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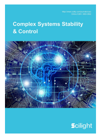

Article

Asymptotic Stability Analysis for Partial Differential Equations Using the Comparison Principle

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How To Cite: Ante, J.E.; Akpan, U.D.; Udofia, E.; et al. Asymptotic Stability Analysis for Partial Differential Equations Using the Comparison Principle *Complex Systems Stability & Control* **2026**, *2*(1), 1. https://doi.org/10.53941/cssc.2026.100001

Received: 31 December 2025
Revised: 10 February 2026
Accepted: 11 February 2026
Published: 9 March 2026

Abstract: This paper assesses the stability results for partial differential equations by means of the comparison principle. By using the python software, numerical example is given to illustrates the rapid convergence of the system solution which goes a long way to show the effectiveness of the adopted approach.

Keywords: stability; partial differential equations; comparison principle; Lyapunov functions

2020 MSC Classification: Primary: 34A08; Secondary: 34A34; 34D20; 34N05

1. Introduction

The theory of partial differential equations (PDEs), closely related to that of ordinary differential equations (ODEs), is a fundamental area of mathematical analysis (see [1]). In particular, PDEs with variable coefficients have been extensively studied due to their relevance in both theory and applications. Significant contributions include existence results for systems of partial derivatives [2], stability analyses for impulsive evolution equations [3], and foundational work on the stability of PDEs (see [4]). Moreover, existence results for Darboux problems associated with partial functional equations with infinite delay in Banach spaces were established in [5]. The broad applicability of PDEs across scientific disciplines is well recognized. For a more comprehensive treatments on the subject, (see [6,7]).

In [8], stability results for systems of nonlinear differential equations with piecewise (constant) arguments were established using the comparison principle. The author further argued that the piecewise (constant) argument can exert a stabilizing influence in certain cases where the underlying systems are unstable. [9] proposed a method for deriving stability results for a wide range of linear PDEs in one spatial dimension. The approach leveraged the Lyapunov functional method in conjunction with the comparison principle to establish exponential stability of the systems under consideration, while [10] derived fundamental stability results for certain nonlinear parabolic equations using maximum principles to obtain solution bounds through comparison techniques. Further stability results covering variety of areas in mathematics, utilizing the comparison arguments were established in [11].

A broad class of numerical and semi-analytical methods has been developed for the solution of PDEs, including the homotopy method, the variational iteration method (VIM), the Adomian decomposition method (ADM), the finite element method (FEM), and Laplace transform–based techniques. Among these, VIM—introduced by Ji-Huan He—has been successfully applied to a wide range of nonlinear ODEs and PDEs (see [12,13]). Subsequent extensions to nonlinear fractional-order systems were proposed in [14,15], and numerous studies have demonstrated the rapid convergence of VIM approximations (see [1,12,16–18]).

ADM has likewise been applied to linear and nonlinear PDEs, offering advantages such as the avoidance of discretization and perturbation [19]. However, its reliance on restrictive analytical assumptions may limit the representativeness of the resulting solutions [20]. Comparative investigations indicate that VIM frequently outperforms ADM in terms of convergence speed [14,15,21–23].

Motivated by these developments, this paper examines the asymptotic stability and convergence of the trivial solution for a class of PDEs via the comparison principle. Suffice to say that, inspite of its relevance, the approach adopted in this paper has not been extensively explored. Hence, this paper seeks to close this gap. The remainder of this paper is organized as follows: Section 2 presents preliminary results; Section 3 establishes the main stability theorems; Section 4 provides numerical simulations; and Section 5 concludes the study.

