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Review

Cosmological Standard Timers: Framework and Perspectives

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Abstract: Cosmological standard timers (CSTs) are novel cosmological probes that aim to measure the cosmic time-redshift relation, which quantifies the size of the Universe with respect to cosmic time and further constrains cosmic evolution. With an initial statistical distribution of the dynamical systems as a standard reference, we can independently measure cosmic time from the time evolution of this statistical distribution. By combining the measured cosmic time with redshift extracted from observables, we can construct the cosmic time-redshift relation. Following this idea, we develop the framework of CSTs and discuss their perspectives in potential dynamical systems such as dark matter halo mass function, primordial black hole (PBH) mass function and PBH binaries as illustrative examples to demonstrate their feasibility in constructing the cosmic time-redshift relation.

Keywords: cosmological probe; cosmic time-redshift relation; primordial black hole

1. Introduction

With decades of development in cosmological probes, the modern cosmology has been expanded to the precise frontier. Precise measurements on cosmic microwave background (CMB) [1–4], large scale structure (LSS) [5], Type Ia supernovae (SNe) [6,7] have revealed the energy components of the Universe and led to the establishment of the concordance cosmology model, the Λ-cold dark matter (ΛCDM) model [8–11]. The success of the ΛCDM model in describing cosmic evolution encourages us to explore the hidden physics behind these observations.

On top of these achievements, further measurements have uncovered intrinsic inconsistencies among different observations. The measurement of Hubble parameter, which characterizes cosmology at the background level, has a 5σ discrepancy between CMB and Type Ia SNe observations, this is well-known Hubble tension [12–14]. The S_8 parameter that describes the first-order cosmological evolution shows a 2–3σ tension between CMB and LSS probes [4,15]. Furthermore, the underlying assumption of the ΛCDM model, the cosmological principle, is challenged by the measurement of dipole structure in CMB and quasar distributions with a 5σ tension [16,17]. To address these cosmological crises, various theoretical explanations have been proposed [18–27], indicating that the ΛCDM model is not the final story. Moreover, the nature of dark matter (DM) and dark energy (DE) remain elusive. DM properties in the small scale [28] and the dynamical behavior of DE in recently DESI results [29] are waiting for theoretical interpretation.

To reveal the nature of cosmic tensions and dark sectors, precise cosmological probes are crucial. We have developed several cosmological probes to trace cosmic evolution from the local Universe to high redshifts. For example, the absolute magnitude of Type Ia SNe serves as a standard candle to construct the luminosity distance-redshift relation up to redshift $z{\sim}1$ [30,31]; the sound scale of baryon acoustic oscillations (BAO) in LSS gives a standard ruler to determine the angular diameter-redshift relation around $z{\sim}2$–3 [29,32–36]; and gravitational waves (GWs) from binary black holes (BBHs) with their electromagnetic (EM) counterparts can work together as a standard siren to measure the luminosity distance-redshift relation up to redshift $z{\sim}10$ [37–40]. For a deeper understanding of cosmic dynamics, it is essential to develop a novel cosmological probe that can not only place an independent constraint on cosmic evolution but also offer a window into high-redshift Universe.

In this work, we review the framework and perspectives of a novel cosmological probe, cosmological standard timers (CSTs), which uses the time evolution of dynamical systems in the Universe as an independent measurement of cosmic time. By measuring the statistical distribution of such dynamical systems and comparing it with their identical initial distribution, a cosmic time-redshift relation can be determined. This cosmic time-redshift relation provides insight into the size of the Universe with respect to the cosmic time, and place an independent constraint on cosmic evolution. We have studied two dynamical systems in the Universe, primordial black hole (PBH) clustering [41] and PBH binaries [42], studying their the Hawking radiation [43] and GW emission [44] make them potential candidates for CSTs. For a detailed understanding of CSTs, we discuss the perspectives of CSTs in DM halo mass function, PBH mass function and PBH binaries.

This review is organized as follows. In Section 2, we review the current cosmological probes at late Universe, including standard candles, standard rulers, standard sirens and cosmic chronometers. In Section 3, we review the framework of the CST on constructing the cosmic time-redshift relation based on its initial state and dynamics. In Section 4, we focus on potential dynamical systems in the Universe that can be CST candidates and discuss their perspectives. In Section 5, we summarize the framework and perspective of CSTs.

2. Cosmological Probes

To measure the dynamical evolution of the Universe, the basic idea is to quantify its size as a function of cosmic time. The size of the Universe can be described by the scale factor a, however, the scale factor is not a directly measurable quantity in observations. Instead, we measure the cosmic redshift to quantify the relative scale of the Universe. With the expansion of the Universe, the wavelengths of cosmic signals are stretched, leaving a redshift signature in their spectrum. Therefore, extracting the redshift from observables can be used to quantify the scale factor as

$$1 + z = \frac{a(t_0)}{a(t_e)} \tag{1}$$

where $a(t_0)$ is the scale factor at the present Universe that the value is set as $a(t_0) = 1$, and $a(t_e)$ is the scale factor when this signal was emitted at cosmic time t_e. Since there are no CSTs at different redshifts, obtaining cosmic time t_e is not straightforward. To quantify this cosmic time t_e, we can evaluate cosmic time interval Δt between t_e and t_0 instead. During the propagation of signals from the source to the observer, the propagation distance depends on this cosmic time interval Δt. Thus we can use the cosmic distance between the signal source and the observer to infer Δt. Once the cosmic distance-redshift relation is measured, it provides key information about the size of the Universe at a given cosmic time, while simultaneously place a constraint on how the Universe evolves from a scale factor $a(t_e)$ at cosmic time t_e to $a(t_0)$ at the present Universe.

At present, a variety of methods have been well developed and reveal the behavior of the Universe with respect to the redshift, such as standard candles, standard rulers, standard sirens, and cosmic chronometers. In this section, we review how these cosmological probes are used to construct the cosmic distance-redshift and cosmic time-redshift relations.

2.1. Standard Candles

A fundamental observable in cosmic distance measurement is the source luminosity. With the luminosity fixed, the cosmic distance between the source and observer determines the apparent brightness as

$$F = \frac{L}{4\pi d_L^2(z)} \tag{2}$$

where F denotes observed photon flux and L represents the stellar luminosity. The luminosity distance $d_L(z)$ in a flat Friedmann–Lemaître–Robertson–Walker (FLRW) Universe is expressed as,

$$d_L(z) = (1+z)\int_0^z \frac{c}{H(z')}\mathrm{d}z' \tag{3}$$

Here, c represents the speed of light, and $H(z)$ denotes the Hubble expansion rate at redshift z. Under the standard flat ΛCDM paradigm, the Hubble parameter evolves as $H(z) = H_0\sqrt{\Omega_\gamma(1+z)^4 + \Omega_m(1+z)^3 + \Omega_\Lambda}$, where H_0 is the present-day Hubble constant. The density parameters are defined as $\Omega_X \equiv \rho_X/\rho_c$, where the subscripts $X = \gamma, m, \Lambda$ correspond to radiation, matter, and dark energy components, respectively, relative to the critical density ρ_c. Consequently, measurements of $d_L(z)$ constrain the cosmic expansion history from the emission epoch t_e to the present t_0. Furthermore, the redshift z is determined by identifying spectral shifts in the observed light relative to the source's rest-frame spectrum.

When using the luminosity distance to quantify the cosmic evolution, the Hubble parameter $H(z)$ provides a model-dependent constraint on cosmological parameters. For instance, the measured value of Hubble parameter and matter density depends on the assumption of the cosmological models, different cosmological models like ΛCDM or dynamical DE model produce various Hubble parameters and matter density [29]. Luminosity distance can also provides constraints on other cosmological models, such as modified gravity. By imposing $H(z)$ from modified gravity into the luminosity distance, the parameters in modified gravity can be naturally constrained by the measurement of the luminosity distance [45,46].

It is evident that for a source with a fixed intrinsic luminosity, the apparent brightness is governed by its luminosity distance. Proximate stars yield a high observed photon flux and appear bright, whereas distant objects appear faint. In this framework, the luminosity distance is derived by comparing the measured photon flux against the known stellar luminosity, while the redshift is independently determined from the stellar spectrum. This procedure allows for the construction of the luminosity distance-redshift relation, a technique widely known as the standard candles method [47–50].

In astrophysical measurements, the intrinsic luminosity of a celestial object is frequently parameterized by its absolute magnitude. This quantity corresponds to the apparent magnitude the object would exhibit if placed at a standard distance of $10\,\mathrm{pc}$. Consequently, the relationship linking the observed apparent magnitude m and the absolute magnitude M is formulated as,

$$\mu = m - M = 5\log_{10}\frac{d_L(z)}{\mathrm{Mpc}} + 25 \tag{4}$$

where μ represents the distance modulus, defined as the difference between the apparent and absolute magnitudes. This parameter directly quantifies the luminosity distance $d_L(z)$ between the distant source and the observer.

The application of the standard candle method relies on identifying celestial tracers with a constant absolute magnitude. To date, several such indicators have been established, including Cepheid variables [47,48], Type Ia SNe [50,51], and the tip of the red giant branch [52,53]. This section primarily focuses on Cepheid variables as a representative example.

Cepheids are variable stars characterized by radial pulsations, maintaining stable periodicities and amplitudes in both diameter and temperature. The initial observation of a Cepheid is credited to Edward Pigott [54]. Later in 1908, Henrietta Swan Leavitt made the critical breakthrough that established Period-Luminosity relation through her analysis of 1777 Cepheid variables in the Magellanic clouds [47].

Currently, Cepheids are categorized into four distinct classes: Classical (Type I), Type II [55,56], Anomalous [56], and Double-mode Cepheids [57]. Among these, Classical Cepheids are particularly valuable for distance measurement duo to their highly regular pulsation cycles. The relationship between their luminosity and pulsation period is described by [58],

$$M = -2.76(\log_{10}(P/1\,\mathrm{day}) - 1.0) - 4.16 \tag{5}$$

where M denotes the absolute magnitude of the Cepheid and P represents its pulsation period in days.

By observing the pulsation period, Equation (5) determines the absolute magnitude. Combining this with the apparent magnitude allows Equation (4) to yield the luminosity distance d_L. Concurrently, the redshift is derived from the spectral lines of the host galaxies. Consequently, the luminosity distance-redshift relation for Cepheids is established, which is then applied in Equation (3) to constrain the Hubble parameter and cosmological parameters. Historically, Hubble employed this relation in 1929 to demonstrate the expansion of the Universe, initially estimating the Hubble parameter at approximately $500\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ [59]. With the advent of precision measurement, Cepheid variables now offer high-precision distance measurement, resulting in a refined Hubble constant $H_0 = 74.03 \pm 1.42\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ [60].

