has the ability to approximate any smooth switching nonlinear system on a compact set. Specifically, by reasonably constructing fuzzy rules, all switching systems composed of subsystems characterized by smooth nonlinear functions can be uniformly characterized by* (12).

Theorem 2. *Consider switched T-S fuzzy system CFOSNS* (12)*, given constants* $\lambda_i < 0$*,* $\mu_i > 1$*,* $0 < \eta_i < 1$*,* $\forall i \in \mathcal{S}$ *and* $\lambda_c > 0$*,* $0 < \mu_c < 1$*,* $0 < \eta_c < 1$*,* $\forall i \in \mathcal{U}$*, if there exists a set of matrices* $P_i^j > 0$*,* $i \in \mathcal{M}, j \in \mathcal{G}_{\sigma(t)}$*, such that conditions (**A21**)–(**A41**) hold, then for any switched signal* σ *that satisfies* (2)*, the system* (12) *is GUES with marginal* γ (3)*.*

Proof. We select the MDLF for each subsystem as shown in (10).

Based on Lemma 1 and the condition (**A21**), we are able to secure

$$
\begin{aligned}
&{}_{t_0}^{C}D_t^{\alpha} V_\xi^j\left(x\left(t\right)\right)\\
&\leq {}_{t_0}^{C}D_t^{\alpha}\left(x^T\left(t\right)P_\xi^j x\left(t\right)\right)\\
&\leq 2x^T\left(t\right)P_\xi^j {}_{t_0}^{C}D_t^{\alpha} x\left(t\right)\\
&= 2x^T\left(t\right)P_\xi^j\left(\sum_{m=1}^{\gamma_p} h_{\xi m}(t)A_{\xi m}(x(t))\right)\\
&= \left(\sum_{m=1}^{\gamma_p} h_{\xi m}(t)A_{\xi m}(x(t))\right)^T P_\xi^j x\left(t\right) + x^T\left(t\right)P_\xi^j\left(\sum_{m=1}^{\gamma_p} h_{\xi m}(t)A_{\xi m}(x(t))\right)\\
&= x^T\left(t\right)\left(\sum_{m=1}^{\gamma_p} h_{\xi m}(t)A_{\xi m}^T(x(t))P_\xi^j + P_\xi^j\sum_{m=1}^{\gamma_p} h_{\xi m}(t)A_{\xi m}\right)x\left(t\right)\\
&= x^T\left(t\right)\left[\sum_{m=1}^{\gamma_p} h_{\xi m}(t)\left(A_{\xi m}^T(x(t))P_\xi^j + P_\xi^j A_{\xi m}\right)\right]x\left(t\right).
\end{aligned}
$$

So, it's easy to get

$$
\begin{aligned}
&{}_{t_0}^{C}D_t^{\alpha} V_\xi^j\left(x\left(t\right)\right) - \lambda_\xi^j V_\xi^j\left(x\left(t\right)\right)\\
&\leq x^T\left(t\right)\left[\sum_{m=1}^{\gamma_p} h_{\xi m}(t)\left(A_{\xi m}^T(x(t))P_\xi^j + P_\xi^j A_{\xi m}\right)\right]x\left(t\right) - \lambda_\xi^j\left(x^T\left(t\right)P_\xi^j x\left(t\right)\right)\\
&= x^T\left(t\right)\left[\sum_{m=1}^{\gamma_p} h_{\xi m}(t)\left(A_{\xi m}^T(x(t))P_\xi^j + P_\xi^j A_{\xi m}\right)\right]x\left(t\right) - x^T\left(t\right)\left(\sum_{m=1}^{\gamma_p} h_{\xi m}(t)\lambda_\xi^j P_\xi^j\right)x\left(t\right)\\
&= x^T\left(t\right)\left[\sum_{m=1}^{\gamma_p} h_{\xi m}(t)\left(A_{\xi m}^T(x(t))P_\xi^j + P_\xi^j A_{\xi m} - \lambda_\xi^j\right)\right]x\left(t\right)\\
&\leq 0.
\end{aligned}
$$

The remainder of the proof is left out here since it is comparable to the Theorem 1. According to Lemma 4, the system (12) is GUES if the switching signal meets (2) under any initial condition. □

In (S2), because all subsystems perform MDADT slow switching, corresponding to multiple discontinuous Lyapunov function V_i^j.

