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Dynamical Behavior and Soliton Solutions of the (2+1)-Dimensional Novikov-Veselov System of Equations

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Abstract: In this study, the third-order nonlinear (2+1)-dimensional Novikov-Veselov system of equations with constant coefficients has been investigated using an appropriate traveling wave transformation. The extended rational $\sin-\cos$ technique and the modified exponential function method are two reliable and powerful methods that have been used for the specified nonlinear system. The main goal is to get valuable, exact traveling waves, periodic waves, and soliton solutions. The resulting solutions are expressed as a variety of trigonometric functions, including hyperbolic trigonometric functions, exponential functions, and rational functions. In that they provide light on the pertinent facets of the physical phenomenon, the suggested solutions are both innovative and significant. The properties of the solutions have been illustrated in a variety of figures, including two- and three-dimensional ones, to ensure the best visual assessment. Furthermore, two-dimensional graphs demonstrated how temporal development affects solution structures. The most powerful and efficient technologies are the computer software tools we use to create solutions and graphs.

Keywords: modified exponential function method; Extended rational $\sin$-$\cos$ technique; third-order (2+1)-dimensional novikov-veselov equation; traveling wave transformation; soliton solutions

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1. Introduction

Scholars are drawn to investigations that uncover soliton solutions to nonlinear differential equations, as these findings promote novel methodologies and progress within this subject matter. Many sophisticated techniques for creating multiple solitons and traveling wave solutions have been developed in the scientific and engineering fields to cope with these inquiries. The following are some methodologies that have been used to various mathematical models. The Bernoulli sub-equation function method [1,2], use of the Laplace transformation for the system which involves the Caputo fractional derivatives [3], the Bifurcation analysis, optical solitons, and modulation instability analysis of the complex nonlinear (2+1)-dimensional δ-potential schrödinger equation [4], the modified extended $\tanh$ function method to scientifically deduce semi-analytic traveling wave solutions for the (2+1)-dimensional fourth-order non-linear generalized Hietarinta-type model [5], the extended rational $\sinh$-$\cosh$ technique and the modified extended $\tanh$- function approach have been employed [6], the $\tanh$-function method and it's modified or extended versions [7, 8], the newly modified versions of the rational $\sin$-$\cos$ method and rational $\sinh$-$\cosh$ method applied to to the Fokas system[9], the stander Painlevé analysis approach for investigating integrability of some models, and applying the different integration algorithms [10,11], the method of the inverse scattering [12], the extended and modified versions of rational trigonometric functions are used for the (2+1)-dimensional integro-differential Konopelchenko-Dubrovsky evolution equation [13], the method with the Darboux transformations with it's extended or modified versions [14,15], the improved Bernoulli sub-equation function method and the modified extended $\tanh$- function method applied to fourth-order nonlinear (3+1)-dimensional generalized Kadomtsev–Petviashvili–Benjamin–Bona–Mahony (KP-BBM) equation [16], the Bäcklund transformation method [17], the Lie symmetry analysis [18], the modified exponential function method is employed to the nonlinear (2+1)-dimensional generalized breaking soliton system [19] , and references therein.

The model that analyzed in this study is a nonlinear (2+1)-dimensional Novikov-Veselov system of equations with constant coefficients which is generated by [20] and investigated by utilization of inverse scattering transform technique, aforementioned PDE seems to have the following formulation:

$$\begin{aligned} u_t + \alpha_0 u_{xxx} + \alpha_1 u_{yyy} + \alpha_2 (uv)_x + \alpha_3 (uw)_y = 0, \\ v_y = u_x, \\ w_x = u_y, \end{aligned} \tag{1}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are parameters that have a substantial influence on the behavior of the solutions. Ever since numerous scholars applied alternative techniques for investigating this model in a distinct form including as the linear superposition method [21,22], the Bäcklund transformations method and a nonlinear superposition formula [23], the inverse scattering method [24], the solitary wave ansatz method [25], the Painlevé analysis and Bell polynomials approach [26], the generalized Kudryashov method [27], the Fourier restriction norm method [28], and references therein.

