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Performance-Oriented Metaheuristic PID Tuning for Parametrically Excited Mathieu Systems

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Abstract: This paper investigates metaheuristic-based tuning of classical PID controllers for the performance-oriented regulation of parametrically excited dynamical systems, with a particular focus on the Mathieu equation as a canonical model of time-periodic instability. Unlike many existing studies that emphasize nonlinear or autonomous dynamics, the work explicitly addresses PID tuning under time-periodic parametric excitation with an emphasis on closed-loop performance and bounded response behavior. Three population-based optimization algorithms–Particle Swarm Optimization (PSO), Grey Wolf Optimizer (GWO), and a hybrid PSO–GWO scheme–are employed to determine PID gain parameters under identical computational budgets. The tuning objective combines the Integral of Time-weighted Absolute Error (ITAE) with the Integral of Squared Overshoot (ISO), enabling a unified assessment of transient performance and overshoot suppression. The controlled Mathieu system is examined across multiple excitation regimes, ranging from stable periodic operation to near-resonant and strongly excited conditions. In addition to the metaheuristic approaches, a deterministic baseline PID controller derived from a nominal LTI approximation is included as a practical reference. All controllers are evaluated on the full time-varying Mathieu dynamics using the same performance index, and control-input saturation tests are conducted to assess behavior under practical actuator constraints. Numerical results demonstrate that all considered metaheuristic methods achieve reliable closed-loop stabilization and maintain bounded oscillatory responses despite strong parametric excitation. In contrast, the deterministic baseline exhibits a pronounced initial overshoot and higher steady-state oscillation amplitudes, resulting in significantly higher performance-index values. While the hybrid PSO–GWO approach occasionally yields smoother transients, it does not provide a consistent performance advantage over standalone PSO or GWO when normalized by computational effort. Overall, the findings indicate that, for low-dimensional PID tuning problems in parametrically excited systems, solution quality is governed primarily by adequate search-space coverage and population diversity rather than algorithmic hybridization. Well-configured single-population metaheuristics therefore offer an effective and practically relevant framework for stability-oriented PID control of time-varying systems subject to parametric excitation.

Keywords: metaheuristic optimization; PID control; parametrically excited systems; Mathieu equation; stability analysis; Particle Swarm Optimization (PSO); Grey Wolf Optimizer (GWO)

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1. Introduction

Proportional–Integral–Derivative (PID) controllers remain one of the most widely used feedback control structures in engineering practice due to their simplicity, robustness, and ease of implementation. Despite their conceptual simplicity, PID controllers have demonstrated remarkable effectiveness across a broad range of linear and mildly nonlinear systems, particularly when the plant dynamics are of low order and time-invariant [1,2]. For this reason, PID control continues to serve as a fundamental reference framework in both industrial applications and control-oriented research.

A class of systems that poses a distinct challenge to classical PID control is represented by parametrically excited dynamical systems. In such systems, key parameters vary periodically in time, leading to non-autonomous dynamics that may exhibit instability, resonance amplification, or complex transient behavior even when the underlying system is linear in its state variables [3]. Among these systems, the Mathieu equation occupies a central role as a canonical model of time-periodic instability and parametric resonance, with applications spanning structural mechanics, rotating machinery, electrical circuits, and micro- and nano-electromechanical systems.

The stability properties of the Mathieu equation are governed by a delicate balance between damping and periodic parameter modulation. Small variations in excitation amplitude or frequency may drive the system across stability boundaries, resulting in rapid growth of oscillations or sustained amplitude modulation. From a control perspective, this sensitivity makes the Mathieu system an ideal benchmark for assessing the practical sensitivity and stabilization capability of feedback controllers operating under time-varying conditions. While linear control theory provides valuable insight into the underlying instability mechanisms, practical stabilization often relies on appropriately tuned feedback gains that must account for both transient performance and steady-state oscillatory behavior.

Tuning PID controllers for parametrically excited systems remains a nontrivial task, as the associated performance landscape is shaped by time-periodic dynamics, competing stability requirements, and the need to balance rapid convergence with effective overshoot suppression. In this context, metaheuristic optimization techniques have emerged as attractive tools for controller tuning, as they do not rely on linearization, convexity assumptions, or explicit gradient information. Instead, these methods explore the gain space through population-based stochastic search, making them well suited for control problems involving complex or time-varying dynamics.

Among the most widely adopted metaheuristic algorithms are Particle Swarm Optimization (PSO) [4–7] and the Grey Wolf Optimizer (GWO) [8]. PSO is known for its rapid convergence driven by social–cognitive learning mechanisms, whereas GWO emphasizes global exploration through hierarchical leadership and encircling strategies. Recent surveys [9] discuss the use of these methods in a variety of optimization and control settings, including parallel, sequential, and embedded hybrid strategies.

Motivated by these considerations, the present study investigates the metaheuristic tuning of classical PID controllers for the regulation of a parametrically excited Mathieu system, with particular emphasis on closed-loop performance and bounded response behavior. Particle Swarm Optimization (PSO), Grey Wolf Optimizer (GWO), and a cooperative PSO–GWO hybrid strategy are employed to optimize the PID gains under identical computational budgets, enabling a fair comparison of convergence behavior and closed-loop performance. The analysis considers multiple excitation regimes, ranging from stable periodic operation to near-resonant and strongly modulated conditions, thereby capturing the key dynamical challenges associated with parametrically excited systems.

Rather than proposing a new control structure or optimization algorithm, the objective of this work is to assess how different metaheuristic strategies influence PID tuning outcomes in explicitly time-periodic systems. In contrast to many existing studies that focus primarily on nominal tracking performance, the present work adopts a deliberately practical perspective. In particular, attention is given to aspects that are often treated only marginally in the literature, such as control-effort requirements, actuator constraints, and computational cost under comparable optimization budgets.

The study reports both positive and negative outcomes, including cases in which the hybrid strategy does not consistently outperform the individual metaheuristic methods. In addition to standard tracking-based performance indices, practical measures such as peak control effort and the effect of actuator saturation are explicitly included. Overall, the goal is to provide a balanced and application-oriented assessment of metaheuristic PID tuning for parametrically excited Mathieu-type systems.

Novelty and Contribution of the Paper

The main contributions of this paper can be summarized as follows:

- A systematic investigation of metaheuristic-based PID tuning for a parametrically excited linear time-varying system, using the Mathieu equation as a canonical benchmark for time-periodic instability. While metaheuristic control design has been extensively studied for nonlinear and autonomous systems, its application to performance-oriented PID tuning under explicit parametric excitation remains relatively underexplored.
- A unified, performance-based comparison of Particle Swarm Optimization (PSO), Grey Wolf Optimizer (GWO), and a cooperative PSO–GWO hybrid strategy under strictly identical computational budgets. By enforcing the same population size and number of iterations for all methods, the study isolates the influence of algorithmic structure from sampling density and evaluation-budget effects.
- A regime-based analysis encompassing stable, near-resonant, and strongly excited operating conditions, providing insight into how increasing excitation intensity affects closed-loop behavior, control effort, and the optimization process in time-periodic systems.
- The inclusion of a deterministic baseline PID controller derived from a nominal time-averaged LTI approximation of the Mathieu system, serving as a computationally inexpensive and fully deterministic reference. The baseline is evaluated on the full time-varying dynamics and compared directly with metaheuristic-tuned controllers, highlighting the limitations of nominal designs in suppressing overshoot and sustained oscillations under parametric excitation.
- A practical assessment of controller behavior under actuator constraints through control-input saturation tests. These tests complement the nominal performance metrics and illustrate how different tuning strategies respond to realistic input limitations without conflating design-time optimization with post-design implementation effects.
- Empirical evidence demonstrating that, for low-dimensional PID tuning problems in parametrically excited systems, solution quality is governed primarily by adequate search-space coverage and population diversity rather than by algorithmic hybridization. In particular, the hybrid PSO–GWO strategy does not exhibit a consistent performance advantage over well-configured standalone metaheuristics when the computational budget is held constant.