Lemma 5. *Consider CFOSS* (1)*, for* $\forall i \in \mathcal{M}$ *there are constants* λ_i*,* $\mu_i > 1$*,* $0 < \eta_i < 1$ *and two postive constants* α_1*,* α_2*, and let* $\forall\sigma\left(t_k\right) = p$*,* $\sigma\left(t_k^-\right) = q$*. Assume that there exist* C^1 *non-negative functions* $V_i^j(x(t))$*:* $\mathbb{R}^n \to \mathbb{R}$*,* $\forall i \in \mathcal{M}$*,* $j \in \mathcal{G}_{\sigma(t)}$*, such that*

(B1) $\alpha_1\|x(t)\|^2 \leq V_i^j(x(t)) \leq \alpha_2\|x(t)\|^2, \quad \forall i \in \mathcal{M},$

(B2) $\begin{cases} {}_{t_0}^{C}D_t^{\alpha} V_i^j(x(t)) \leq \lambda_i V_i^j(x(t)), & \forall i \in \mathcal{S}, \\ {}_{t_0}^{C}D_t^{\alpha} V_c^j(x(t)) \leq \lambda_c V_c^j(x(t)), & \forall i \in \mathcal{U}, \end{cases}$

(B3) $V_i^j(x(t_k + J_i^j)) \leq \eta_i V_i^{j-1}(x(t_k + J_i^j)),\ \forall i \in \mathcal{M}, j \neq 0,$

(B4) $V_p^0\left(x\left(t_k\right)\right) \leq \mu_p V_q^{G_q - 1}\left(x\left(t_k^-\right)\right),\ \forall\left(p,\ q\right) \in \mathcal{M} \times \mathcal{M},$

then for the switching signal that meets the following conditions

$$\begin{cases} \dfrac{\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)+\bar{G}_i\ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)}{\tau_{ai}}<0, \quad \forall i\in\mathcal{S}, \\ \dfrac{\ln\left(\frac{\mu_i}{\alpha}\right)+\bar{G}_i\ln\left(\frac{\eta_i}{\alpha}\right)}{\tau_a i}+\lambda_i^{\frac{1}{\alpha}}>0, \quad \forall i\in\mathcal{U}, \\ \dfrac{T^-}{T^+}>-\dfrac{\gamma^++2\gamma}{\gamma^-+2\gamma}, \quad (\gamma^-<-2\gamma<0<\gamma^+), \end{cases} \tag{13}$$

where the overall running times of the unstable and stable subsystems are denoted by $T^+=\sum_{i\in\mathcal{U}}T_i(t_0,t)$ *and* $T^-=\sum_{i\in\mathcal{S}}T_i(t_0,t)$*, respectively, moreover*

$$\begin{cases} \gamma^-=\max_{i\in\mathcal{S}}\left(\dfrac{\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)+\bar{G}_i\ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)}{\tau_{ai}}\right), \\ \gamma^+=\max_{i\in\mathcal{U}}\left(\dfrac{\ln\left(\frac{\mu_i}{\alpha}\right)+\bar{G}_i\ln\left(\frac{\eta_i}{\alpha}\right)}{\tau_a i}+\lambda_i^{\frac{1}{\alpha}}\right), \end{cases}$$

the system is GUES with marginal γ*, and the MDADT of* $V_i(x(t))$ *is denoted by* τ_{ai}*.*

Proof. The proof and Lemma 4 are comparable, therefore using (4)–(6), one may obtain