The different versions of the model Equation (1) have been studied by the scholars using different approach for instance the $\tanh$ method, the extended $\tanh$ method, and the $\cosh$-$\sinh$ method [29], the exponential function method [30], the extended $\tanh$- function method based on the mapping method [31], the Bäcklund transformation and the variable separation approach [32], the binary Darboux transformation technique [33], and so on.

This article is organized as follows. Section 1 focuses on the literature review relevant to this study. Section 2 presents the algorithms of the applicable method. Section 3 details the implementation of the proposed approach for deriving specific exact solutions of Equation (1). Finally, Section 4 includes the discussion and conclusion of the research.

2. Structures of the Applied Methods

Within each of the mentioned approaches, there is a fundamental step that focuses on it significantly, this common step is shortened as follows:

Step 1. Examine the subsequent non-linear PDE in which $u = u(x, y, t)$.

$$\phi(u, u_x, u_y, u_t, u_{yy}, u_{xx}, \cdots) = 0, \tag{2}$$

setting:

$$u(x, y, t) = U(\mathscr{Q}),\ \mathscr{Q} = \mathscr{I}_1 x + \mathscr{I}_2 y - \mathscr{I}_3 t, \tag{3}$$

where $\mathscr{I}_1, \mathscr{I}_2, \mathscr{I}_3$ are non zero arbitrary parameters, $\mathscr{I}_3$ represents the wave number. By subbing Equation (3) into Equation (2) the next non-linear ODE is resulted:

$$\Phi\big(U, U', U'', U^2, \cdots\big) = 0, \tag{4}$$

where $U = U(\mathscr{Q}),\ U' = \frac{dU}{d\mathscr{Q}},\ U'' = \frac{d^2U}{d\mathscr{Q}^2},\ \cdots$.

2.1. Properties of Modified Exponential Function Method (MEFM)

assuming that step one took place, then the Modified exponential function technique (MEFM) is described in the following steps:

Step 2. Let us consider that Equation (4) has a solution expressed subsequently:

$$\begin{aligned} U(\mathscr{Q}) &= \frac{\sum\limits_{i=0}^{N} \mathscr{E}_i \big(e^{-\chi(\mathscr{Q})}\big)^i}{\sum\limits_{j=0}^{M} \beta_j \big(e^{-\chi(\mathscr{Q})}\big)^j} \\ &= \frac{\mathscr{E}_0 + \mathscr{E}_1 e^{-\chi(\mathscr{Q})} + \cdots + \mathscr{E}_N e^{-N\chi(\mathscr{Q})}}{\beta_0 + \beta_1 e^{-\chi(\mathscr{Q})} + \cdots + \beta_M e^{-M\chi(\mathscr{Q})}}, \end{aligned} \tag{5}$$

where β_j , $\mathscr{E}_i$ $(j = 0, \cdots, M,\ i = 0, \cdots,\ N)$ are parameters that ought to be identified later, such that $\mathscr{E}_N \neq 0$

and $\beta_M \neq 0$.

Remark 1. *1. The positive integers N and M may be identified through the application of the homogeneous balance technique, which involves analyzing the relationship between the highest-order derivatives and the elevated degree of nonlinear terms present in Equation (4).*

2. It is essential that $\chi(\mathscr{Q})$ fulfills the subsequent ordinary differential equation:

$$\chi'(\mathscr{Q}) = e^{-\chi(\mathscr{Q})} + \mathscr{H} e^{\chi(\mathscr{Q})} + \mathscr{G}. \tag{6}$$

Solutions of the Equation (6) are disclosed as follows:

- If $\mathscr{H} \neq 0,\ \mathscr{G}^2 - 4\mathscr{H} > 0$, then

$$\chi(\mathscr{Q}) = \ln\left(\frac{-\sqrt{\mathscr{G}^2 - 4\mathscr{H}}}{2\mathscr{H}} \tanh\left(\frac{\sqrt{\mathscr{G}^2 - 4\mathscr{H}}}{2}(\mathscr{Q} + \wp)\right) - \frac{\mathscr{G}}{2\mathscr{H}}\right). \tag{7}$$

- If $\mathscr{H} \neq 0,\ \mathscr{G}^2 - 4\mathscr{H} < 0$, then

$$\chi(\mathscr{Q}) = \ln\left(\frac{\sqrt{-\mathscr{G}^2 + 4\mathscr{H}}}{2\mathscr{H}} \tan\left(\frac{\sqrt{-\mathscr{G}^2 + 4\mathscr{H}}}{2}(\mathscr{Q} + \wp)\right) - \frac{\mathscr{G}}{2\mathscr{H}}\right). \tag{8}$$

- If $\mathscr{H} = 0,\ \mathscr{G} \neq 0,\ \mathscr{G}^2 - 4\mathscr{H} > 0$, then

$$\chi(\mathscr{Q}) = -\ln\left(\frac{\mathscr{G}}{e^{\mathscr{G}(\mathscr{Q}+\wp)}} - 1\right). \tag{9}$$

- If $\mathscr{H} \neq 0,\ \mathscr{G} \neq 0,\ \mathscr{G}^2 - 4\mathscr{H} = 0$, then

$$\chi(\mathscr{Q}) = \ln\left(\frac{2\lambda(\mathscr{Q} + \wp) + 4}{\mathscr{G}^2(\mathscr{Q} + \wp)}\right). \tag{10}$$

- If $\mathscr{H} = 0,\ \mathscr{G} = 0,\ \mathscr{G}^2 - 4\mathscr{H} = 0$, then

$$\chi(\mathscr{Q}) = \ln(\mathscr{Q} + \wp). \tag{11}$$

in every specific case, $\wp$ represents a non-zero constant.

Step 3. By substituting Equations (5) and (6) into Equation (4) and setting all coefficients of $e^{-\chi(\mathscr{Q})}$ that correspond to identical powers of $e^{-\chi(\mathscr{Q})}$ to zero, we derive the subsequent system of algebraic equations for the coefficients.

$$\Psi_j = 0, j = 0, \cdots, k \tag{12}$$

here k is a greatest power of $e^{-\chi(\mathscr{Q})}$.

Step 4. Identifying the parameters by applying different computer algorithms to solve the generated system (12). When results are combined with one of the Equations (7)–(11), the exact solution for the mathematical model under study is obtained.

2.2. *Properties of Extended Rational* sin-cos *Method (ERSCM)*

Now state the algorithm of the second method in this research, the extended rational sin-cos, which is described in the following steps, where step one is identical to the first step in the above process.

Step 2. Assume that the solution to (4) has the following specifications:

$$U(\mathscr{Q}) = \frac{\gamma_0 \sin(\mathscr{H}\mathscr{Q})}{\gamma_2 + \gamma_1 \cos(\mathscr{H}\mathscr{Q})},\ \cos(\mathscr{H}\mathscr{Q}) \neq -\frac{\gamma_2}{\gamma_1}, \tag{13}$$

or in the form,

$$U(\mathscr{Q}) = \frac{\gamma_0 \cos(\mathscr{H}\mathscr{Q})}{\gamma_2 + \gamma_1 \sin(\mathscr{H}\mathscr{Q})},\ \sin(\mathscr{H}\mathscr{Q}) \neq -\frac{\gamma_2}{\gamma_1}, \tag{14}$$

where γ_i for $i = 0, 1, 2$ indicates parameters that will be determined later, and $\mathscr{H}$ is a non-zero wave number.