2. Basic Notations and Definitions

Let $\alpha, \beta \in [J, R]$, and suppose that $\alpha(t) < \beta(t), t \in J$. Assume $\alpha'(t), \beta'(t)$ exists and are continuous on J. For $t_0 \in J$, we define the following sets:

$$E = \{(t, x) : t_0 \leq t < \infty,\ \alpha(t) \leq x \leq \beta(t)\},$$

$$E_0 = \{(t, x) : t_0 < t < \infty,\ \alpha(t) \leq x \leq \beta(t)\},$$

$$\partial E_1 = \{(t, x) : t = t_0,\ \alpha(t_0) \leq x \leq \beta(t)\},$$

$$\partial E_2 = \{(t, x) : t_0 \leq < \infty,\ x = \alpha(t)\}.$$

$$\partial E_3 = \{(t, x) : t_0 < t < \infty,\ x = \beta(t)\},$$

where,

- E (the main set): Points (t, x) where t is in $[t_0, \infty)$ and x is between $\alpha(t)$ and $\beta(t)$ (inclusive). Think of E as a region in the $t - x$ plane bounded by $\alpha(t)$ and $\beta(t)$, starting from t_0 and going to infinity.
- E_0 (E interior): Same as E, but $t > t_0$ (excludes $t = t_0$).
- ∂E_1 (boundary part 1): Points at $t = t_0$, with x between $\alpha(t_0)$ and $\beta(t_0)$. This is like the "left edge" of E.
- ∂E_2 (boundary part 2): Points where $x = \alpha(t)$, for $t \geq t_0$. This is the "lower curve" of E.
- ∂E_3 (boundary part 3): Points where $x = \beta(t)$, for $t > t_0$. This is the "upper curve" of E.

Let us consider a partial differential equation

$$U_t = f(t, x, u, u_x^i), i = 1, 2, ... \tag{1}$$

where $f \in C[E \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^N]$.

Theorem 1. *[Comparison Result] Suppose that $u \in C[\Omega, R]$, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ exist and are continuous on Ω, where*

$$\Omega = [(t, x) : t_0 \leq t < t_0 + a,\ |x| \leq \beta - M(t - t_0)],$$

and $\beta > M_a$. Let

$$\left|\frac{\partial u}{\partial t}\right| \leq M \left|\frac{\partial u}{\partial x}\right| + g(t, |u|) \text{ on } \Omega. \tag{2}$$

where

$$g \in C[[t_0, t_0 + a) \times \mathbb{R}_+, \mathbb{R}_+] \quad \text{and} \quad g(t, 0) \equiv 0.$$

Assume that $r(t, t_0, y_0)$ is the maximal solution of the comparison system,

$$y' = g(t, y), \quad y(t_0) = y_0 \geq 0, \tag{3}$$

existing on $t_0 \leq t < t_0 + a$.

Then,

$$|u(t_0, x)| \leq y_0 \quad \text{for} \quad |x| \leq \beta \tag{4}$$

implies that

$$|u(t, x)| \leq r(t, t_0, y_0) \quad \text{on} \quad \Omega. \tag{5}$$

Proof. Consider the function,

$$m(t) = \max_{|x| \leq \beta - M(t - t_0)} |u(t, x),$$

and assume that

$$r(t, t_0, y_0) < m(t).$$

This means that $t > t_0$ and $m(t) > 0$. Moreover, $m(t) = u(t, x_0)$.

The point (t, x_0) may be an interior point of Ω or

$$x_0 = \pm[\beta - M(t - t_0)].$$

Suppose that $(t, x_0) \in \text{ int } \Omega$. Then, $\frac{\partial u(t,x_0)}{\partial x} = 0$, and as a consequence, by (2) and Lemmas 9.2.1 and 9.2.2 in [7], we get

$$D_{-}m(t) \leq \left|\frac{\partial u}{\partial t}\right| \leq g(t, m(t)).$$

Suppose now that $x_0 = \beta - m(t - t_0)$. Then there are two possibilities:

(a) $m(t) = u(t, \beta - M(t - t_0))$,

(b) $m(t) = -u(t, \beta - M(t - t_0))$.

