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Article

Dynamic Response of SDoF System with Negative Stiffness—A Relevant Key-Point for Machine Learning

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Abstract: The present article deals with the role of the negative stiffness of structures and the identification of negative stiffness by Artificial Intelligence and Machine Learning (ML). Generally, structures have positive stiffness combined with damping during earthquake-induced oscillations, which leads to energy loss and causes a reduction in the amplitude of the vibration. However, when a structure falls into the nonlinear region of a structure, the appearance of negative stiffness is possible. If this occurs, then the mathematical solution of equation of motion is changed drastically. This is the exact point covered by the present article and a suitable procedure (namely the key-point) for ML is given. This paper aims to define and solve the mathematical equation of motion for a Single Degree of Freedom (SDoF) oscillator with damping and negative stiffness due to various dynamic loading conditions. The derived solutions indicate that, for every case of dynamic loading, oscillation with negative stiffness is absent, and the structure's response increases exponentially. The abovementioned facts are verified by the presentation of a benchmark example that gives the exact values of response of a SDoF system with negative stiffness.

Keywords: machine learning; seismic resistance design; seismic isolation; negative stiffness; negative stiffness system; damping

1. Introduction

The dynamic behavior of a structure depends on its mechanical characteristics as well as the characteristics of the excitation. However, earthquake-induced resonance is inevitable at some point during the structure's lifetime, potentially causing significant damage. As a result, structural engineers have investigated seismic isolation methods by designing seismic isolators that shift the structure's fundamental period out of the range of earthquake frequencies, thereby avoiding resonance between the structure and seismic excitation. However, when seismic excitations have a long duration, it is crucial that structures also possess sufficient ductility. This can be achieved either by increasing the ductility of the structure itself or by installing supplemental damping systems in specific regions of the structure [1–3]. Moreover, seismic isolation often causes large displacements in the region of the structure's seismic joints, thus it is crucial to be eliminated.

At first sight, the application of negative stiffness systems in seismic isolation of civil engineer structures seems to be a bit obscure and even dangerous for the structure's stability. This fact led many researchers to use negative stiffness systems so as to have limited impact on the structure's stiffness, for example the negative stiffness is considered as 20% of the structure's stiffness. The above consideration seems to provide many advantages, according to the relevant studies conducted [4,5]. Also, often the displacements of the negative stiffness system are too large, so in order to not have an unstable state of the system [6], a "temporary limitation" is used in order to eliminate the action of negative stiffness and the system gains its positive stiffness again. The system has positive stiffness for a few seconds, and then it gains negative stiffness again for a few seconds. This procedure is going on until the earthquake incident comes to an end.

The implementation of negative stiffness as a property of structures, in order to provide seismic isolation, is a subject that attracts more and more researchers lately. This is evident through the various studies conducted in order to form innovative seismic isolation devices with negative stiffness mechanisms. However, it is often observed that the proposed systems seem to be functional in experimental examination, but there is no theoretical definition of the response i.e., the mathematical equations of the system's motion are not defined. Furthermore, in some cases, when a structure is investigated through pushover analysis, the stiffness becomes negative when the displacement exceeds a specific value. Indeed, in the studies of Makarios et al. [7–9], where several cases of real structures are examined, the results of the pushover analyses show that when the structures enter the Near Collapse (NC) state, a significant reduction of the system's stiffness occurs, and a descending branch appears to the pushover curves. Consequently, it is crucial in order to fully understand the behavior of such systems, to investigate thoroughly these systems, analytically and mathematically, so as for the correct and efficient design of seismic isolation elements to be possible, that will eliminate structure's oscillation due to dynamic loading.

The consideration and analytical investigation of a Single Degree of Freedom (SDoF) oscillator with negative stiffness undergoing free oscillation is a simple yet effective approach to achieve a deeper understanding of negative stiffness systems. It is important to note that civil engineering structures typically possess multiple degrees of freedom; therefore, a SDoF simulation is only applicable in specific cases, such as water towers or single-story buildings, where columns are considered weightless. Studying the behavior of SDoF systems under dynamic loading enhances our understanding of the key parameters in dynamic problems and provides valuable insights for further investigation of more complex systems [10,11]. A mathematical investigation of a SDoF oscillator with negative stiffness and no damping was conducted in a previous study [12]. However, in reality, all dynamic systems dissipate energy during oscillation and this is the issue of the present study. In buildings, energy dissipation occurs through friction between the superstructure and the foundation, friction between structural elements and infill walls, and other mechanisms. Therefore, this paper focuses on the formulation and solution of the equation of motion for a SDoF oscillator with positive mass, negative stiffness, and damping. Moreover, Machine Learning (ML) techniques are applied in structural engineering aiming to enhance structural health monitoring [13], damage detection [14], structural performance [15], material modeling [16], finite element analysis [17] and vibroacoustic analysis [18]. The idea of implementing ML in structural engineering has gained interest through the last years due to the growing progress in ML techniques [19]. The concept of ML deals with a group of methodologies that can be used to detect patterns in data, in an automatic way and thus can be used to form prediction models as well as to help with decision making under complex conditions [20]. Given that, we can assume if certain parameters are known, it is possible to define the behavior of a structural system under dynamic excitations, even for the case where the system's stiffness becomes negative. Specifically, artificial intelligence and ML can identify the phenomenon of negative stiffness through the capacity curve of the system as it results from the performance of the nonlinear pushover analysis. Indeed, the general procedure that can be applied by the Artificial Machine is the following: (i) firstl, artificial intelligence must ask for the capacity curve -such as given in Figure 1—of the structure until its collapse. (ii) The second step is to calculate the stiffness of the structure at each step analysis as it is given in Figure 1, (iii) the third step is to verify that stiffness has taken a negative value, namely k_N. If the last step is true, then it must inform us that there is an area with negative stiffness (Figure 1) and therefore we are falling in the theoretical gap, because any other common nonlinear response history analysis is incorrect and this is due to the change of the mathematical solution of the motion equation with negative stiffness where now the solution has exponential form (and not harmonic).

The above-mentioned conclusion by Artificial intelligence is true because structures that exhibit negative stiffness in the nonlinear region cannot be subjected to strong dynamic loading, because the equation of motion of the oscillation has a completely different mathematic form. This gap is covered by the present study. The present work gives the exact mathematical solution of the response of a single degree of freedom oscillator with negative stiffness, when it is excited at its base by a group of excitations from many harmonic terms. The study demonstrates that the system with negative stiffness results in greater energy dissipation compared to a SDoF oscillator with positive stiffness. Furthermore, the response displacement increases exponentially over time, indicating the absence of mass oscillation. This above-mentioned procedure constitutes the key-point for a novel machine learning technique for negative stiffness systems.

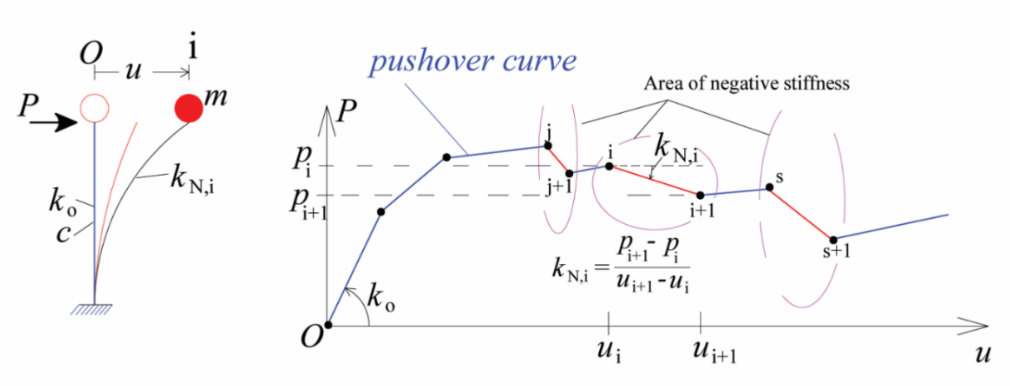


Figure 1. Machine Learning can identify the areas with Negative Stiffness, using the Pushover Curve. Note that point *i* represents every point of the pushover curve for which the stiffness of the system is negative.

2. Analytical and Numerical Investigation of the SDoF System with Damping and Negative Stiffness

2.1. Free Vibration with Damping ($c \neq 0$, $k_N < 0$)

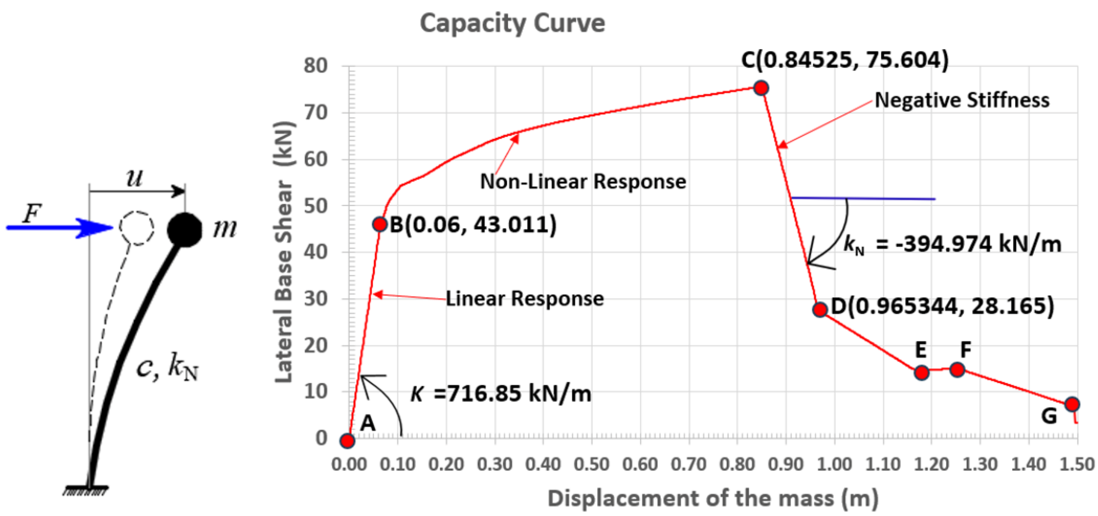
Assume the SDoF oscillator of Figure 2 that due to force *F* reaches at the point C, where it possesses damping and negative stiffness k_N and assume that it performs free vibration. Note that Figure 2 is representative of numerous analyses carried out, in which the occurrence of negative stiffness was prominently observed. Moreover, in the current study it was decided to present only the part that deals with the system's behavior when stiffness becomes negative, in order to highlight the necessity of the modification of the existing algorithms that calculate the response in non-linear region. Thus, the software programs that are used to conduct pushover analyses need to check in every step of the procedure if the stiffness of the system is negative or not. In case that the system's stiffness is positive, the calculations are conducted as usual. However, if the system's stiffness becomes negative, then the mathematical equations presented in this study should be applied for as long as the negative stiffness occurs. This approach represents the central objective of this work.