Step 3. Unknown parameters can be observed by replacing Equation (13) or Equation (14) into Equation (4), collecting all terms with the same powers of $\cos^m(\mathscr{H}\mathscr{Q})$ or $\sin^m(\mathscr{H}\mathscr{Q})$ and equating to zero all the coefficients of $\cos^m(\mathscr{H}\mathscr{Q})$ or $\sin^m(\mathscr{H}\mathscr{Q})$ yields a set of algebraic equations. With certain computer software programs, the solutions to the algebraic system of equations are generated.

Remark 2. *Furthermore, it's possible to collect all the terms with the same powers of* $\cos^{m1}(\mathscr{H}\mathscr{Q})\sin^{m2}(\mathscr{H}\mathscr{Q})$ *where* $m = 0, 1, 2, \cdots \mathscr{G}$ *and equating to zero the summation of all the coefficients which have the same powers, it yields a system of algebraic equations. Solving the resulting system is guaranteed to provide parameters.*

Step 4. By replacing the existing values of $\gamma_0, \gamma_1, \gamma_2$ and $\mathscr{H}$ into Equation (13) or Equation (14), the solution of Equation (4) may be obtained. Combining parameters and re-installing gotten solutions, one may obtain the solution for the required mathematical model Equation (1).

3. Applications of Specified Methods

In this section, the (2+1)-dimensional nonlinear constant coefficient, Novikov-Veselov system of equations, has been analyzed applying the indicated approaches. Let's start with applying the following wave changes to the Equation (1).

$$\begin{gathered} u(x,y,t) = U(\mathscr{Q}), v(x,y,t) = V(\mathscr{Q}), w(x,y,t) = W(\mathscr{Q}) \\ \mathscr{Q} = \mathscr{I}_1 x + \mathscr{I}_2 y - \mathscr{I}_3 t, \end{gathered} \tag{15}$$

here $\mathscr{I}_1, \mathscr{I}_2, \mathscr{I}_3 \neq 0$ since $v_y = u_x$ and $w_x = u_y$ directly $V = \frac{\mathscr{I}_1}{\mathscr{I}_2}U$, and $W = \frac{\mathscr{I}_2}{\mathscr{I}_1}U$ then one gets the following non-linear ODE:

$$-\mathscr{I}_3 U' + \left(\alpha_0 \mathscr{I}_1^3 + \alpha_1 \mathscr{I}_2^3\right)U''' + \left(\frac{2\alpha_2 \mathscr{I}_1^3 + 2\alpha_3 \mathscr{I}_2^3}{\mathscr{I}_1 \mathscr{I}_2}\right)UU' = 0, \tag{16}$$

where $U = U(\mathscr{Q}), U' = \frac{dU}{d\eta}, U'' = \frac{d^2U}{d\eta^2}, \cdots$. Take the integration of Equation (16) one time with zero constant of integration outcome is in the following:

$$-\mathscr{I}_3 U + \left(\alpha_0 \mathscr{I}_1^3 + \alpha_1 \mathscr{I}_2^3\right)U'' + \left(\frac{2\alpha_2 \mathscr{I}_1^3 + 2\alpha_3 \mathscr{I}_2^3}{\mathscr{I}_1 \mathscr{I}_2}\right)U^2 = 0. \tag{17}$$

The equation that appears in Equation (17) is a potential model for verifying the dominant balancing principle on it.

3.1. Application of MEFM to the (2+1)-Dimensional Novikov-Veselov System

In this section, the MEFM is studied to discover some complex exponential and hyperbolic trigonometric function solutions to the (2+1)-dimensional Novikov-Veselov System. After reducing the investigated model into a non-linear ODE (17), immediately applying the homogeneous balance principle, the achieved result is a fundamental relation between M and N as follows:

$$N = M + 2, \text{ where } N, M \in \mathbb{Z}^+. \tag{18}$$

Based on the values of the integers that are satisfied (18), if $M = 1$ then $N = 3$, thus the declared solution gets the following format:

$$\begin{aligned} U(\mathscr{Q}) &= \frac{A_0 + A_1 e^{-\chi} + A_2 e^{-2\chi} + A_3 e^{-3\chi}}{B_0 + B_1 e^{-\chi}} \\ &= \frac{\mathcal{W}(\chi)}{\mathcal{H}(\chi)}, \end{aligned} \tag{19}$$