Consider the case (a). If x is sufficiently close to $x_0 = \beta - M(t - t_0)$, $u(t, x) \leq u(t_0, x_0)$, and therefore, we get

$$\frac{\partial u(t, \beta - M(t - t_0))}{\partial x} = 0.$$

As a result, we obtain the inequality

$$\frac{\partial u}{\partial t} - M\frac{\partial u}{\partial x} \leq g(t, m(t)), \tag{6}$$

at (t, x_0). Furthermore, $u(s, \beta - M(s - t_0)) < M(s)$, $s < t$, and

$$|u(t, \beta - M(t - t_0))| = m(t).$$

Hence, using (6), we deduce using condition (a) that

$$D_{-}m(t) \leq |u(t, \beta - M(t - t_0)),$$

$$\begin{aligned} &\frac{\partial u}{\partial t} - M\frac{\partial u}{\partial x} \\ &\leq \left|\frac{\partial u}{\partial t}\right| - M\frac{\partial u}{\partial x}, \\ &\leq g(t, m(t)). \end{aligned}$$

Suppose that the possibility (b) holds.

Then $m(t) = \max\limits_{|x| \leq \beta - M(t-t_0)} |u(t, x)|$ will imply,

$$-u(t, \beta - M(t - t_0)) \max |u(t, x)|.$$

Hence,

$$-u(t, x) \leq -u(t, \beta - M(t - t_0)),$$

and thus,

$$\frac{\partial u(t, x)}{\partial x} \leq 0,$$

or

$$\frac{\partial u(t, \beta - M(t - t_0))}{\partial x} \leq 0.$$

Since $\left|\frac{\partial u}{\partial t}\right| \leq M\left|\frac{\partial u}{\partial x}\right| + g(t, m(t))$, we get

$$\left|\frac{\partial u}{\partial t}\right| + M\frac{\partial u}{\partial x} \leq g(t, m(t)) \tag{7}$$

at (t, x_0). On the other hand,

$$-u(s, \beta - m(s - t_0)) < m(s), \quad s < t,$$

and

$$-u(t, \beta - m(t - t_0)) = m(t),$$

which implies that

$$D_{-}m(t) \leq -\frac{\partial u}{\partial t} + M\frac{\partial u}{\partial t} \tag{8}$$

at (t, x_0). By the relation (7) and (8) and the fact that

$$-\frac{\partial u}{\partial t} + M\frac{\partial u}{\partial x} \leq \left|\frac{\partial u}{\partial t}\right| + M\frac{\partial u}{\partial x}$$

results in the inequality

$$D_{-}m(t) \leq g(t, m(t)).$$

The proof for the case $x_0 = -[\beta - M(t - t_0)]$ is similar.

Thus, we have shown that the inequality $r(t, t_0, y_0) \leq m(t)$ implies that

$$D_{-}m(t) \leq g(t, m(t)).$$

This yields that $m(t) \leq r((t, t_0, y_0))$ by Theorem 1.4.1 in [7], contradicting our assertion.

Hence, (5) is true and the theorem is proved. □

3. Lyapunov-Like Functions

Let us consider a first order partial differential system of the form,

$$u_t = f(t, x, u, u_x^i), i = 1, 2, ... \tag{9}$$

where $f \in C[E \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^N]$.

We wish to estimate the growth of solutions of (9) by means of a Lyapunov-like function. Thus we have,

Theorem 2. *Assume that*

(i) $V \in C[J \times \mathbb{R}^N, \mathbb{R}_+]$, $V(t, u)$ *possesses continuous partial derivatives with respect to* t *and the components of* u*, and*

$$\frac{\partial V(t, u)}{\partial t} + \frac{\partial V(t, u)}{\partial u} \cdot f(t, x, u, u_x^i) \leq G(t, x, V(t, x), V_x(t, x)),$$

where

$$G \in C[E \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}] \text{ and } V_x(t, u) = \left(\frac{\partial V}{\partial u}\right) \cdot \left(\frac{\partial u}{\partial u}\right);$$

(ii) $G(t, x, z, p) - G(t, x, z, 0) \geq -\alpha'(t)p,\ p \geq 0$ *on* ∂E_2;

(iii) $G(t, x, z, p) - G(t, x, z, 0) \leq -\beta'(t)p,\ p \geq 0$ *on* ∂E_3;

(iv) $G(t, x, z, 0) \leq g(t, z),\ z > 0$ *where* $g \in C[J \times \mathbb{R}_+, \mathbb{R}]$;

(v) *the maximal solution* $r(t, t_0, xy)$ *of* (2) *exists for* $t \geq t_0$.