The remainder of this paper is organized as follows. Section 2 formulates the control problem for the Mathieu equation, introduces the PID control structure, and defines the performance criterion used for optimization. Subsection 2.4 describes the employed metaheuristic algorithms together with their parameter settings and computational budgets. Simulation results across different parametric excitation regimes are presented and discussed in Section 3. Subsection 3.5 introduces the deterministic baseline controller based on a nominal (time-averaged) model, examines control-input saturation effects, and provides a direct comparison with the PSO-based metaheuristic design. Finally, Section 4 summarizes the main findings and outlines directions for future research.

2. Problem Formulation

Consider a general linear dynamical system of order n described by a time-varying ordinary differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, y^{(k)}(t) = u(t), \qquad t \in [0, \infty), \tag{1}$$

where $y(t)$ denotes the system output, $u(t)$ is the control input, and $a_k(t)$ are bounded, continuous coefficient functions that may vary periodically in time. All initial conditions are assumed to be zero,

$$y(0) = \dot{y}(0) = \cdots = y^{(n-1)}(0) = 0, \qquad \dot{} \triangleq \frac{d}{dt}, \tag{2}$$

which allows a clear assessment of transient and steady-state behavior induced by the control action.

This formulation encompasses a broad class of linear time-invariant and linear time-periodic systems. In particular, the Mathieu equation considered in this study represents a special case of (1) with $n = 2$ and periodically modulated stiffness coefficients.

The control objective is to ensure stable tracking of a prescribed reference signal $r(t)$ while maintaining bounded responses, fast convergence, and limited overshoot. The tracking error, which quantifies the instantaneous deviation of the system output from the desired reference, is defined as $e(t) = r(t) - y(t)$.

2.1. PID Control Law

A classical Proportional–Integral–Derivative (PID) controller is employed to regulate the system output,

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \dot{e}(t), \tag{3}$$

where K_p, K_i, and K_d denote the proportional, integral, and derivative gains, respectively.

The PID controller is selected due to its widespread use in industrial practice and its ability to provide a balanced combination of steady-state accuracy, transient shaping, and damping. Although simple in structure, appropriately tuned PID controllers can effectively stabilize low-order linear systems even in the presence of time-periodic parameter variations. In the present work, the PID structure is retained without augmentation in order to isolate the effect of metaheuristic tuning strategies on stability and performance.

2.2. Closed-Loop Dynamics and State-Space Representation

Substituting the control law (3) into the system dynamics (1) yields the closed-loop equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, y^{(k)}(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \dot{e}(t). \tag{4}$$

For numerical simulation and optimization, the closed-loop dynamics is written in an augmented first-order state-space form using the state vector

$$\mathbf{x}(t) = \left[x_1(t), x_2(t), \ldots, x_n(t), x_{n+1}(t)\right]^\top = \left[y(t), \dot{y}(t), \ldots, y^{(n-1)}(t), \textstyle\int_0^t e(\tau)\, d\tau\right]^\top. \tag{5}$$

The state equations become

$$\dot{x}_i(t) = x_{i+1}(t), \qquad i = 1, \ldots, n-1, \tag{6}$$

$$\dot{x}_n(t) = -\sum_{k=0}^{n-1} a_k(t)\, x_{k+1}(t) + K_p\, (r(t) - x_1(t)) + K_i\, x_{n+1}(t) + K_d\, (\dot{r}(t) - x_2(t)), \tag{7}$$

$$\dot{x}_{n+1}(t) = e(t) = r(t) - x_1(t). \tag{8}$$

The reference input is chosen as a unit-step signal,

$$r(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{9}$$

which provides a standard benchmark for evaluating transient response characteristics, including rise time, overshoot, and settling behavior.

For linear time-periodic systems such as the Mathieu oscillator, the unit-step response reveals the interplay between feedback stabilization and parametric excitation. Bounded oscillatory steady states obtained under PID control are therefore interpreted as indicators of closed-loop stability and effective regulation rather than as performance degradation.

Figure 1 illustrates the closed-loop feedback structure employed in this study. The PID controller generates the control input $u(t)$ based on the tracking error $e(t) = r(t) - y(t)$, while the plant represents a general linear time-varying system.

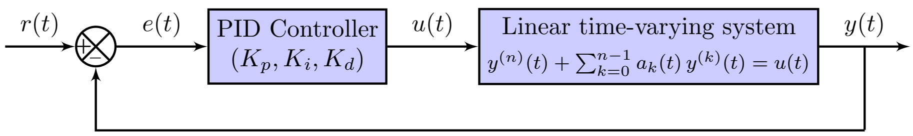

Figure 1. Schematic representation of the closed-loop feedback system with a PID controller, showing the reference input $r(t)$, the controlled output $y(t)$, and the control signal $u(t)$ for a linear time-varying plant.

2.3. Performance Criterion

The tuning of PID controller gains (K_p, K_i, K_d) is formulated as a constrained optimization problem. The objective is to achieve stable reference tracking while suppressing excessive transient oscillations and overshoot induced by time-periodic system coefficients. To this end, a composite performance index is employed,

$$J(K_p, K_i, K_d) = \alpha \int_0^T t\,|e(t)|\,dt + \beta \int_0^T \big(\max(0, -e(t))\big)^2 dt. \tag{10}$$

The two terms in (10) serve complementary and clearly differentiated purposes:

- the Integral of Time-weighted Absolute Error (ITAE) penalizes tracking errors that persist over time, thereby promoting rapid convergence to the reference and effective damping of transient oscillations,
- the Integral of Squared Overshoot (ISO) penalizes only excursions beyond the reference level by acting exclusively on negative tracking error values, which correspond to overshoot in the adopted convention $e(t) = r(t) - y(t)$. In this way, the ISO term specifically discourages excessive amplification induced by parametric excitation, without directly penalizing undershoot or steady-state error.

The weighting coefficients $\alpha, \beta \geq 0$ determine the relative emphasis on reducing transient duration versus limiting overshoot, and T denotes the simulation time horizon chosen sufficiently long to capture both the transient evolution and the steady-state oscillatory response.

This performance criterion is particularly suitable for parametrically excited systems, where stability is reflected not only by asymptotic behavior but also by bounded steady-state oscillations induced by the time-periodic modulation.