Starting from point C of the Capacity Curve of Figure 2, we are writing the equation of motion of the system as following:

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) + k_N \cdot u(t) = 0 \quad (1)$$

where k_N is negative, so we re-write Equation (1) using the absolute value $|k_N|$ and setting the negative sign outside the brackets in order to form a mathematically correct equation:

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) - |k_N| \cdot u(t) = 0 \quad (2)$$



(**a**) (**b**)

Figure 2. (**a**) SDoF system with positive mass *m*, damping *c* and (from point C has) negative stiffness k_N. (**b**) Capacity Curve of SDoF system.

By dividing the two parts of the equation by mass *m* we get:

$$\ddot{u}(t) + \frac{c}{m} \cdot \dot{u}(t) - \frac{|k_N|}{m} \cdot u(t) = 0 \tag{3}$$

We consider the ratio p^2 of two positive quantities as a positive quantity too.

$$\frac{|k_N|}{m} = p^2 \tag{4}$$

By Differential Calculus it is known that the solution of Equation (3) shall be of an exponential form, i.e.:

$$u(t) = e^{\lambda \cdot t} \tag{5}$$

where λ is defined by:

$$\lambda_{1,2} = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 + 4 \cdot p^2}}{2} \tag{6}$$

Note that there is only one case to take into consideration in order to calculate λ, as the discriminant Δ is always positive defined, so there are two real and discreet roots λ_1 and λ_2. In this case the two partial solutions $u_1(t)$ and $u_2(t)$ are:

$$u_1(t) = e^{\lambda_1 \cdot t} \tag{7}$$

$$u_2(t) = e^{\lambda_2 \cdot t} \tag{8}$$

The general solution of the initial differential Equation (3) is defined:

$$u(t) = A \cdot u_1(t) + B \cdot u_2(t) = A \cdot e^{\lambda_1 \cdot t} + B \cdot e^{\lambda_2 \cdot t} \tag{9}$$

where *A* and *B* are arbitrary constants that will be defined later. As it is shown the extracted general solution does not depict a harmonic oscillation but shows clearly that the SDoF system with damping has exponential displacement for every time step which is given by the following equation:

$$u(t) = A \cdot e^{\lambda_1 \cdot t} + B \cdot e^{\lambda_2 \cdot t}$$

Because it is known that:

$$\Delta = \left(\frac{c}{m}\right)^2 + 4 \cdot p^2 > 0$$

At this point, it is useful to define system's damping *c* as a function of the critical damping, $c = \zeta \cdot c_{cr}$ where ζ is a damping ratio with a range of values, either greater or minor than one.

$$c = \zeta \cdot c_{cr} = 2\zeta mp \tag{10}$$

The two real and discreet roots are given by:

$$\lambda_{1,2} = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 + 4 \cdot p^2}}{2} = \frac{-2\zeta p}{2} \pm \frac{\sqrt{(2\zeta p)^2 + 4p^2}}{2} = -\zeta p \pm p\sqrt{\zeta^2 + 1} \tag{11}$$

In order to calculate constants *A* and *B*, we assume that at the beginning of time (i.e., when *t*=0) the system possesses an initial velocity $\dot{u}(0) \neq 0$ (or $\dot{u}_0 \neq 0$) and an initial displacement $u(0) \neq 0$ (or $u_0 \neq 0$). By using these initial conditions, we get the following expressions:

$$A = \frac{\lambda_2 \cdot u(0) - \dot{u}(0)}{\lambda_2 - \lambda_1}$$

$$B = \frac{\dot{u}(0) - \lambda_1 \cdot u(0)}{\lambda_2 - \lambda_1}$$

Thus, Equation (9) is fully defined and describes a non-periodical motion, which means that the parameter *p* is not possible to describe an eigenfrequency:

$$u(t) = \frac{\lambda_2 \cdot u(0) - \dot{u}(0)}{\lambda_2 - \lambda_1} \cdot e^{\lambda_1 \cdot t} + \frac{\dot{u}(0) - \lambda_1 \cdot u(0)}{\lambda_2 - \lambda_1} \cdot e^{\lambda_2 \cdot t} \tag{12}$$

The velocity of mass is given by:

$$\dot{u}(t) = \lambda_1 \cdot \frac{\lambda_2 \cdot u(0) - \dot{u}(0)}{\lambda_2 - \lambda_1} \cdot e^{\lambda_1 \cdot t} + \lambda_2 \cdot \frac{\dot{u}(0) - \lambda_1 \cdot u(0)}{\lambda_2 - \lambda_1} \cdot e^{\lambda_2 \cdot t} \tag{13}$$

And the acceleration of mass $\ddot{u}(t)$ by:

$$\ddot{u}(t) = \lambda_1^2 \cdot \frac{\lambda_2 \cdot u(0) - \dot{u}(0)}{\lambda_2 - \lambda_1} \cdot e^{\lambda_1 \cdot t} + \lambda_2^2 \cdot \frac{\dot{u}(0) - \lambda_1 \cdot u(0)}{\lambda_2 - \lambda_1} \cdot e^{\lambda_2 \cdot t} \tag{14}$$

Figures 3–5 show the plots of the displacement, velocity and acceleration of a SDoF oscillator with mass m = 120 t, negative stiffness $k_N = -100$ kN/m, initial conditions $u_0 = 0.03$ m, $\dot{u}_0 = 0.07$ m/s and damping ratio ζ = 0.05, which performs free vibration. Note that despite damping, the system does not oscillate, and the displacement, velocity and acceleration are increasing exponentially by time.

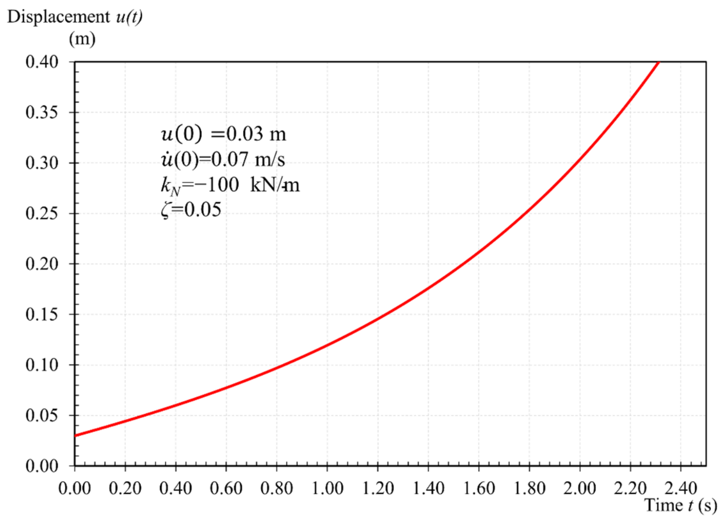


Figure 3. Displacements of the mass of the SDoF oscillator with damping and negative stiffness due to free vibration.

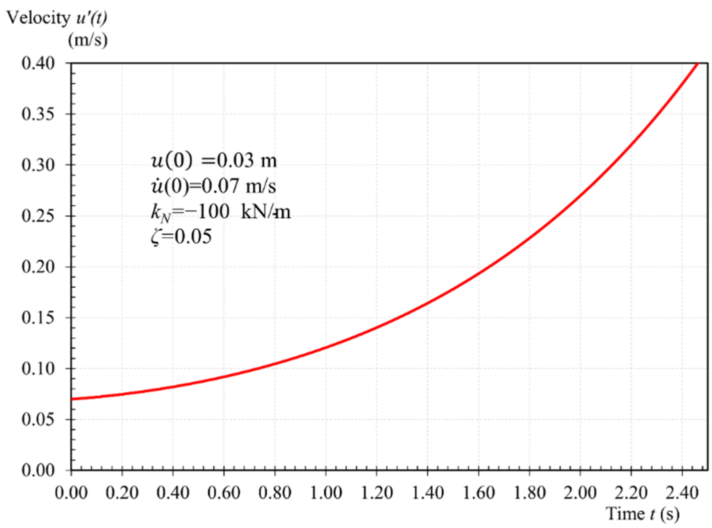


Figure 4. Velocities of the mass of the SDoF oscillator with damping and negative stiffness due to free vibration.

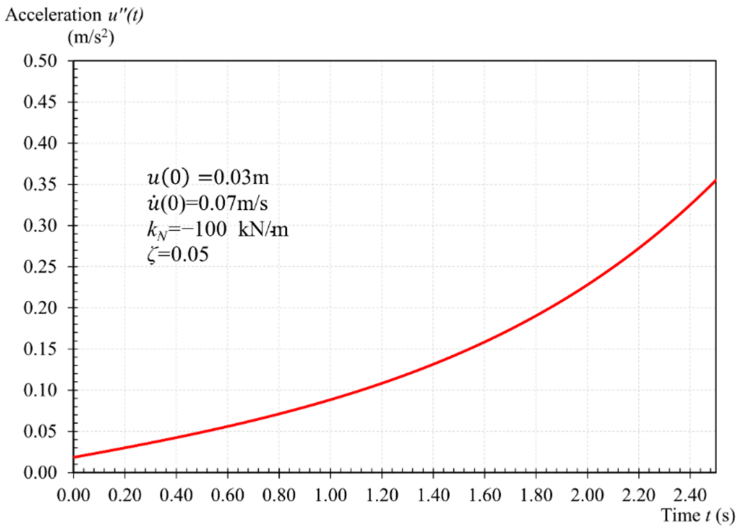


Figure 5. Accelerations of the mass of the SDoF oscillator with damping and negative stiffness due to free vibration.