2.2. Standard Rulers

As the standard candles, imaging we have a standard size on the sky, it could also provide a measurement on the size of the Universe at different cosmic time. For a fixed size r on the sky, we can have an observed angular size θ as

$$\theta = \frac{r}{d_A(z)} \tag{6}$$

where $d_A(z)$ is physical angular diameter distance that also characterize the cosmic distance between the source and the observer. In a flat FLRW cosmology, it can be calculated in following form

$$d_A(z) = \frac{c}{1+z} \int_0^z \frac{1}{H(z')} \mathrm{d}z' \tag{7}$$

There is a clear relation between d_L and d_A in Equations (3) and (7) as $d_L(z) = (1+z)^2 d_A(z)$. Consequently, the angular diameter distance encodes the same cosmic expansion history as the luminosity distance.

In the context of angular size measurements, for an object with a fixed intrinsic linear dimension, the observed angular scales are inversely proportional to the angular diameter distance. Naturally, nearby structures gives a larger measurable angle, whereas distant ones appear smaller. By comparing the observed angular size against the known physical length via Equation (6), one can derive the angular diameter distance. When combined with redshift measurements from spectral analysis, this process yields the angular diameter distance-redshift relation, which serves as a powerful probe for constraining cosmic evolution. This method is referred to as standard rulers.

In practice, employing the standard ruler method requires identifying a candidate with a fixed physical dimension on the sky. While several candidates exist, such as ultra-compact radio [61,62] and double-lobed radio sources [63], the BAO scale [64] serves as the quintessential standard ruler for cosmological distance measurements. Physically, the BAO scale represents the maximum distance sound waves propagated prior to recombination-a length that becomes frozen once photons decouple from the baryon-photon plasma. This characteristic scale leaves a distinct imprint on the distribution of galaxy clusters. By analyzing the two-point correlation function in the LSS, this feature can be extracted at redshifts around $z{\sim}2$–3. Furthermore, the BAO signature is not limited to the LSS but is also encoded in the angular power specturm of the CMB temperature perturbation at redshift $z = 1089$. Consequently, BAO measurements provide a robust probe of cosmic evolution from the early universe to the present.

The theoretical prediction for the BAO scale within the angular power spectrum of the CMB temperature perturbation C_ℓ^{TT}, is formulated as follows,

$$C_\ell^{TT} = \frac{2}{\pi} \int \mathrm{d}k\, k^2 P(k) \Delta_{T\ell}^2(k) \tag{8}$$

where $P(k)$ denotes the scale-invariant primordial power spectrum and $\Delta_{T\ell}(k)$ represents the transfer function calculated via Boltzmann codes. In the context of the CMB anisotropy power spectrum, the BAO peak originates from the baryonic oscillations driven by the interplay between the gravitational pull of dark matter and the counteracting photon pressure. Initially, within overdense regions, intense photon pressure drives the baryons outward at the sound speed. This expansion continues until photon decoupling occurs, at which point the baryonic distribution freezes, imprinting the characteristic BAO scale. Analyses of the CMB angular power specturm constrain the sound horizon at the drag epoch to be [4], $r_d = 147.09 \pm 0.26\,\mathrm{Mpc}$.

In LSS surveys, the BAO scale is derived from the galaxy two-point correlation function as described by [65],

$$\xi(r) = \frac{1}{2\pi^2} \int \mathrm{d}k\, k^2 P(k) \frac{\sin(kr)}{kr} \tag{9}$$

where $P(k)$ represents the power spectrum of the galaxy distribution. The Sloan Digital Sky Survey (SDSS) reported the first detection of the BAO signal in 2005, which manifested as a distinct peak in the galaxy two-point correlation function, yielding a sound horizon of $r_d = 100\ h^{-1}$ Mpc [66]. When combined with the Planck 2018 Hubble parameter, this corresponds to a sound horizon of $r_d = 148.37 \pm 1.10\,\mathrm{Mpc}$.

With the release of subsequent galaxy survey datasets [29,67–70], the precision of BAO measurements has significantly improved. By combining these observations with Big Bang Nucleosynthesis (BBN) constraints, the Hubble parameter is determined with high accuracy, as illustrated in Table 1 [71], thereby establishing a precise standard ruler.

Table 1. The measured Hubble parameter from the galaxy survey SDSS DR11, DR12, DR14, DESI DR2 and CMB power spectrum Planck 2018.

Datasets	Ω_m	r_d [Mpc]	H_0 [km s^{-1} Mpc^{-1}]
DESI DR2 BAO + BBN	0.2977 ± 0.0086	148.21 ± 1.65	68.51 ± 0.58
DR14 BAO + BBN	$0.302^{+0.017}_{-0.020}$	149.0 ± 3.2	67.6 ± 1.1
DR14 BAO + BBN	0.300 ± 0.018	148.0 ± 3.1	68.1 ± 1.1
DR12 BAO + BBN	0.290 ± 0.018	150.0 ± 3.5	67.5 ± 1.2
DR11 BAO + BBN	$0.289^{+0.016}_{-0.021}$	$150.3^{+3.7}_{-3.3}$	67.4 ± 1.2
Planck 2018	0.3153 ± 0.0073	147.09 ± 0.26	67.4 ± 0.5

2.3. Standard Sirens

Since the detection of the first GW event (GW150914) in 2015 [72] by LIGO [73], standard sirens have emerged as a viable tool for cosmology. This approach utilizes GW signals from compact binaries, often combined with EM counterparts, to measure the luminosity distance-redshift relation. The basic framework, proposed by Bernard Schutz in 1986 [37], and named as "standard sirens" in Ref. [38]. It works as follows.

Assuming a binary system with component masses m_1 and m_2, the strain amplitude of the two GW polarizations h_+ and $h_\times$ are expressed as [38],

$$h_+ = \frac{2(G\mathcal{M}_z)^{5/3}(\pi f)^{2/3}}{c^4 d_L(z)}[1+(\hat{L}\cdot\hat{n})^2]\cos[\Phi(t)]\ , \quad h_\times = \frac{4(G\mathcal{M}_z)^{5/3}(\pi f)^{2/3}}{c^4 d_L(z)}(\hat{L}\cdot\hat{n})\sin[\Phi(t)]\ . \tag{10}$$

Here, $\mathcal{M}_z \equiv (1+z)(m_1 m_2)^{3/5}/(m_1+m_2)^{1/5}$ represents the redshifted chirp mass and f denotes the observed GW frequency. The unit vector $\hat{n}$ aligns with the line of sight from the observer to the source, while $\hat{L}$ represents the unit vector parallel to the binary's orbital angular momentum. G is the gravitational constant. The GW phase evolution $\Phi(t)$ is determined by the system's intrinsic properties, including mass and spin [74], related to the frequency via $f(t) = (1/2\pi)\mathrm{d}\Phi/\mathrm{d}t$. Consequencely, by analyzing $\Phi(t)$, $f(t)$, and the frequency derivative $\dot{f}$, the redshifted chirp mass can be derived as [75,76],

$$\mathcal{M}_z = \left(\frac{5}{96}\right)^{3/5} \frac{c^3}{G\pi^{8/5}} \frac{\dot{f}^{3/5}}{f^{11/5}} \tag{11}$$

Upon deriving the redshifted chirp mass, the luminosity distance can be inferred from the GW waveform amplitude via Equations (10) and (11). However, due to the inherent degeneracy between mass and redshift, GW signals alone-specifically from binary black holes, cannot independently constrain the redshift, rendering them dark siren. To utilize these sources for Hubble parameter measurements, statistical methods relying on the intrinsic mass function of compact objects have been proposed [77,78].

To break this degeneracy and acquire redshift information, two primary strategies exist. The first involves identifying an EM counterpart. By analyzing the spectral shift of the EM emission relative to its intrinsic spectrum, the redshift can be precisely determined. A landmark application of this method was the detection of GW170817 in 2017 [39] by aLIGO and aVirgo. This event, originating from a binary neutron star inspiral, was followed by EM wave detected by Fermi-GBM $1.7\,\mathrm{s}$ [79]. This coincident detection yielded a Hubble constant measurement of $H_0 = 70^{+12}_{-8}\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, leading us to a new era of multi-messenger astronomy. Notably, the inclusion of an EM counterpart not only yields the redshift but also improves the precision of the luminosity distance measurement to $\delta d_L/d_L \sim 0.1\%$ [38].

The second approach involves statistically identifying the host galaxy of the GW source [80,81], from which the redshift can be spectroscopically derived. With the development of GW detectors in low frequency and high sensitivity frontier, such as DECIGO [82] and LISA [83], will be capable of observing the inspiral phase with sufficient precision to localize source accurately. This will facilitate the identification of host galaxies and the construction of the luminosity distance-redshift relation, thereby enabling detailed studies of cosmic evolution.

2.4. Cosmic Chronometers

Above cosmological probes are trying to measure the cosmic dynamics via constructing a distance-redshift relation. The cosmic chronometer aims to build the cosmic time-redshift relation, which can measure the Hubble parameter at different redshifts as follows

$$H(z) = \frac{\dot{a}}{a} = -\frac{1}{(1+z)}\frac{\mathrm{d}z}{\mathrm{d}t} \tag{12}$$

Hence, if we can obtain cosmic time interval independent of cosmic redshift z, and the resulting $\mathrm{d}z/\mathrm{d}t$ relation can directly produce a Hubble parameter at the observed redshift.

The idea of cosmic chronometers is proposed in Ref. [84], and the astrophysical system that they choose as a candidate of cosmic chronometers is passively–evolving galaxies. Such a kind of galaxies has already experienced a rapid star formation rate at high redshifts. This rapid star formation causes the lacking of gas and further suppresses star formation rate. Since almost stars in the passively-evolving galaxies formed at early time, a measurement on the age of stars can be an independent measurement on cosmic time. To estimate the differential cosmic time-redshift relation $\mathrm{d}z/\mathrm{d}t$, we should pick two passively-evolving galaxies at two closing redshifts z_1 and z_2. This gives a redshift interval $\Delta z = z_1 - z_2$. There are several kinds of methods like 4000 Å break and full spectrum fitting, can be used in determining the age of stars. We use 4000 Å break as an example [85], the age of stars can be estimated as

$$D4000 = A(Z, \mathrm{SFH})\, t_{\mathrm{age}} + B \tag{13}$$

where $D4000$ is a discontinuity in the spectrum of passively-evolving galaxies appearing at 4000 Å restframe wavelength, $A(Z, \mathrm{SFH})$ is a parameter depends on the metallicity of the population Z and star formation history (SFH), and B is another fitting parameter. Then we can obtain the cosmic time interval by deriving Equation (13), which produces $\Delta t = \Delta D4000/A(Z, \mathrm{SFH})$, the Hubble parameter is further obtained via

$$H(\bar{z}) \simeq -\frac{A(Z, \mathrm{SFH})}{(1+\bar{z})} \frac{z_1 - z_2}{\Delta D4000} \tag{14}$$

Here, $\bar{z} = (z_1 + z_2)/2$ is the mean redshift between two passively-evolving galaxies. With above discussion steps, the cosmic chronometer provides a determination of the Hubble parameter with an accuracy of around 5% at $z{\sim}0.5$ up to $\mathcal{O}(10\%)$ at $z{\sim}2$ [85].