The PID tuning problem is thus stated as

$$\min_{K_p, K_i, K_d} \; J\big(K_p, K_i, K_d\big), \tag{11}$$

subject to the parameter bounds

$$0 \leq K_p,\, K_i,\, K_d \leq K_{\max}, \tag{12}$$

where $K_{\max}$ defines the admissible gain search range.

Although actuator limits may in practice impose an additional constraint $|u(t)| \leq u_{\max}$, this constraint is not explicitly enforced during the optimization process. Instead, control-input saturation is evaluated as a post-design test to assess the behavior of the tuned controllers under bounded actuation. The gain bounds indirectly limit the magnitude of the control input during the optimization stage.

In the Mathieu system simulations presented in this study, $K_{\max} = 15$ is used to provide sufficient search-space coverage while avoiding excessively large controller gains.

2.4. Optimization Algorithms

PID controller gains are optimized using three population-based metaheuristic algorithms: Particle Swarm Optimization (PSO), Grey Wolf Optimizer (GWO), and a hybrid PSO–GWO strategy. Each individual in the population encodes a candidate PID gain vector (K_p, K_i, K_d). At each iteration, candidate controllers are evaluated through closed-loop numerical simulation of the parametrically excited system, and their fitness is assessed using the performance index (10). The populations are subsequently updated according to the respective search mechanisms of the individual algorithms.

These metaheuristic approaches are well suited to PID tuning for time-varying systems, as they do not rely on linearization, convexity assumptions, or gradient information. Instead, the gain space is explored through stochastic sampling and collective learning, which allows effective navigation of complex stability boundaries and multimodal objective landscapes arising from parametric excitation.

The complete tuning procedure, including cost evaluation, simulation-based fitness assessment, and population-based search updates, is summarized in the pseudocode provided in Appendix A.

Particle Swarm Optimization (PSO). The `OriginalPSO` implementation from the `mealpy-3.0.3` (Python) library is employed. A swarm of $N_p = 30$ particles evolves over $E = 20$ epochs. Velocity updates use an inertia weight $\omega = 0.4$ and acceleration coefficients $c_1 = c_2 = 2.05$, corresponding to a constriction-type PSO formulation commonly adopted for continuous control parameter optimization.

Table 1. PSO parameters.

Parameter	Symbol	Value
Population size	N_p	30
Inertia weight	ω	0.4
Cognitive coefficient	c_1	2.05
Social coefficient	c_2	2.05
Epochs	E	20

This configuration provides a suitable balance between convergence speed and global exploration and is well suited for low-dimensional PID tuning problems.

Grey Wolf Optimizer (GWO). The `OriginalGWO` algorithm from `mealpy-3.0.3` (Python) is applied with a population of $N_p = 30$ wolves and $E = 20$ epochs. The optimization process follows the standard Alpha–Beta–Delta leadership hierarchy and encircling mechanism, which promotes global exploration in the early stages and gradual convergence toward promising regions of the search space.

Table 2. GWO parameters.

Parameter	Symbol	Value
Population size	N_p	30
Epochs	E	20
Leader weights	–	default

Compared to PSO, GWO typically exhibits stronger global sampling behavior, which can reduce the risk of premature convergence in multimodal performance landscapes.

Hybrid PSO–GWO. The hybrid PSO–GWO algorithm follows a sequential cooperative strategy that combines the population-based search mechanisms of PSO and GWO without modifying their internal update rules. The total population is partitioned into two equal subgroups ($15 + 15$ individuals), which are updated independently according to PSO and GWO dynamics in alternating short optimization bursts.

Elite information exchange is implemented by tracking and propagating the current global best solution across the PSO and GWO subpopulations between successive optimization bursts. This coordination is performed without explicit migration of individuals between the subpopulations and while maintaining a fixed total population size of $N_p = 30$ and a fixed number of epochs $E = 20$, ensuring that the hybrid strategy operates under the same computational budget as the individual metaheuristic methods.

The hybrid strategy is intended to retain global search diversity while allowing periodic local refinement through alternating PSO and GWO updates. In the present study, all algorithmic parameters are selected from commonly used, standard settings and are kept fixed across all experiments, as moderate variations of these values are not expected to fundamentally alter algorithmic behavior in low-dimensional PID tuning problems. As demonstrated by the simulation results, hybridization does not necessarily lead to superior performance in such settings, where search-space coverage and the available iteration budget play a more decisive role than fine adjustment of algorithm-specific parameters.

In summary, the proposed metaheuristic framework enables automated and stability-oriented PID tuning for linear time-varying systems without model simplification. By focusing exclusively on the classical PID structure, the study isolates the impact of optimization strategy on closed-loop stability and performance under parametric excitation.

3. Mathieu Equation: Parametric Excitation and Closed-Loop Control Analysis

As discussed in the Introduction, parametrically excited systems represent a distinct class of time-varying dynamics in which stability is governed by the interplay between damping and periodic parameter modulation. To investigate stability-oriented PID tuning under such conditions, this section focuses exclusively on the Mathieu equation, which serves as a canonical linear model of parametric excitation and time-periodic instability.

Unlike nonlinear systems where complexity arises from state-dependent feedback, the Mathieu system is linear in its state variables but exhibits rich dynamic behavior due to its periodically varying coefficients. This property makes it particularly suitable for assessing the practical sensitivity and effectiveness of PID control under time-varying conditions without confounding nonlinear effects.

3.1. Mathematical Formulation

The controlled Mathieu oscillator considered in this paper is described by the linear time-periodic differential equation

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2\big[1 + m\cos(\Omega t)\big]y(t) = u(t) + d(t), \qquad t \in [0, \infty), \tag{13}$$

where ζ denotes the damping ratio, ω_n the nominal natural frequency, m the amplitude of parametric modulation, and Ω the modulation frequency. Throughout all simulations, the nominal frequency is fixed at $\omega_n = 1.0$. An external harmonic disturbance $d(t) = \sin(2t)$ is applied to examine the closed-loop response under persistent forcing.

The control input $u(t)$ is generated by the PID law (3), with the tracking error defined as $e(t) = r(t) - y(t)$ and the reference signal chosen as a unit step. Zero initial conditions, $y(0) = \dot{y}(0) = 0$, are imposed on all state variables, consistent with standard evaluations of transient tracking performance.

Equation (13) represents a damped, parametrically excited linear oscillator with both additive and multiplicative forcing. Although linear in structure, its stability properties depend sensitively on the parameter set (ζ, m, Ω), making it an effective benchmark for evaluating controller performance under time-periodic excitation.

3.2. Simulated Stability Regimes

To systematically examine the influence of parametric excitation on control performance, three representative operating regimes are considered. These regimes correspond to increasing modulation strength and decreasing damping, thereby spanning stable, near-resonant, and marginally stable dynamics:

(1) Regime I—Stable periodic response: $\zeta = 0.08$, $m = 0.2$, $\Omega = 2.0$. The uncontrolled system exhibits bounded, nearly harmonic oscillations with low-amplitude modulation, as illustrated in Figure 2. In this nominally stable regime, the PID controller primarily reduces the steady-state oscillation amplitude.