where $\chi(\mathscr{Q})$ is satisfies the (6). Take the derivatives of (19); the outcomes are the following:

$$U'(\mathscr{Q}) = \frac{\mathcal{H}\mathcal{W}' - \mathcal{W}\mathcal{H}'}{\mathcal{H}^2}, \tag{20}$$

and

$$U''(\mathscr{Q}) = \frac{\mathcal{H}^3\mathcal{W}'' - \mathcal{H}^2(\mathcal{W}'\mathcal{H}' + \mathcal{W}\mathcal{H}'') - \mathcal{H}^2\mathcal{W}'\mathcal{H}' + 2\mathcal{W}\mathcal{H}(\mathcal{H}')^2}{\mathcal{H}^4}. \tag{21}$$

Currently, replacing Equations (18)–(21) into (17) one fetches some different powers of $e^{-\chi(\mathscr{Q})}$. Collecting all the representations with the same power of $e^{-\chi(\mathscr{Q})}$ and equating coefficients to zero given a system of algebraic equations as follows

$$\left.\begin{aligned}
&Eq1: 6\alpha_0 A_3 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 + \alpha_2 A_3^2 B_1 \mathscr{I}_1^3 + 6\alpha_1 A_3 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 + \alpha_3 A_3^2 B_1 \mathscr{I}_2^3 = 0,\\
&Eq2: 10\alpha_0 A_3 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G} + 10\alpha_1 A_3 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G} + 2\alpha_0 A_2 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 + 16\alpha_0 A_3 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 + \alpha_3 A_3^2 B_0 \mathscr{I}_2^3\\
&\quad + \alpha_2 A_3^2 B_0 \mathscr{I}_1^3 + 2\alpha_2 A_2 A_3 B_1 \mathscr{I}_1^3 + 2\alpha_1 A_2 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 + 16\alpha_1 A_3 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 + 2\alpha_3 A_2 A_3 B_1 \mathscr{I}_2^3 = 0,\\
&Eq3: \alpha_0 A_2 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 + 11\alpha_0 A_3 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 + \alpha_1 A_2 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2 + 11\alpha_1 A_3 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2\\
&\quad + 6\alpha_0 A_3 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H} + 6\alpha_1 A_3 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H} + 21\alpha_0 A_3 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G} + 9\alpha_0 A_2 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\\
&\quad + 21\alpha_1 A_3 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G} + 9\alpha_1 A_2 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G} + 2\alpha_0 A_2 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H} + 22\alpha_0 A_3 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H}\\
&\quad + 2\alpha_1 A_2 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H} + 22\alpha_1 A_3 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H} + 6\alpha_0 A_2 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 + \alpha_2 A_2^2 B_0 \mathscr{I}_1^3\\
&\quad + 2\alpha_2 A_1 A_3 B_0 \mathscr{I}_1^3 + 2\alpha_2 A_1 A_2 B_1 \mathscr{I}_1^3 + 2\alpha_2 A_0 A_3 B_1 \mathscr{I}_1^3 + 6\alpha_1 A_2 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 - 2A_3 B_0 B_1 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1\\
&\quad + \alpha_3 A_2^2 B_0 \mathscr{I}_2^3 + 2\alpha_3 A_1 A_3 B_0 \mathscr{I}_2^3 + 2\alpha_3 A_1 A_2 B_1 \mathscr{I}_2^3 + 2\alpha_3 A_0 A_3 B_1 \mathscr{I}_2^3 - A_2 B_1^2 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1 = 0,\\
&Eq4: 4\alpha_0 A_3 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 + 4\alpha_1 A_3 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2 + 3\alpha_0 A_2 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G} + 27\alpha_0 A_3 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\\
&\quad + 3\alpha_1 A_2 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G} + 27\alpha_1 A_3 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G} + 8\alpha_0 A_3 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H} + 8\alpha_1 A_3 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H} + \alpha_3 A_2^2 B_1 \mathscr{I}_2^3\\
