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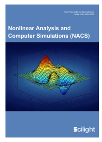

Article

Non-Radially Symmetric Solutions for Some Sublinear Defocusing Elliptic Problems

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Abstract: In this paper, we study the action of the orthogonal group $O(n)$ on the set of solutions of some sub-linear elliptic problem of the form $\Delta u + u - \mid u \mid^{-2\theta} u = 0$ on the unit ball B of $\mathbb{R}^n$, $n \geq 2$ and $0 < 2\theta < 1$. For suitable subgroups G of the orthogonal group $O(n)$, we show the existence of non-radially symmetric solutions, which are G-invariant. We precisely give a necessary and sufficient condition on G for the existence of non-radially symmetric but G-invariant solutions. Besides, we develop complete proofs for the existence and uniqueness of radial solutions. Recall that a majority of studies of this type of problem were focusing on radial solutions which are obviously invariant by the group $O(n)$. On the other hand, the question of the existence of non-radial solutions remaines somehow abandoned. In this paper, we studied this question and provided computer numerical simulations for radial solutions and the estimate of the Sobolev norm of the new non radial solutions.

Keywords: Group invariance; nonlinear elliptic equations; variational method; Sobolev norm

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1. Introduction

The paper is devoted to the study of the orthogonal group $O(n)$ action on the solutions of the sublinear elliptic equation

$$\begin{cases} \Delta u + u - \mid u \mid^{-2\theta} u = 0 & \text{in} \quad B, \\ u = 0 & \text{on} \quad \partial B, \end{cases} \tag{1}$$

and some derived forms. B is an open ball centered at 0 and with radius R in $\mathbb{R}^n$, $n \geq 2$, and θ is a real number parameter with $0 < 2\theta < 1$.

Problem (1) is part of the general framework of studies focusing on nonlinear models of type

$$\Delta u + f(u) = 0, \tag{2}$$

which in turn can be seen as stationary versions of the time-dependent problem

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \qquad \text{or} \qquad i\frac{\partial u}{\partial t} = \Delta u + f(u). \tag{3}$$

Many studies focused on these problems for the existence of solutions, positivity, unicity, and also numerical solutions and simulations. In a major part, the existing studies focused on the radial solutions, where the elliptic partial differential equation was converted into a second order ordinary differential equation. In [1], the authors considered a superlinear $\mathcal{C}^1$ model $f(u)$ satisfying at least two main hypothesis,

- $0 < f'(u) \leq C(1 + |u|^s)$ for some suitable $s > 0$,
- $f'(u) > \dfrac{f(u)}{u} > 0,\ u \neq 0.$

Both assumptions are not satisfied for our model. Atkinson, Brezis and Peletier in [2] investigated the radial solutions of a similar case with $f(u) = u + |u|^{p_c-1}u$ where $p_c = \dfrac{n+2}{n-2}$ is the critical Sobolev exponent. The main tool used in the recalled related works as well as in our present work is the critical point theory and variational methods, originally developed in [3–5].

The same nonlinearity as in our model (1) was studied in [6] for nodal solutions. Next, it was reconsidered in [7] for a classification of radially symmetric solutions according to the initial value. In [8], problem (1) was investigated for existence and uniqueness of the radial solutions without use of the variational methods. In [9], a first numerical study was carried on for problem (1) using a generalized Lyapunov-Sylvester approach for approximating the solution without serving of being radial or not. Cortazar and Elgueta studied in [10] a special model which intersects the present one in the property of the non-lipschitzian nonlinearity. This fact yields as usual difficulties in the existence and uniqueness tasks.

In [11,12], the authors considered a nonlinear model $f(u)$ composed of two nonlinearities, one is superlinear convex and the second is concave sublinear such as

$$\Delta u + |u|^{p-1}u + \lambda |u|^{q-1}u = 0,\ 0 < q < 1 < p, \tag{4}$$

on suitable domains. Existence, uniqueness, nodal solutions, and singularities of solutions have been investigated, especially for radial solutions. Notice that problem (1) may be obtained from (4) by chosing $p = 1$, $q = 1 - 2\theta$ and $\lambda = -1$. The parameter λ has many roles according to the equation and the nonlinearity. For $q = 1$ in (4), it will induce a comparison with the Laplacian eigenvalues. Otherwise, in the case of Schrödinger equation (3), it leads to the focusing and defocusing cases. These concepts are strongly related and strongly affect the behavior of the solution. We may speak about blow-up phenomenon. Furthermore, we may understand problem (1) as a problem limit of (4) when $p \longrightarrow 1$. This lead to the interest of investigating such an asymptotic problem as studied in [13] where the authors investigated the asymptotic problem due to (4) according to the exponents p and q when approaching the linear frontier 1.

In [14], some numerical approaches have been developed to approximate the solution of the evolutionary problems

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \text{ on } \Omega \times \mathbb{T} \tag{5}$$

on some suitable space-time domain $\Omega \times \mathbb{T}$ and suitable initial-boundary conditions and where the nonlinear term $f(u)$ is the same as in problem (4).