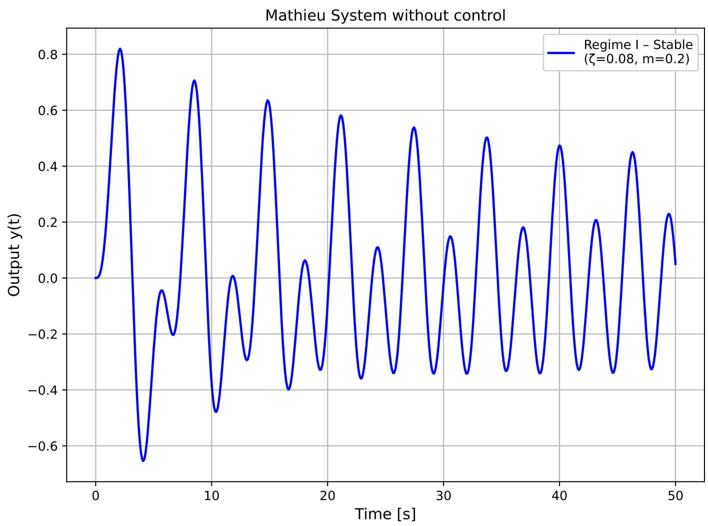

Figure 2. Uncontrolled Mathieu oscillator ($\zeta = 0.08$, $m = 0.2$, $\Omega = 2.0$): the system exhibits bounded, nearly harmonic oscillations with low-amplitude modulation, corresponding to a stable periodic regime.

(2) Regime II—Near parametric resonance: $\zeta = 0.04$, $m = 0.6$, $\Omega = 2.0$. The excitation frequency approaches the primary parametric resonance condition $\Omega \approx 2\omega_n$, leading to slow amplitude growth and pronounced modulation in the uncontrolled response, as shown in Figure 3. This regime challenges the controller's ability to suppress quasi-resonant amplification.

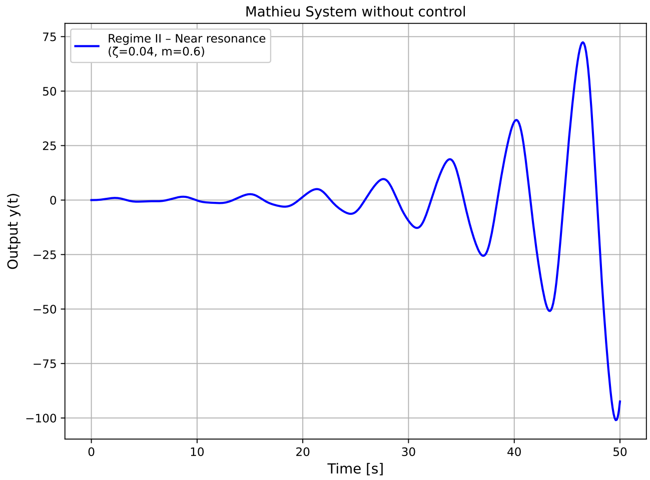

Figure 3. Uncontrolled Mathieu oscillator ($\zeta = 0.04$, $m = 0.6$, $\Omega = 2.0$): the oscillations display slow amplitude modulation and gradual transient growth due to near-resonant parametric excitation.

(3) Regime III—Strong excitation and marginal stability: $\zeta = 0.02$, $m = 0.9$, $\Omega = 2.0$. Low damping combined with strong modulation places the system close to parametric instability, resulting in alternating growth and decay phases with significantly increased oscillation energy, as illustrated in Figure 4. In this regime, stabilization requires a careful balance between damping enhancement and control effort.

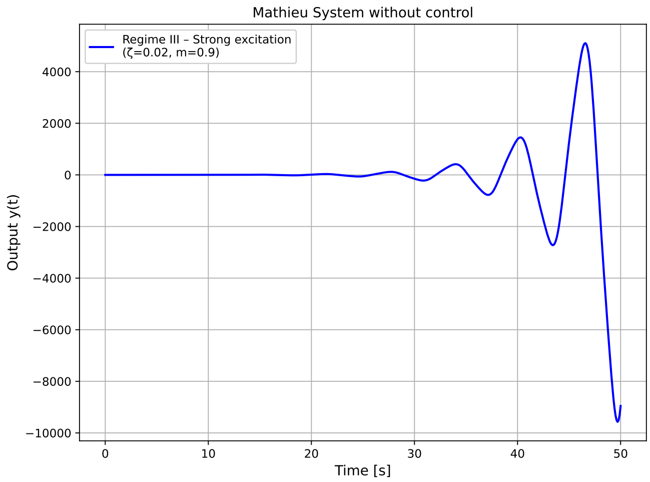

Figure 4. Uncontrolled Mathieu oscillator under strong excitation ($\zeta = 0.02$, $m = 0.9$, $\Omega = 2.0$): the motion alternates between growth and decay phases, reflecting marginally stable behavior and strong parametric amplification.

Although the uncontrolled system trajectories in Regimes II and III may appear qualitatively similar, their underlying stability mechanisms differ substantially. Regime II operates near the boundary of the first instability tongue of the Mathieu stability diagram [3], where damping partially compensates for periodic energy injection. In contrast, Regime III lies deeper within the unstable region, where parametric amplification dominates dissipation, leading to rapid amplitude escalation. This progression highlights the pronounced sensitivity of parametrically excited systems to small variations in excitation intensity and damping.

From a control-design perspective, the considered progression from mildly excited to strongly unstable regimes can be interpreted as a structured variation of the system parameters that spans the principal dynamical behaviors of the Mathieu equation. Although not intended as a formal robustness analysis, this regime-based variation provides a practical indication of controller behavior across qualitatively different stability mechanisms inherent to parametrically excited systems.

All regimes are simulated under identical external forcing and with the same PID controller structure. Controller gains are optimized using PSO, GWO, and hybrid PSO–GWO algorithms with a fixed population size $N_p = 30$ and $E = 20$ epochs, ensuring a uniform computational budget across all cases. The gain bounds $K_p, K_i, K_d \in [0, 15]$

are selected to provide sufficient flexibility for stabilization while maintaining numerical well-posedness during the optimization process.

3.3. Simulation Results for the Mathieu Oscillator

This section presents the numerical results of metaheuristic PID tuning for the parametrically excited Mathieu oscillator across the three stability regimes defined in Subsection 3.2. All simulations were performed using a fixed time horizon of $T = 50$, and each optimization scenario was repeated over five independent runs (Run 1–5) to assess statistical consistency and reduce the influence of stochastic variability.

For all considered regimes and optimization algorithms, the closed-loop response exhibits a short transient phase following the unit-step change of the reference signal $r(t)$ from zero to one. After the transient, the system settles into a steady-state characterized by bounded, low-amplitude oscillations around the reference value. Such residual oscillations are an inherent feature of linearly damped, parametrically excited systems under periodic forcing and cannot be completely eliminated by linear PID feedback.