$$\begin{aligned} V_{\sigma(t)}^{g(t)}(x(t)) &\le \prod_{i=0}^{m}\left\{\eta_{\sigma(t_i)}^{\bar{G}_{\sigma(t_i)}}\mu_{\sigma(t_i)}\prod_{j=1}^{\bar{G}_{\sigma(t_i)}+1}E_\alpha\left[\lambda_{\sigma(t_i)}\left(H_{\sigma(t_i)}^j\right)^\alpha\right]\right\}V_{\sigma(t_0)}^0\left(x\left(0\right)\right) \\ &\le \prod_{\sigma(t_i)\in\mathcal{S}}\left\{\eta_{\sigma(t_i)}^{\bar{G}_{\sigma(t_i)}}\mu_{\sigma(t_i)}\prod_{j=1}^{\bar{G}_{\sigma(t_i)}+1}E_\alpha\left[\lambda_{\sigma(t_i)}\left(H_{\sigma(t_i)}^j\right)^\alpha\right]\right\} \\ &\quad\times\prod_{\sigma(t_i)\in\mathcal{U}}\left\{\eta_{\sigma(t_i)}^{\bar{G}_{\sigma(t_i)}}\mu_{\sigma(t_i)}\prod_{j=1}^{\bar{G}_{\sigma(t_i)}+1}E_\alpha\left[\lambda_{\sigma(t_i)}\left(H_{\sigma(t_i)}^j\right)^\alpha\right]\right\}V_{\sigma(t_0)}^0\left(x\left(0\right)\right). \end{aligned}$$

Lemma 2 and Assumption 1 imply that

$$\begin{aligned} &V_{\sigma(t)}^{g(t)}(x(t)) \\ &\le \prod_{\sigma(t_i)\in\mathcal{S}}\left[\eta_{\sigma(t_i)}^{\bar{G}_{\sigma(t_i)}}\mu_{\sigma(t_i)}\prod_{j=1}^{\bar{G}_{\sigma(t_i)}+1}E_\alpha\left(\lambda_{\sigma(t_i)}\theta^\alpha\right)\right]V_{\sigma(t_0)}^0\left(x\left(0\right)\right) \\ &\quad\times\prod_{\sigma(t_i)\in\mathcal{U}}\left\{\eta_{\sigma(t_i)}^{\bar{G}_{\sigma(t_i)}}\mu_{\sigma(t_i)}\prod_{j=1}^{\bar{G}_{\sigma(t_i)}+1}\frac{1}{\alpha}\exp\left\{\lambda_{\sigma(t_i)}^{\frac{1}{\alpha}}H_{\sigma(t_i)}^j\right\}\right\} \\ &= V_{\sigma(t_0)}^0\left(x\left(0\right)\right)\prod_{\sigma(t_i)\in\mathcal{S}}\left\{\eta_{\sigma(t_i)}^{\bar{G}_{\sigma(t_i)}}\mu_{\sigma(t_i)}\left[E_\alpha\left(\lambda_{\sigma(t_i)}\theta^\alpha\right)\right]^{\bar{G}_{\sigma(t_i)}+1}\right\} \\ &\quad\times\prod_{\sigma(t_i)\in\mathcal{U}}\left\{\eta_{\sigma(t_i)}^{\bar{G}_{\sigma(t_i)}}\mu_{\sigma(t_i)}\left(\frac{1}{\alpha}\right)^{\bar{G}_{\sigma(t_i)}+1}\exp\left\{\lambda_{\sigma(t_i)}^{\frac{1}{\alpha}}(t_{i+1}-t_i)\right\}\right\} \\ &= \prod_{i\in\mathcal{S}}\left\{\eta_i^{\bar{G}_i}\mu_i\left[E_\alpha\left(\lambda_i\theta^\alpha\right)\right]^{\bar{G}_i+1}\right\}^{N_{\sigma i}(t,t_0)}V_{\sigma(t_0)}^0\left(x\left(0\right)\right) \\ &\quad\times\exp\left\{\sum_{i\in\mathcal{U}}\lambda_i^{\frac{1}{\alpha}}T_i(t,t_0)\right\}\prod_{i\in\mathcal{U}}\left\{\eta_i^{\bar{G}_i}\mu_i\left(\frac{1}{\alpha}\right)^{\bar{G}_i+1}\right\}^{N_{\sigma i}(t,t_0)} \end{aligned} \tag{14}$$