The two problems (1) and (4) intersect in the common characteristics of the existence of a sublinear non lipchitzian term, and differ by the other part of $f(u)$. Compared to the famous Brezis-Nirenberg problem on positive solutions of $\Delta u + u^p + \lambda u = 0$, in problem (1), $p > 1$, the convex nonlinear term u^p is replaced by a nonlinear non locally lipschitzian odd term $|u|^{q-1}u$ or $-|u|^{-2\theta}u$, which yields new difficulties, new techniques, and also different behavior of the solution. These problems or their derved forms may be found widely in interesting real-world cases such as nonlinear waves, plasma, optics, condensed matter, Bose-Einstein condensation and the stabilized solitons, nonlinear Shrödinger equation, chemical reaction models, population genetics models, and so on. (See for example [15]). In some cases of nonlinear Schrödinger equation, the solutions may be explicitely expressed by means of solitons as in [16–18]. Related to our present study, the model (4) was studied for non radially symmetric but group invariant solutions in [19]. In [20], a similar problem to (4) from the point of view of mixed concave and convex nonlinearities was tackled for radially symmetric solutions.

A natural problem may be now stated as: Given a subgroup G of $O(n)$, the main question deals with the effect of the G-action on the solutions. A natural and immediate case is when the group G acts transitively on the sphere $\mathcal{S}^{n-1} = \{x \in \mathbb{R}^n,\ \|x\|_2 = 1\}$, where for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. In this case, it is immediate that any G-invariant solution can be seen as radial. So, one looks for suitable conditions on G for the existence of non-radially symmetric but G-invariant solution(s). We mean by G-invariant solution any solution u of problem (1) satisfying

$$u(gx) = u(x),\ \forall\ g \in G \text{ and } \forall\ x \in B. \tag{6}$$

Such question has been studied by many authors such as Kajikiya in [21,22], where the author focused on orthogonal group invariant solutions of the problem

$$\begin{cases} \Delta u + \mid u \mid^{p-1} u = 0 & \text{in} \quad B, \\ u(gx) = u(x) & \text{for} \quad (g,x) \in G \times B, \\ u = 0 & \text{on} \quad \partial B, \end{cases} \tag{7}$$

with a closed subgroup G of $O(n)$ and a sub-superlinear power p. In [21,22], it is already noticed that we may consider G as a subset of $O(n)$ and not necessarily a closed subgroup. In this case, we just consider the group generated by the subset G and its closure $\overline{< G >}$. Then G-invariance is equivalent to $\overline{< G >}$ one. A comprehensive study and more details about transitive groups may be found in [23].

In the present paper, we assume that G is a closed non-transitive subgroup of $O(n)$, and propose to establish some results about the existence of G-invariant solutions to the problem

$$\begin{cases} \Delta u + u - \mid u \mid^{-2\theta} u = 0 & \text{in} \quad B, \\ u(gx) = u(x) & \text{for} \quad (g,x) \in G \times B, \\ u = 0 & \text{on} \quad \partial B, \end{cases} \tag{8}$$

where B is simply the unit ball in $\mathbb{R}^n$, and $0 < 2\theta < 1$.

2. Motivations

There are many motivations behind this work. The first, in our knowledge, is the lack of studies about the question of non-radial solutions of this type of problems. In this context, there are few references having investigated the non-radial solutions. These are resumed in the following items according to the nonlinear term $f(u)$,

i. problem (4) with $\lambda > 0$ and $0 < q < p < 1$ [19] .
ii. problem (4) with $\lambda = 0$ and $0 < p < 1$ [21] .
iii. problem (4) with $\lambda = 0$ and $p > 1$ [22].

Compared to our case, in the present work, we have $f(u) = u - |u|^{-2\theta} u$ which no longer satisfies conditions i–iii above.

A second part of the existing works was based on the development of numerical approximations applying known methods such as finite differences, finite volumes, finite elements, Fourier modes, and so on. These studies also intersect in the absence of the case of non-radial solutions invariant by subgroups of $O(n)$. Readers may refer essentially to [9,13,14,16–18].

Another motivation and difference with existing studies evoking the problem of invariant solutions by the subgroups of $O(n)$, is again the absence of associated numerical simulations in view of the norms estimations. This is in some way due to the possible difficulties behind it, even if the theory is well done. See [19,21,22].

Now, to highlight our results as well as previous studies in the same context of non radial solutions which are group invariant, we recall some cases of subgroups of the orthogonal group $O(n)$ which are not transitive, and which induce in a natural way invariant solutions. Starting with the simple case due to the reflection $x \longmapsto -x$. In such a case, we get notice that the function $x \longmapsto u(-x)$ is evidently a solution of problem (1). This case may be generalized to any partial reflection $x = (x_1, x_2, \ldots, x_i, \ldots, x_n) \longmapsto (x_1, x_2, \ldots, -x_i, \ldots, x_n)$, for any $i = 1, 2, \ldots, n$, or any composition of a finite number of these reflections.

To be more esier, restrict to $\mathbb{R}^3$, another example may be obtained by acting the symmetric group, or in other words by permutating the coordinates of x in any way. Consider for simplicity the permutation $(x_1, x_2, x_3) \longmapsto (x_3, x_2, x_1)$, and $\widetilde{u}(x) = u(x_3, x_2, x_1)$. Here also we get a solution $\widetilde{u}$ of problem (1). This example may be extended to other permutations and to the general space $\mathbb{R}^n$.