Regime I. In the first regime, corresponding to lightly damped stable oscillations, the Mathieu oscillator behaves as a harmonically modulated linear system with bounded response amplitude. The optimized PID controllers effectively attenuate oscillations and maintain the output close to the reference value with negligible steady-state error. As illustrated in Figure 5, all three optimization algorithms–PSO, GWO, and PSO–GWO–exhibited nearly identical control performance, yielding well-damped and regular closed-loop responses. Although the hybrid algorithm occasionally showed marginally faster convergence, this advantage was counterbalanced by a higher number of cost function evaluations due to its combined exploration–exploitation structure. Overall, the results indicate that for weakly excited parametrically driven systems, standalone PSO or GWO methods offer an efficient and reliable PID tuning strategy.

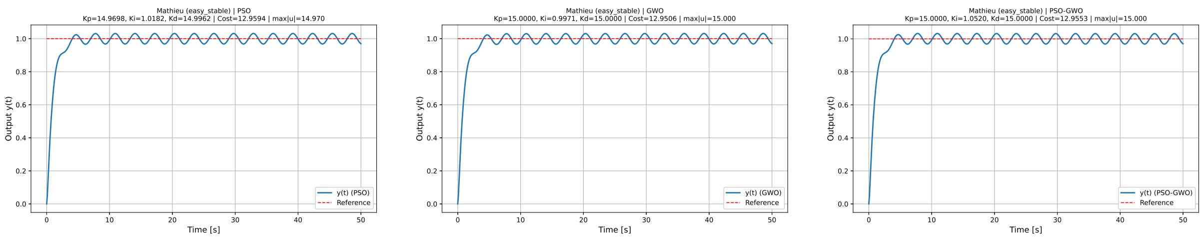

Figure 5. Closed-loop responses of the Mathieu oscillator in the stable parametric excitation regime (Regime I), obtained using PID controllers optimized by (**left**) Particle Swarm Optimization (PSO), (**center**) Grey Wolf Optimizer (GWO), and (**right**) the hybrid PSO–GWO algorithm. The parameters K_p^*, K_i^*, and K_d^* denote the optimal proportional, integral, and derivative gains, respectively, corresponding to the minimum value of the performance index (10) with weighting coefficients $\alpha = 0.5$ and $\beta = 1.0$. The reported value $\max|u(t)|$ represents the peak control effort produced by the controller during the simulation and is not a saturation limit.

Regime II. The second regime corresponds to a near-resonant operating condition, in which the modulation frequency approaches twice the natural frequency of the oscillator. As a result, the uncontrolled system exhibits slow amplitude modulation and transient growth, requiring the controller to suppress quasi-resonant oscillations. As shown in Figure 6, all three optimization algorithms successfully stabilized the system and ensured bounded steady-state behavior. The hybrid PSO–GWO algorithm achieved control performance comparable to that of PSO and GWO, albeit at the expense of increased computational effort due to its two-phase update strategy. The close agreement of results across all methods indicates that the optimization landscape in this regime remains relatively smooth and well conditioned.

Regime III. In the third regime, the Mathieu oscillator operates under strong parametric excitation combined with very low damping, leading to large transient amplitudes and alternating phases of energy growth and decay in the uncontrolled response. Despite this highly demanding operating condition, all optimized PID controllers maintained stable closed-loop behavior and reduced oscillation amplitudes to acceptable levels. As illustrated in Figure 7, the hybrid PSO–GWO algorithm again achieved control performance comparable to that of the standalone methods, but at a significantly higher computational cost due to the increased number of cost function evaluations per run. This result further confirms that algorithmic hybridization does not necessarily improve efficiency when the optimization problem is already adequately explored by simpler population-based strategies. In such cases, increased algorithmic complexity does not translate into tangible performance gains.

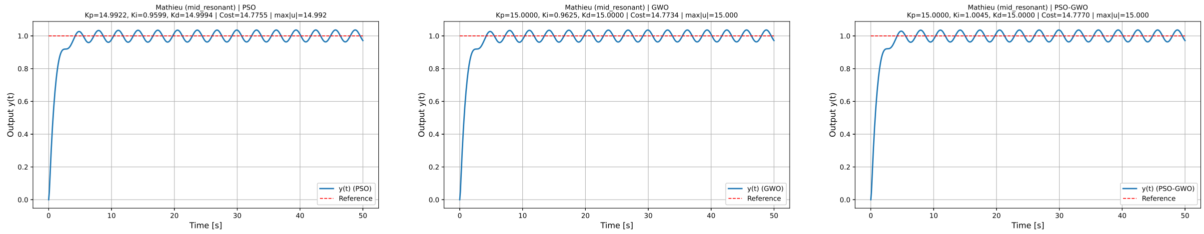

Figure 6. Closed-loop responses of the Mathieu oscillator in the near-resonant parametric excitation regime (Regime II), obtained using PID controllers optimized by (**left**) Particle Swarm Optimization (PSO), (**center**) Grey Wolf Optimizer (GWO), and (**right**) the hybrid PSO–GWO algorithm. The parameters K_p^*, K_i^*, and K_d^* denote the optimal proportional, integral, and derivative gains, respectively, corresponding to the minimum value of the performance index (10) with weighting coefficients $\alpha = 0.5$ and $\beta = 1.0$.

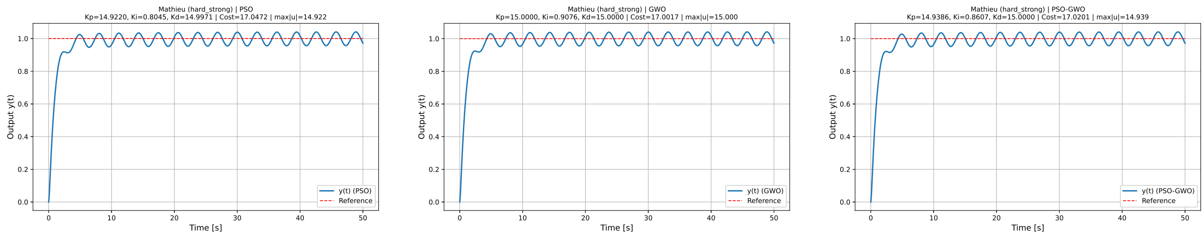

Figure 7. Closed-loop responses of the Mathieu oscillator in the strongly excited parametric excitation regime (Regime III), obtained using PID controllers optimized by (**left**) Particle Swarm Optimization (PSO), (**center**) Grey Wolf Optimizer (GWO), and (**right**) the hybrid PSO–GWO algorithm. The parameters K_p^*, K_i^*, and K_d^* denote the optimal proportional, integral, and derivative gains, respectively, corresponding to the minimum value of the performance index (10) with weighting coefficients $\alpha = 0.5$ and $\beta = 1.0$.

Across all considered regimes, the numerical results summarized in Table 3 demonstrate that the three metaheuristic optimizers achieve closely comparable control performance. Differences between PSO, GWO, and the hybrid PSO–GWO approach are statistically small, indicating consistent convergence toward similar PID parameter sets in all cases. The gradual increase of the performance index with increasing modulation strength reflects the additional control effort required to suppress parametric amplification under stronger excitation. Nevertheless, all optimized controllers produce bounded oscillatory responses over the simulation horizon, indicating effective regulation of the time-periodic dynamics across the examined regimes.