Then any solution $u(t, x)$ *of* (1) *satisfying*

$$V(t_0, \phi(x)) \leq y_0 \text{ on } \partial E_1, \tag{10}$$

allows the estimate

$$V(t, u(t, x)) \leq r(t, t_0, y_0) \text{ on } E. \tag{11}$$

Proof. Let $u(t, x)$ be any solution of (1) such that (10) holds.

Consider the function,

$$m(t, x) = V(t, u(t, x)).$$

By assumption of (i) we have

$$\frac{\partial M(t, x)}{\partial t} = \frac{\partial V(t, u(t, x))}{\partial t} + \frac{\partial V(t, u(t, x))}{\partial t} \cdot f(t, x, u(t, x), u_x^i(t, x)) \leq G(t, x, m(t, x), m_x(t, x)),$$

and

$$m(t_0, x) \leq y_0 \text{ on } \partial E_1.$$

It is evident that the hypothesis of Theorem 9.2.3 in [7] are fulfilled, and as a consequence,

$$v(t, u(t, x)) = m(t, x) \leq r(t, t_0, y_0) \text{ on } E,$$

where $r(t, t_0, y_0)$ is the maximal solution of the system (3). This completes the proof. □

Let us assume the existence of solution of (1).
Suppose also that the system (1) has a trivial solution $u = 0$.
We proceed now to formulate the definition of stability of the trivial solution of (1).

Definition 1. *The trivial solution $u \equiv 0$ of (1) is said to be stable if for every $\epsilon > 0$ and $t_0 \in J$, there exists a $\delta > 0$ such that $\|\phi(x)\| < 0$ of ∂E_1, implies*

$$\|u(t, x)\| < \epsilon \text{ on } E_1,$$

where $u(t, x)$ is any solution of (1) with $u(t_0, x) = \phi(x)$ on ∂E_1.

Definition 2. *The trivial solution $u \equiv 0$ of (1) is said to be asymptotically stable if it is stable and for every $\epsilon > 0$, $t_0 \in J$, there exists a positive number δ_0 and T such that $\|\phi(x)\| < \delta_0$ on ∂E_1 implies $\|u(t, x)\| < \epsilon$, $t \geq t_0$.*

$$\alpha(t) \leq x \leq \beta(t)$$

Theorem 3. *[Stability] Let the assumptions of Theorem 2 hold. Suppose also that*

$$b(\|u\|) \leq V(t, u) \leq a(\|u\|), \tag{12}$$

where $a, b \in K$.

Then, the stability of the trivial solution $y \equiv 0$ of the system (3) implies the stability of the trivial solution of the system (1).

Proof. Suppose that the trivial solution of (1) is stable.
Let $\epsilon > 0$ and $t_0 \in J$.
Then given $b(\epsilon) > 0$ and $t_0 \in J$, there exists a $\delta > 0$ such that $y_0 \leq \delta$ implies

$$y(t, t_0, y_0) < \epsilon, \ t \geq t_0, \tag{13}$$

where $y(t, t_0, y_0)$ is any solution of (3).
By Theorem 2,

$$V(t, u(t, x)) \leq r(t, t_0, y_0) \text{ on } E, \tag{14}$$

for any solution $u(t, x)$ of (1), where $r(t, t_0, y_0)$ is the maximal solution of (3).
Choose a positive number δ_1 such that $a(\delta_1) = \delta$ and assume that $\|\phi(x)\| \leq \delta_1$.
This implies that,