Consider in $\mathbb{R}^3$, the subgroup of rotations R_j with angle $\dfrac{2\pi}{3} j$, $j = 0, 1, 2$ around the z-axis. We get here the function $x = (x_1, x_2, x_3) \longmapsto \widehat{u}(x) = u(x_1 \cos(\frac{2\pi}{3} j) - x_2 \sin(\frac{2\pi}{3} j)\, ,\, x_1 \sin(\frac{2\pi}{3} j) + x_2 \cos(\frac{2\pi}{3} j)\, ,\, x_3)$ which is also a solution of problem (1).

3. Preliminaries

To tackle our problem, we introduce firstly the functional framework. Denote $V = \mathcal{H}_0^1(B)$. Recall that we have already the inclusion $L^{2(1-\theta)}(B) \subset V$. Consider on V the norm

$$\|u\|_v = \|u\|_{\mathcal{H}_0^1(B)}, \; \forall\, u \in \; V.$$

Obviously, $(V, \|.\|)$ is a Banach space. For a subgroup G of $O(n)$, write

$$V_G(B) = \Big\{ u \in V,\ u \text{ satisfying } (6) \Big\}, \tag{9}$$

and $V_G^{\perp}(B)$ its orthogonal supplement in V,

$$V = V_G(B) \oplus^{\perp} V_G^{\perp}(B). \tag{10}$$

Let $x \in \mathcal{S}^{n-1}$, and $G(x)$ its orbit relatively to the G-action,

$$G(x) = \Big\{ gx;\ g \in G \Big\}. \tag{11}$$

Denote also

$$m = \max \big\{ \, dim\, G(x)\,;\ x \in \mathcal{S}^{n-1} \big\} \tag{12}$$

the principal orbit dimension. Define next the functional I on $V_G(B)$ by

$$I(u) = \int_{\Omega} (\frac{1}{2} |\nabla u|^2 - F_{\theta}(u)) dx, \tag{13}$$

where

$$F_{\theta}(u) = \frac{1}{2} u^2 - \frac{1}{2(1-\theta)} |\ u\ |^{2-2\theta}. \tag{14}$$

It consists of a $\mathcal{C}^1$ function of u.

We also introduce the concept of G-invariant critical points and values of the energy functional I.

Definition 1.

- *$u \in V_G(B)$ is called a G-invariant critical point of I if $I'(u) = 0$.*
- *$\alpha \in \mathbb{R}$ is called a G-invariant critical value of I if there exists a critical point $u \in V_G(B)$ such that $I(u) = \alpha$.*

Finally, we need also to recall the concept of the genus.

Definition 2. *Let F be the closed subset of $V_G(B)$ with $\mathbb{Z}_2$-symmetry group and $0 \notin F$. The genius $\gamma(F)$ is the smallest integer n for which there exists $h \in \mathcal{C}(F, \mathbb{R}^n \setminus \{0\})$. $\gamma(F) = \infty$ if there exists no finite such n and $\gamma(\emptyset) = 0$.*

4. Main Results

In the first step, we propose to come back to some radial versions related to the problem (1) such as

$$\begin{cases} u'' + \dfrac{n-1}{r} u' + u - |\ u\ |^{-2\theta}\ u = 0 & , \quad r \in (0, \infty), \\ u(0) = a & , \quad u'(0) = 0, \end{cases} \tag{15}$$

where $a \in \mathbb{R}$. It has been widely investigated by many authors such as [6–8] for existence, uniqueness, phase plane and nodal solutions. Remark that the solution of (15) which satisfies $u(1) = 0$ is, in fact, natural $O(n)$-invariant solutions.

The first result concerns the existence and uniqueness of solutions. Unfortunately, there is no complete proof of these results in the references that investigated the same problem. For this reason, we will provide full proofs here, and also different from the existing essays.

Theorem 1. *Denote $p = \dfrac{1}{(1-\theta)^{\frac{1}{2\theta}}}$. For all $a \neq 0$, problem (15) has a unique solution u. Furthermore, for all $a \in (0, p)$, the solution u remains nonnegative on $(0, \infty)$.*

Next, we recall the following result which provides a classification of the radial solutions according to their initial values ([6–8]).

Theorem 2. *[6–8] Denote* $p = \dfrac{1}{(1-\theta)^{\frac{1}{2\theta}}}$. *The following assertions hold.*

1. *For* $0 < a < 1$, *the solution* u *of the problem (15) is oscillating around* 1 *with no zeros in* $(0, \infty)$.
2. *For* $1 < a < p$, *the solution* u *of the problem (15) is oscillating around 1 with no zeros in* $(0, \infty)$.
3. *For* $a > p$, *the solution* u *of the problem (15) is oscillating around* ± 1 *with a finite number of zeros in* $(0, \infty)$ *or oscillating around* 0 *with a finite number of zeros in its support being compact.*

Our second main result relates the solutions of the problem (8) to the G-invariant critical values of the energy functional I, and is stated as follows.