Table 3. Statistical summary of PID tuning results for the parametrically excited Mathieu system across three excitation regimes. Reported values correspond to the mean and standard deviation of the performance index (10) over five independent runs for the metaheuristic methods. The deterministic baseline is obtained via a one-shot analytical design and therefore exhibits negligible variance; its role and limitations are discussed separately in Subsection 3.5. In addition to the achieved performance, the table reports the mean design-time computational cost, illustrating the difference between optimization-based tuning and classical deterministic design.

Regime	Algorithm	Mean Cost	Std Dev	Mean Time [s]
Regime I	PSO	12.9868	0.0166	107.23
	GWO	12.9506	0.0000	100.95
	PSO–GWO	13.0681	0.0834	143.39
	Baseline	26.2279	0.0000	0.15
Regime II	PSO	14.8282	0.0445	115.02
	GWO	14.7734	0.0000	109.74
	PSO–GWO	15.0289	0.3053	152.60
	Baseline	29.8663	0.0000	0.16
Regime III	PSO	17.0931	0.0547	120.41
	GWO	17.0017	0.0000	114.14
	PSO–GWO	17.1093	0.0721	158.44
	Baseline	34.3693	0.0000	0.15

3.4. Discussion of Mathieu Optimization Results

The simulation results further indicate that PSO, GWO, and the hybrid PSO–GWO algorithm provide comparable stabilization quality across all parametric excitation regimes. For the Mathieu oscillator, the PID tuning problem is characterized by a smooth and low-dimensional optimization landscape, which facilitates rapid and consistent convergence of all three methods. Consequently, no systematic performance advantage of the hybrid approach over the standalone algorithms is observed when normalized by computational effort (Table 3).

As the excitation intensity increases from the stable to the strongly excited regime, the performance index rises accordingly, reflecting the increased control energy required to counteract stronger parametric modulation and near-resonant effects. Despite the time-periodic variation of system parameters, the optimized PID controllers successfully prevent excessive amplitude growth by introducing sufficient effective damping.

Overall, these results confirm that classical population-based metaheuristics such as PSO and GWO offer stable, repeatable, and computationally efficient PID tuning for parametrically excited linear systems. In this context, adequate population diversity and search-space coverage play a more decisive role than algorithmic hybridization, particularly under fixed computational budgets.

Physical and Practical Significance. The Mathieu equation serves as a canonical model for parametric instability in a wide range of engineering applications, including vibrating structures, rotating machinery, and periodically excited electrical and electromechanical systems. Although linear in structure, its time-periodic coefficients can induce instability and resonance phenomena that pose nontrivial challenges for feedback control.

From a control-design perspective, the Mathieu oscillator provides a demanding benchmark for evaluating optimization-based PID tuning methods under explicitly time-varying conditions. The results obtained in this study show that metaheuristically tuned PID controllers are capable of regulating the system response across different excitation regimes, including near-resonant and strongly modulated cases, using a common controller structure and performance criterion.

Practical Interpretation of Steady-State Oscillations. The presence of small, regular oscillations in the steady-state response should be interpreted as an inherent consequence of the system's periodic excitation rather than as a limitation of the controller. According to the Internal Model Principle [10], exact rejection of harmonic disturbances would require the controller to embed an internal model of the excitation frequency, which is not the case for classical PID control.

Consequently, the observed residual oscillations reflect a steady energy balance between periodic excitation and feedback dissipation. From an engineering standpoint, such bounded and predictable oscillatory behavior is often preferable to aggressive tuning aimed at complete cancellation, which may lead to excessive control effort, actuator saturation, or undesirable transient amplification.

The controlled responses obtained in this study therefore represent a practically relevant compromise between stabilization, regulation accuracy, and control effort. Low-amplitude steady-state oscillations can be viewed as a characteristic outcome of effective regulation in parametrically excited systems, indicating that the feedback loop limits response growth without excessive compensation.

Remark on stochastic variability of optimization results. Repeated optimization runs occasionally converged to different PID gain vectors while yielding nearly identical values of the performance index. Such variability is an inherent characteristic of stochastic metaheuristic algorithms and arises from random population initialization, probabilistic update rules, and finite sampling of the search space.

From a statistical perspective, this behavior reflects the presence of multiple near-optimal solutions forming a shallow basin or plateau in the objective landscape rather than a single isolated global minimum. For low-dimensional PID tuning problems such as the Mathieu system, different gain combinations may therefore induce comparable closed-loop dynamics and nearly indistinguishable performance metrics. Variations in the optimized parameters should thus be interpreted as a consequence of solution non-uniqueness in the gain space rather than as an indication of numerical or algorithmic issues.

To ensure a fair and reproducible evaluation, all algorithms were assessed over multiple independent runs, and conclusions were drawn based on the consistency of the achieved performance values rather than on individual realizations. This statistical treatment follows established practice in metaheuristic optimization, where performance distributions and repeatability across runs provide a more informative assessment than single-run results.

To place the metaheuristic tuning results into a broader control-design context, it is instructive to compare them against a simple deterministic baseline derived from a nominal plant model. Such a baseline reflects what may be obtained using classical design procedures when time-periodic effects are neglected at the controller design stage. Although not expected to perform optimally for the full parametrically excited dynamics, it provides a useful reference for interpreting the benefits and limitations of optimization-based PID tuning.

3.5. Deterministic Baseline Tuning for a Nominal (Time-Averaged) Model

For parametrically excited Mathieu-type dynamics, the plant is linear but explicitly time-varying (LTV) through the term $1 + m\cos(\Omega t)$. Consequently, standard PID tuning rules derived for time-invariant (LTI) models (e.g., Ziegler–Nichols) are not expected to be uniformly reliable across the full parametric excitation cycle, especially in regimes close to parametric resonance. Nevertheless, a deterministic baseline is useful as a practical reference representing what an engineer might obtain from a simple nominal model when metaheuristic optimization is not used.

3.5.1. Nominal (Time-Averaged) Baseline Model

We construct a nominal deterministic baseline by time-averaging the parametric stiffness modulation, $\langle\cos(\Omega t)\rangle = 0$, where $\langle\cdot\rangle$ denotes averaging over one excitation period, which yields the LTI second-order approximation

$$y''(t) + 2\zeta\omega_n y'(t) + \omega_n^2 y(t) = u(t) + d(t), \qquad t \geq 0 \tag{14}$$

where $d(t)$ denotes the external forcing/disturbance used in our simulations. This approximation preserves the same (ζ, ω_n) as the original system while removing the explicit time dependence. The resulting controller is therefore deterministic and computationally inexpensive, but it is not expected to be optimal for the full LTV Mathieu dynamics. Its role is strictly to provide a classical benchmark.