$$
\begin{aligned}
&= \prod_{i\in\mathcal{S}} \left\{ \left[\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right]\left[\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right]^{\bar{G}_i} \right\}^{N_{\sigma i}(t,t_0)} V^0_{\sigma(t_0)}\left(x\left(0\right)\right) \\
&\quad \times \prod_{i\in\mathcal{U}} \left[\frac{\mu_i}{\alpha}\left(\frac{\eta_i}{\alpha}\right)^{\bar{G}_i}\right]^{N_{\sigma i}(t,t_0)} \exp\left\{\sum_{i\in\mathcal{U}} \lambda_i^{\frac{1}{\alpha}} T_i(t,t_0)\right\} \\
&= \exp\left\{ \sum_{i\in\mathcal{S}} N_{\sigma i}(t,t_0)\left[\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) + \bar{G}_i \ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)\right]\right. \\
&\quad \left. + \sum_{i\in\mathcal{U}} \left[N_{\sigma i}(t,t_0)\left(\ln\left(\frac{\mu_i}{\alpha}\right) + \bar{G}_i\ln\left(\frac{\eta_i}{\alpha}\right)\right) + \lambda_i^{\frac{1}{\alpha}} T_i(t,t_0)\right]\right\} \\
&\quad \times V^0_{\sigma(t_0)}\left(x\left(0\right)\right) \\
&\le M_2 M_3 \exp\left\{ \sum_{i\in\mathcal{S}} \frac{\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) + \bar{G}_i\ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)}{\tau_{ai}} T_i(t,t_0)\right. \\
&\quad \left. + \sum_{i\in\mathcal{U}} \left(\frac{\ln\left(\frac{\mu_i}{\alpha}\right) + \bar{G}_i\ln\left(\frac{\eta_i}{\alpha}\right)}{\tau_a i} + \lambda_i^{\frac{1}{\alpha}}\right) T_i(t,t_0)\right\} V^0_{\sigma(t_0)}\left(x\left(0\right)\right),
\end{aligned}
$$

where $T_i(t_0, t)$ indicates the ith subsystem's running time, and

$$
M_2 = \begin{cases}
\exp\left\{\sum_{i\in\mathcal{S}} \left[N_{0i}(t,t_0)\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) - N_{0i}(t,t_0)\bar{G}_i\ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)\right]\right\}, \\
\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) > 0, \\
\exp\left\{\sum_{i\in\mathcal{S}} \left[-N_{0i}(t,t_0)\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) - N_{0i}(t,t_0)\bar{G}_i\ln\left(\eta_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right)\right]\right\}, \\
\ln\left(\mu_i E_\alpha\left(\lambda_i\theta^\alpha\right)\right) < 0,
\end{cases}
$$

$$
M_3 = \begin{cases}
\exp\left\{\sum_{i\in\mathcal{U}} N_{0i}(t,t_0)\ln\left(\frac{\mu_i}{\alpha}\right) - N_{0i}(t,t_0)\bar{G}_i\ln\left(\frac{\eta_i}{\alpha}\right)\right\}, & \ln\left(\frac{\eta_i}{\alpha}\right) < 0, \\
\exp\left\{\sum_{i\in\mathcal{U}} N_{0i}(t,t_0)\ln\left(\frac{\mu_i}{\alpha}\right) + N_{0i}(t,t_0)\bar{G}_i\ln\left(\frac{\eta_i}{\alpha}\right)\right\}, & \ln\left(\frac{\eta_i}{\alpha}\right) > 0,
\end{cases}
$$

and let the inequality $\gamma^- < -2\gamma < 0$ holds. According to (13) and (14), which easily leads to

$$
\begin{aligned}
V^{g(t)}_{\sigma(t)}(x(t)) &\le M_2 M_3 \exp\left\{\sum_{i\in\mathcal{S}} \gamma^- T_i(t,t_0) + \sum_{i\in\mathcal{U}} \gamma^+ T_i(t,t_0)\right\} \\
&\le \exp\left\{T^-\gamma^- + T^+\gamma^+\right\} \\
&< \exp\left\{(T^- + T^+)(-2\gamma)\right\} \\
&= \mathrm{e}^{-2\gamma(t-t_0)}.
\end{aligned} \tag{15}
$$

From (**B1**) and (15), it can be inferred that

$$
\alpha_1 \left\|x(t)\right\|^2 \le M_2 M_3 \mathrm{e}^{-2\gamma(t-t_0)} \alpha_2 \left\|x(t_0)\right\|^2,
$$

thus

$$
\begin{aligned}
\|x(t)\| &\le \sqrt{\frac{\alpha_2}{\alpha_1} M_2 M_3} \mathrm{e}^{-\gamma(t-t_0)} \left\|x(t_0)\right\| \\
&\le K\mathrm{e}^{-\gamma(t-t_0)} \left\|x\left(t_0\right)\right\|, \quad \left(K = \sqrt{\frac{\alpha_2}{\alpha_1} M_2 M_3}\right).
\end{aligned}
$$