$$V(t_0, \phi(x)) \leq \|\phi(x)\| \leq a(\delta_1) = \delta.$$

Choose $y_0 = \sup_{x \in \partial E_1} V(t_0, \phi(x))$. It then follows by the relations (12), (13) and (14) that

$$b(\|u(t, u)) \leq V(t, u(t, x)) \leq r(t, t_0, y_0) < b(\epsilon) \text{ on } E,$$

which leads to further inequality
$\|u(t, u)\| < \epsilon$ on E, provided $\|\phi(x)\| \leq \delta_1$.
This proves the stability of the trivial solution of (1). □

Now, suppose the trivial solution of the comparison system (3) is asymptotically stable.

Theorem 4. *Let the assumptions of Theorem 3 hold together with that of Theorem 2. Then, the asymptotic stability of the trivial solution of (3) implies the asymptotic stability of the trivial solution of (1).*

Proof. Let $\epsilon > 0$. Then, given $b(\epsilon) > 0$ and $t_0 \in J$, there exist two positive numbers δ_0 and T such that $y_0 \leq \delta_0$ implies $y(t, t_0, y_0) < b(\epsilon)$, $t \geq t_0 + T$.

Let us choose

$$y_0 = \sup_{x \in \partial E_1} V(t_0, \phi(x)).$$

Furthermore, let $a(\delta_0) = \delta$ and assume that $\|\phi(x)\| \leq \delta_0$.

These considerations show that, as previously,

$$b(\|u(t, x)\|) \leq V(t, u(t, x)) \leq r(t, t_0, y_0) < b(\epsilon),$$

for $t \geq t_0 + T$ and $\alpha(t) \leq x \leq \beta(t)$.

From this follows the inequality

$\|u(t, x)\| < \epsilon$, $t \geq t_0 + T$, $\alpha(t) \leq x \leq \beta(t)$, provided $\|\phi(x)\| \leq \delta_0$.

Hence, the system (1) is asymptotically stable in the sense of Lyapunov. □

4. Application

4.1. Application 1

Let us consider a one dimensional heat (diffusion) equation of the form

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < L,\ t > 0 \tag{15}$$

where $u(x, t)$ is the state variable (e.g., temperature, concentration) and $\alpha > 0$ is the diffusion coefficient. with initial conditions,

$$u(x, 0) = u_0(x) \quad 0 \leq x \leq L,$$

and boundary conditions (Dirichlet)

$$u(x, t) = 0 \quad u(L, t) = 0\ t \geq 0.$$

Define the Lyapunov functional

$$V(t) = \int_L^0 u^2(x, t)dx.$$

Differentiating with respect to t and applying integration by parts yields,

$$\frac{dV}{dt} = -2\alpha \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \leq 0.$$

Hence, we conclude that (15) is Lyapunov stable and infact, the solution is asymptotically stable, that is, $\lim_{t\to\infty} u(x, t) = 0$.

Using separation of variables, the solution admits the Fourier expansion

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\alpha \frac{n^2\pi^2}{L^2} t\right).$$

Each mode decays exponentially proving uniform convergence in time, exponential decay rate and numerical convergence under standard finite difference or finite element schemes.

The initial condition shown in Figure 1a defines a smooth spatial distribution that vanishes at the domain boundaries. This choice is consistent with the imposed boundary constraints and ensures that the solution remains admissible for all subsequent times. The profile excites the fundamental diffusion mode, which dominates the system dynamics in the absence of external forcing.

Figure 1b confirms the enforcement of the Dirichlet boundary conditions throughout the evolution. The solution remains identically zero at both ends of the spatial domain for all times considered, demonstrating numerical and conceptual consistency with the mathematical model.

The solution at an early time, displayed in panel Figure 1c, closely resembles the initial profile, with only a modest reduction in amplitude. This behavior indicates that diffusion has begun to redistribute the state variable while preserving the overall spatial symmetry of the system. At this stage, the influence of higher-order diffusive effects remains limited.