Theorem 3. *Let* $u \in V_G(B)$. *It holds that* u *is a weak solution of (8) if and only if* $I'(u) = 0$.

The next result concerns the G-invariant critical values and their estimation.

Theorem 4. *There exists a sequence* $(\alpha_k)_k$ *strictly increasing of G-invariant critical values satisfying*

(i) *If* $\alpha_k = \alpha_{k+1} = \ldots = \alpha_{k+l} = \alpha$, *then* $\gamma(\mathcal{K}_\alpha) \geq l + 1$ *with*

$$\mathcal{K}_\alpha = \{\, u \in V_G(B);\ I(u) = \alpha,\ I'(u) = 0\}.$$

(ii) *The sequence of critical values* $(\alpha_k)_k$ *satisfies*

$$\alpha_k \sim k^{p(\theta)} \text{ where } p(\theta) = \frac{2(1-\theta)}{n-m} + \frac{2\theta(2-\theta)}{1-\theta}.$$

To show that, effectively, the problem investigated here has G-invariant but non-radially symmetric solutions, we propose in the next part to prove that the set of radially symmetric solutions in strictly contained in the set of G-invariant solutions. To do it, we investigate and/or characterize the radially symmetric critical values.

Denote $\mathcal{H}^{0,1}_{rad}(B)$ the subspace of $\mathcal{H}^1_0(B)$ composed of radial elements. A radially symmetric critical value is a critical value of the energy functional I restricted to $\mathcal{H}^{0,1}_{rad}(B)$. The following result holds.

Theorem 5. *There exists an unbounded strictly increasing sequence* $(\beta_k)_k$ *of radially symmetric critical values such that*

$$\beta_k \sim k^{p(\theta)} \ \ with \ \ p(\theta) = 2(1 - \frac{1}{\theta}).$$

5. Proof of Main Results

5.1. Proof of Theorem 1

If u is a solution of (15), then $(-u)$ is also a solution of (15). So, we only study the case where $a > 0$. Let

$$M_a = \Big\{ u \in C((0,\delta));\ a \leq u(r) \leq 2a,\ \ \forall r \in (0,\delta) \Big\},$$

where

$$0 < \delta < \min\left\{ \sqrt{\frac{2}{|f'(\frac{a}{2})|n}}, \sqrt{\frac{2}{n}}, \sqrt{\frac{an}{2|f(p)|}} \right\}.$$

We notice that if u is solution of (15), it satisfies

$$u(r) = a - r^2 \int_0^1 \int_0^1 x s^{n-1} f(u(rxs)) ds dx.$$

Denote next $\Phi : M_a \to M_a \cap C^2$ defined by

$$\Phi(u(r)) = a - r^2 \int_0^1 \int_0^1 x s^{n-1} f(u(rxs)) ds dx.$$

Φ is well defined, because of the fact that

$$|\Phi(u(r)) - a| \leq |r^2 \int_0^1 \int_0^1 x s^{n-1} f(u(rxs)) ds dx| \leq \frac{\delta^2}{n} |f(p)| < \frac{a}{2}.$$

Hence, $\Phi(u) \in M_a$. On the one hand,

$$\Phi(u(r)) = a - \int_0^r \int_0^1 ts^{n-1}(f(u(ts))ds)dt$$

is a primitive of

$$\varphi(t) = -\frac{1}{t^{n-1}} \int_0^t x^{n-1} f(u(x))dx,$$

which is C^2 on $(0, \delta)$ because of the continuity of the function ψ defined by

$$\psi(x) = x^{n-1} f(u(x)), \ \ x \in (0, \delta).$$

We will prove that Φ satisfies the fixed point theorem. Indeed, let $u, v \in M_a$. We may write that

$$\parallel \Phi(u) - \Phi(v) \parallel_\infty \leq K \parallel u - v \parallel_\infty r^2 \int_0^1 \int_0^1 xs^{n-1} dsdx \leq K \frac{n\delta^2}{2} \parallel u - v \parallel_\infty,$$

where $K = \max(|f'\left(\frac{a}{2}\right)|, 1)$. Hence, Φ is contractive. Thus, Φ has a unique fixed point $u \in M_a$. Consequently, for all $a \neq 0$, problem (15) has a unique solution u.

It remains to show that this solution is global on $(0, +\infty)$. Assume by the contrast that there exists $t_0 \in (0, +\infty)$ such that

$$\lim_{t \to t_0} |u(t)| = \infty.$$

Consider the energy function

$$E(r) = \frac{1}{2} u'^2(r) + F(u(r)).$$

It is straightforward that $E(r)$ is non-increasing, then

$$F(u(r)) \leq E(r) \leq E(0) = F(a) < 0, \ \ \forall r > 0,$$

which yields that

$$F(a) > \lim_{t \to t_0} F(|u(t)|) = \infty,$$

which is a contradiction.

Finally, it remains to check that for any initial value $a = u(0)$ in $(0, p)$, the solution u remains nonnegative. Indeed, for $a \in (0, p)$, we get

$$F(u(r)) \leq E(r) \leq E(0) = F(a) < 0, \ \ \forall r > 0.$$

Since F is even and coercive, there exists a unique positive $\alpha \neq a$ which satisfies

$$F(a) = F(\alpha) \ \text{ and } \ \alpha \leq u(r) \leq a \ \text{ or } \ a \leq u(r) \leq \alpha.$$

Hence, the result follows.