System (1) is clearly GUES with margin γ since $\lim_{t\to+\infty} x(t) = 0$. □

Theorem 3. *Consider CFOSLS* (9)*, for* $\forall i \in \mathcal{M}$ *there are constants* λ_i*,* $\mu_i > 1$*,* $0 < \eta_i < 1$*, and let* $\forall \sigma(t_k) = p$*,* $\sigma\left(t_k^-\right) = q$*. If there exists a set of matrices* $P_i^j > 0$*,* $i \in \mathcal{M}, j \in \mathcal{G}_{\sigma(t)}$*, such that*

(B21) $A_i^T P_i^j + P_i^j A_i \le \lambda_i P_i^j \begin{cases} \lambda_i < 0, \forall i \in \mathcal{S}, \\ \lambda_i > 0, \forall i \in \mathcal{U}, \end{cases}$

(B31) $P_i^j \le \eta_i P_i^{j-1}, \quad \forall i \in \mathcal{S}, j \ne 0,$

(B41) $P_p^0 \le \mu_p P_q^{G_q - 1}, \quad \forall (p,\ q) \in \mathcal{M} \times \mathcal{M},$

then system (9) *is GUES with marginal* γ *when the switching signal* $\sigma(t)$ *satisfies* (13)*.*

Proof. This process is similar to the previous Theorem 1. □

Theorem 4. *Consider CFOSNS* (12)*, given constants* λ_i*,* $\mu_i > 1$*,* $0 < \eta_i < 1$*, for* $\forall i \in \mathbf{M}$ *and if there exists a set of matrices* $P_i^j > 0$*,* $i \in \mathcal{M}, j \in \mathcal{G}_{\sigma(t)}$*, such that conditions* ***(B21)–(B41)*** *hold, then for any switched signal* σ *that satisfies* (13)*, the system* (12) *is GUES with marginal* γ*.*

Proof. The proof can be derived using analogous methods to those in Theorem 2. □

Remark 9. *Lemmas 4 and 5 clearly reveal that by changing the parameters* $G_{\sigma(t)}$ *and* $\eta_{\sigma(t)}$*, it is possible to achieve strict constraints on MDADT, thereby enhancing the adaptability of practical applications. However, it is significant to note that the increase of parameter will significantly increase the computational complexity, and too small a value will easily lead to the failure to meet the constraints. Therefore, parameters need to be designed in conjunction with system characteristics and computing resources. In particular, choosing* $\eta_{\sigma(t)} = 1$ *and* $G_{\sigma(t)} = 1$ *will convert MDLF to conventional MLF. Thus, our results may be less conservative than those based on MLF method.*

4. Numerical Example

In order to illustrate the validity of our major conclusions, two numerical simulation examples are offered in this section to discuss the stability of CFOSLS (9) and CFOSNS (12) under fast-slow switching behavior, respectively. Considering that each switched system has two subsystems, a stable subsystem and an unstable subsystem.

4.1. Example 1

Study the Caputo fractional switched system below, in which unstable subsystems cannot be continually activated:

$$ {}_{t_0}^{C}D_t^{\alpha} x(t) = A_{\sigma(t)}(x(t)), \quad x(t_0) = x_0, \tag{16} $$

where $\sigma(t) \in \mathbf{M} = \{1, 2\}$. Furthermore, the following is the matrix that corresponds to the two subsystems:

$$ A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},\ A_2 = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}. $$

Calculate the eigenvalues of the above two matrices, and obtain $\lambda_{A_1} = \frac{3 \pm \sqrt{5}}{2}$ and $\lambda_{A_2} = -2 \pm i$, which is obviously unstable for subsystem 1 corresponding to matrix A_1 and stable for subsystem 2 corresponding to matrix A_2. Thus, subsystems 1 and 2 should thus adhere to the fast and slow switching strategies, respectively. The exponential stability of system (16) is analyzed using Theorem 1.