Figure 1d illustrates the solution at an intermediate time. A clear attenuation of the amplitude is observed, reflecting increased thermal dissipation across the domain. Despite this decay, the spatial structure remains unchanged, indicating that the solution evolves primarily through amplitude damping rather than shape deformation.

In Figure 1e, the solution at a later time exhibits further amplitude reduction, approaching a near-uniform zero state. This progression demonstrates the long-time behavior of the system, where diffusion drives the solution toward equilibrium while continuously satisfying the boundary constraints.

Figure 1f isolates the temporal decay at the midpoint of the domain. The monotonic decrease confirms the expected diffusive damping and provides a clear temporal signature of stability. The absence of oscillations or growth indicates that the system is well-posed and stable under the chosen initial and boundary conditions.

The results validate the qualitative behavior of the heat equation model. The solution exhibits smooth spatial evolution, strict boundary adherence, and stable temporal decay, all of which are hallmark features of classical diffusion dynamics.

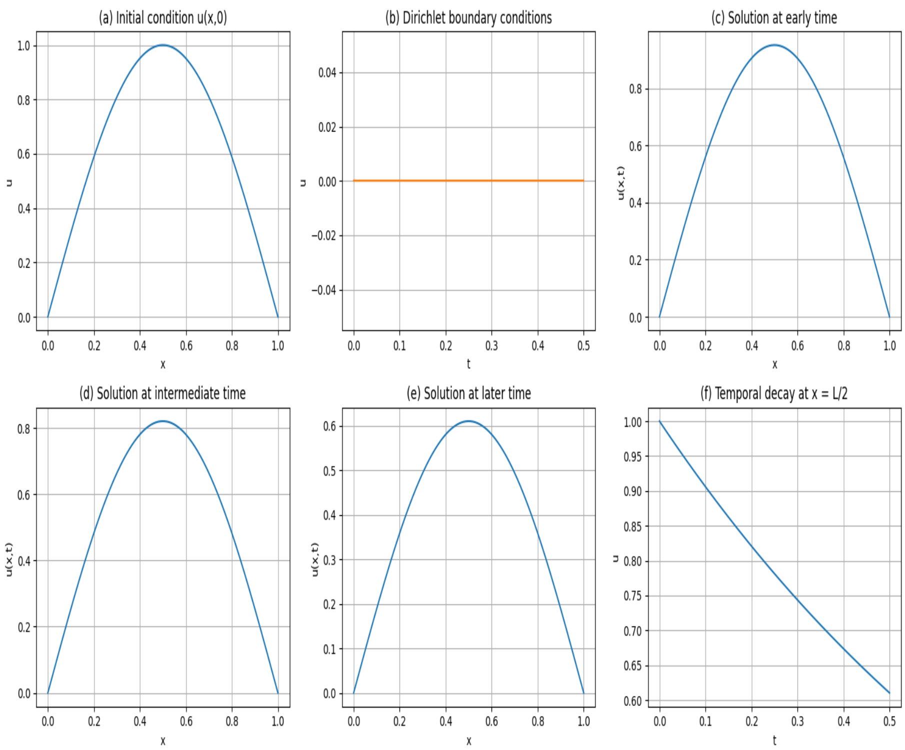

Figure 1. Diffusive evolution and stability of the one-dimensional heat equation solution this illustrates the solution behavior of the one-dimensional heat equation with homogeneous Dirichlet boundary conditions. Panel (**a**) shows the initial spatial distribution. Panel (**b**) confirms the enforced boundary conditions. Panels (**c**–**e**) present the solution at early, intermediate, and later times, respectively, highlighting progressive diffusive decay. Panel (**f**) depicts the temporal evolution at the domain midpoint, illustrating stable monotonic dissipation toward equilibrium.