Since the equations of displacement $u(t)$, velocity $\dot{u}(t)$ and acceleration $\ddot{u}(t)$ are defined it is possible to calculate the elastic force P_e, the damping force P_d and the inertial force P_a of the SDoF system respectively. Indeed:

$$P_e = -|k_N|\, u(t) \tag{15}$$

$$P_d = c\, \dot{u}(t) \tag{16}$$

$$P_a = m\, \ddot{u}(t) \tag{17}$$

Note that the above three forces are constantly in equilibrium according to D'Alembert's principle.

$$P_a + P_d + P_e = 0 \tag{18}$$

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) = |k_N| \cdot u(t) \tag{19}$$

This means that in every time moment *t* of the negative-stiffness-SDoF mass motion, the elastic force P_e is competitive to the sum of the other two forces (i.e., inertial force P_a and damping force P_d). Figure 6 shows the plots of the elastic force P_e, damping force P_d and inertial force P_a, respectively, for a SDoF system with mass m = 120 t, negative stiffness k_N = −100 kN/m, initial conditions $u_0 =$ 0.03 m, $\dot{u}_0 =$ 0.07 m/s and damping ratio ζ = 0.05. Observe that for every time moment *t* the sum of the inertial force P_a and damping force P_d is equal to the elastic force P_e, which means that in every time moment *t* the three forces are in equilibrium state.

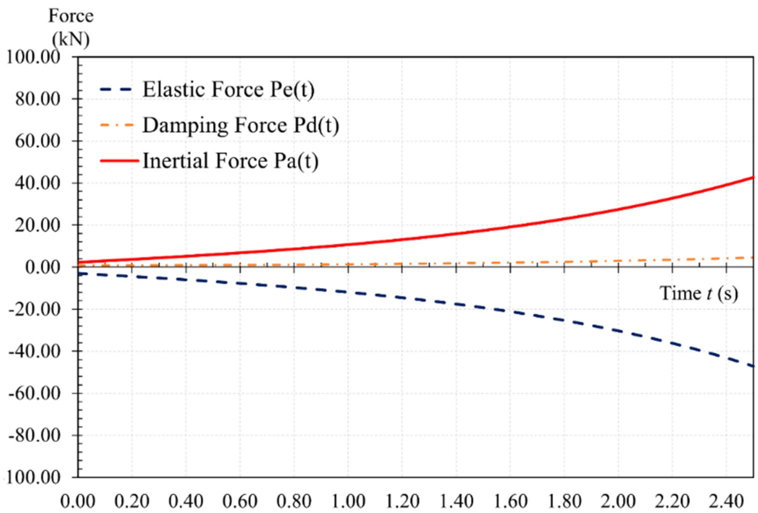


Figure 6. Plot of the elastic force, damping force and inertial force of the SDoF system with damping and negative stiffness due to free vibration.

2.2. Response of the SDoF System with Damping and Negative Stiffness Due to Harmonic Loading ($c \neq 0$, $k_N < 0$)

We consider again the SDoF system of Figure 7 with damping *c* and negative stiffness k_N which now is loaded with a dynamic harmonic load $F = F_o \cdot \sin(\Omega t)$.

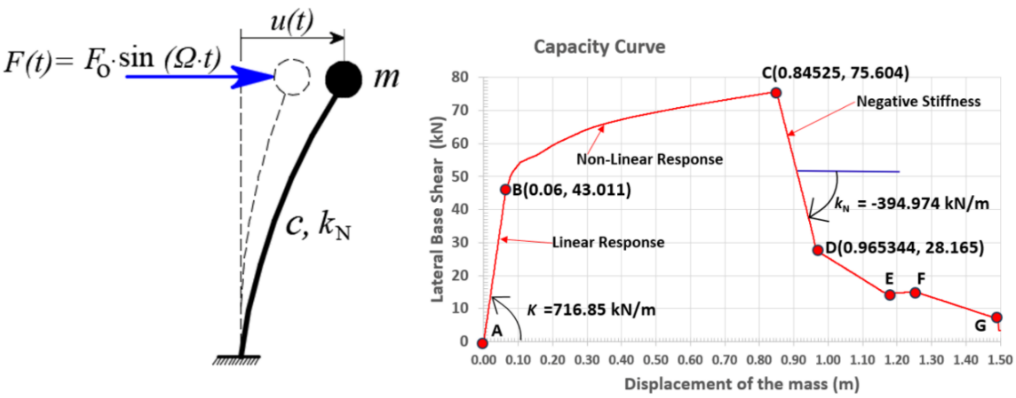


Figure 7. SDoF system with positive mass *m*, damping *c* and (from the point C has) negative stiffness k_N loaded with a dynamic harmonic load *F*.

Starting from point C of the Capacity Curve of Figure 7, we are writing the equation of motion of the system as following:

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) + k_N \cdot u(t) = F_o \cdot \sin(\Omega t) \tag{20}$$

However, as k_N is negative, we need to rewrite Equation (20) using the absolute value $|k_N|$ and considering the negative sign outside the brackets, in order to form a mathematically correct equation:

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) - |k_N| \cdot u(t) = F_o \cdot \sin(\Omega t) \tag{21}$$

According to Differential Calculus, the general solution of Equation (21) is the sum of two separate solutions: (a) the partial solution of the homogenous equation:

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) - |k_N| \cdot u(t) = 0$$

the solution of the above equation is:

$$u(t) = A \cdot e^{\lambda_1 \cdot t} + B \cdot e^{\lambda_2 \cdot t}$$

where constants *A* and *B* will be defined below, while λ_1 and λ_2 are calculatedas:

$$\lambda_1 = -\zeta p + p\sqrt{\zeta^2 + 1}$$

$$\lambda_2 = -\zeta p - p\sqrt{\zeta^2 + 1}$$

and (b) the partial solution due to the external harmonic loading $F_o \cdot \sin(\Omega t)$ which is known that is given by the following type:

$$u(t) = C\sin(\Omega t) + D\cos(\Omega t) \tag{22}$$

where constants *C* and *D* are defined as:

$$C = \frac{F_o}{|k_N|} \cdot \frac{-(1+\gamma^2)}{(1+\gamma^2)^2 + (2\zeta\gamma)^2} \tag{23}$$

$$D = \frac{F_o}{|k_N|} \cdot \frac{-2\zeta\gamma}{(1+\gamma^2)^2 + (2\zeta\gamma)^2} \tag{24}$$

where parameter *γ* is described as the ratio of the excitation's frequency *Ω* to the parameter p of the SDoF system, i.e., $\gamma = \frac{\Omega}{p}$.

So, the sum of the two partial solutions (a) and (b) gives us the general solution of Equation (21), as it follows:

$$u(t) = A\, e^{\lambda_1 \cdot t} + B\, e^{\lambda_2 \cdot t} + C\, \sin(\Omega t) + D\, \cos(\Omega t) \tag{25}$$

Differentiating Equation (25) once, the equation of velocity is extracted:

$$\dot{u}(t) = A\lambda_1 e^{\lambda_1 \cdot t} + B\lambda_2 e^{\lambda_2 \cdot t} + \Omega C\, \cos(\Omega t)\, - \Omega D\, \sin(\Omega t) \tag{26}$$

We now have a system of two equations available for the calculation of the unknown constants $A,\ B$. Setting as time $t = 0$ and solving this system, we get:

$$A = \frac{\lambda_2\, u(0) - D\lambda_2 - \dot{u}(0) + \Omega C}{\lambda_2 - \lambda_1} \tag{27}$$

$$B = \frac{\dot{u}(0)\, - \Omega C - \lambda_1\, u(0) + D\lambda_1}{\lambda_2 - \lambda_1} \tag{28}$$

Also, in order to define the equation of acceleration $\ddot{u}(t)$ of mass m we differentiate once again with respect to time t the equation of velocity and we get:

$$\ddot{u}(t) = A{\lambda_1}^2 e^{\lambda_1 \cdot t} + B{\lambda_2}^2 e^{\lambda_2 \cdot t} - \Omega^2 C\, \sin(\Omega t) - \Omega^2 D\, \cos(\Omega t) \tag{29}$$

Figures 8–10 show the plots of the displacement, velocity and acceleration of the system according to time t with a mass m = 120 t, negative stiffness k_N = −100 kN/m, initial conditions $u_0 =$ 0.03 m, $\dot{u}_0 =$ 0.07 m/s and damping ζ = 0.05, which oscillates due to a harmonic loading $F = 10 \cdot \sin(9t)$, where $\Omega = 9\frac{\text{rad}}{\text{s}}, F_o =$ 10 kN. Observe that despite the fact that the system has damping, the system does not oscillate, and the displacement, velocity and acceleration of the system are increasing exponentially by time.

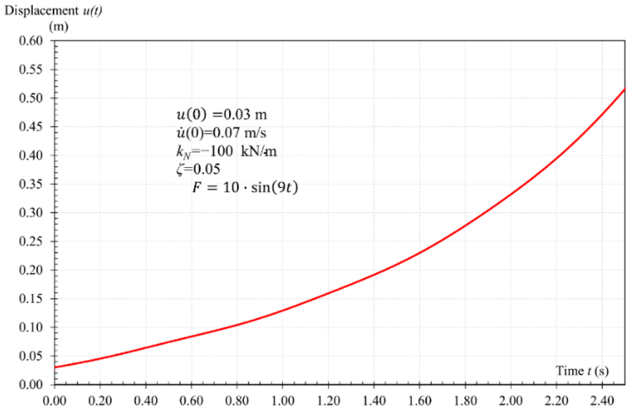


Figure 8. Displacements of the mass of the SDoF oscillator with damping and negative stiffness due to harmonic loading.

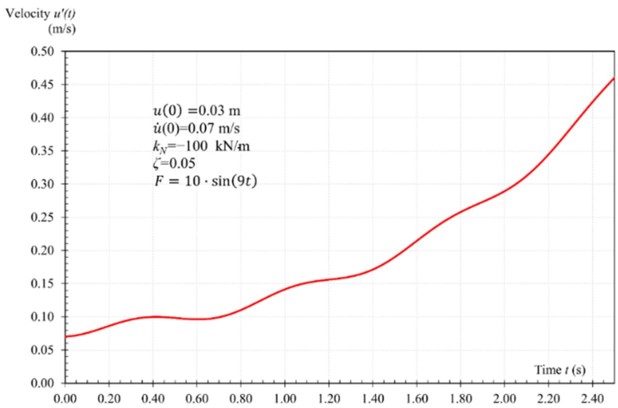


Figure 9. Velocities of the mass of the SDoF oscillator with damping and negative stiffness due to harmonic loading.