3. The Framework of Cosmological Standard Timers

A cosmological probe targets a recording of the size of the Universe with respect to cosmic time. Previous probes mainly focus constructing the cosmic distance-redshift relation to constrain cosmic evolution. On top of them, we would like to build a CST based on the cosmic time-redshift relation which directly track the size of the Universe as a function of cosmic time.

The fundamental principle of the CST framework is described as follows. The Universe host numerous dynamical systems that evolve according to their intrinsic mechanisms. Over a long time, this dynamical evolution significantly alters their physical states. By modeling these intrinsic mechanism, one can derive the elapsed physical time between an initial state and a later state. Since this internal evolution is decoupled from the cosmic evolution, their physical evolution time serves as an independent cosmic time measurement. During this evolution, the systems emit signals, such as EM waves and GWs, which are redshifted by the cosmic expansion. Consequencely, observations yield two measurements: cosmic redshift and the physical evolution time. As a result, we can construct a cosmic time-redshift relation, which directly constrains the cosmic expansion history $H(z)$ through the following equation

$$t = \int_z^\infty \frac{1}{(1+z')} \frac{\mathrm{d}z'}{H(z')} \tag{15}$$

Within this framework, dynamical systems work as CSTs in the Universe, which records the state of the Universe across different physical time.

The construction of CSTs has three requirements,

1. There exists an identical initial state as a standard reference for cosmic time measurement.
2. The relation between the observables and time evolution of the dynamical systems is explicit.
3. The redshift is measurable in the redshifted observables.

Based on these requirements, the cosmic chronometer satisfies all of them. The stars in passively-evolving galaxies have almost the identical formation time as a standard reference for cosmic time measurement. The observable $D4000$ depends on the age of stars in passively-evolving galaxies. The redshift of the passively-evolving galaxies can be extracted from their spectrum. Hence, cosmic chronometers can be viewed as a model of the CST.

However, cosmic chronometers as CSTs are not enough to understand the Hubble tension. Because the passively evolving galaxies can only provide Hubble parameter measurement within a narrow redshift band $z < 2$ [85], while Hubble tension is a measurement inconsistency between CMB measurement at redshift $z \simeq 1100$ and local cosmic probe measurement within redshift $z < 1$. To probe the underlying physics behind the Hubble tension, we should develop a cosmic probe at cosmic epoch within redshift $1 < z < 1100$ to study the evolution of Hubble parameter. CSTs provides a framework to build cosmic probes on the dynamical systems at high redshifts, including, DM halos and PBHs, which we will introduce in Section 4. Follow the fundamental principle of CSTs, we extend the CST to a general framework in dynamical systems. In this section, we discuss on the framework of CSTs in constructing the cosmic time-redshift relation.

3.1. The Cosmological Standard Timer from Single-Parameter Dynamical Systems

To implement CSTs effectively, a specific reference condition must be defined. Just as standard candles utilize the consistent brightness of Type Ia SNe and standard rulers depend on the fixed BAO scale, CSTs require a similar anchor. Although the initial condition of dynamical system are subject to Gaussian fluctuations, the ensemble's statistical distribution is determined by the underlying physical mechanism and thus remains unique. By adopting this statistical distribution as the standard reference, it becomes possible to derive the evolution time and redshift from observational data, which is essential for calibrating the cosmic time function $z(t)$.

For simplicity, we start with a single-parameter dynamical system, whose time evolution follows

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -f(M)T(t) \tag{16}$$

Here M denotes the observable physical parameter characterizing the dynamical systems. Its time evolution is governed by the derivative $-f(M)T(t)$, where $f(M)$ and $T(t)$ are only physical parameter-dependent and time-dependent, respectively. Consequently, the statistical distribution of single-parameter dynamical systems $S(M;t)$ at cosmic time t can be expressed as

$$S(M;t) = \frac{\mathrm{d}N}{\mathrm{d}M_t} \tag{17}$$

where $M_t \equiv M(t)$ denotes the value of physical parameter M at cosmic time t, while N represents the statistic of the dynamical system distribution. For example, N is the number density in DM halo mass function. To reconstruct the historical evolution of these dynamical systems, establishing a standard initial distribution is essential. Therefore, Equation (17) can be further expressed relative to its initial state as,

$$S(M;t) = \frac{\mathrm{d}N}{\mathrm{d}M_\mathrm{i}}\frac{\mathrm{d}M_\mathrm{i}}{\mathrm{d}M_t} \tag{18}$$

Here, $\mathrm{d}N/\mathrm{d}M_\mathrm{i}$ represents the initial statistical distribution $S(M;t_\mathrm{i})$. The term $\mathrm{d}M_\mathrm{i}/\mathrm{d}M_t$ characterizes the dynamical evolution of the system, which is governed by its time evolution $\mathrm{d}M/\mathrm{d}t = -f(M)T(t)$, which leads to

$$\int_{M_\mathrm{i}}^{M_t}\frac{\mathrm{d}M}{f(M)} = g(M_t) - g(M_\mathrm{i}) = -\int_{t_\mathrm{i}}^{t} T(t)\mathrm{d}t = \mathcal{T}(t_\mathrm{i}) - \mathcal{T}(t) \tag{19}$$

Here, we introduce $g(M)$ as the integral of $1/f(M)$ and $\mathcal{T}(t)$ as the integral of $T(t)$. Based on these definitions, the evolution of dynamical systems can be expressed as

$$\frac{\mathrm{d}M_\mathrm{i}}{\mathrm{d}M_t} = \frac{g'(M_t)}{g'(M_\mathrm{i})} = \frac{g'(M_t)}{g'(g^{-1}(g(M_t)+\Delta\mathcal{T}(t)))} \tag{20}$$

Here, $g'(M) \equiv \mathrm{d}g(M)/\mathrm{d}M$ and g^{-1} denotes the inverse function of $g(M)$. $\Delta\mathcal{T}(t) \equiv \mathcal{T}(t) - \mathcal{T}(t_\mathrm{i})$. Hence, we can write Equation (18) as

$$S(M;t) = \frac{\mathrm{d}N}{\mathrm{d}M_\mathrm{i}}\frac{g'(M_t)}{g'(g^{-1}(g(M_t)+\Delta\mathcal{T}(t)))} \tag{21}$$

The physical evolution interval $\Delta\mathcal{T}(t)$ can be derived by specifying the initial statistical distribution $\mathrm{d}N/\mathrm{d}M_\mathrm{i}$ in Equation (21).

From an observational aspect, the statistical distribution of the system appears deformed because the physical parameter M is redshifted by cosmic expansion. This results in the observational distribution $S_o(M_z;t)$ (subscript o denotes the observational quantity) as

$$S_o(M_z;t) = \frac{\mathrm{d}N}{\mathrm{d}M_\mathrm{i}(z)}\frac{\mathrm{d}M_\mathrm{i}(z)}{\mathrm{d}M_z} = \frac{\mathrm{d}N}{\mathrm{d}M_\mathrm{i}(z)}\frac{g'(M_z)}{g'(g^{-1}(g(M_z)+\Delta\mathcal{T}(t_z)))} \tag{22}$$

Here, M_z denotes the redshifted physical parameter, exemplified by redshifted photon energy $E_z = E/(1+z)$ or the redshifted chirp mass in BBH systems $\mathcal{M}_z = (1+z)\mathcal{M}$. The term $\mathrm{d}N/\mathrm{d}M_\mathrm{i}(z)$ characterizes the influence of cosmic redshift on the initial statistical distribution. Consistent with Equation (19), the relation is given by $g(M_\mathrm{i}(z)) = g(M_z) + \Delta\mathcal{T}(t_z)$.

To derive the cosmic time-redshift relation, we analyze Equation (22) under two distinct regimes. In the first regime, where $g(M_z) \gg \Delta\mathcal{T}(t_z)$, the intrinsic time evolution is negligible. Consequently, the redshift can be determined by directly comparing the redshifted physical parameter M_z with the initial physical parameter M_i in the initial statistical distribution. The second regime is $g(M_z) \ll \Delta\mathcal{T}(t_z)$, where the physical evolution term $\Delta\mathcal{T}(t_z)$ dominates, leading to the approximation $g(M_\mathrm{i}(z)) \simeq \Delta\mathcal{T}(t_z)$. Then $\Delta\mathcal{T}(t_z)$ can be extracted in following expression, which independently measures cosmic time t.

$$S_o(M_z;t) \simeq \begin{cases} \dfrac{\mathrm{d}N}{\mathrm{d}M_\mathrm{i}(z)} & , \ g(M_z) \gg \Delta\mathcal{T}(t_z) \\ \dfrac{\mathrm{d}N}{\mathrm{d}g^{-1}(\Delta\mathcal{T}(t_z))}\dfrac{g'(M_z)}{g'(g^{-1}(\Delta\mathcal{T}(t_z)))} & , \ g(M_z) \ll \Delta\mathcal{T}(t_z) \end{cases} \tag{23}$$

Consequently, the cosmic time-redshift relation can be calibrated at two parameter regimes in this single-parameter CST.

3.2. The Cosmological Standard Timer from Multi-Parameter Dynamical Systems

For a general consideration, we study multi-parameter dynamical systems as CST, their statistical distributions are described as

$$S(\mathbf{M};t) = \frac{\mathrm{d}N}{\mathrm{d}^n\mathbf{M}_t} \tag{24}$$

where $\mathbf{M}$ denotes a n-dimensional physical parameter vector which describes the dynamical system state. By introducing the initial statistical distribution as a standard reference, we can write Equation (24) as

$$S(\mathbf{M};t) = \frac{\mathrm{d}N}{\mathrm{d}^n\mathbf{M}_\mathrm{i}} \det \mathbf{J}(\mathbf{M}, \Delta t) \tag{25}$$

Here, the Jacobian of dynamical systems is defined as $\mathbf{J}_{ij} \equiv \partial M_i(t_\mathrm{i})/\partial M_j(t)$. With an initial statistical distribution as a standard reference, we can extract the physical evolution time Δt from the determinant of the Jacobian $\det \mathbf{J}(\mathbf{M}, \Delta t)$. To calculate the Jacobian, we study the time evolution of this multi-parameter dynamical system as

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = -\mathbf{f}(\mathbf{M})T(t) \tag{26}$$

where $-\mathbf{f}(\mathbf{M})T(t)$ is its time derivative function and $\mathbf{f}(\mathbf{M})$ and $T(t)$ are only physical parameter-dependent and time-dependent, respectively. Because, multiple physical parameters are strongly coupled with each other in its time evolution, it is difficult to obtain a general analytical expression of the Jacobian element $\partial M_i(t_\mathrm{i})/\partial M_j(t)$, which pushes us to find the numerical solution of $\det \mathbf{J}(\mathbf{M}, \Delta t)$ in extracting the physical evolution time Δt.