4.2. Application II

Consider the one-dimensional reaction–diffusion equation

$$u_t(t,x) = Du_{xx}(t,x) - ku(t,x), \quad x \in (0,L),\ t > 0, \tag{16}$$

subject to homogeneous Dirichlet boundary conditions

$$u(t,0) = u(t,L) = 0,$$

where $D > 0$ is the diffusion coefficient and $k > 0$ denotes a damping parameter. Equation (16) is a special case of (1) with

$$f(t,x,u,u_x) = Du_{xx} - ku.$$

The trivial solution $u \equiv 0$ represents the equilibrium state of the system.

Define the Lyapunov functional

$$V(t,u) = \frac{1}{2}\int_0^L u^2(t,x)\,dx. \tag{17}$$

Clearly, $V(t,u)$ satisfies the conditions

$$b(\|u\|_{L^2}) \le V(t,u) \le a(\|u\|_{L^2}), \tag{18}$$

where

$$a(s) = \frac{1}{2}s^2, \quad b(s) = \frac{1}{2}s^2.$$

Differentiating (17) along solutions of (16) yields

$$\begin{aligned}\dot{V}(t) &= \int_0^L u(t,x)u_t(t,x)\,dx \\ &= \int_0^L u(Du_{xx} - ku)\,dx.\end{aligned}$$

Applying integration by parts and using the boundary conditions, we obtain

$$\int_0^L uu_{xx}\,dx = -\int_0^L (u_x)^2\,dx \le 0.$$

Consequently,

$$\dot{V}(t) \le -k\int_0^L u^2(t,x)\,dx = -2kV(t). \tag{19}$$

Consider the scalar comparison system

$$y'(t) = -2ky(t), \quad y(t_0) = V(t_0) \ge 0. \tag{20}$$

The maximal solution of (20) is given explicitly by

$$y(t) = V(t_0)e^{-2k(t-t_0)}, \quad t \ge t_0.$$

It follows that the trivial solution $y \equiv 0$ of (20) is globally asymptotically stable.
From inequality (19) and the comparison principle, we have

$$V(t) \le y(t), \quad t \ge t_0.$$

Since $y(t) \to 0$ as $t \to \infty$, it follows that

$$V(t) \to 0 \quad \text{as } t \to \infty,$$

which implies

$$\|u(t,\cdot)\|_{L^2} \to 0 \quad \text{as } t \to \infty.$$

Therefore, by Theorem 4, the asymptotic stability of the trivial solution of the comparison system (20) guarantees the asymptotic stability of the trivial solution of the partial differential equation (16). This result demonstrates that diffusion combined with linear damping ensures convergence of solutions to the equilibrium state without the need for explicit solutions of the governing partial differential equation.

To visually support the analytical results obtained above, we present the following numerical simulations for the comparison system and the associated Lyapunov functional of system (16).

Figure 2 illustrates the solution of the scalar comparison system

$$y'(t) = -2ky(t), \quad y(0) = y_0 > 0.$$

The solution is explicitly given by

$$y(t) = y_0 e^{-2kt},$$

which decays exponentially to zero as $t \to \infty$. The monotonic decrease of $y(t)$ confirms that the trivial solution

$y \equiv 0$ is asymptotically stable. This behaviour satisfies the hypothesis of the asymptotic comparison theorem and serves as the reference trajectory for bounding the Lyapunov functional of the partial differential equation.