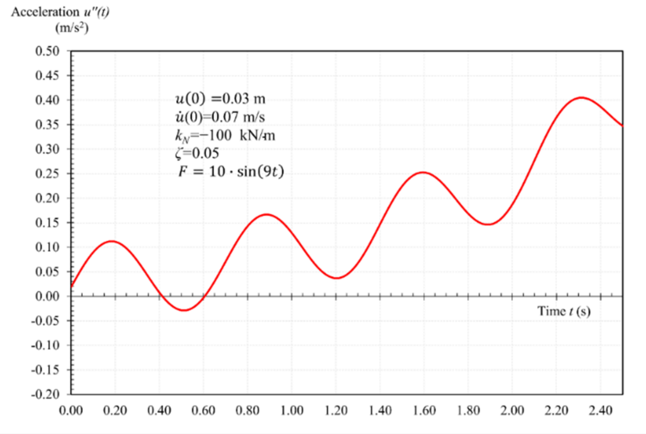


Figure 10. Accelerations of the mass of the SDoF oscillator with damping and negative stiffness due to harmonic loading.

Since we have defined displacement $u(t)$ by Equation (25), velocity $\dot{u}(t)$ by Equation (26), acceleration $\ddot{u}(t)$ by Equation (29), we can easily calculate in every time moment the elastic force P_e, damping force P_d and inertial force P_a, respectively, of the SDoF system. Indeed:

$$P_e = -|k_N|\ u(t) \tag{30}$$

$$P_d = c\,\dot{u}(t) \tag{31}$$

$$P_a = m\,\ddot{u}(t) \tag{32}$$

Moreover, note that the impact of the external harmonic excitation $F_o \sin(\Omega t)$ should be taken under consideration. These four forces, that are acting on mass m of the system, are always in equilibrium state, according to D'Alembert's principle.

Figure 11 shows the plots of the elastic force P_e, damping force P_d and inertial force P_a, respectively, of the SDoF system with mass m = 120 t, negative stiffness k_N = −100 kN/m, initial conditions $u_0 =$ 0.03 m, $\dot{u}_0 =$ 0.07 m/s and damping ratio ζ = 0.05 which is loaded with a harmonic force $F = 10 \cdot \sin(9t)$, where $\Omega =$ 9 rad/s, $F_o = 10$ kN. Observe that in every time moment, the sum of the inertial force P_a, damping force P_d and elastic force P_e equals to the value of the external excitation F which leads to the equilibrium state of the forces in every time step.

In order to prove this, we choose two-time moments $t_1 =$ 0.24 s and $t_2 = T_D = \frac{2\pi}{\omega_D} = 0.786$ s and we calculate the sum of the forces for each time moment.

For time moment t_1:

$$P_{e,1} = -|k_N|\,u(t_1) = -100 \cdot (0.04883) \ = -4.883 \text{ kN}$$

$$P_{d,1} = c\,\dot{u}(t_1) = 10.95 \cdot (0.090) = 0.9911 \text{ kN}$$

$$P_{a,1} = m\,\ddot{u}(t_1) = 120 \cdot 0.1017 = 12.2057 \text{ kN}$$

$$F_1 = F_O \cdot \sin(\Omega t) = 10 \cdot sin(9 \cdot 0.24) = 8.3138 \text{ kN}$$

The sum of the above forces, considering them as vectors, is calculated:

$$P_{e,1} + P_{d,1} + P_{a,1} = -4.883 \text{ kN} + 0.9911 \text{ kN} + 12.2057 \text{ kN} = 8.3138 \text{ kN} = F_1$$

Thus, the forces are in equilibrium state.

For time moment t_2:

$$P_{e,2} = -|k_N|\,u(t_2) = -100 \cdot 0.102479 \ = -10.2479 \text{ kN}$$

$$P_{d,2} = c\,\dot{u}(t_2) = 10.95 \cdot 0.1082 = 1.1855 \text{ kN}$$

$$P_{a,2} = m\,\ddot{u}(t_2) = 120 \cdot 0.1347 = 16.1716 \text{ kN}$$

$$F_2 = F_O \cdot \sin(\Omega t) = 100 \cdot \sin(9 \cdot 0.786) = 7.1092 \text{ kN}$$

The sum of the above forces, considering them as vectors, is calculated:

$$P_{e,2} + P_{d,2} + P_{a,2} = -10.2479 \text{ kN} + 1.1855 \text{ kN} + 16.1716 \text{ kN} = 7.1092 \text{ kN}$$

Thus, the forces are in equilibrium state. Now it is crucial to examine how displacement changes when the excitations frequency is equal to parameter p. For $p = \Omega$ we get $\gamma = \frac{\Omega}{p} = 1$ and constants C and D are defined:

$$C = \frac{F_o}{|k_N|} \cdot \frac{-(1+1^2)}{(1+1^2)^2 + (2\zeta \cdot 1)^2} = -\frac{F_o}{|k_N|} \cdot \frac{2}{4+4\zeta^2}$$

$$D = \frac{F_o}{|k_N|} \cdot \frac{-2 \cdot \zeta \cdot 1}{(1+1^2)^2 + (2 \cdot 1)^2} = -\frac{F_o}{|k_N|} \cdot \frac{\zeta}{4}$$

Given that constants A and B are relevant to constants C and D:

$$A = \frac{\lambda_2\,u(0) - D\lambda_2 - \dot{u}(0) + \Omega C}{\lambda_2 - \lambda_1}$$

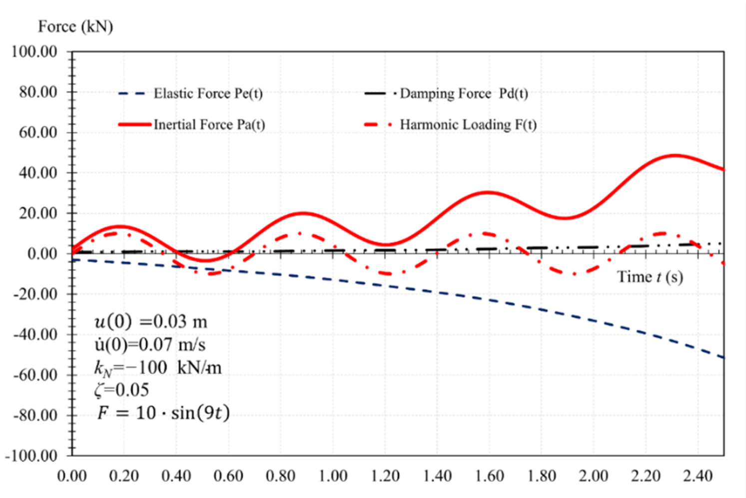


Figure 11. Plot of the elastic force P_e, the damping force P_d and the inertial force P_a, respectively of the SDoF system due to the harmonic loading $F = F_O \cdot \sin(\Omega t)$.

$$B = \frac{\dot{u}(0) - \Omega C - \lambda_1\,u(0) + D\lambda_1}{\lambda_2 - \lambda_1}$$

And given that the equation of the system's displacement is:

$$u(t) = A\,e^{\lambda_1 \cdot t} + B\,e^{\lambda_2 \cdot t} + C\sin(\Omega t) + D\cos(\Omega t)$$

Then, considering a SDoF system with mass m = 120 t, negative stiffness k_N = −100 kN/m, initial conditions $u_0 = 0.0$ m, $\dot{u}_0 = 0.0$ m/s and damping ratio ζ = 0.05 which oscillates due to an external harmonic loading $F = 5 \cdot \sin(2t)$, where $p = \Omega = 2$ rad/s, $F_o = 10$ kN the following plot (Figure 12) is formed. Observe that in contrast with the case where the SDoF had positive stiffness k_{p1}, there is no oscillation now that the system has negative stiffness. Thus, it is not possible for the phenomenon of resonance to happen in systems with negative stiffness.

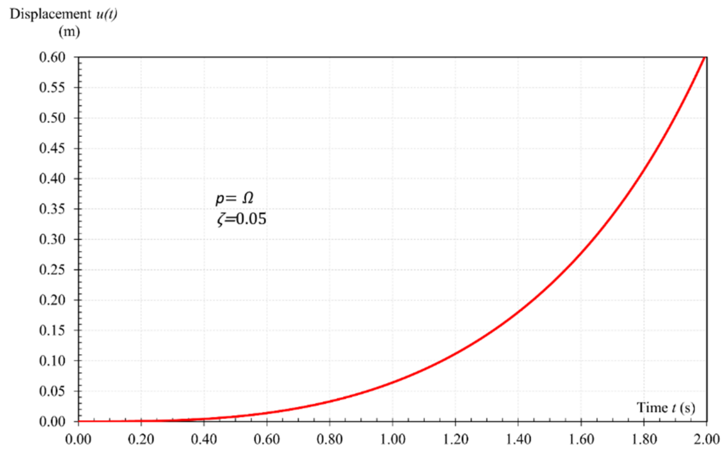


Figure 12. Displacements of the SDoF system's mass with negative stiffness due to harmonic loading and considering $p = \Omega$.

2.3. Response of the SDoF System with Damping and Negative Stiffness Due to Seismic Excitation ($c \neq 0$, $k_N < 0$)

We consider a SDoF oscillator of mass m, with damping c and negative stiffness k_N which is shown in Figure 13 and we are going to examine this oscillator for a horizontal motion of its base that is described by a harmonic (sinusoidal) time function. Firstly, we are assuming that the SDoF's base is in equilibrium state and at the time $t = 0$ begins to move horizontally for a specific time duration, such as the base displacement $u_g(t)$ is described by the following equation:

$$u_g(t) = \Pi \cdot \sin(\Omega t) \tag{33}$$

where Π is the ground oscillation amplitude (in m) and Ω is the circular frequency of the harmonic motion of the base, in rad/s. The first derivative of Equation (33) with respect to time gives the equation of velocity at the base:

$$\dot{u}_g(t) = \Omega\Pi \cdot \cos(\Omega t) \tag{34}$$

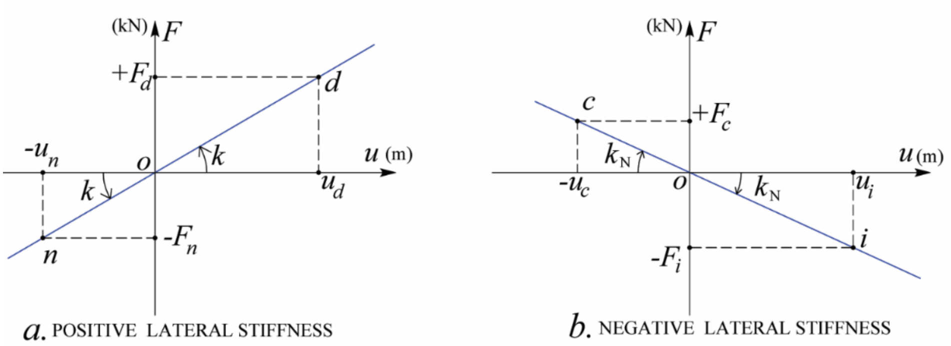


Figure 13. Positive (**a**) and Negative (**b**) lateral Stiffness.