From observational aspects, the redshift term induced by the cosmic expansion also appears in the statistical distribution of multi-parameter dynamical systems as following

$$S_o(\mathbf{M}_z;t) = \frac{\mathrm{d}N}{\mathrm{d}^n\mathbf{M}_\mathrm{i}(z)} \det \mathbf{J}(\mathbf{M}_z, \Delta t_z) \tag{27}$$

where $\mathbf{M}_z$ represents the redshifted n-dimensional physical parameter vector, while $\mathrm{d}N/\mathrm{d}^n\mathbf{M}_\mathrm{i}(z)$ describes the redshifted initial statistical distribution.

Consistent with the framework in Section 3.1, we consider two distinct limiting cases for establishing the cosmic time-redshift relation. In the region of the parameter space where intrinsic time evolution is negligible relative to the inital values, the Jacobian determinant satisfies $\det \mathbf{J}(\mathbf{M}_z, \Delta t_z) \simeq 1$. Then redshift z can be determined by matching the observed statistical distribution $S_o(\mathbf{M}_z;t) \simeq \mathrm{d}N/\mathrm{d}^n\mathbf{M}_\mathrm{i}(z)$ with the initial one. The other regime is that in the parameter space where the time evolution of parameters becomes significant in statistical distributions, where physical evolution time Δt is extracted by numerically solving the equation of $\det \mathbf{J}(\mathbf{M}_z, \Delta t_z)$. Moreover, the statistical distribution of CSTs may not be an observable, which requires us to obtain it by solving an inverse problem from the observables. We discuss this scenario in Appendix A.

3.3. Uncertainties in Cosmological Standard Timers

The uncertainty is a crucial part in applying CSTs to constrain cosmic evolution. There are two main kinds of uncertainties appearing in CSTs, one is the uncertainty in the initial statistical distribution and the other one is the measurement uncertainty in the late statistical distribution. These uncertainties would be reflected as an error in the determination of the cosmic time.

To study the impact of uncertainties on the cosmic time, we consider a single-parameter dynamical system as an example and study its statistical distribution in rapid evolved region, which follows Equation (23)

$$S(M;t) = \frac{S(g^{-1}(\Delta\mathcal{T}(t));t_{\rm i})}{g'(g^{-1}(\Delta\mathcal{T}(t)))}g'(M) \tag{28}$$

where as defined in above discussion $S(M;t)$ is a statistical distribution of dynamical systems at a late time t that can be inferred from observed distribution. $S(M;t_{\rm i})$ is the initial statistical distribution of dynamical systems that is uncertain. If we give a wrong initial distribution of $S_X(M;t_{\rm i})$ in analyzing the cosmic time, Equation (28) can be written as

$$S(M;t) = \frac{S_X(g^{-1}(\Delta\mathcal{T}(t));t_{\rm i})}{g'(g^{-1}(\Delta\mathcal{T}(t)))}\frac{S(g^{-1}(\Delta\mathcal{T}(t));t_{\rm i})}{S_X(g^{-1}(\Delta\mathcal{T}(t));t_{\rm i})}g'(M) = \alpha(t)\frac{S_X(g^{-1}(\Delta\mathcal{T}(t));t_{\rm i})}{g'(g^{-1}(\Delta\mathcal{T}(t)))}g'(M) \tag{29}$$

where we define a ratio factor $\alpha(t) \equiv S(g^{-1}(\Delta\mathcal{T}(t));t_{\rm i})/S_X(g^{-1}(\Delta\mathcal{T}(t));t_{\rm i})$ to quantify the ratio difference between the true initial distribution and our modeled initial distribution. To clearly demonstrate how uncertainty in the initial distribution affects the uncertainty of cosmic time, we define a factor function F as

$$F(t) = \frac{S_X(g^{-1}(\Delta\mathcal{T}(t));t_{\rm i})}{g'(g^{-1}(\Delta\mathcal{T}(t)))} \tag{30}$$

From Equation (28), we can obtain a cosmic time-dependent factor $S(g^{-1}(\Delta\mathcal{T}(t));t_{\rm i})/g'(g^{-1}(\Delta\mathcal{T}(t)))$. Because we don't know the true initial distribution $S(M;t_{\rm i})$, it would introduce a cosmic time error Δt in the factor function F, which satisfies

$$F(t+\Delta t) = \alpha(t)F(t) \tag{31}$$

Assuming $\Delta t \ll t$, we can expand $F(t)$ to the linear order as $F(t+\Delta t) \simeq F(t) + F'(t)\Delta t$ with $F' \equiv {\rm d}F/{\rm d}t$, this gives an estimation of cosmic time error as

$$\Delta t = (\alpha(t)-1)\frac{F(t)}{F'(t)} \tag{32}$$

Meanwhile, the other uncertainty would appear in late statistical distribution due to the measurement uncertainty as

$$S(M;t) = S_p(M;t)(1+\delta(M)) \tag{33}$$

where $\delta(M)$ is the uncertainty in a late statistical distribution and $S_p(M;t)$ is the true physical distribution. Combine with two uncertainties and apply them in Equation (28), we can obtain

$$S_p(M;t) = \frac{\alpha(t)}{1+\delta(M)}F(t)g'(M) \tag{34}$$

The factor $\alpha(t)F(t)/1+\delta(M)$ at a given mass M would determine a mass-dependent cosmic time error $\Delta t(M)$ in function F, which satisfies

$$F(t+\Delta t(M)) = \frac{\alpha(t)}{1+\delta(M)}F(t) \tag{35}$$

Consider the linear order approximation, we have $F(t+\Delta t) \simeq F(t) + F'(t)\Delta t$, which results in a cosmic time error as

$$\Delta t(M) = \left(\frac{\alpha(t)}{1+\delta(M)} - 1\right)\frac{F(t)}{F'(t)} \tag{36}$$

If we consider $\delta(M) \ll 1$, this approximation can be imposed as $\alpha(t)/1+\delta(M) \simeq \alpha(t)(1-\delta(M))$. Then we pick up a number of masses to calculate its mean value as

$$\langle \Delta t \rangle = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\alpha(t)}{1+\delta(M_i)} - 1\right)\frac{F(t)}{F'(t)} \simeq \left(\alpha(t) - 1 - \alpha(t)\frac{\sum \delta(M_i)}{N}\right)\frac{F(t)}{F'(t)} \tag{37}$$

4. The Perspective of Cosmological Standard Timers in Dynamical Systems

To develop CSTs in dynamical systems, they should satisfy three requirements: identical initial states, explicit time evolution and a redshift-dependent observable. This ensures a construction of the cosmic time-redshift relation in above discussed framework. Moreover, we expect such dynamical systems could record the historical evolution of the Universe to high redshifts where current cosmological probes can hardly measure. Based on this consideration, we consider three typical dynamical systems, DM halo mass function, PBH mass function and PBH binaries. In following parts, we discuss their perspectives on constructing CSTs.

4.1. Dark Matter Halo Mass Function

The DM halo is a promising candidate of CSTs. They were formed from the gravitational collapse of DMs at redshift $z{\sim}10$. The DM halo mass function provides an initial standard state as a standard reference in CSTs. The DM halo mass function is defined as the comoving number density of DM halos in the mass range $(M, M + \mathrm{d}M)$. It can be calculated as

$$S(M;t_\mathrm{i}) = \frac{\mathrm{d}n}{\mathrm{d}M_\mathrm{i}} = f(\sigma_M)\frac{\rho_m}{M_\mathrm{i}}\frac{\mathrm{d}\log(\sigma_M^{-1})}{\mathrm{d}M_\mathrm{i}} \tag{38}$$

where $\rho_m(z) = \rho_{m,0}(1+z)^3$ is the cosmic matter density with the present matter density $\rho_{m,0} = 39.7\, M_\odot\ \mathrm{kpc}^{-3}$. σ_M is the linear root-mean-square fluctuation of the density field on the scale of M. $f(\sigma_M)$ accounts for the geometry of the collapse overdense region and can be estimated as

$$f(\sigma_M) = A\left(1 + \left(\frac{\sigma_M}{b}\right)^{-a}\right)\exp\left(-\frac{c}{\sigma_M^2}\right) \tag{39}$$

For a spherical collapse, the parameter $(A,\ a,\ b,\ c) = (0.213,\ 1.8,\ 1.85,\ 1.57)$ [86].

With an initial DM halo mass function, we can study its time evolution to extract the encoded cosmic time. The halo mass growth rate is fitted in Ref. [87] as

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 46.1\, M_\odot\ \mathrm{yr}^{-1}\left(\frac{M}{10^{12}\, M_\odot}\right)^{1.1}(1+1.11z)\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda} \tag{40}$$

We should notice that Equation (40) is obtained from the DM halo mass growth with respect to redshift $\mathrm{d}M/\mathrm{d}z$, and multiply $\mathrm{d}M/\mathrm{d}z$ with $\mathrm{d}z/\mathrm{d}t$ relation, which includes an assumption of the cosmic time-redshift relation. In order to independently measure cosmic time from the time evolution of DM halo mass function, a redshift-independent halo mass growth rate is required in future study. In this work, we use Equation (40) as an example to show the perspective of DM halos as a candidate of CSTs. Although the DM halo mass growth law strongly depends on the background cosmology model, we can still combine it with other cosmological probes to test various cosmological models. For instance, we have cosmic chronometers to independently measure the cosmic time at $z < 2$, meanwhile a cosmic time measurement can also be obtained in studying the DM halo mass evolution under the assumption of a cosmic time-redshift relation in a cosmological model. Only when the cosmic time-redshift relation that we impose in DM halo mass evolution is correct, it produces the same cosmic time as that from cosmic chronometers.

The time evolution of DM halo mass can be simply written as $\mathrm{d}M/\mathrm{d}t = \beta(t)M^{1.1}$, this gives a DM halo mass at cosmic time t as

$$\frac{1}{M_t^{0.1}} - \frac{1}{M_\mathrm{i}^{0.1}} = -\frac{1}{10}\int_{t_\mathrm{i}}^{t}\beta(t')\mathrm{d}t' = h(t_\mathrm{i}) - h(t) \tag{41}$$

where function $h(t)$ is an antiderivative of function $\beta(t)/10$. Then the time evolution of DM halo mass function can be described as

$$S(M;t) = \frac{\mathrm{d}n}{\mathrm{d}M_t} = \frac{\mathrm{d}n}{\mathrm{d}M_\mathrm{i}}\frac{\mathrm{d}M_\mathrm{i}}{\mathrm{d}M_t} = S\left(\frac{M_t}{[1+M_t^{0.1}(h(t)-h(t_\mathrm{i}))]^{10}};t_\mathrm{i}\right)\frac{1}{[1+M_t^{0.1}(h(t)-h(t_\mathrm{i}))]^{11}} \tag{42}$$

Consider a large mass limit where $1/M_t^{0.1} \ll h(t) - h(t_\mathrm{i})$, the DM halo mass function can be approximated as

$$S(M;t) \simeq \frac{S\left([h(t)-h(t_\mathrm{i})]^{-10};t_\mathrm{i}\right)}{[h(t)-h(t_\mathrm{i})]^{11}} \frac{1}{M_t^{1.1}} \tag{43}$$

This equation shows that DM halo mass function scales as $M^{-1.1}$ in large mass limit, and its factor depends on the initial DM halo mass function $S(M;t_\mathrm{i})$ and time evolution term of DM halo mass $h(t) - h(t_\mathrm{i})$. Hence apply an initial DM halo mass function into Equation (43), we can obtain cosmic time t from $h(t) - h(t_\mathrm{i})$.