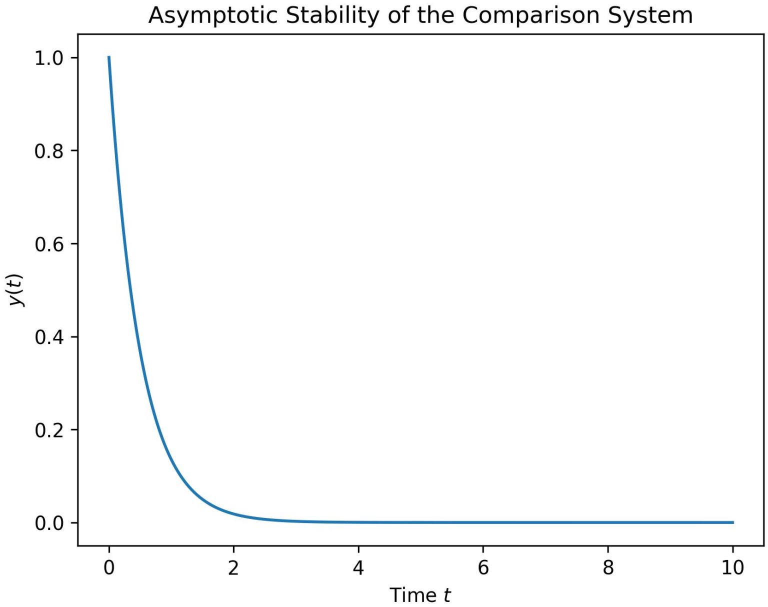

Figure 2. Asymptotic decay of the solution of the comparison system $y'(t) = -2ky(t)$.

Figure 3 depicts the time evolution of the Lyapunov functional

$$V(t) = \frac{1}{2}\int_0^L u^2(t,x)\,dx,$$

associated with the reaction–diffusion equation. The exponential decay of $V(t)$ reflects the dissipation of energy in the system due to the combined effects of diffusion and damping. As $V(t) \to 0$, it follows that

$$\|u(t,\cdot)\|_{L^2} \to 0 \quad \text{as } t \to \infty,$$

which confirms the asymptotic stability of the trivial solution of the partial differential equation.

These plots coincides with the theoretical analysis and clearly demonstrate the applicability of our results.

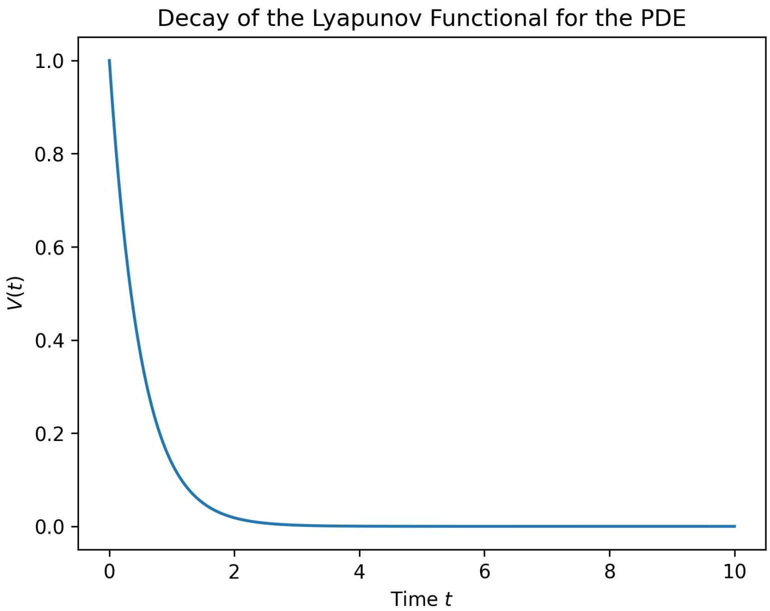

Figure 3. Exponential decay of the Lyapunov functional $V(t)$ for the partial differential equation.

5. Conclusions

This paper examined the asymptotic stability properties of a class of PDEs via the comparison principle, which provides a rigorous analytical framework for characterizing the qualitative behavior of the system. The theoretical results are substantiated by a numerical example implemented in Python, demonstrating rapid convergence of the solution. The strong agreement between analytical predictions and numerical outcomes confirms the effectiveness and practical applicability of the proposed approach. These findings establish the comparison principle as a reliable tool for stability analysis in PDE models.

Author Contributions

All authors contributed equally to the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding

This research received no external funding.

Institutional Review Board Statement

Not applicable.

Informed Consent Statement

Not applicable.

Data Availability Statement

The data that support the findings of this study are available on request from the corresponding author.

Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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