5.2. Proof of Theorem 3

Let $u \in V_G(B)$ be a critical value of the functional I, that is $I'(u) = 0$, and let $v \in \mathcal{H}_0^1(B)$. We shall show that

$$\int_B (\nabla u \nabla v - uv + | u |^{-2\theta} uv) dx = 0. \tag{16}$$

Indeed, consider the decomposition of v as a unique sum $v = v_1 + v_2$ due to (10), where $v_1 \in V_G(B)$, and $v_2 \in V_G^{\perp}(B)$. It follows easily that

$$\int_B (\nabla u \nabla v_1 - f(u)v_1) dx = 0.$$

Moreover, as since $v_2 \in V_G^{\perp}(B)$, we also have

$$\int_B \nabla u \nabla v_2 dx = 0.$$

To achieve the proof, we shall show finally that

$$\int_B (u - | u |^{-2\theta} u) v_2 dx = 0.$$

Here-also, we know that

$$\int_B u v_2 dx = 0.$$

So, it remains to show that

$$\int_B | u |^{-2\theta} u v_2 dx = 0.$$

It is easily seen that $|u|^{-2\theta}u$ satisfies (6) whenever u satisfies it too. As a consequence, there exists a sequence of functions $(f_k)_k$ in $\mathcal{C}_0^\infty(B, G)$ convergent to $|u|^{-2\theta}u$ in $L^{\frac{2n}{n+2}}(B, G)$. By considering for each k (or each f_k), the unique solution $\Phi_k \in V_G(B)$ of the problem

$$\begin{cases} \Delta\Phi_k + f_k = 0 & \text{in} \quad B, \\ \Phi_k = 0 & \text{on} \quad \partial B, \end{cases} \tag{17}$$

we obtain

$$\int_B f_k v_2 dx = -\int_B \Delta\Phi_k v_2 dx =< \Phi_k, v_2 >_{\mathcal{H}_0^1(B)} = 0.$$

This yields by letting $k \to \infty$,

$$\int_B | u |^{-2\theta} u v_2 dx = 0.$$

5.3. Proof of Theorem 4

Denote firstly for $k \in \mathbb{N}$, λ_k to be the k^{th} eigenvalue, associated to the unitary eigenvector Φ_k of the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in} \quad B, \\ u = 0 & \text{on} \quad \partial B, \\ u(gx) = u(x) & \text{in} \quad B \times G. \end{cases} \tag{18}$$

Denote $V_k(G) = spann(\Phi_i;\, 1 \leq i \leq k)$. It is a finite-dimensional vector space. Therefore, there exists constants $C_k > 0$ for which

$$C_k^{-1}\|u\| \leq \|u\|_{2(1-\theta)} \leq C_k\|u\|,$$

and

$$C_k^{-1}\|u\| \leq \|u\|_2 \leq C_k\|u\|.$$

As a result, we get

$$I(u) \approx \begin{cases} \dfrac{C_k^{2(\theta-1)}}{2(1-\theta)}\|u\|n^{2(1-\theta)} & \text{for} \quad \|u\| \to 0^+, \\ \dfrac{1}{2}(1 - C_k^{-2})\|u\|^2 & \text{for} \quad \|u\| \to \infty. \end{cases}$$

Consequently, there exists a sequence $R_k > 0$ non-decreasing, such that

$$I(u) \approx \begin{cases} \dfrac{C_k^{2(\theta-1)}}{2(1-\theta)} R_k^{2(1-\theta)} & \text{for} \quad \|u\| \leq R_k,\ u \in V_k(G), \\ \dfrac{1}{2}(1 - C_k^{-2}) R_k^2 & \text{for} \quad \|u\| \geq R_k,\ u \in V_k(G). \end{cases}$$

Denote next

$$B_k = \{u \in V_k(G);\ ||u|| \leq R_k\},$$

and define

$$\Lambda_k = \{h \in \mathbf{C}(B_k, V_G(B));\ h \text{ is odd and } h(u) = u \text{ on } \partial B_k\},$$

$$\mathcal{V} = \{A \subset V_G(B);\ A \text{ is closed, } 0 \notin A \text{ and } -u \in A,\ \forall u \in A\},$$

and

$$\Gamma_k = \{h(\overline{B_i \setminus A});\ h \in \Lambda_i,\ i \geq k,\ A \in \mathcal{V} \text{ and } \gamma(A) \leq i-k\}.$$