The detection of DM halo is around $z < 10$, where CSTs in DM halo mass function is applicable. The measurement of its mass function can be achieved via various channels, such as X-ray [88], SZ cluster abundance [89], weak lensing [90], etc. The direct mass measurement for DM halo is the weak lensing, it uses the lensing signals to obtain the density profile of DM halos, which infers the DM halo mass [91]. The X-ray method is to measure the luminosity of the intracluster medium in DM halo and use the relation between the DM halo mass and this luminosity to determine the DM halo mass [88]. The SZ method is based on the temperature fluctuation induced by the interaction between CMB photon and hot electron in the galaxy, where we can obtain the DM halo mass [92]. Since each DM halo locates at a specific redshift, which can be obtained from the spectrum of the galaxies in DM halos. Then we can obtain the cosmic time-redshift relation from DM halos. By studying the time evolution of DM halo mass function with a future precise measurements on DM halo mass function, a CST can be well developed in DM halos to study the evolution of the Universe.

4.2. Primordial Black Hole Mass Function

PBHs are hypothetical objects that were born from primordial perturbation in highly overdense regions by gravitational collapse. When these overdense fluctuations re-enter the Hubble horizon, PBHs form in the radiation-dominated era with a mass of

$$M \sim \frac{c^3 t}{G} \sim 5.03 \times 10^4 \left(\frac{t}{1\,\mathrm{s}}\right) M_\odot \tag{44}$$

where t represents cosmic time after the hot big bang. For a stellar mass PBH, corresponding cosmic time is around $\sim\mathcal{O}(10^{-4}\,\mathrm{s})$. Due to Hawking radiation effects, the mass of PBHs is not a constant and keeps evaporating. For PBHs with an initial mass less than 10^{15} g, they have already evaporated by the present. Therefore, PBHs with mass larger than 10^{15} g can exist from the early Universe to the present Universe. This ensures PBHs as a candidate of the CST, can record cosmic evolution for the almost cosmic history.

As a candidate of the CST, PBHs need an initial statistical state as a standard reference for measuring cosmic time. Their primordial origin provides an identical initial state, the initial PBH mass function. In inflationary scenario, an initial PBH mass function $S(M;t_\mathrm{i})$ characterizes the comoving number density of PBH within a mass range $(M, M + \mathrm{d}M)$. It can be described as a log-normal type mass function as

$$S(M;t_\mathrm{i}) = \frac{\mathrm{d}n}{\mathrm{d}M_\mathrm{i}} = \frac{1}{\sqrt{2\pi}\sigma M_\mathrm{i}} \exp\left(-\frac{\log^2(M_\mathrm{i}/M_c)}{2\sigma^2}\right) \tag{45}$$

where $\mathrm{d}n$ is the comoving number density of PBHs in the mass range $(M, M + \mathrm{d}M)$, σ is the mass width of the log-normal distribution, and M_c is the characteristic PBH mass in the log-normal distribution. This unique PBH mass distribution can serve as a standard reference of cosmic time.

The next step is to study the time evolution of the PBH mass distribution, which provides an independent measurement on cosmic time. The PBH mass can evolve via two mechanisms, Hawking radiation and matter accretion. The mass change rate via Hawking radiation and matter accretion can be described as follows

$$\frac{\mathrm{d}M_\mathrm{HW}}{\mathrm{d}t} = -\frac{\hbar c^4}{15360\pi G^2} \frac{1}{M^2}\,, \quad \frac{\mathrm{d}M_\mathrm{acc}}{\mathrm{d}t} = 4\pi\lambda\rho_\mathrm{m} \frac{G^2 M^2}{v_\mathrm{eff}^3} \tag{46}$$

Here, $\hbar$ is the reduced Planck constant. ρ_m is the surrounding matter density of PBHs, v_eff is the effective velocity of PBHs in surrounding environments. λ is the accretion parameter that depends on the density ρ_m. For two mass change mechanism, Hawking radiation effect is proportional to M^{-2} and matter accretion effect is proportional to M^2. Therefore, Hawking radiation effect governs the evolution of small PBH mass, while matter accretion effect controls the large PBH mass evolution. In this section, we mainly discuss how the Hawking radiation influences the evolution of PBH mass function. Then the PBH mass evolves as

$$M_t^3 = M_{\rm i}^3 - \delta^3(\Delta t) \tag{47}$$

where $\delta^3(\Delta t) \equiv \hbar c^4 \Delta t/5120\pi G^2$, and Δt is the cosmic time interval between cosmic time t and PBH formation time $t_{\rm i}$. Since $t_{\rm i} \ll t$, $\Delta t \simeq t$. The PBH mass function at cosmic time t follows

$$S(M;t) = \frac{{\rm d}n}{{\rm d}M_t} = \frac{{\rm d}n}{{\rm d}M_{\rm i}}\frac{{\rm d}M_{\rm i}}{{\rm d}M_t} = S((M_t^3 + \delta^3(\Delta t))^{1/3}; t_{\rm i})\frac{M_t^2}{(M_t^3 + \delta^3(\Delta t))^{2/3}} \tag{48}$$

Consider a small mass limit where $M_t \ll \delta(\Delta t)$, PBH mass function at cosmic time t can be approximated as

$$S(M;t) \simeq \frac{S(\delta(\Delta t); t_{\rm i})}{\delta^2(\Delta t)} M_t^2 \tag{49}$$

This equation shows that the PBH mass function scales as M^2 in small mass limit, and the its factor depends on the initial PBH mass function $S(M;t_{\rm i})$ and cosmic time interval Δt. Hence apply the initial PBH mass function into Equation (49), we can obtain cosmic time $t \simeq \Delta t$.

The last one requirement for this PBH CST is a measurable redshift from observables. Since efficient Hawking radiation occurs on small mass PBHs, and the luminosity of Hawking radiation for a single PBH is not large enough to be detected. However, a PBH clustering changes the story, the luminosity of Hawking radiation would be enhanced by a large number of PBHs in the clustering. In Ref. [43], we show that $\mathcal{O}(10^{23})$ PBHs with mass around 10^{15} g can be detected by Fermi LAT up to redshift $z{\sim}100$. This ensures the detectability of PBHs via Hawking radiation, and the redshift information is encoded in the Hawking radiation spectrum as

$$F(E;z) = \frac{L(E(1+z);z)}{4\pi d_L^2(z)} \tag{50}$$

Here, $L(E;z)$ is the intrinsic luminosity emitted from PBH clustering at redshift z, and $F(E;z)$ is the observed photon flux from a PBH clustering located at redshift z. By imposing the PBH mass function at redshift z, we can obtain their intrinsic luminosity, and comparing with the $F(E;z)$, it would determine the redshift z (see [41,43] for more details). Combine redshift with previously obtained cosmic time t, we can construct the cosmic time-redshift relation. This would place a constraint on cosmic evolution.

4.3. Primordial Black Hole Binaries

In PBH scenarios, not only PBH mass function provides an identical initial state, but also PBH binaries have an standard initial state. That is the probability distribution of orbital parameters of PBH binary. For stellar mass PBHs, they would form PBH binaries around the matter-radiation equality [93,94] with an initial probability distribution on orbital parameter, semimajor axis a and eccentricity e as,

$$S(a,e;t_{\rm i}) = \frac{{\rm d}P}{{\rm d}a{\rm d}e} = \frac{3}{4} f_{\rm PBH}^{3/2} \frac{a^{1/2}}{\bar{x}^{3/2}} \frac{e}{(1-e^2)^{3/2}} \tag{51}$$

where $f_{\rm PBH}$ is the energy density fraction of PBHs in the DM at present Universe and $\bar{x}$ is the mean physical separation of PBHs at the matter-radiation equality. Therefore, this unique initial probability distribution of orbital parameters in PBH binaries can serve as a standard reference for a later evolving distribution to extract cosmic time. This is a two-dimension probability distribution. For a simple and clear demonstration of its feasibility as a CST, we only consider a circular orbit PBH binary with an single-parameter initial state $S(a;t_{\rm i})$ in following discussion. This choice is reasonable, because the circularization of PBH binaries could be achieved via various mechanism before entering the GW detection frequency band [95]. For a discussion of a multi-parameter PBH binaries, see Ref. [42] for more details.

In order to study the dynamical evolution of this single-parameter PBH binary, GWs become very important that determines the intrinsic dynamics of PBH binary and its detectability. As a candidate of CSTs, it requires a construction of the cosmic time-redshift relation, which means we need to study the evolution of probability distribution of PBH binaries from its initial state to a later state and this later state should come from the same redshift. However, as what we introduced in standard sirens, redshift cannot be determined from GW itself, due to the mass-redshift degeneracy in GWs. This difficulty can be resolved by an assumption of a monochromatic mass function. From GW waves, we can extract its redshifted mass M_z, which is a combination of the cosmic redshift and the intrinsic mass of PBHs $M_z = (1+z)M_{\rm PBH}$. If all the PBHs have the same intrinsic mass $M_{\rm PBH}$, the PBHs from the same redshift z should have the same redshifted mass M_z. Therefore, we can distinguish whether a number of PBHs comes from the same redshift based on their redshifted mass.

The GW emission drives the shrinkage of PBH binaries and decreasing the semimajor axis, its dynamics can be described as [96]

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{128}{5}\frac{G^3 M_{\rm PBH}^3}{c^5 a^3} \tag{52}$$

where $M_{\rm PBH}$ is the monochromatic mass of PBHs. We can integrate Equation (52) from the initial cosmic time $t_{\rm i}$ to later cosmic time t to obtain the evolution of the semimajor axis a at cosmic time t as

$$a_{\rm i}^4 = a_t^4 + \delta^4(\Delta t)\ , \quad \delta^4(\Delta t) \equiv \frac{512 G^3}{5c^5} M_{\rm PBH}^3 \Delta t \tag{53}$$

Here, we define the symbol δ^4 for a later simplified expression. Δt is the physical evolution time between two cosmic time $t_{\rm i}$ and t as $\Delta t = t - t_{\rm i}$. Based on Equation (53), we can study the evolution of the probability distribution of orbital parameters in PBH binaries as

$$S(a;t) = \frac{\mathrm{d}P}{\mathrm{d}a_t} = \frac{\mathrm{d}P}{\mathrm{d}a_{\rm i}}\frac{\mathrm{d}a_{\rm i}}{\mathrm{d}a_t} = S(a;t_{\rm i})\frac{a_t^3}{(a_t^4+\delta^4(\Delta t))^{3/4}} = S((a_t^4+\delta^4(\Delta t))^{1/4};t_{\rm i})\frac{a_t^3}{(a_t^4+\delta^4(\Delta t))^{3/4}} \tag{54}$$

This equation shows that the physical evolution time Δt can be determined, once we compare a later probability distribution $S(a;t)$ with an initial probability distribution $S(a;t_{\rm i})$. Since this physical evolution time Δt only depends on the intrinsic mechanism of PBH binaries, it can be viewed as an independent measurement of the cosmic time interval.