&\quad + 12\alpha_0 A_3 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 + 6\alpha_0 A_2 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 + 2\alpha_2 A_2 A_3 B_0 \mathscr{I}_1^3 + \alpha_2 A_2^2 B_1 \mathscr{I}_1^3 + 2\alpha_3 A_1 A_3 B_1 \mathscr{I}_2^3\\
&\quad + 2\alpha_2 A_1 A_3 B_1 \mathscr{I}_1^3 + 12\alpha_1 A_3 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 + 6\alpha_1 A_2 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 + 2\alpha_3 A_2 A_3 B_0 \mathscr{I}_2^3 - A_3 B_1^2 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1 = 0,\\
&Eq5: 9\alpha_0 A_3 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 + 3\alpha_0 A_2 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 + 9\alpha_1 A_3 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2 + 3\alpha_1 A_2 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2\\
&\quad + \alpha_0 A_2 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H} + 17\alpha_0 A_3 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H} + \alpha_1 A_2 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H} + 17\alpha_1 A_3 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H}\\
&\quad + 10\alpha_0 A_2 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G} + \alpha_0 A_0 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G} - \alpha_0 A_1 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G} + 10\alpha_1 A_2 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\\
&\quad + \alpha_1 A_0 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G} - \alpha_1 A_1 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G} + 2\alpha_0 A_3 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H}^2 + 2\alpha_1 A_3 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H}^2\\
&\quad + 18\alpha_0 A_3 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H} + 6\alpha_0 A_2 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H} + 18\alpha_1 A_3 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H} + 6\alpha_1 A_2 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H}\\
&\quad + 2\alpha_0 A_1 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 - 2\alpha_0 A_0 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 + 2\alpha_2 A_1 A_2 B_0 \mathscr{I}_1^3 + 2\alpha_2 A_0 A_3 B_0 \mathscr{I}_1^3 - A_3 B_0^2 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1\\
&\quad + \alpha_2 A_1^2 B_1 \mathscr{I}_1^3 + 2\alpha_2 A_0 A_2 B_1 \mathscr{I}_1^3 + 2\alpha_1 A_1 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 - 2\alpha_1 A_0 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 - A_1 B_1^2 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1\\
&\quad + 2\alpha_3 A_1 A_2 B_0 \mathscr{I}_2^3 + 2\alpha_3 A_0 A_3 B_0 \mathscr{I}_2^3 + \alpha_3 A_1^2 B_1 \mathscr{I}_2^3 + 2\alpha_3 A_0 A_2 B_1 \mathscr{I}_2^3 - 2A_2 B_0 B_1 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1 = 0,\\
&Eq6: 4\alpha_0 A_2 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 + \alpha_0 A_0 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 - \alpha_0 A_1 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 + 4\alpha_1 A_2 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2\\
&\quad + \alpha_1 A_0 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2 - \alpha_1 A_1 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2 + 15\alpha_0 A_3 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H} + 3\alpha_0 A_2 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H}\\
&\quad + 15\alpha_1 A_3 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H} + 3\alpha_1 A_2 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H} + 3\alpha_0 A_1 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G} - 3\alpha_0 A_0 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\\