We select $t_0 = 0$, $x_0 = (2.5, -1.5)$, $\alpha = 0.9$, $\lambda_c = 3.5$, $\mu_c = 0.57$, $\eta_c = 0.6$ and $\lambda_1 = -2.5$, $\mu_1 = 2$, $\eta_1 = 0.6$, $\theta = 0.4172$, $G_1 = G_2 = 2$ then by substituting these parameters from condition (**A21**)–(**A41**) and (2), we can find the MDLF that match the criteria, and the system's switched law requires: $\tau_{a_i} \ge 0.8344$, $\tau_a^c \le 0.1136$.

If the above conditions are satisfied, Figures 2 and 3 show the CFOLSS's switching signal and trajectory of state, respectively, under the the initial condition $x_0 = (2.5, -1.5)$.

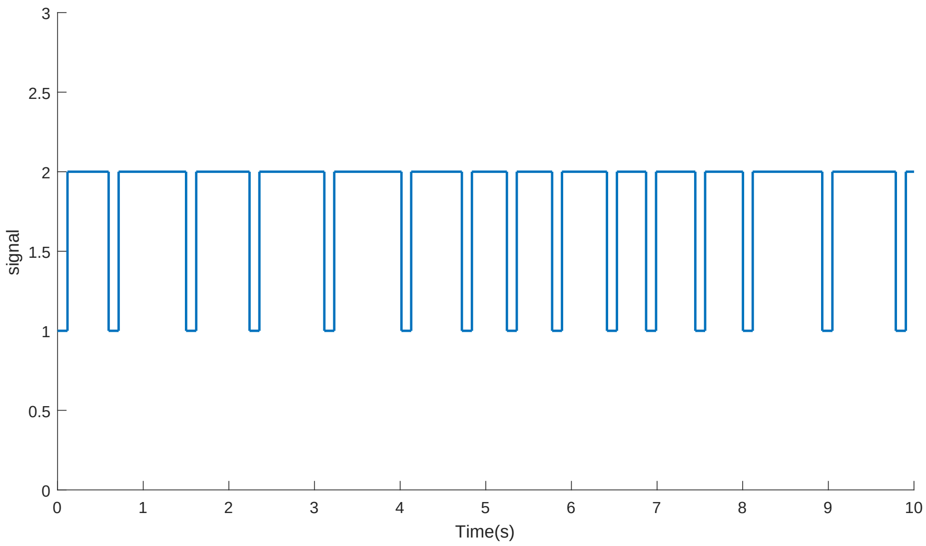

Figure 2. The switching signal of system (16).

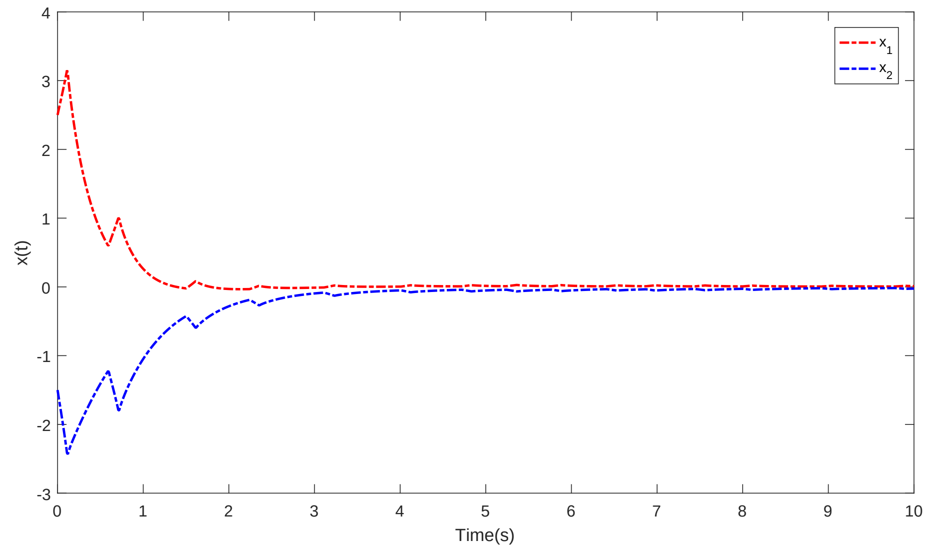

Figure 3. The state trajectory of system (16).