We set finally

$$\alpha_k = \inf_{B \in \Gamma_k} \max_{u \in B} I(u). \tag{19}$$

(i) Suppose in the contrast that $\gamma(\mathcal{K}_\alpha) < l+1$. There exists $\varepsilon > 0$ such that $\gamma(\mathcal{K}_{\alpha,\varepsilon}) \leq l$, where $\mathcal{K}_{\alpha,\varepsilon}$ is the ε-neighborhood of $\mathcal{K}_\alpha$. So, $\gamma(\mathcal{K}_{\alpha,\varepsilon} \cap \mathcal{S}^{n-1}) \leq l$. Hence, there exists $\eta \in (0,1)$ and $h \in \mathcal{C}([0,1] \times \mathcal{S}^{n-1}, \mathcal{S}^{n-1})$ with $h(t,u)$ odd in u and such that

$$h\bigg(1, (\overline{\mathcal{K}}_{\alpha+\eta} \cap \mathcal{S}^{n-1}) \setminus (\mathcal{K}_{\alpha,\varepsilon} \cap \mathcal{S}^{n-1})\bigg) \subset \overline{\mathcal{K}}_{\alpha-\eta} \cap \mathcal{S}^{n-1}, \tag{20}$$

where $\overline{\mathcal{K}}_\alpha = \{u;\ I(u) \leq \alpha\}$. Let $A \in \Gamma_{k+l}$ such that $\max\limits_{u \in A} I(u) \leq \alpha+\eta$. One has immediately $\overline{A \setminus (\mathcal{K}_{\alpha,\varepsilon} \cap \mathcal{S}^{n-1})} \in \Gamma_k$ and $h\bigg(1, \overline{A \setminus (\mathcal{K}_{\alpha,\varepsilon} \cap \mathcal{S}^{n-1})}\bigg) \in \Gamma_k$. Therefore, by equation (20), we obtain

$$\alpha \leq \max_u I(u) \leq \alpha - \eta,$$

where the maximum is taken over the set $h\bigg(1, \overline{A \setminus (\mathcal{K}_{\alpha,\varepsilon} \cap \mathcal{S}^{n-1})}\bigg)$. The last relation is contradictory.

(ii) Observe firstly that for $u \in V_k(G)$, we have

$$\|u\|_2^2 \leq Ck^{\theta/(1-\theta)} \|u\|_{2(1-\theta)}^2,\ \forall\, u \in V_k(G), \tag{21}$$

and

$$\|\nabla u\|_2^2 \leq Ck^{2/(n-m)+\theta/(1-\theta)} \|u\|_{2(1-\theta)}^2,\ \forall\, u \in V_k(G). \tag{22}$$

It follows from (21) and (22) that for $u \in V_k(G)$,

$$I(u) \approx \frac{1}{2}(1 - Ck^{\theta/(1-\theta)})\|u\|^2 + \frac{C^{-1}k^{-2\theta-2(1-\theta)/(n-m)}}{2(1-\theta)}\|u\|^{2(1-\theta)}. \tag{23}$$

Now, a simple computation yields that

$$\alpha_k \sim \frac{C^3\theta}{2(1-\theta)} k^{p(\theta)},$$

where

$$p(\theta) = \frac{2(1-\theta)}{n-m} + \frac{2\theta(2-\theta)}{1-\theta}.$$

5.4. Proof of Theorem 5

Consider for $k \in \mathbb{N}$, the nodal solution u_k of the problem (15) with exactly $k-$ zero in $(0,1)$. Denote next $\widetilde{u}_k(r) = z_k^\alpha u_k(z_k r)$ with $\alpha = \dfrac{-1}{\theta}$. We immediately observe that

$$I(\widetilde{u}_k) = \frac{1}{2}\bigg(z_k^{-(1+2\theta)/\theta} - 1\bigg)\|\widetilde{u}_k\|^2 + \frac{\theta}{2(1-\theta)}\|\widetilde{u}_k\|^{2(1-\theta)},$$

which in turn yields that

$$I(\widetilde{u}_k) \sim Ck^{-2(1-\theta)/\theta}.$$

We thus take $\beta_k = I(\widetilde{u}_k)$.

6. Further Discussions

To further emphasize the theoretical results developed and proved in the previous sections, we provide in the present section some computer simulations provided with graphical illustrations. The following figures (Figures 1–4) illustrate graphically the result of Theorems 1 and 2. We recall that radial solutions are already invariant via the action of the orthogonal group $O(n)$. Figure 1 illustrates the case $\theta = 0.45$, for whch the zero of $F(u)$ is $p = 1.941$.

It shows easily that the radial solutions are infinitely oscillating around 1 being nonnegative on the whole interval. Figure 2 Shows the solutions oscillating around ± 1 whenever the initial value $u(0) = a > p = 1.85$, and $\theta = 0.35$. Figure 3 shows a similar behavior of the radial solutions as in Figure 2 but for a large initial value $u(0) = a > p = 1.94$ and $\theta = 0.45$.