3.5.2. Deterministic PID Gains via Coefficient Matching

For unity-feedback tracking with a PID controller

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau + K_d \dot{e}(t), \qquad e(t) = r(t) - y(t), \tag{15}$$

and constant reference $r(t) = 1$ for $t \geq 0$, we use $\dot{e}(t) = -\dot{y}(t)$. For the nominal model (14), the closed-loop characteristic polynomial can be written as

$$s^3 + (2\zeta\omega_n + K_d)\,s^2 + \left(\omega_n^2 + K_p\right)s + K_i = 0. \tag{16}$$

We choose a standard third-order target polynomial as a product of a dominant second-order factor and an additional real pole,

$$(s+\alpha)\left(s^2 + 2\zeta_d\omega_d s + \omega_d^2\right) = s^3 + (2\zeta_d\omega_d + \alpha)s^2 + (\omega_d^2 + 2\alpha\zeta_d\omega_d)s + \alpha\omega_d^2, \tag{17}$$

and match coefficients between (16) and (17), yielding the deterministic gains

$$K_d = (2\zeta_d\omega_d + \alpha) - 2\zeta\omega_n, \tag{18}$$

$$K_p = (\omega_d^2 + 2\alpha\zeta_d\omega_d) - \omega_n^2, \tag{19}$$

$$K_i = \alpha\omega_d^2. \tag{20}$$

In the deterministic baseline design, the desired closed-loop dynamics are specified by fixed design constants $(\zeta_d, \omega_d, \alpha)$, which are selected identically for all regimes and are not tuned on a case-by-case basis. These constants are chosen as problem-independent values intended to yield a reasonable and well-behaved nominal closed-loop response rather than to optimize performance for any particular excitation regime.

The damping ratio is set to $\zeta_d = 0.7$, corresponding to a moderately damped dominant second-order response that limits overshoot while maintaining acceptable transient speed. The desired natural frequency is selected as $\omega_d = 2.0$, which results in a nominal closed-loop bandwidth higher than the plant natural frequency $\omega_n = 1.0$ and reflects a typical choice in classical PID design to improve tracking responsiveness.

The additional real pole is placed at $\alpha = 3\omega_d = 6.0$, sufficiently far from the dominant complex-conjugate pair so that it does not significantly influence the dominant transient dynamics. This choice ensures a clear separation of time scales between the dominant response and the higher-order dynamics introduced by the integral action. All design constants are selected once and kept fixed across all excitation regimes, yielding a fully deterministic baseline controller that serves as a consistent reference rather than a tuned competitor to the metaheuristic designs.

The corresponding PID gains (K_p, K_i, K_d) are computed by direct coefficient matching between the nominal closed-loop characteristic polynomial and the chosen third-order target polynomial.

The obtained gains are then applied without modification to all three considered regimes of the parametrically excited Mathieu system, and the resulting closed-loop responses are evaluated using the full time-varying dynamics. To ensure a fair and numerically comparable benchmark with the metaheuristic-based tuning approaches, the gains are clipped, if necessary, to the same admissible bounds used in the optimization-based design, namely $K_p, K_i, K_d \in [0, 15]$. This clipping serves only to prevent unrealistically large controller gains and is not used to improve the baseline performance.

The corresponding baseline responses for Regimes I–III are illustrated in Figure 8. When using the deterministic baseline controller derived from the nominal time-averaged LTI model via coefficient matching, the closed-loop responses do not achieve performance comparable to the metaheuristic-tuned controllers. In particular, the baseline exhibits a pronounced overshoot at the beginning of the regulation process, and the resulting steady-state oscillations attain higher amplitudes than those observed for the metaheuristic-based designs.

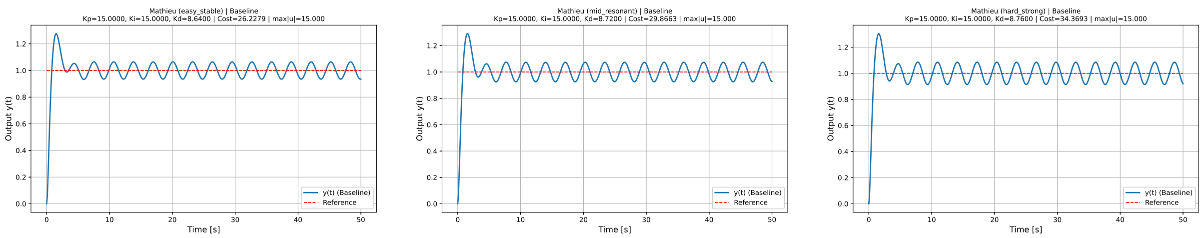

Figure 8. Closed-loop responses obtained with the deterministic baseline PID controller for Regimes I–III of the parametrically excited Mathieu system. Although the baseline controller yields a performance cost approximately twice that of the metaheuristic-tuned controllers, this increase reflects the use of a nominal time-averaged design rather than optimization on the full time-varying dynamics. At the same time, the design-time computational cost of the baseline is negligible and remains fractional compared to the optimization-based approaches, as also summarized in Table 3.

3.5.3. Comparison against Metaheuristic Tuning and Practical Constraints

The deterministic baseline controller is evaluated under the same simulation horizon, external forcing, and performance index as the metaheuristic-based tuning approaches, ensuring a consistent basis for comparison. In addition to the achieved cost value, we report the peak control effort $\max_t |u(t)|$ as an indicator of control aggressiveness.

To reflect practical actuator limitations, both the baseline and the metaheuristic-tuned controllers are further evaluated under control-input saturation, $u(t) \in [-u_{\max}, u_{\max}]$, using the same saturation level across all methods. The saturated responses are not used to recompute the performance index, but serve as a post-design test to assess how the nominally designed controllers behave when actuator constraints are imposed. The resulting saturated closed-loop responses are illustrated in Figure 9 for the PSO-tuned controller and in Figure 10 for the deterministic baseline.

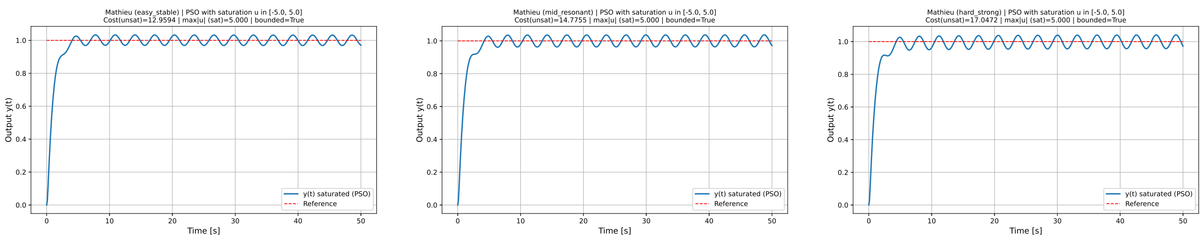

Figure 9. Closed-loop responses obtained with the PSO-tuned PID controller for Regimes I–III of the parametrically excited Mathieu system under control-input saturation. A bounded response (bounded = True) indicates that the output and control input remain finite over the entire simulation horizon and does not imply a formal stability guarantee. The reported value Cost(unsat) corresponds to the nominal performance index computed without input saturation and is included only to identify the controller. For the PSO-based design, the saturated and unsaturated responses are practically indistinguishable across all three regimes, indicating that the imposed saturation level does not significantly affect the closed-loop behavior in this case.