While the second derivative of Equation (34) with respect to time gives the equation of acceleration at the base:

$$\ddot{u}_g(t) = -\Omega^2\, \Pi \cdot \sin(\Omega t) \tag{35}$$

Setting $t = 0$ (beginning of the excitation) in Equations (33)-(35) we get the initial conditions at the base of the oscillator:

$$u_g(0) = \Pi \cdot \sin(0) = 0$$

$$\dot{u}_g(0) = \Omega\Pi \cdot \cos(0) = \Omega\Pi$$

$$\ddot{u}_g(0) = -\Omega^2\, \Pi \cdot \sin(0) = 0$$

According to Figure 14, for a random time moment *t* the SDoF oscillator's mass has a displacement equal to $u_{\text{tot}}(t)$ which is calculated as:

$$u_{\text{tot}}(t) = u_{\text{g}}(t) - u(t) \tag{36}$$

This happens because we assume that the oscillator possesses a negative stiffness system at its base that reverses the direction of the ground motion when the seismic incident happens. Figure 14 shows a SDoF oscillator in the initial equilibrium state *O* and when the ground displacement has positive direction i.e., when the motion occurs along the positive semiaxis *Ox*, the oscillator's base is located at point *O'* due to the reverse of the ground motion direction. We also assume that the mass has a positive relative displacement $+u$ compared to the oscillator's base. Thus, the real distance that mass *m* executes is given by Equation (36) and as a result the inertial force P_a has a positive direction along the positive semiaxis *Ox*, while the damping force P_d has a positive direction along the negative semiaxis *Ox*. Last, the elastic force P_e that is acting upon mass, has a positive direction along the positive semiaxis *Ox* due to the action of negative stiffness. The value of the above forces, is calculated as it is demonstrated below. Mass m of the system has a total acceleration $u_{\text{tot}}(t)$ which leads to an inertial force P_a (Newton law) acting on mass, equal to (Figure 13):

$$P_a = m\,\ddot{u}_{\text{tot}}(t) = m\left[\ddot{u}_{\text{g}}(t) - \ddot{u}(t)\right] = m \cdot \ddot{u}_{\text{g}}(t) - m \cdot \ddot{u}(t) \tag{37}$$

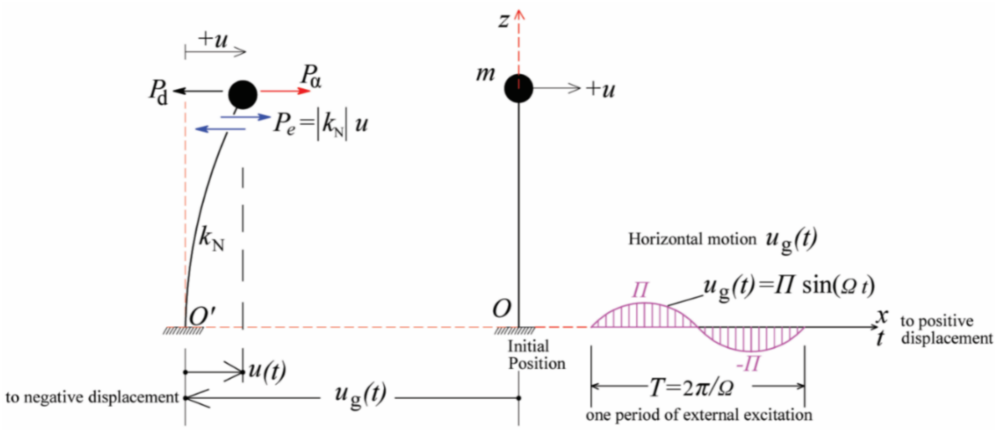


Figure 14. SDoF oscillator with positive mass *m*, damping *c* and negative stiffness k_N. under seismic excitation.

Simultaneously, there is another force acting on mass, the damping force P_d (Kelvin-Voight law) with opposite sign from the inertial force P_a. Also, there is the elastic force P_e (Hooke law) with the same sign to the inertial force P_a:

$$P_d = c \cdot \dot{u}(t) \tag{38}$$

$$P_e = |k_N| \cdot u(t) \tag{39}$$

Thus, according to D'Alembert principle and considering the positive axis to the right, we get:

$$P_a - P_d + P_e = 0 \tag{40}$$

The equation of motion of mass is written:

$$m \cdot \ddot{u}_{\text{g}}(t) - m \cdot \ddot{u}(t) - c \cdot \dot{u}(t) + |k_N| \cdot u(t) = 0$$

$$-m \cdot \ddot{u}_{\text{g}}(t) + m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) - |k_N| \cdot u(t) = 0$$

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) - |k_N| \cdot u(t) = m \cdot \ddot{u}_{\text{g}}(t) \tag{41}$$

By inserting Equation (35) into Equation (41):

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) - |k_N| \cdot u(t) = -m\,\Omega^2\,\Pi \cdot \sin(\Omega t) \tag{42}$$

Also, if we consider that the sinus term of the right part of Equation (42) represents a harmonic loading *F*, i.e.:

$$F = -m\,\Omega^2 \Pi \tag{43}$$

then Equation (42) can be written as:

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) - |k_N| \cdot u(t) = F \cdot \sin(\Omega t) \tag{44}$$

Consequently, we get an equation of motion similar to the one we would obtain as if the oscillator was loaded by a harmonic loading on its mass. Therefore, we consider again the ideal SDoF system of Figure 15 with damping and negative stiffness k_N which now is loaded with a dynamic harmonic lateral load $F = F_o \cdot \sin(\Omega t)$.

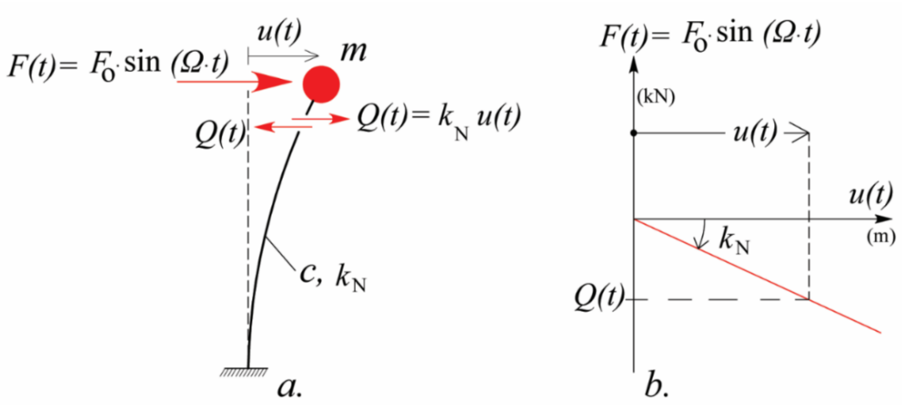


Figure 15. (**a**) SDoF system with (positive) mass *m*, damping *c*and negative stiffness k_N loaded with a dynamic harmonic lateral load *F(t)*. (**b**) graphical presentation of the harmonic lateral load *F(t)*

The equation of motion for this system is written as:

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) + k_N \cdot u(t) = F_0 \cdot \sin(\Omega t) \tag{45}$$

However, as k_N is negative, we need to rewrite Equation (45) using the absolute value $|k_N|$ and considering the negative sign outside the brackets, in order to form a mathematically correct equation:

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) - |k_N| \cdot u(t) = F_0 \cdot \sin(\Omega t) \tag{46}$$

According to Differential Calculus, the general solution of Equation (46) is the sum of two separate solutions: (a) the partial solution of the homogenous equation:

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) - |k_N| \cdot u(t) = 0$$

the solution of the above equation is (Chatzikonstantinou et al. [21]):

$$u(t) = A \cdot e^{\lambda_1 \cdot t} + B \cdot e^{\lambda_2 \cdot t}$$

where constants *A* and *B* will be defined below, while λ_1 and λ_2 are calculated by:

$$\lambda_1 = -\zeta p + p\sqrt{\zeta^2 + 1}$$

$$\lambda_2 = -\zeta p - p\sqrt{\zeta^2 + 1}$$

and (b) the partial solution due to the external harmonic lateral load $F_0 \cdot \sin(\Omega t)$ which will be defined immediately. Indeed, in order to calculate the partial solution, first we divide both parts of Equation (46) by mass *m* and replace *c* by $2\zeta mp$:

$$\ddot{u}(t) + \frac{c}{m} \cdot \dot{u}(t) - \frac{|k_N|}{m} \cdot u(t) = \frac{F_0 \cdot \sin(\Omega t)}{m} \tag{47}$$

We consider the ratio p^2 of two positive quantities as a positive quantity too, that cannot represent frequency, as we demonstrated above (because vibration does not exist):

$$\frac{|k_N|}{m} = p^2 \tag{48}$$

At this point, in order to have a comparable solution to the case of positive stiffness and without loss of generality, it is useful to define system's damping c as a function of the of the quantity $2mp$ (where in the case of the positive stiffness is known as the «critical damping», $c_{cr} = 2mp$), where ζ is a damping ratio with a range of values, either greater or minor than one. We also consider:

$$c = 2\zeta mp \tag{49}$$

By replacing Equations (48) and (49) into Equation (47) we get:

$$\ddot{u}(t) + 2\zeta p \cdot \dot{u}(t) - p^2 \cdot u(t) = \frac{F_o}{m} \cdot \sin(\Omega t) \tag{50}$$

From differential Calculus it is known that the partial solution of Equation (50) is given by:

$$u(t) = C\sin(\Omega t) + D\cos(\Omega t) \tag{51}$$

The first two derivatives of Equation (51), with respect to time *t* are calculated as:

$$\dot{u}(t) = \Omega C\cos(\Omega t) - \Omega D\sin(\Omega t) \tag{52}$$

$$\ddot{u}(t) = -\Omega^2 C\sin(\Omega t) - \Omega^2 D\cos(\Omega t) \tag{53}$$

By inserting Equations (52) and (53) into Equation (50) we get:

$$(-\Omega^2 C\sin(\Omega t) - \Omega^2 D\cos(\Omega t)) + 2\zeta p(\Omega C\cos(\Omega t) - \Omega D\sin(\Omega t)) - p^2(C\sin(\Omega t) + D\cos(\Omega t)) = \frac{F_o}{m} \cdot \sin(\Omega t) \Rightarrow$$

$$(-p^2 C - \Omega^2 C - 2\zeta p\Omega D)\sin(\Omega t) + (-p^2 D - \Omega^2 D + 2\zeta p\Omega C)\cos(\Omega t) = \frac{F_o}{m} \cdot \sin(\Omega t) + 0 \cdot \cos(\Omega t) \tag{54}$$