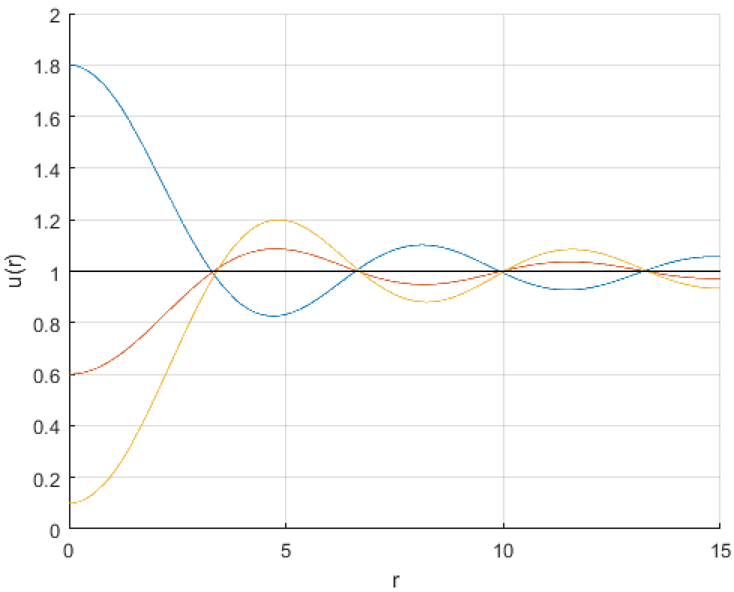

Figure 1. Some solutions of (15) with $a \in (0, p)$, $\theta = 0.45$, $p = 1.94$.

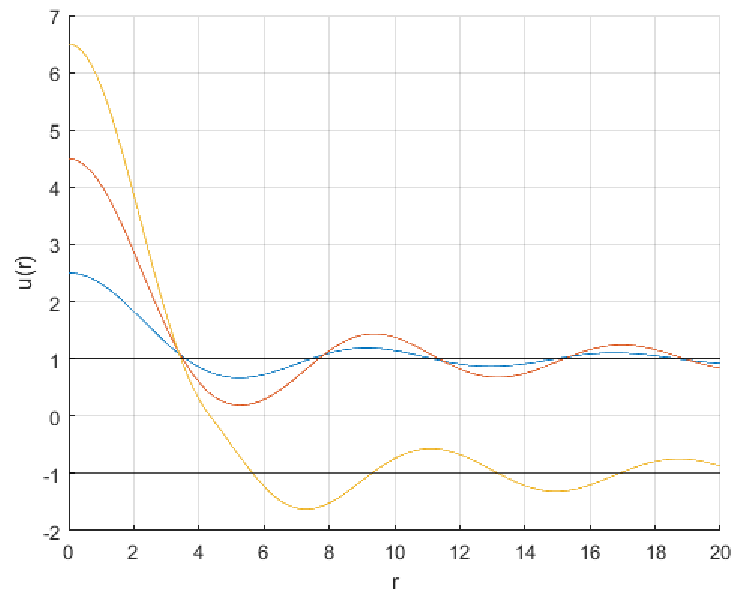

Figure 2. Oscillations around ± 1, with $u(0) = a > p$, $\theta = 0.35$, $p = 1.85$.

Next, as it is done for the theoretical results raised in Theorems 1 and 2, we provide her after some computer simulations relative to the results in Theorems 4 and 5. We firstly plotted in Figure 5 below the behavior of $\alpha_k = f(k)$ as a log-log curve for some values of the parameters θ, n and m.

Notice easily that the ponits $(\log \alpha_k, \log k)$ follow the same direction and are quite strategically located on the line $y = p(\theta)x$ for $k \geq 10$.

As previously, we plotted in Figure 6 below the behavior of $\beta_k = f(k)$ as a log-log curve for some values of the parameters θ, n and m. We notice here also that the points $(\log \beta_k, \log k)$ follow the same direction and are quite strategically located on the line $y = p(\theta)x$ for $k \geq 10$.

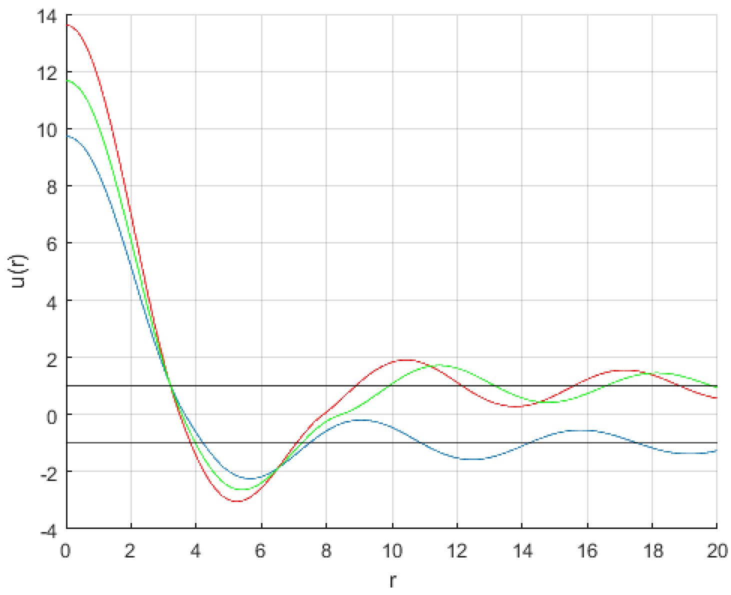

Figure 3. Some solutions of (15) with $u(0) = a > p$, $\theta = 0.45$, $p = 1.94$.