The unstable and stable subsystems are alternately switched during the duration $t = 10s$, as the pictures illustrate. In Figure 3, the system trajectory confirms that under the switched signal of MDADT, the stable subsystem uses slow switching while the unstable subsystem uses fast switching. The stable subsystem's long dwell time can counteract the effect on the unstable subsystem, and the switched system's state trajectory eventually converges to 0, indicating that under $\sigma(t)$ of Figure 2, the system (16) is GUES.

Remark 10. *Compared with the MLF method* [37], *the MDLF method does not require each subsystem to be asymptotically stable, nor does it impose overly strict constraints on the average dwell time of the switching signal. By designing a customized Lyapunov function for each mode, the MDLF method can accurately characterize the memory effect and high-frequency switching characteristics of fractional-order systems, ensuring the exponential stability of the system even under the high-frequency impulsive switching shown in Figure 2, while obtaining less conservative stability criteria and finally achieving the convergence of the state trajectories in Figure 3.*

4.2. Example 2

Think about a switched T-S fuzzy system (12) that has two subsystems with two fuzzy rules each, where Subsystem 1

$$A_{11} = \begin{pmatrix} 1.2 & -1 \\ -0.5 & 1.8 \end{pmatrix},\ A_{12} = \begin{pmatrix} 1.5 & -1.2 \\ -0.8 & 2 \end{pmatrix}.$$

Subsystem 2

$$A_{21} = \begin{pmatrix} -2 & 1.2 \\ -0.8 & -1.5 \end{pmatrix},\ A_{22} = \begin{pmatrix} -1.5 & 1 \\ -0.5 & -2 \end{pmatrix}.$$

The fuzzy membership functions are set to

$$h_{11} = 0.5[1 - \sin(x_1(t))],\ h_{12} = 1 - h_{11},$$
$$h_{21} = 0.5[1 - \cos(x_2(t))],\ h_{22} = 1 - h_{21}.$$

According to the eigenvalues of the coefficient matrix $A_{11} - A_{22}$, subsystem 1 is unstable and subsystem 2 is stable, so let subsystem 1 adhere to the fast switching scheme and subsystem 2 to the slow switching scheme. Next, using the Theorem 1 to analyze the exponential stability of system.

The initial condition is supposed to be $t_0 = 0$, $x_0 = (2, -1.2)$. In the case of condition (**A21**)–(**A41**) and (2), we can select $\alpha = 0.85$, $\lambda_c = 3.8$, $\mu_c = 0.4$, $\eta_c = 0.8$ and $\lambda_1 = -2.4$, $\mu_1 = 2.9$, $\eta_1 = 0.6$, $\theta = 0.3725$, $G_1 = G_2 = 2$. The MDLF that meets the conditions is found and the switching law requirement MDADT of the system is obtained by calculation, where $\tau_{a_i} \geq 0.7450$, $\tau_a^c \leq 0.1819$.

When the above conditions are met, Figures 4 and 5 is the switching signal of CFONSS and the system state response diagram respectively.

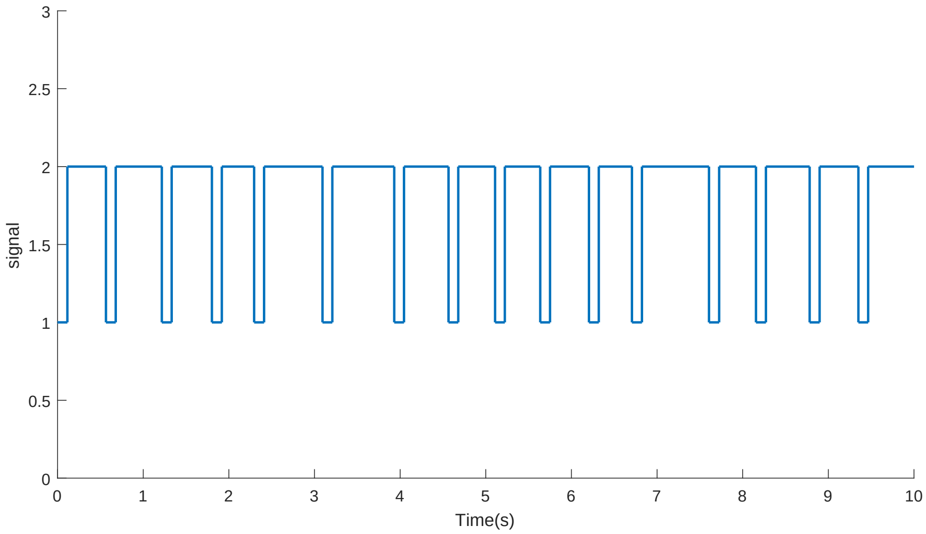

Figure 4. The switching signal of system (12).