After determining the cosmic time, we need to extract the redshift from the observed quantity. With the cosmic expansion, the observed semimajor axis a_z is redshifted by $a_z = (1+z)a$, which can be found in the Kepler's third law $a_z \sim (GM_z)^{1/3} f_z^{-2/3} \sim (1+z)(GM)^{1/3} f^{-2/3}$, where M_z and f_z are the redshifted mass and GW frequency of binary systems, respectively. Therefore, the observed probability distribution follows

$$S_o(a_z;t) = \frac{\mathrm{d}P}{\mathrm{d}a_{\rm i}(z)}\frac{a_z^3}{(a_z^4+\delta^4(\Delta t_z))^{3/4}} \tag{55}$$

As Equation (53), we have the relation $a_{\rm i}^4(z) = a_z^4 + \delta^4(\Delta t_z)$, where Δt_z depends on the redshift of PBHs binaries. In order to obtain the physical evolution time Δt from Δt_z, we should first obtain the correct redshift from observed probability distribution. We consider the parameter regime $a_z^4 \gg \delta^4(\Delta t_z)$, which indicates $a_{\rm i}(z) \simeq a_z$. In this large semimajor axis limit, the time evolution of semimajor axis is negligible compared with its initial value. Therefore, the observed distribution of the semimajor axis will only change with the expansion of the Universe. It gives an observed probability distribution in large semimajor axis limit regime as,

$$S_o(a_z;t) \simeq S_o(a_{\rm i}(1+z);t) = \frac{\mathrm{d}P}{\mathrm{d}a_{\rm i}(z)} = \frac{1}{1+z}\frac{\mathrm{d}P}{\mathrm{d}a_{\rm i}} = \frac{1}{1+z}S(a;t_{\rm i}) \tag{56}$$

In Equation (56), we can obtained an equation on the redshift. Consider a specific semimajor axis a_L at the limit of large semimajor axis, we have

$$(1+z)S_o(a_L(1+z);t) = S(a_L;t_{\rm i}) \tag{57}$$

Here, $S_o(a_z;t)$ in known from the observations and the initial probability distribution $S(a;t_{\rm i})$ have been well studied in theoretical work [93,94]. Then, cosmic redshift can be numerically solved in Equation (57). With the determination of the redshift, intrinsic probability distributions can be found from the redshifted ones via $S(a;t) = (1+z)S_o(a(1+z);t)$ and the mass of PBHs can be solved from the redshifted mass as $M_{\rm PBH} = M_z/(1+z)$. Then, we can obtain the physical evolution time in the parameter regime $a_t^4 \ll \delta^4(\Delta t)$, which infers $a_{\rm i} \simeq \delta(\Delta t)$. In this small semimajor axis limit, we have the probability distribution from Equation (54) as

$$S(a;t) \simeq \frac{S(\delta(\Delta t);t_{\rm i})}{\delta^3(\Delta t)} a_t^3 \tag{58}$$

It shows that the probability distribution is proportional to a_t^3 in the small semimajor axis regime, and the factor of a_t^3 only depends on the initial probability distribution and time evolution term $\delta(\Delta t)$, where $\delta(\Delta t)$ can be extracted. Since $\delta(\Delta t)$ depends on the mass of PBHs M_{PBH} and physical evolution time Δt. In above calculation, the mass of PBHs has been obtained after deriving redshift in Equation (57). Then, we can obtain the physical evolution time Δt from obtained $\delta(\Delta t)$. With the redshift and physical evolution time, the calibration between redshift and cosmic time can be constructed in this single-parameter PBH binary systems.

Above discussion is based on the detection of PBH binaries in future GW observations, which is uncertain at present. With the construction of the next-generation GW detectors, LISA is able to detect BBH events up to redshift $z{\sim}1000$ [44], which ensures the application of CSTs in PBH binaries to probe cosmic evolution up to CMB epoch that helps us understand the Hubble parameter evolution from CMB epoch to Type Ia SNe epoch. Meanwhile the high sensitivity of low-frequency GW detectors decreases the measurement uncertainty in BBH parameters that enhances the precision of CSTs in PBH binaries.

In this section, we mainly discuss the perspectives of CSTs in three typical dynamical systems as examples. In general, once a dynamical system owns an identical initial state and a intrinsic dynamics, it can naturally be a candidate of CSTs. For instance, Population III (Pop III) stars have an unique initial mass function (IMF) and a star formation rate (SFR), once we can measure IMF and SFR accurately, Pop III stars can also be a potential CST. With a further study on astrophysical systems, more special dynamical systems would be found to help us understand the historical evolution of the Universe.

Meanwhile, some additional physical effects would cause uncertainties in this independent measurement of cosmic time, such as baryonic effects in DM halos [97], different PBH formation models [98], and dynamical friction in binary evolution [99,100]. These effects either contribute a modification on initial statistical distribution or dynamical mechanism, which would lead to an error in cosmic time determination as what we discuss in Section 3.3. For example, baryonic and dynamical effects changes the time evolution of DM halo mass and PBH orbital parameters, various PBH formation models produce distinct initial PBH mass functions. These uncertainties cause a cosmic time meansurement error and lead to a misinterpretation on cosmic evolution. To distinguish the origin of measured cosmic time-redshift relation is from measurement uncertainties or background cosmology, we should perform a consistent test on the cosmic time measurements on CSTs from high redshifts to the present.

5. Summary

To summarize, we propose a novel cosmological probe, cosmological standard timer (CST). By studying the time evolution of dynamical systems in the Universe, we can have an independent measurement on cosmic time, while the redshift can be obtained from the observables of dynamical systems. Once the cosmic time-redshift relation is determined within a dynamical system, we can infer the size of the Universe with respect to cosmic time, thereby placing constraints on cosmic evolution and the corresponding cosmological parameters. As a result, dynamical systems in the Universe can work as cosmological probes to measure the Universe as other cosmological probes [101].

To build a CST, we first propose a general framework of the CST that presents a dynamical evolution of a statistical distribution from an initial state of the dynamical system can provide an independent measurement of cosmic time. By combining the measured cosmic time with the redshift encoded in observables, a CST can be constructed. We then discuss the perspectives of three potential candidates of CSTs: DM halo mass function, PBH mass function and PBH binaries. The dynamical evolution of their mass function and orbital parameters could enable them as CSTs. For a more detailed understanding of the CST, see Refs. [41,42].

The framework of CSTs is based on the statistical distribution of dynamical systems, which provides an identical initial state as a standard reference for cosmic time. However, constructing a statistical distribution requires a large number of observational events as samples, this causes the difficulty in obtaining a precise measurement on the cosmic time-redshift relation. To improve the feasibility of CSTs, the detection capabilities of multi-messenger detectors should be enhanced to extend astrophysical observations to the high-sensitivity frontier. Moreover, the intrinsic mechanisms of dynamical systems cannot be simply described by differential equations, which introduces uncertainties in extracting cosmic time from observables and reduces the precision of the obtained cosmic time-redshift relation. For further development of CSTs, an improvement on detecting dynamical systems and analyzing their intrinsic mechanisms are essential.

With the development of astrophysical observations, constructing CSTs becomes feasible. For instance, James Webb Space Telescope improves the detectability of high-redshift galaxies [102], the LIGO-Virgo-KAGRA network [73,103,104] and next-generation GW detectors Einstein Telescope [105] enable the detection of PBH populations in GW channels. By combining CSTs with other cosmological probes, we can provide a strong constraint on cosmic evolution and shed light on current cosmic tensions.

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Data Availability Statement

Not applicable.

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Conflicts of Interest

The author declare no conflict of interest.

Use of AI and AI-Assisted Technologies

During the preparation of this work, the author used ChatGPT to rephrase parts of the content. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

Appendix A. The Cosmological Standard Timer with Indirect Observables

In the main text, we focus on the CST with a direct-observed statistical distribution $S(M;t)$. However, this framework does not apply to the case that M is not an observable of dynamical systems. In this case, the observable of dynamical systems, e.g., EM waves and GWs, is related with the physical parameter M via following integral equation,

$$P(E;t) = \int_0^\infty K(E,M)S(M;t)\,\mathrm{d}M \tag{A1}$$

Here, $K(E,M)$ is the kernel function which transfers an unobservable distribution $S(M;t)$ to an observable distribution $P(E;t)$. For example, GW background is an integration of BBH merger rate distribution over a redshift range [106]. $S(M;t)$ can be inferred by an inverse integral equation

$$S(M;t) = \int_0^\infty K^{-1}(E,M)P(E;t)\,\mathrm{d}E \tag{A2}$$

where $K^{-1}(E,M)$ denotes the inverse of the kernel function of $K(E,M)$. As a consequence of cosmic expansion, the physical parameter E is redshifted to E_z. Consequently, the observable quantity is expressed as

$$P_o(E_z;t) = \int_0^\infty K_o(\mathcal{Z}_1(E_z),M)S(M;t)\,\mathrm{d}M \tag{A3}$$

where the function $\mathcal{Z}_1$ describes the influence of redshift on the observable E_z. To construct the cosmic time-redshift relation, a redshift term need to appear in $S(M;t)$, which requires the interchange of the redshift scaling factor between E and M in the kernel function. For example, the primary Hawking radiation kernel follows $H(E(1+z),M) = H(E,M(1+z))$ [41]. Therefore, we assume the kernel function follows

$$K_o(\mathcal{Z}_1(E_z),M) = K_o(E_z,\mathcal{Z}_2(M)) \tag{A4}$$

Here, the functions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ characterize how the redshift term is mapped from E_z to M in kernel function. Accordingly, Equation (A3) can be expressed as

$$P_o(E_z;t) = \int_0^\infty K_o(E_z,\mathcal{Z}_2(M))S_o(\mathcal{Z}_2(M);t)\,\mathrm{d}\mathcal{Z}_2(M) \tag{A5}$$

As a result, $S_o(\mathcal{Z}_2(M);t)$ is given by an inverse integral equation as

$$S_o(\mathcal{Z}_2(M);t) = \int_0^\infty K_o^{-1}(E_z,\mathcal{Z}_2(M))P_o(E_z;t)\,\mathrm{d}E_z \tag{A6}$$