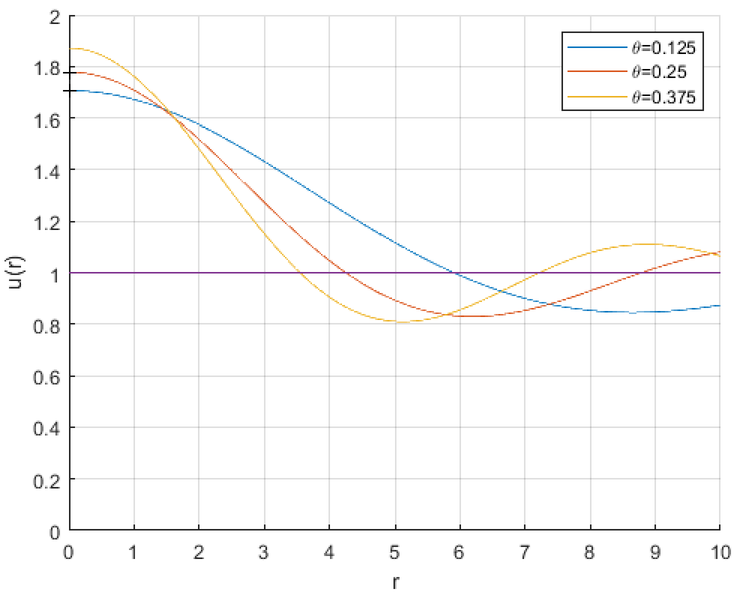

Figure 4. Some solutions of (15) for different values of θ and $u(0) = a \in (1, p)$.

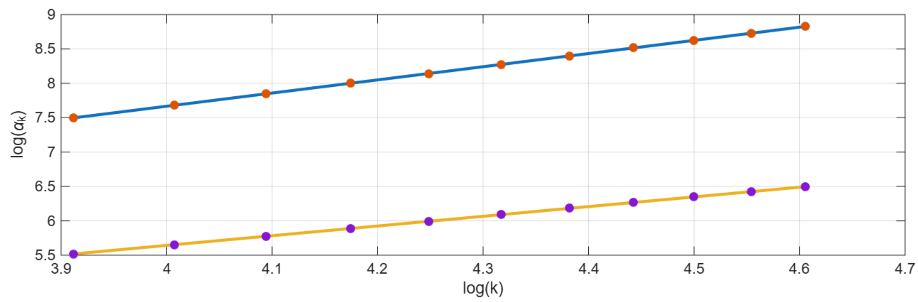

Figure 5. $\log \alpha_k = p(\theta) \log k$, for $\theta = 0.25$ et the top and $\theta = 0.125$ at the bottom ($n = 3$, $m = 1$).

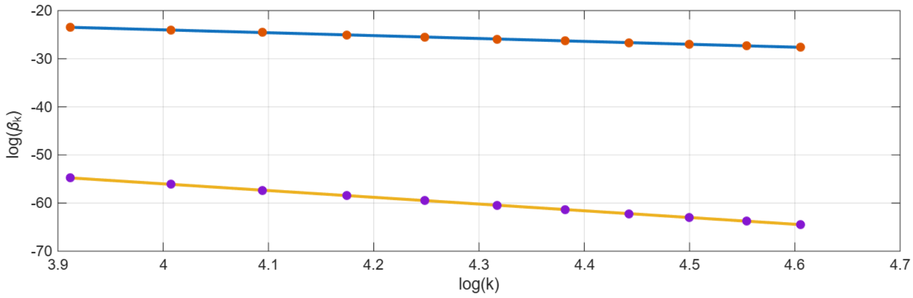

Figure 6. $\log \beta_k = p(\theta) \log k$, for $\theta = 0.25$ at the top and $\theta = 0.125$ at the bottom ($n = 2$).

7. Conclusions

In this paper, we investigated the action of the orthogonal group on the set of solutions of a sub-linear elliptic problem. Specifically, We focused on subgroups of the orthogonal group O(n) and showed the existence of non-radially symmetric solutions that exhibit invariance under these subgroups. Notably, we provided a necessary and sufficient condition regarding the subgroup G for the existence of such non-radially symmetric solutions. The paper is also reinforced with computer simulations which illustrate graphically the theoretical results.

More precisely, we focused on the problem of existence of non-radial solutions which are invariant by specific subgroups of $O(n)$ such as non-transitive subgroups. In addition, we have shown using the theory of the critical point on suitable spaces and the properties of the nonlinear part $f(u)$ the existence of sequences of non-radial solutions, and we have estimated their Sobolev norm. Such an estimate clearly shows that these solutions are indeed non-radial (comparing it to the estimate of the norm of radial solutions). Computer simulations associated to the theoretical results are also given via graphical illustrations which confirm the developed theory. Figures 1–4) illustrate the result of Theorems 1 and 2 dealing with the behavior or the classes of the solutions in the radial case. Figures 5 and 6 illustrate the results of Theorems 4 and 5 on the estimation of the Sobolev norm of the sequence of solutions due to the critical point theory application. We notice easily the adeuqcy between the theoretical results and the computer simulations. Compared to the existing studies, the last simulations were no longer provided in our knowledge, especially for similar problems as ours.

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Conflicts of Interest

The author declares no conflict of interest.

Use of AI and AI-Assisted Technologies

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