&\quad + 3\alpha_1 A_1 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G} - 3\alpha_1 A_0 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G} + 6\alpha_0 A_3 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H}^2 + 6\alpha_1 A_3 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H}^2\\
&\quad + 8\alpha_0 A_2 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H} + 2\alpha_0 A_0 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H} - 2\alpha_0 A_1 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H} + 8\alpha_1 A_2 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H} - A_2 B_0^2 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1\\
&\quad + 2\alpha_1 A_0 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H} - 2\alpha_1 A_1 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H} + \alpha_2 A_1^2 B_0 \mathscr{I}_1^3 + 2\alpha_2 A_0 A_2 B_0 \mathscr{I}_1^3 - A_0 B_1^2 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1\\
&\quad + 2\alpha_2 A_0 A_1 B_1 \mathscr{I}_1^3 + \alpha_3 A_1^2 B_0 \mathscr{I}_2^3 + 2\alpha_3 A_0 A_2 B_0 \mathscr{I}_2^3 + 2\alpha_3 A_0 A_1 B_1 \mathscr{I}_2^3 - 2A_1 B_0 B_1 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1 = 0,\\
&Eq7: \alpha_0 A_1 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 - \alpha_0 A_0 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}^2 + \alpha_1 A_1 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2 - \alpha_1 A_0 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}^2 + \alpha_2 A_0^2 B_1 \mathscr{I}_1^3\\
&\quad + 6\alpha_0 A_2 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H} + 3\alpha_0 A_0 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H} - 3\alpha_0 A_1 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H} + 6\alpha_1 A_2 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H}\\
&\quad + 3\alpha_1 A_0 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H} - 3\alpha_1 A_1 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H} + 6\alpha_0 A_3 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H}^2 + 6\alpha_1 A_3 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H}^2\\
&\quad + 2\alpha_0 A_1 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H} - 2\alpha_0 A_0 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H} + 2\alpha_1 A_1 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H} - 2\alpha_1 A_0 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H}\\
&\quad + 2\alpha_2 A_0 A_1 B_0 \mathscr{I}_1^3 + 2\alpha_3 A_0 A_1 B_0 \mathscr{I}_2^3 + \alpha_3 A_0^2 B_1 \mathscr{I}_2^3 - A_1 B_0^2 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1 - 2A_0 B_0 B_1 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1 = 0,\\
&Eq8: \alpha_0 A_1 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H} - \alpha_0 A_0 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{G}\mathscr{H} + \alpha_1 A_1 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H} + 2\alpha_1 A_2 B_0^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H}^2 + \alpha_2 A_0^2 B_0 \mathscr{I}_1^3\\
&\quad + 2\alpha_0 A_2 B_0^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H}^2 + 2\alpha_0 A_0 B_1^2 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H}^2 - 2\alpha_0 A_1 B_0 B_1 \mathscr{I}_2 \mathscr{I}_1^4 \mathscr{H}^2 - A_0 B_0^2 \mathscr{I}_2 \mathscr{I}_3 \mathscr{I}_1\\
&\quad + 2\alpha_1 A_0 B_1^2 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H}^2 - 2\alpha_1 A_1 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{H}^2 + \alpha_3 A_0^2 B_0 \mathscr{I}_2^3 - \alpha_1 A_0 B_0 B_1 \mathscr{I}_2^4 \mathscr{I}_1 \mathscr{G}\mathscr{H} = 0,
\end{aligned}\right\} \tag{22}$$