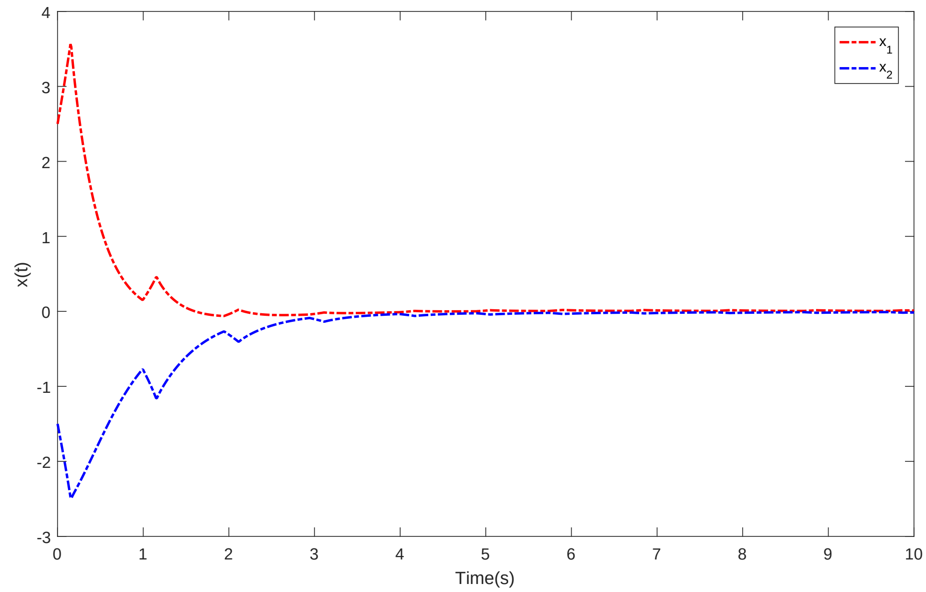

Figure 5. The state trajectory of system (12).

Although the existence of unstable subsystems will affect the system, as the figures show, the system state trajectory can still approach 0 by alternating between the unstable and stable subsystems using the switching rule depicted in Figure 4, which means that the T-S fuzzy system (12) is GUES.

5. Conclusions

This work primarily discusses the exponential stability of a CFOSS (1) that has both unstable and stable subsystems. Firstly, the fast/slow switching mechanism of MDADT is designed to reduce the conservatism of traditional ADT technology, and together with the MDLF approach, then the stability's sufficient conditions of the CFOSS are given. Further, the general switched system is simplified into CFOSLS (9) and CFOSNS (12) to apply the existing lemmas, in which we model the switched nonlinear system studied into the switched T-S fuzzy system through the T-S fuzzy. The results show that: (1) The unstable subsystems and the stable subsystems carry out fast switching and slow switching of MDADT correspondingly. To ensure the system stability, the former must switched faster than the latter; (2) All subsystems carry out MDADT slow switching, in contrast to the unstable subsystems, however, the stable subsystems have a sufficiently long dwell time, which can offset the effects of the unstable subsystems. Eventually, two illustrative numerical examples of fractional order switched system are provided to show the validity of the research findings.

Event-triggered control is widely applied in engineering-related fields, and extensive research has been conducted on the qualitative analysis such as exponential stability and robustness of complex switched system based on event-triggered control [49–51]. Inspired by these works, our next research focus will be on the exponential stability of a class of fractional switched system with event-triggered control. The memory characteristic described by fractional system has unique advantages in depicting some complex physical phenomena; meanwhile, event-triggered control exerts a ground breaking impact on the research of switched system' stability, making fractional switched system with event-triggered control more applicable in engineering scenarios with limited resources, complex dynamics, and strong disturbances. The research on their exponential stability is of great significance for improving the relevant theoretical system and promoting the practical application of engineering.

Author Contributions

Q.L.: conceptualization, methodology, software, data curation, writing—original draft preparation, visualization, investigation; D.L.: supervision, funding acquisition, writing—reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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