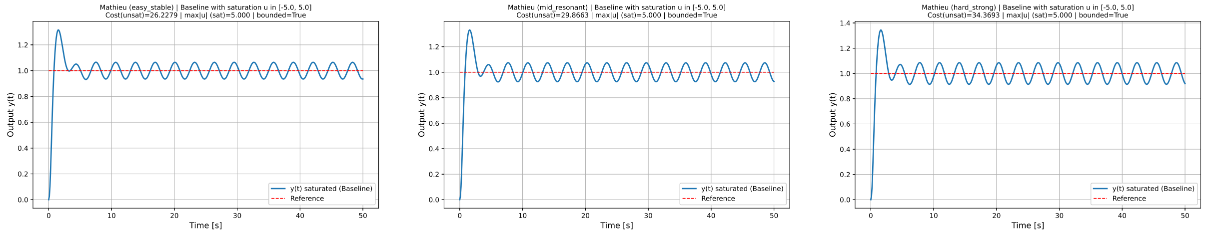

Figure 10. Closed-loop responses obtained with the deterministic baseline PID controller for Regimes I–III of the parametrically excited Mathieu system. The baseline gains are derived from a nominal time-averaged LTI model and evaluated on the full time-varying dynamics. A bounded response (bounded = True) indicates that the output and control input remain finite over the entire simulation horizon and does not imply a formal stability guarantee. The reported value Cost(unsat) corresponds to the nominal performance index computed without input saturation and is included only to identify the controller. For the baseline controller, the saturated and unsaturated output responses are practically indistinguishable across all three regimes, indicating that the imposed input saturation does not significantly alter the closed-loop trajectories. Despite this, the baseline exhibits a pronounced initial overshoot and higher steady-state oscillation amplitudes compared to the metaheuristic-tuned controllers.

To complement the nominal simulations, we additionally include a compact parameter-variation study around each excitation regime by perturbing selected system parameters (such as ζ, m, and Ω) within a limited range around their nominal values while keeping the controller gains fixed. Across all tested variations, the closed-loop responses remained bounded and the observed peak control effort was consistent with the imposed gain and saturation limits. This analysis is intended to examine the sensitivity of the closed-loop response to moderate parameter variations beyond a single nominal trajectory, without pursuing formal robustness analysis or stability certification.

Overall, the deterministic baseline serves as a computationally inexpensive and fully deterministic reference derived from a nominal model, whereas the metaheuristic methods directly optimize the selected performance index on the full time-varying Mathieu dynamics and can better accommodate challenging operating regimes and nonstandard performance objectives.

4. Conclusions

This paper investigated metaheuristic-based tuning of classical PID controllers for the stabilization and regulation of parametrically excited dynamical systems, with a particular focus on the Mathieu equation as a canonical model of time-periodic instability. By restricting attention to a linear time-varying system, the study isolated the influence of the optimization strategy on closed-loop stability and performance without confounding nonlinear effects.

A comparative assessment of Particle Swarm Optimization (PSO), Grey Wolf Optimizer (GWO), and a hybrid PSO–GWO strategy demonstrated that all three metaheuristic approaches are capable of delivering stable and bounded closed-loop responses across a range of parametric excitation regimes. For the low-dimensional PID tuning problem considered here, standalone metaheuristics such as PSO and GWO converged reliably and achieved performance comparable to that of the hybrid scheme. No consistent advantage of algorithmic hybridization was observed when the computational budget was held constant.

To provide a practical reference, the metaheuristic designs were compared against a deterministic baseline PID controller derived from a nominal LTI approximation of the Mathieu system. While the baseline offers a computationally inexpensive, one-shot design, its closed-loop responses exhibited noticeably inferior performance when evaluated on the full time-varying dynamics. In particular, a pronounced initial overshoot was observed, and the resulting steady-state oscillations attained higher amplitudes compared to those obtained with metaheuristic-tuned controllers. These differences directly translate into higher values of the selected performance index.

Practical actuator constraints were further examined through control-input saturation tests. For the considered saturation level, both the deterministic baseline and the metaheuristic-tuned controllers exhibited closed-loop responses that were practically indistinguishable from the unsaturated case. Although short-lived peaks in the control input may occur in the absence of saturation, their limitation did not lead to noticeable changes in the output trajectories or performance metrics. This observation indicates that, for the examined scenarios, the imposed saturation level does not play a dominant role in shaping the closed-loop behavior, while post-design saturation tests remain a useful consistency check in practical control assessments.

Overall, the findings indicate that the effectiveness of PID tuning for parametrically excited systems is governed primarily by search-space coverage, population diversity, and sufficient iteration budgets, rather than by algorithmic complexity. From a practical perspective, well-configured standalone metaheuristic methods provide a favorable balance between achievable performance and computational effort, particularly in challenging regimes where nominal deterministic designs struggle to adequately suppress overshoot and sustained oscillations.

Future Work

Future research will extend the present framework toward adaptive and frequency-aware control strategies capable of explicitly addressing time-periodic excitation. Potential directions include gain-scheduled and adaptive PID schemes, fractional-order controllers, and hybrid approaches that combine offline metaheuristic tuning with online adaptation or predictive control. Such developments aim to further improve regulation accuracy and transient behavior in systems operating under uncertainty and strong parametric modulation.

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No data was used for the research described in the paper.

Conflicts of Interest

The author declares no conflict of interest.

Use of AI and AI-assisted Technologies

No AI tools were utilized for this paper.

Appendix A. Metaheuristic PID Tuning Procedure for the Mathieu System

Algorithm A1: Pseudocode of the metaheuristic PID tuning framework applied to the Mathieu oscillator. The cost function combines the integral time-weighted absolute error (ITAE) and integral of squared overshoot (ISO) with weights $\alpha = 0.5$ and $\beta = 1.0$. Each candidate PID triple is evaluated by numerical simulation of the time-varying Mathieu dynamics, and the best gains (K_p^*, K_i^*, K_d^*) are obtained after E population-based search epochs.

Input: Population size N_p, number of epochs E,
PID bounds $\mathcal{B} = [K_p, K_i, K_d]$,
simulation horizon T, grid size N,
cost weights α, β in (10)
Output: Optimal PID gains (K_p^*, K_i^*, K_d^*)
Initialize random seed for reproducibility;
`/* --- Define cost function --- */`
Function `Cost` (K_p, K_i, K_d)
    Simulate PID-controlled Mathieu system over $[0, T]$ using solve_ivp;
    Compute tracking error $e(t) = 1 - y(t)$;
    Evaluate performance index;
        $ITAE = \int_0^T t\,|e(t)|\,dt$
        $ISO = \int_0^T \max(0, -e(t))^2\,dt$;
        $J = \alpha\, ITAE + \beta\, ISO$;
    return J;

Algorithm A1: *Cont.*

/* --- Optimization loop (PSO/GWO/Hybrid) --- */

Initialize population of candidate PID vectors inside $\mathcal{B}$;

Evaluate J for all candidates;

for $epoch = 1$ **to** E **do**

  Update candidate population using selected metaheuristic:

    PSO: velocity and position update;

    GWO: leader hierarchy and hunting update;

    Hybrid PSO–GWO: cooperative burst updates + refinement;

  Enforce bounds and evaluate new costs J;

  Store best solution so far;

end

Return $(K_p^*, K_i^*, K_d^*) = \arg\min J$;

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