In order Equation (54) to be accurate for every time moment *t* the sinus and cosines coefficients of the first part of the equation need to be equal to the respective ones of the second part of the equation, so:

$$-p^2 C - \Omega^2 C - 2\zeta p\Omega D = \frac{F_o}{m} \tag{55}$$

$$-p^2 D - \Omega^2 D + 2\zeta p\Omega C = 0 \tag{56}$$

Then, Equations (55) and (56) are rewritten as:

$$\left(-1 - \frac{\Omega^2}{p^2}\right)C - \frac{2\zeta\Omega}{p}D = \frac{\mathrm{F_o}}{mp^2}$$

$$\frac{2\zeta\Omega}{p}C + \left(-1 - \frac{\Omega^2}{p^2}\right)D = 0$$

These last two equations are a system of two equations with two unknown parameters *C* and *D*, so by using equation $|k_N| = mp^2$ to solve the system in order to define these factors, and by inserting the term γ as the ratio of the excitation's frequency Ω to the parameter p of the SDoF system, i.e., $\gamma = \Omega/p$, constants *C* and *D* are defined as:

$$C = \frac{\mathrm{F_o}}{|k_N|} \cdot \frac{-(1+\gamma^2)}{(1+\gamma^2)^2 + (2\zeta\gamma)^2} \tag{57}$$

$$D = \frac{\mathrm{F_o}}{|k_N|} \cdot \frac{-2\zeta\gamma}{(1+\gamma^2)^2 + (2\zeta\gamma)^2} \tag{58}$$

So, the sum of the two partial solutions (a) and (b) gives us the general solution of Equation (46), as it follows:

$$u(t) = A\,e^{\lambda_1 \cdot t} + B\,e^{\lambda_2 \cdot t} + C\sin(\Omega t) + D\cos(\Omega t) \tag{59}$$

If we consider time $t = 0$ in Equation (59), then we get:

$$\mathrm{u}(0) = A \cdot e^0 + B \cdot e^0 + C\sin(0) + D\cos(0)$$

$$A + B = u(0) - D \tag{60}$$

Also, the derivative of Equation (59) once, the equation of velocity is extracted:

$$\dot{u}(t) = A\lambda_1 e^{\lambda_1 \cdot t} + B\lambda_2 e^{\lambda_2 \cdot t} + \Omega C\cos(\Omega t) - \Omega D\sin(\Omega t) \tag{61}$$

Next, we consider time $t = 0$ in Equation (61), so we get:

$$\dot{\mathrm{u}}(0) = A\lambda_1 e^0 + B\lambda_2 e^0 + \Omega C\cos(0) - \Omega D\sin(0)$$

$$A\lambda_1 + B\lambda_2 = \dot{u}(0) - \Omega C \tag{62}$$

We now have the following system of two equations (Equations (60) and (62)) available for the calculation of the unknown constants $A,\ B$:

$$A = \frac{\lambda_2\,\mathrm{u}(0) - D\lambda_2 - \dot{\mathrm{u}}(0) + \Omega C}{\lambda_2 - \lambda_1} \tag{63}$$

$$B = \frac{\dot{\mathrm{u}}(0) - \Omega C - \lambda_1\,\mathrm{u}(0) + D\lambda_1}{\lambda_2 - \lambda_1} \tag{64}$$

Also, in order to define the equation of acceleration $\ddot{u}(t)$ of mass m we get the derivative once again with respect to time t the equation of velocity:

$$\ddot{u}(t) = A{\lambda_1}^2 e^{\lambda_1 \cdot t} + B{\lambda_2}^2 e^{\lambda_2 \cdot t} - \Omega^2 C\sin(\Omega t) - \Omega^2 D\cos(\Omega t) \tag{65}$$

Since we have defined displacement $u(t)$ by Equation (59), velocity $\dot{u}(t)$ by Equation (61), acceleration $\ddot{u}(t)$ by Equation (65), we can easily calculate in every time moment the elastic force P_e, damping force P_d and inertial force P_a, respectively, of the SDoF system. Indeed:

$$P_e = -|k_N|\,u(t) \tag{66}$$

$$P_d = c\,\dot{u}(t) \tag{67}$$

$$P_a = m\,\ddot{u}(t) \tag{68}$$

This means that for the ideal case that the ground motion consists of a harmonic motion, then the response of the SDoF oscillator with negative stiffness is identical to the displacement that the system would have if its mass *m* was loaded to a harmonic force. Given that the random seismic excitation consists of a sum of *N* harmonic terms, as it is proven through Fast Fourier Transform (FFT) analysis of the ground motion, where each of the terms is identical to Equation (33), it is possible to perform the above procedure for each one of these harmonic terms, find the solution for each one, and finally calculate the sum of them. In this way the precise solution of the displacement is extracted. This is very important, because when the precise displacement of a SDoF system with damping and negative stiffness is known, one can easily check the efficiency and stability of the various numerical methods, as well as to see if they converge to the correct values.

3. Numerical Example

Consider a SDoF oscillator of mass m = 100 t, negative stiffness k_N= −5 kN/m, where $p = \sqrt{\frac{|k_N|}{m}} = 0.2236$ and a damping ratio ζ = 0.05, so as $c = 2 \cdot \zeta \cdot m \cdot p = 2.23606$. The base of the oscillator is moving due to random seismic excitation. Assuming that the seismic loading (Figure 16) consists of the sum of 46 harmonic terms of Table 1, each of them creating an inertial harmonic force p_n to the oscillator's mass and is described by the following expression:

$$p_n = -m\,\Omega_n^2 \Pi_n \cdot \sin(\Omega_n t - \varphi_n)$$

where $P_n = -m\,\Omega_n^2 \Pi_n$, $\Omega_n = 2 \cdot \pi \cdot f_n$ is the circular frequency of the base excitation, T_n is period, Π_n is the amplitude and φ_n is the phase of the seismic excitation at its base. The values of each term's amplitude Π_n are chosen arbitrarily, between a range from 0.001 m to 0.00001 m. Period T_n of each harmonic term was chosen from a range between 1.00 s to 0.10 s and each harmonic term differs by 0.02 s relatively to the previous and the fore coming term. Phase φ_n was chosen from a range between 0.15 rad to 1.05 rad, and each harmonic term differs by 0.02 rad relatively to the previous and the fore coming term. These assumptions where set in order to form harmonic terms that if combined will result to a realistic form of a base ground excitation.

Table 1. Characteristics of the 46 harmonic terms that combine the base ground excitation.

A/A	Amplitude Π_i (m)	Phase φ_i (rad)	Frequency f_i (Hz)	Period T_i (s)	Frequency Ω_i (rad/s)
1	0.0003	0.15	1	1	6.28319
2	0.00025	0.17	1.02041	0.98	6.41143
3	0.0002	0.19	1.04167	0.96	6.54501
4	0.00015	0.21	1.06383	0.94	6.68424
5	0.0009	0.23	1.08696	0.92	6.82957
6	0.0009	0.25	1.11111	0.9	6.98131
7	0.00075	0.27	1.13636	0.88	7.13996
8	0.0007	0.29	1.16279	0.86	7.30603
9	0.0006	0.31	1.19048	0.84	7.48001
10	0.0005	0.33	1.21951	0.82	7.66241
11	0.0004	0.35	1.25	0.8	7.85398
12	0.00015	0.37	1.28205	0.78	8.05536
13	0.001	0.39	1.31579	0.76	8.26735
14	0.0035	0.41	1.35135	0.74	8.49078
15	0.003	0.43	1.38889	0.72	8.72665
16	0.0005	0.45	1.42857	0.7	8.97597
17	0.00026	0.47	1.47059	0.68	9.23999
18	0.0007	0.49	1.51515	0.66	9.51997
19	0.0006	0.51	1.5625	0.64	9.81748
20	0.0005	0.53	1.6129	0.62	10.13415
21	0.0003	0.55	1.66667	0.6	10.472
22	0.0009	0.57	1.72414	0.58	10.83309
23	0.0007	0.59	1.78571	0.56	11.21995
24	0.0008	0.61	1.85185	0.54	11.63552
25	0.0005	0.63	1.92308	0.52	12.08307
26	0.0007	0.65	2	0.5	12.56637
27	0.0004	0.67	2.08333	0.48	13.08995
28	0.0002	0.69	2.17391	0.46	13.65908
29	0.0003	0.71	2.27273	0.44	14.27998
30	0.00025	0.73	2.38095	0.42	14.95995
31	0.0001	0.75	2.5	0.4	15.70796
32	0.00055	0.77	2.63158	0.38	16.5347
33	0.0004	0.79	2.77778	0.36	17.45331
34	0.0006	0.81	2.94118	0.34	18.47998
35	0.0005	0.83	3.125	0.32	19.63495
36	0.0004	0.85	3.33333	0.3	20.94393
37	0.0003	0.87	3.57143	0.28	22.43996
38	0.00085	0.89	3.84615	0.26	24.16607
39	0.00075	0.91	4.16667	0.24	26.17996
40	0.00065	0.93	4.54545	0.22	28.5599
41	0.0002	0.95	5	0.2	31.41593
42	0.00001	0.97	5.55556	0.18	34.90661
43	0.00002	0.99	6.25	0.16	39.26991
44	0.00003	1.01	7.14286	0.14	44.87991
45	0.00004	1.03	8.33333	0.12	52.35986
46	0.00001	1.05	10	0.1	62.83185

From the summation of the above 46 harmonic terms of the above Table 1 we get the following accelerogram of the seismic excitation:

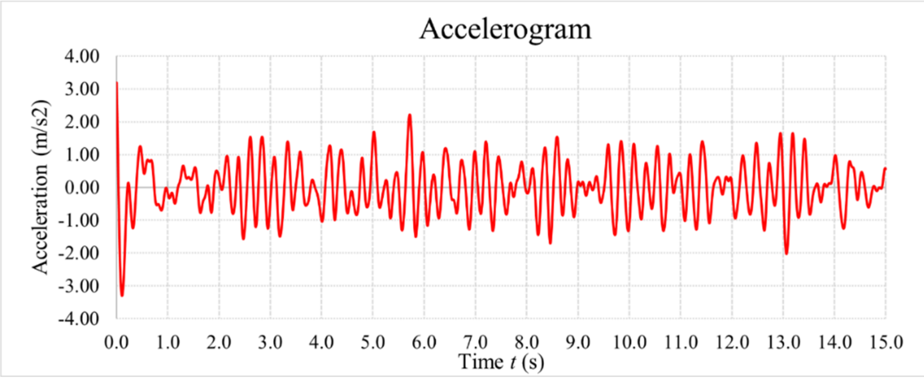


Figure 16. The accelerogram of the seismic excitation.