As we discuss in Equation (23), $S_o(\mathcal{Z}_2(M);t)$ can be approximately expressed in two regimes in Equation (A7), which further gives the cosmic time-redshift relation.

$$S_o(\mathcal{Z}_2(M);t) \simeq \begin{cases} \dfrac{\mathrm{d}N}{\mathrm{d}\mathcal{Z}_2(M_\mathrm{i})} & , \ g(\mathcal{Z}_2(M)) \gg \Delta\mathcal{T}(t_z) \\ \dfrac{\mathrm{d}N}{\mathrm{d}g^{-1}(\Delta\mathcal{T}(t_z))}\dfrac{g'(\mathcal{Z}_2(M))}{g'(g^{-1}(\Delta\mathcal{T}(t_z)))} & , \ g(\mathcal{Z}_2(M)) \ll \Delta\mathcal{T}(t_z) \end{cases} \tag{A7}$$

References

1. Gamow, G. The Origin of Elements and the Separation of Galaxies. *Phys. Rev.* **1948**, *74*, 505–506.
2. Alpher, R.A.; Herman, R.C. On the Relative Abundance of the Elements. *Phys. Rev.* **1948**, *74*, 1737–1742.
3. Alpher, R.A.; Herman, R.C. Remarks on the Evolution of the Expanding Universe. *Phys. Rev.* **1949**, *75*, 1089–1095.
4. Planck Collaboration. Planck 2018 results. VI. Cosmological parameters. *Astron. Astrophys.* **2020**, *641*, A6.
5. Peebles, P.J.E. Large scale background temperature and mass fluctuations due to scale invariant primeval perturbations. *Astrophys. J. Lett.* **1982**, *263*, L1–L5.
6. Perlmutter, S.; Aldering, G.; Goldhaber, G.; et al. Measurements of Ω and Λ from 42 high redshift supernovae. *Astrophys. J.* **1999**, *517*, 565–586.
7. Riess, A.G.; Filippenko, A.V.; Challis, P.; et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.* **1998**, *116*, 1009–1038.
8. Rubin, V.C.; Ford, W.K., Jr. Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions. *Astrophys. J.* **1970**, *159*, 379–403.
9. Faber, S.M.; Gallagher, J.S. Masses and mass-to-light ratios of galaxies. *Ann. Rev. Astron. Astrophys.* **1979**, *17*, 135–183.
10. Turner, M.S.; Steigman, G.; Krauss, L.M. Flatness of the Universe: Reconciling Theoretical Prejudices with Observational Data. *Phys. Rev. Lett.* **1984**, *52*, 2090–2093.
11. Frenk, C.S.; White, S.D.M.; Efstathiou, G.; et al. Cold dark matter, the structure of galactic haloes and the origin of the Hubble sequence. *Nature* **1985**, *317*, 595–597.
12. Perivolaropoulos, L.; Skara, F. Challenges for ΛCDM: An update. *New Astron. Rev.* **2022**, *95*, 101659.
13. Abdalla, E.; Abellán, G.F.; Aboubrahim, A.; et al. Cosmology intertwined: A review of the particle physics, astrophysics, and cosmology associated with the cosmological tensions and anomalies. *J. High Energy Astrophys.* **2022**, *34*, 49–211.
14. Di Valentino, E.; Mena, O.; Pan, S.; et al. In the realm of the Hubble tension—A review of solutions. *Class. Quant. Grav.* **2021**, *38*, 153001.
15. Nunes, R.C.; Vagnozzi, S. Arbitrating the S8 discrepancy with growth rate measurements from redshift-space distortions. *Mon. Not. R. Astron. Soc.* **2021**, *505*, 5427–5437.
16. Secrest, N.J.; von Hausegger, S.; Rameez, M.; et al. A Test of the Cosmological Principle with Quasars. *Astrophys. J. Lett.* **2021**, *908*, L51.
17. Secrest, N.J.; von Hausegger, S.; Rameez, M.; et al. A Challenge to the Standard Cosmological Model. *Astrophys. J. Lett.* **2022**, *937*, L31.
18. Freedman, W.L. Cosmology at a Crossroads. *Nat. Astron.* **2017**, *1*, 0121.
19. Rameez, M.; Sarkar, S. Is there really a Hubble tension? *Class. Quant. Grav.* **2021**, *38*, 154005.
20. Poulin, V.; Smith, T.L.; Karwal, T.; et al. Early Dark Energy Can Resolve The Hubble Tension. *Phys. Rev. Lett.* **2019**, *122*, 221301.
21. Bernal, J.L.; Verde, L.; Riess, A.G. The trouble with H_0. *J. Cosmol. Astropart. Phys.* **2016**, *10*, 019.
22. Lin, M.X.; Benevento, G.; Hu, W.; et al. Acoustic Dark Energy: Potential Conversion of the Hubble Tension. *Phys. Rev. D* **2019**, *100*, 063542.
23. Kenworthy, W.D.; Scolnic, D.; Riess, A. The Local Perspective on the Hubble Tension: Local Structure Does Not Impact Measurement of the Hubble Constant. *Astrophys. J.* **2019**, *875*, 145.
24. Ding, Q.; Nakama, T.; Wang, Y. A gigaparsec-scale local void and the Hubble tension. *Sci. China Phys. Mech. Astron.* **2020**, *63*, 290403.
25. Vagnozzi, S. New physics in light of the H_0 tension: An alternative view. *Phys. Rev. D* **2020**, *102*, 023518.
26. Fung, L.W.; Li, L.; Liu, T.; et al. The Hubble Constant in the Axi-Higgs Universe. *arXiv* **2021**, arXiv:astro-ph.CO/2105.01631.
27. Cai, T.; Ding, Q.; Wang, Y. Reconciling cosmic dipolar tensions with a gigaparsec void. *Phys. Rev. D* **2025**, *111*, 103502.
28. Bullock, J.S.; Boylan-Kolchin, M. Small-Scale Challenges to the ΛCDM Paradigm. *Ann. Rev. Astron. Astrophys.* **2017**, *55*, 343–387.
29. Karim, M.A.; Aguilar, J.; Ahlen, S.; et al. DESI DR2 results. II. Measurements of baryon acoustic oscillations and cosmological constraints. *Phys. Rev. D* **2025**, *112*, 083515.
30. Fernie, J.D. The Period-Luminosity Relation: A Historical Review. *Publ. Astron. Soc. Pac.* **1969**, *81*, 707.
31. Scolnic, D.M.; Jones, D.O.; Rest, A.; et al. The Complete Light-curve Sample of Spectroscopically Confirmed SNe Ia

from Pan-STARRS1 and Cosmological Constraints from the Combined Pantheon Sample. *Astrophys. J.* **2018**, *859*, 101.
32. Beutler, F.; Blake, C.; Colless, M.; et al. The 6dF Galaxy Survey: baryon acoustic oscillations and the local Hubble constant. *Mon. Not. R. Astron. Soc.* **2011**, *416*, 3017–3032.
33. Ross, A.J.; Samushia, L.; Howlett, C.; et al. The clustering of the SDSS DR7 main Galaxy sample—I. A 4 per cent distance measure at $z = 0.15$. *Mon. Not. R. Astron. Soc.* **2015**, *449*, 835–847.
34. Font-Ribera, A.; Kirkby, D.; Miralda-Escudé, J.; et al. Quasar-Lyman α Forest Cross-Correlation from BOSS DR11 : Baryon Acoustic Oscillations. *JCAP* **2014**, *05*, 027.
35. Delubac, T.; Bautista, J.E.; Rich, J.; et al. Baryon acoustic oscillations in the Lyα forest of BOSS DR11 quasars. *Astron. Astrophys.* **2015**, *574*, A59.
36. Alam, S.; Ata, M.; Bailey, S.; et al. The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological analysis of the DR12 galaxy sample. *Mon. Not. R. Astron. Soc.* **2017**, *470*, 2617–2652.
37. Schutz, B.F. Determining the Hubble Constant from Gravitational Wave Observations. *Nature* **1986**, *323*, 310–311.
38. Holz, D.E.; Hughes, S.A. Using gravitational-wave standard sirens. *Astrophys. J.* **2005**, *629*, 15–22.
39. Abbott, B.P.; Abbott, R.; Abbott, T.D.; et al. GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral. *Phys. Rev. Lett.* **2017**, *119*, 161101.
40. The LIGO Scientific Collaboration; The Virgo Collaborationl; The 1M2H Collaboration; et al. A gravitational-wave standard siren measurement of the Hubble constant. *Nature* **2017**, *551*, 85–88.
41. Cai, Y.F.; Chen, C.; Ding, Q.; et al. Cosmological Standard Timers from Unstable Primordial Relics. *arXiv* **2021**, arXiv:astro-ph.CO/2112.10422.
42. Ding, Q. Toward cosmological standard timers in primordial black hole binaries. *Phys. Rev. D* **2023**, *108*, 023514.
43. Cai, Y.F.; Chen, C.; Ding, Q.; et al. Ultrahigh-energy Gamma Rays and Gravitational Waves from Primordial Exotic Stellar Bubbles. *Eur. Phys. J. C* **2022**, *82*, 464.
44. Ding, Q. Detectability of primordial black hole binaries at high redshift. *Phys. Rev. D* **2021**, *104*, 043527.
45. Hu, W.; Sawicki, I. Models of f(R) Cosmic Acceleration that Evade Solar-System Tests. *Phys. Rev. D* **2007**, *76*, 064004.
46. Yang, Y.; Ren, X.; Wang, Q.; et al. Quintom cosmology and modified gravity after DESI 2024. *Sci. Bull.* **2024**, *69*, 2698–2704.
47. Leavitt, H.S. 1777 variables in the Magellanic Clouds. *Harv. Obs. Ann.* **1908**, *60*, 87–108.
48. Leavitt, H.S.; Pickering, E.C. Periods of 25 Variable Stars in the Small Magellanic Cloud. *Harv. Obs. Circ.* **1912**, *173*, 1–3.
49. Branch, D.; Tammann, G.A. Type ia supernovae as standard candles. *Ann. Rev. Astron. Astrophys.* **1992**, *30*, 359–389.
50. Phillips, M.M. The absolute magnitudes of Type IA supernovae. *Astrophys. J. Lett.* **1993**, *413*, L105–L108.
51. Baade, W.; Zwicky, F. On Super-Novae. *Proc. Nat. Acad. Sci. USA* **1934**, *20*, 254–259.
52. Baade, W. The Resolution of Messier 32, NGC 205, and the Central Region of the Andromeda Nebula. *Astrophys. J.* **1944**, *100*, 137–146.
53. Lee, M.G.; Freedman, W.L.; Madore, B.F. The Tip of the Red Giant Branch as a Distance Indicator for Resolved Galaxies. *Astrophys. J.* **1993**, *417*, 553.
54. Pigott, E. VII. Observations of a new variable star. In a letter from Edward Pigott, Esq. to Sir. HC Englefield, Bart. FRS and AS. *Philos. Trans. R. Soc. Lond.* **1785**, *75*, 127–136.
55. Wallerstein, G. The Cepheids of Population II and related stars. *Publ. Astron. Soc. Pac.* **2002**, *114*, 689.
56. Soszynski, I.; Udalski, A.; Szymanski, M.K.; et al. The Optical Gravitational Lensing Experiment. The OGLE-III Catalog of Variable Stars. II. Type II Cepheids and Anomalous Cepheids in the Large Magellanic Cloud. *Acta Astron.* **2008**, *58*, 293.
57. Smolec, R.; Moskalik, P. Double-mode classical Cepheid models—revisited. *Acta Astron.* **2008**, *58*, 233.
58. Ferrarese, L.; Freedman, W.L.; Hill, R.J.; et al. The extragalactic distance scale key project. Iv. The discovery of Cepheids and a new distance to M100 using the Hubble Space Telescope. *Astrophys. J.* **1996**, *464*, 568.
59. Hubble, E. A relation between distance and radial velocity among extra-galactic nebulae. *Proc. Natl. Acad. Sci. USA* **1929**, *15*, 168–173.
60. Riess, A.G.; Casertano, S.; Yuan, W.; et al. Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond ΛCDM. *Astrophys. J.* **2019**, *876*, 85.
61. Kellermann, K.I. The cosmological deceleration parameter estimated from the angular-size/redshift relation for compact radio sources. *Nature* **1993**, *361*, 134–136.
62. Gurvits, L. Apparent milliarcsecond sizes of active galactic nuclei and the geometry of the universe. *Astrophys. J.* **1994**, *425*, 442–449.
63. Buchalter, A.; Helfand, D.J.; Becker, R.H.; et al. Constraining ω0 with the angular size-redshift relation of double-lobed quasars in the first survey. *Astrophys. J.* **1998**, *494*, 503.
64. Bassett, B.A.; Hlozek, R. Baryon Acoustic Oscillations. In *Dark Energy: Observational and Theoretical Approaches*; Cambridge University Press: Cambridge, UK, 2009.
65. Peebles, P. *The Large-Scale Structure of the Universe*; Princeton University Press: Princeton, NJ, USA, 1980.
66. Eisenstein, D.J.; Zehavi, I.; Hogg, D.W.; et al. Detection of the Baryon Acoustic Peak in the Large-Scale Correlation Function of SDSS Luminous Red Galaxies. *Astrophys. J.* **2005**, *633*, 560–574.