The maximum acceleration is 3.30 m/s^2 which is a little smaller than the value of the acceleration of seismic zone III as it is defined by the Greek Code for Seismic Resistant Structures 2000. Note that for the specific seismic zone the value of the acceleration is equivalent to 0.36g:

$$0.36 \cdot 9.81\,\mathrm{m/s^2} = 3.53\,\mathrm{m/s^2}$$

According to the accelerogram, the acceleration response spectrum is formed as below (Figure 17):

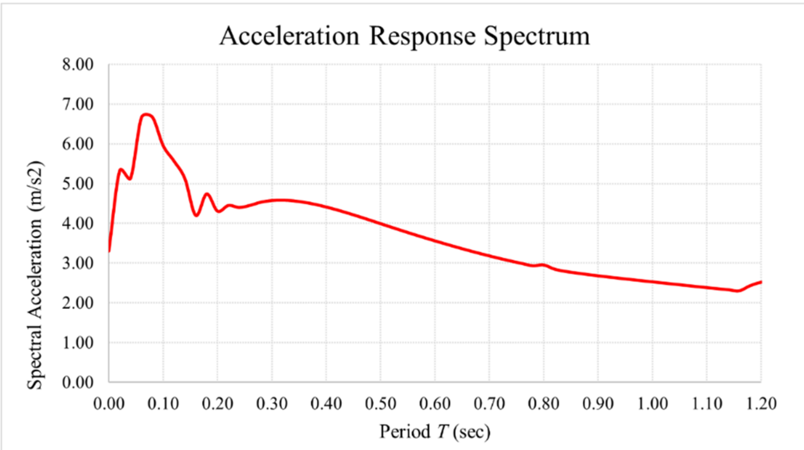


Figure 17. The acceleration response spectrum for the given accelerogram.

Now, the definition of the response for the first harmonic term will be presented. This term has the following characteristics: $\Omega_1 = 2 \cdot \pi \cdot f_1 = 6.28319\ \mathrm{rad/s}$, $\Pi_1 = 0.0003\ \mathrm{m}$, $\varphi_1 = 0.15\ \mathrm{rad}$, $P_1 = -m\,\Omega_1^2 \Pi_1 = -100 \cdot (6.28319)^2 \cdot 0.0003 = -1.18435\ \mathrm{kN}$, so the first harmonic term of the combination of the harmonic terms that describe the random base ground excitation of the numerical example, is written as:

$$u_{g,1}(t) = \Pi_1 \cdot \sin(\Omega_1 t - \varphi_1) = 0.0003 \cdot \sin(6.28319t - 0.15)$$

And the initial conditions are given as:

$$u_1(0) = u_{tot}(0) - u_{g,1}(0) = 0 - \Pi_1 \cdot \sin(\Omega_1 \cdot 0 - \varphi_1) = 4.48 \cdot 10^{-5}\mathrm{m}$$

and

$$\dot{u}_1(0) = \dot{u}_{tot}(0) - \dot{u}_{g,1}(0) = 0 - \Pi_1 \cdot \Omega_1 \cdot \cos(\Omega_1 \cdot 0 - \varphi_1) = -0.00186379\ \mathrm{m/s}$$

and

$$\ddot{u}(0) = \ddot{u}_{tot}(0) - \dot{u}_{g,1}(0) = 0 - (-1) \cdot \Omega_1^2 \cdot \Pi \cdot \sin(\Omega_1 \cdot 0 - \varphi_1) = -0.001769874\ \mathrm{m/s^2}$$

Thus, the response values (displacement $u_{1,a}$, velocity $\dot{u}_{1,a}$, acceleration $\ddot{u}_{1,a}$) of the first harmonic term of the excitation will be calculated by the following equations:

$$u_{1,a}(t) = Ae^{\lambda_1 \cdot t} + Be^{\lambda_2 \cdot t} + C\sin(\Omega_n t - \varphi_n) + D\cos(\Omega_n t - \varphi_n)$$

$$\dot{u}_{1,a}(t) = A\lambda_1 e^{\lambda_1 \cdot t} + B\lambda_2 e^{\lambda_2 \cdot t} + \Omega_n C\cos(\Omega_n t - \varphi_n) - \Omega_n D\sin(\Omega_n t - \varphi_n)$$

$$\ddot{u}_{1,a}(t) = A{\lambda_1}^2 e^{\lambda_1 \cdot t} + B{\lambda_2}^2 e^{\lambda_2 \cdot t} - \Omega_1^2 C\sin(\Omega_1 t - \varphi_1) - \Omega_1^2 D\cos(\Omega_1 t - \varphi_1)$$

where:

$$\lambda_1 = -\zeta p + p\sqrt{\zeta^2+1} = -0.05 \cdot 0.2236 + 0.2236\sqrt{0.05^2+1} = 0.212705$$

$$\lambda_2 = -\zeta p - p\sqrt{\zeta^2+1} = -0.05 \cdot 0.2236 - 0.2236\sqrt{0.05^2+1} = -0.235066$$

$$\gamma_1 = \frac{\Omega_1}{p} = 28.0992$$

$$C = \frac{F}{|k_N|} \cdot \frac{-(1+\gamma_1^2)}{(1+\gamma_1^2)^2 + (2\zeta\gamma_1)^2} = \frac{-m\Omega^2\Pi_i}{|k_N|} \cdot \frac{-(1+\gamma_1^2)}{(1+\gamma_1^2)^2 + (2\zeta\gamma_1)^2} = 0.0002996$$

$$D = \frac{F}{|k_N|} \cdot \frac{-2\zeta\gamma_1}{(1+\gamma_1^2)^2 + (2\zeta\gamma_1)^2} = \frac{-m\Omega^2\Pi_i}{|k_N|} \cdot \frac{-2\zeta\gamma_1}{(1+\gamma_1^2)^2 + (2\zeta\gamma_1)^2} = 1.06 \cdot 10^{-6}$$

$$A = \frac{\lambda_2 u_1(0) - D\lambda_2 - \dot{u}_1(0) + \Omega_1 C}{\lambda_2 - \lambda_1} = -0.00834$$

$$B = \frac{\dot{u}(0) - \Omega C - \lambda_1 u(0) + D\lambda_1}{\lambda_2 - \lambda_1} = 0.00839$$

Selecting a time step equal to $\Delta t = 0.005$ s and a total duration of the excitation equal to $t = 15$ s the values of displacement, velocity, acceleration are defined for every time moment for the first harmonic term of the 46 terms that contribute to the random seismic excitation. The plots of the response values of displacement, velocity and acceleration with respect to time are presented in Figures 18, 19 and 20 respectively.

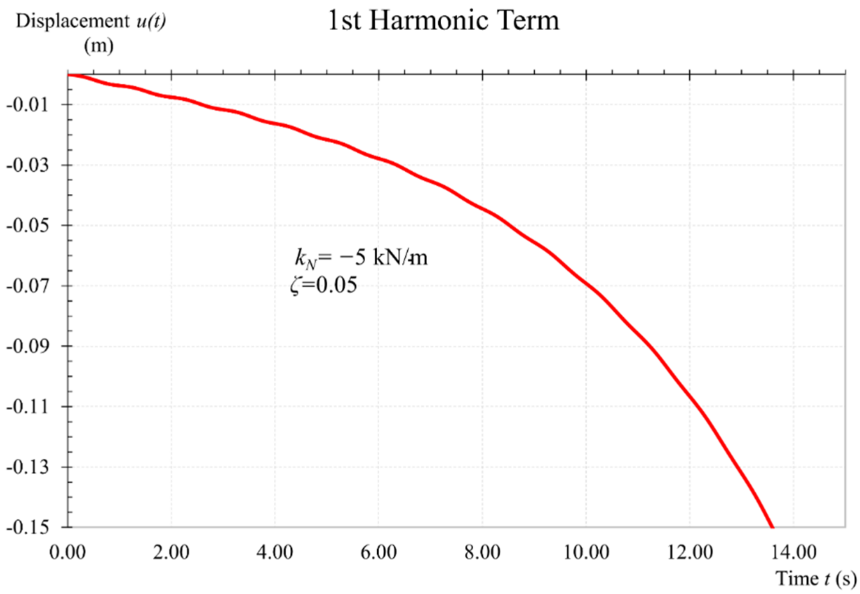


Figure 18. Displacements for the 1st harmonic term of the seismic base excitation of the SDoF oscillator with damping and negative stiffness.

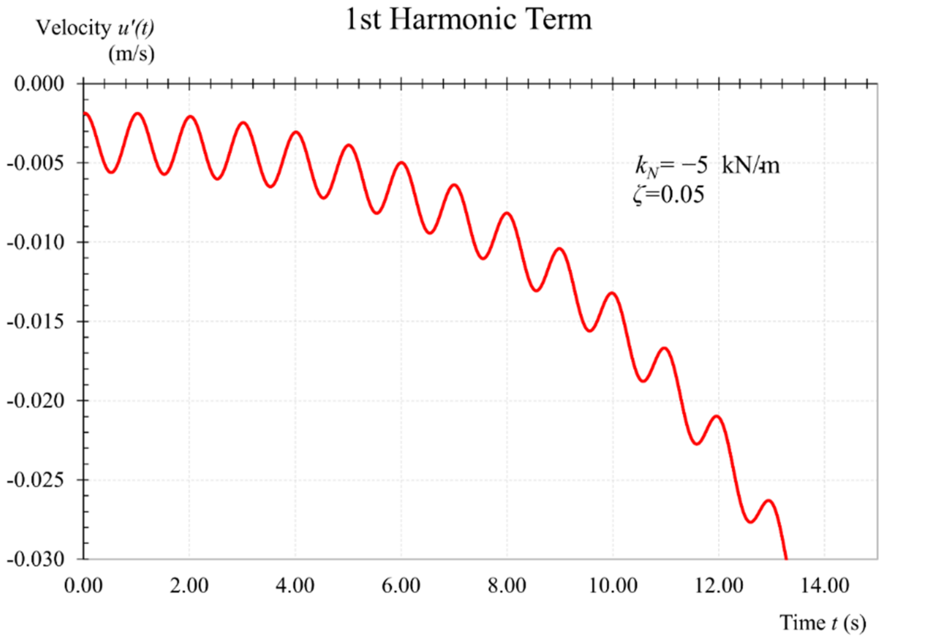


Figure 19. Velocities for the 1st harmonic term of the seismic base excitation of the SDoF oscillator with damping and negative stiffness.

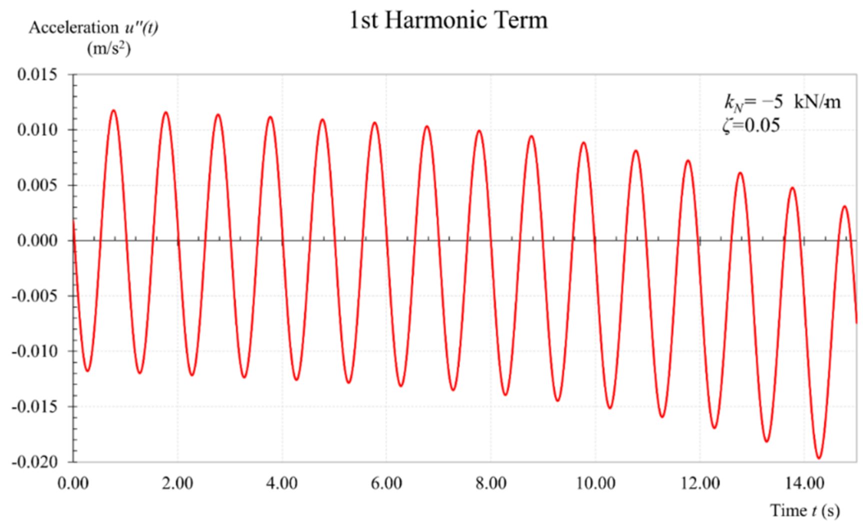


Figure 20. Accelerations for the 1st harmonic term of the seismic base excitation of the SDoF oscillator with damping and negative stiffness.

Consequently, by calculating for each harmonic term the response values $u(t)$, $\dot{u}(t)$, $\ddot{u}(t)$ using the abovementioned equations, the response of the SDoF oscillator under random seismic base excitation at time moment t is defined as the sum of the values of all the harmonic terms. This procedure was conducted for all the harmonic terms and then the values of response $u(t)$, $\dot{u}(t)$, $\ddot{u}(t)$ was defined for every time moment t, and they were used to create the plots presented in Figures 21–23. The main observation of these figures is that when a system possesses negative stiffness there is no oscillation. The motion is characterized by exponential growth of the displacements (compared to the case where the stiffness is positive) which leads to the development of large displacements that cannot be absorbed by the structure and as a result it finally collapses.

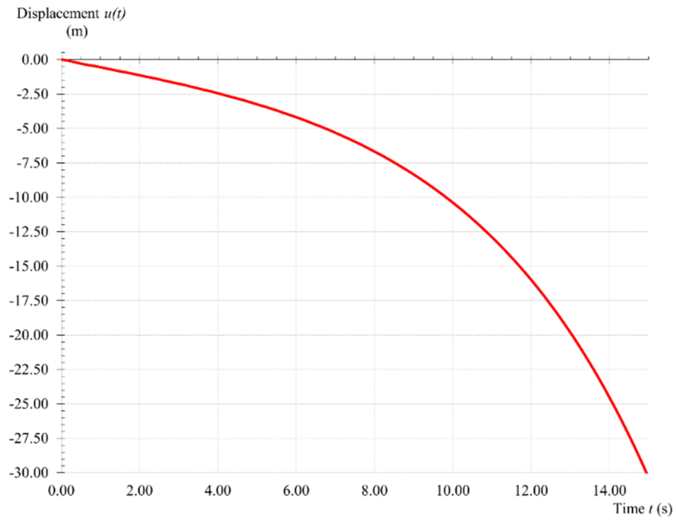


Figure 21. Displacements of the mass of the SDoF oscillator with damping and negative stiffness due to seismic base excitation defined as the sum of the values of all the harmonic terms (note the calculations were conducted for very large ideal displacements in order to ensure better oversight of the results).

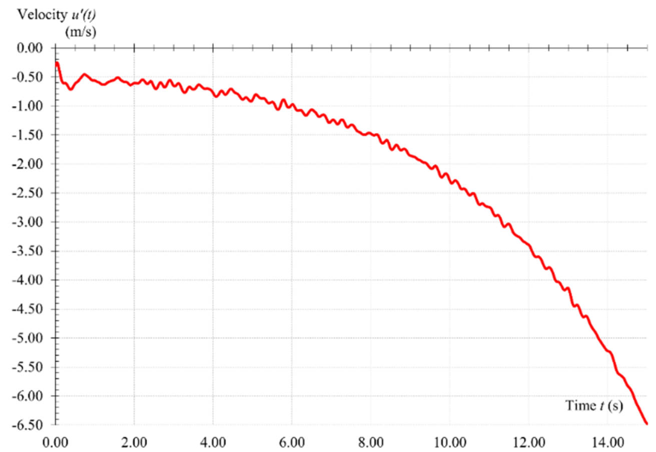


Figure 22. Velocities of the mass of the SDoF oscillator with damping and negative stiffness due to seismic base excitation defined as the sum of the values of all the harmonic terms.

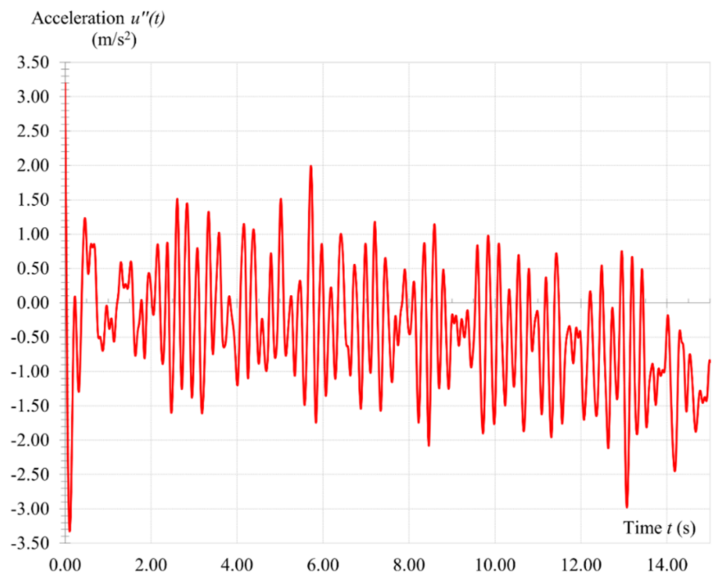


Figure 23. Accelerations of the mass of the SDoF oscillator with damping and negative stiffness due to seismic base excitation defined as the sum of the values of all the harmonic terms.

4. Conclusions

In this study, an attempt was made to use the advantages of Machine Learning techniques, in order to demonstrate a method that implements Machine Learning to identify the negative stiffness of a Single Degree of Freedom system. The concept of this key-point strategy has been presented through an analytical investigation of the behavior of a theoretical Single Degree of Freedom (SDoF) oscillator. The system has positive mass, damping, and negative stiffness, and its behavior was investigated under the following loading conditions: (i) free vibration, (ii) forced vibration due to dynamic harmonic loading, and (iii) forced vibration due to ground motion. The primary objective was to derive the theoretical/analytical equations that describe the system's response (displacement, velocity, acceleration) of the mass with respect to time. These equations, which are the core of the key-point and possess primary role (since it is a benchmark example that gives the exact values of response of a SDoF system with negative stiffness) represent the precise theoretical solution of the system's motion. Additionally, the study explored how the behavior of a negative stiffness system differs from that of a system with positive stiffness. The main conclusions are summarized below.

- The analytical investigation of the mass equation of motion revealed that the response of a SDoF system with damping and negative stiffness is described by an exponential equation that does not depict oscillation of the system's mass, regardless of whether damping is present. However, in order to be useful, the above conclusion has to remain true exclusively under the following condition: A restriction must be placed on the exponentially increasing displacements, and this can only be achieved if the period of negative stiffness lasts for a few seconds (2 to 3 s) and is immediately followed by a return to positive stiffness for another 2 to 3 s. Then, the negative stiffness reappears, and this cycle repeats for as long as the strong motion of the earthquake persists. This suggests that such systems could potentially be used in seismic isolation. It is important to note that this finding applies to both free vibration and forced vibration scenarios. The experimental justification of the above is a subject of great interest for further investigation.
- The forces acting on the free oscillating SDoF system with damping and negative system, i.e., the elastic force P_e, the damping force P_d and the inertial force P_a are in equilibrium state for every time moment t. When the SDoF system with damping and negative stiffness is under forced vibration, when there is also the harmonic lateral load F, the sum of the elastic force P_e, damping force P_d and inertial force P_a are equal to the value of the harmonic lateral load F, and of opposite direction, for every time step t, so the forces are in equilibrium state. This means that the fundamental D'Alembert principle is true for negative stiffness systems.
- In the present study, the investigation of a SDoF system with damping and negative stiffness was chosen, due to various loading conditions, as it is considered the simplest yet most effective way to achieve an intuitive understanding of the system's behavior as well as the basic principles of dynamic response. There are already some primary attempts to expand the mathematical solution to multi-degree-of-freedom systems (MDoF). It is known that multi-degree-of-freedom systems in their non-linear response, cannot be decomposed into single-degree-of-freedom oscillators as it is possible in the elastic region. Negative stiffness in MDoF systems appears when the stiffness matrix of the system is not positive defined, i.e., when a diagonal term of the matrix has negative sign. However, the investigation of such systems is the objective of future study.
- The seismic excitation example used in the present study, constitutes of arbitrary harmonic terms of known characteristics, which is an idealization of a real seismic excitation. The realistic case where the seismic excitation is described by an actual ground motion accelerogram, is part of a separate study that is being developed by the authors of the present paper.
- Last but not least, the exact response values by the presented benchmark example (of a SDoF with negative stiffness) are a useful tool to evaluate any new numerical method for calculating the dynamic response of a SDoF system with negative stiffness.

Author Contributions

Conceptualization, N.C.; methodology, investigation, validation, formal analysis, software, data curation, writing-original draft preparation. T.M.; conceptualization, methodology, visualization, writing-review and editing, project administration. All authors have read and agreed to the published version of the manuscript

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Conflicts of Interest

The authors declare no conflict of interest

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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