67. Bautista, J.E.; Guy, J.; Rich, J.; et al. Measurement of baryon acoustic oscillation correlations at $z = 2.3$ with SDSS DR12 Lyα-Forests. *Astron. Astrophys.* **2017**, *603*, A12.
68. Des Bourboux, H.D.M., Le Goff, J.M., Blomqvist, M.; et al. Baryon acoustic oscillations from the complete SDSS-III Lyα-quasar cross-correlation function at $z = 2.4$. *Astron. Astrophys.* **2017**, *608*, A130.
69. Blomqvist, M.; Des Bourboux, H.D.M.; de Sainte Agathe, V.; et al. Baryon acoustic oscillations from the cross-correlation of Lyα absorption and quasars in eBOSS DR14. *Astron. Astrophys.* **2019**, *629*, A86.
70. de Sainte Agathe, V.; Ball, C.; Des Bourboux, H.D.M.; et al. Baryon acoustic oscillations at z = 2.34 from the correlations of Lyα absorption in eBOSS DR14. *Astron. Astrophys.* **2019**, *629*, A85.
71. Cuceu, A.; Farr, J.; Lemos, P.; et al. Baryon Acoustic Oscillations and the Hubble Constant: Past, Present and Future. *J. Cosmol. Astropart. Phys.* **2019**, *10*, 044.
72. Abbott, B.P.; Abbott, R.; Abbott, T.D.; et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* **2016**, *116*, 061102.
73. Abbott, B.P.; Abbott, R.; Adhikari, R.; et al. LIGO: The Laser interferometer gravitational-wave observatory. *Rep. Prog. Phys.* **2009**, *72*, 076901.
74. Poisson, E.; Will, C.M. Gravitational waves from inspiraling compact binaries: Parameter estimation using second-post-Newtonian wave forms. *Phys. Rev. D* **1995**, *52*, 848–855.
75. Finn, L.S.; Chernoff, D.F. Observing binary inspiral in gravitational radiation: One interferometer. *Phys. Rev. D* **1993**, *47*, 2198–2219.
76. Cutler, C.; Flanagan, E.E. Gravitational waves from merging compact binaries: How accurately can one extract the binary's parameters from the inspiral wave form? *Phys. Rev. D* **1994**, *49*, 2658–2697.
77. Ding, Q. Merger rate of primordial black hole binaries as a probe of the Hubble parameter. *Phys. Rev. D* **2024**, *110*, 063542.
78. Fung, L.W.H.; Broadhurst, T.; Smoot, G.F. Pure gravitational wave estimation of Hubble's constant using neutron star–black hole mergers. *Mon. Not. R. Astron. Soc.* **2025**, *538*, 711–725.
79. Goldstein, A.; Veres, P.; Burns, E.; et al. An Ordinary Short Gamma-Ray Burst with Extraordinary Implications: Fermi-GBM Detection of GRB 170817A. *Astrophys. J. Lett.* **2017**, *848*, L14.
80. Soares-Santos, M.; Palmese, A.; Hartley, W.; et al. First Measurement of the Hubble Constant from a Dark Standard Siren using the Dark Energy Survey Galaxies and the LIGO/Virgo Binary–Black-hole Merger GW170814. *Astrophys. J. Lett.* **2019**, *876*, L7.
81. Abbott, B.P.; Abbott, R.; Abbott, T.D.; et al. A Gravitational-wave Measurement of the Hubble Constant Following the Second Observing Run of Advanced LIGO and Virgo. *Astrophys. J.* **2021**, *909*, 218.
82. Kawamura, S., Nakamura, T., Ando, M.; et al. The Japanese space gravitational wave antenna DECIGO. *Class. Quantum Gravity* **2006**, *23*, S125–S132.
83. Bender, P.; Brillet, A.; Ciufolini, I.; et al. *Lisa Pre-Phase a Report*; Max-Planck Institut für Quantenoptik: Garching bei München, Germany, **1998**.
84. Jimenez, R.; Loeb, A. Constraining cosmological parameters based on relative galaxy ages. *Astrophys. J.* **2002**, *573*, 37–42.
85. Moresco, M. Measuring the expansion history of the Universe with cosmic chronometers. *arXiv* **2024**, arXiv:astro-ph.CO/2412.01994.
86. Tinker, J.L.; Kravtsov, A.V.; Klypin, A.; et al. Toward a halo mass function for precision cosmology: The Limits of universality. *Astrophys. J.* **2008**, *688*, 709–728.
87. Fakhouri, O.; Ma, C.P.; Boylan-Kolchin, M. The merger rates and mass assembly histories of dark matter haloes in the two Millennium simulations. *Mon. Not. R. Astron. Soc.* **2010**, *406*, 2267–2278.
88. Vikhlinin, A.; Kravtsov, A.V.; Burenin, R.A.; et al. Chandra Cluster Cosmology Project III: Cosmological Parameter Constraints. *Astrophys. J.* **2009**, *692*, 1060–1074.
89. Ade, P.A.R., Aghanim, N., Arnaud, M.; et al. Planck 2015 results. XXIV. Cosmology from Sunyaev-Zeldovich cluster counts. *Astron. Astrophys.* **2016**, *594*, A24.
90. Umetsu, K. Cluster—galaxy weak lensing. *Astron. Astrophys. Rev.* **2020**, *28*, 7.
91. Johnston, D.E.; Sheldon, E.S.; Wechsler, R.H.; et al. Cross-correlation Weak Lensing of SDSS galaxy Clusters II: Cluster Density Profiles and the Mass–Richness Relation. *arXiv* **2007**, arXiv:astro-ph/0709.1159.
92. Ade, P.A.R.; Aghanim, N.; Armitage-Caplan, C.; et al. Planck 2013 results. XX. Cosmology from Sunyaev—Zeldovich cluster counts. *Astron. Astrophys.* **2014**, *571*, A20.
93. Ali-Haïmoud, Y.; Kovetz, E.D.; Kamionkowski, M. Merger rate of primordial black-hole binaries. *Phys. Rev. D* **2017**, *96*, 123523.
94. Sasaki, M.; Suyama, T.; Tanaka, T.; et al. Primordial Black Hole Scenario for the Gravitational-Wave Event GW150914. *Phys. Rev. Lett.* **2016**, *117*, 061101.
95. Franciolini, G.; Cotesta, R.; Loutrel, N.; et al. How to assess the primordial origin of single gravitational-wave events with mass, spin, eccentricity, and deformability measurements. *Phys. Rev. D* **2022**, *105*, 063510.
96. Peters, P.C. Gravitational Radiation and the Motion of Two Point Masses. *Phys. Rev.* **1964**, *136*, B1224–B1232.
97. Somerville, R.S.; Davé, R. Physical Models of Galaxy Formation in a Cosmological Framework. *Ann. Rev. Astron.*

Astrophys. **2015**, *53*, 51–113.

98. Carr, B.; Kohri, K.; Sendouda, Y.; et al. Constraints on Primordial Black Holes. *arXiv* **2020**, arXiv:astro-ph.CO/2002.12778.
99. Ding, Q.; He, M.; Takhistov, V.; et al. Dark-matter-independent orbital decay bounds on ultralight bosons from OJ287. *Phys. Rev. D* **2025**, *112*, 103051.
100. Ding, Q.; He, M.; Zhu, H.Y. Extracting Properties of Dark Dense Environment Around Black Holes From Gravitational Waves. arXiv **2025**, arXiv:gr-qc/2510.27424.
101. Moresco, M.; Amati, L.; Amendola, L.; et al. Unveiling the Universe with emerging cosmological probes. *Living Rev. Relativ.* **2022**, *25*, 6.
102. Gardner, J.P.; Mather, J.C.; Clampin, M.; et al. The James Webb Space Telescope. *Space Sci. Rev.* **2006**, *123*, 485
103. Acernese, F.; Agathos, M.; Agatsuma, K.; et al. Advanced Virgo: a second-generation interferometric gravitational wave detector. *Class. Quant. Grav.* **2015**, *32*, 024001.
104. Akutsu, T.; Ando, M.; Arai, K.; et al. KAGRA: 2.5 Generation Interferometric Gravitational Wave Detector. *Nat. Astron.* **2019**, *3*, 35–40.
105. Abac, A., Abramo, R., Albanesi, S.; et al. The Science of the Einstein Telescope. *arXiv* **2025**, arXiv:gr-qc/2503.12263.
106. Phinney, E.S. A Practical theorem on gravitational wave backgrounds. *arXiv* **2001**, arXiv:astro-ph/0108028.