The following cases are produced by solving the previously acquired system:

CASE 1. The following is a list of the obtained coefficients:

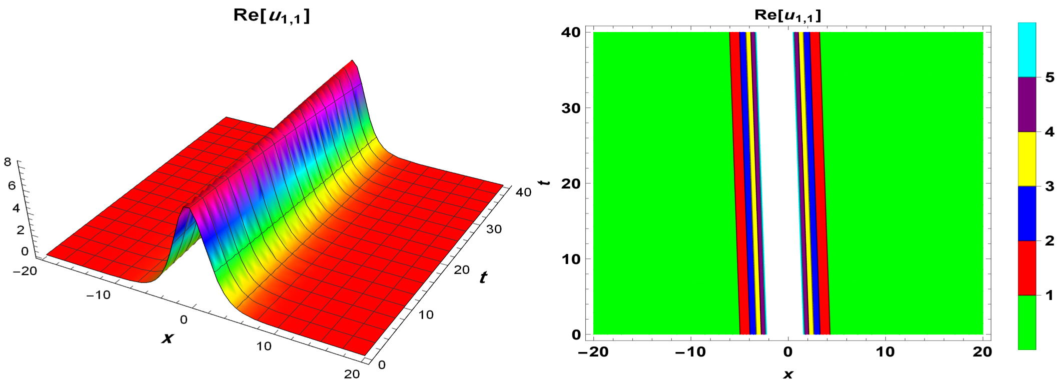
$$\begin{aligned} &A_0 = \frac{\mathscr{H}(A_1 - A_3\mathscr{H})}{\mathscr{G}}; \mathscr{I}_3 = \left(\alpha_0\mathscr{I}_1^3 + \alpha_1\mathscr{I}_2^3\right)\left(\mathscr{G}^2 - 4\mathscr{H}\right); \\ &B_0 = \frac{B_1(A_1 - A_3\mathscr{H})}{A_3\mathscr{G}}; A_2 = \frac{A_3\left(\mathscr{G}^2 - \mathscr{H}\right) + A_1}{\mathscr{G}}; \\ &\alpha_3 = -\frac{\mathscr{I}_1\left(\alpha_2 A_3\mathscr{I}_1^2 + 6B_1\mathscr{I}_2\left(\alpha_0\mathscr{I}_1^3 + \alpha_1\mathscr{I}_2^3\right)\right)}{A_3\mathscr{I}_2^3}. \end{aligned} \tag{23}$$

Using parameters Equation (23), based on the provided solutions Equations (7)–(11) for the described Equation (6) one obtains the following sub-cases for the solution of the offered model (1)

Case 1.1. The following solution to Equation (1) is obtained using Equations (23) and (7) together.

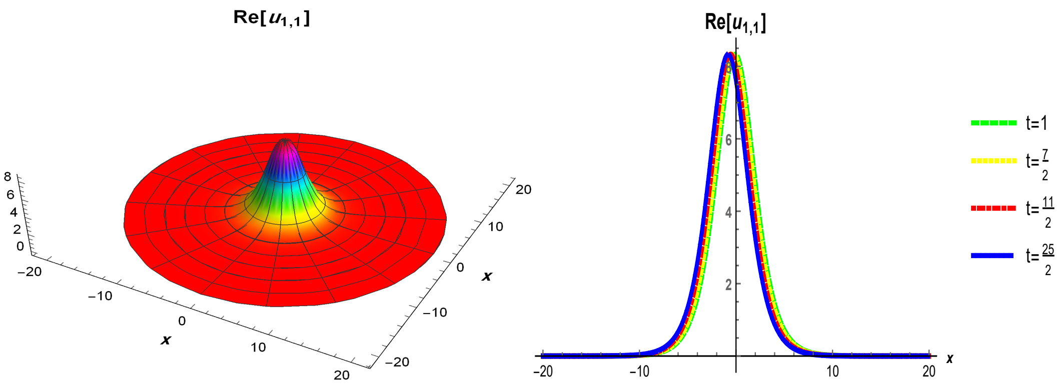
$$u_{1,1} = \frac{A_3\mathscr{H}\left(4\mathscr{H} - \mathscr{G}^2\right)}{B_1\left(\mathcal{A}\sinh\left(\frac{1}{2}(\wp + \mathscr{Q})\mathcal{A}\right) + \mathscr{G}\cosh\left(\frac{1}{2}(\wp + \mathscr{Q})\mathcal{A}\right)\right)^2}, \tag{24}$$

where $\mathcal{A} = \sqrt{\mathscr{G}^2 - 4\mathscr{H}}$, $\mathscr{Q} = \mathscr{I}_1 x + \mathscr{I}_2 y - \mathscr{I}_3 t$. Graphs of the bright soliton solutions to (24) are shown in Figures 1 and 2, where: $\wp = \frac{2}{3}; \mathscr{G} = \frac{3}{2}; \mathscr{H} = \frac{1}{4}; A_3 = -\frac{2}{3}; B_1 = \frac{1}{3}; \mathscr{I}_1 = -\frac{2}{3}; \mathscr{I}_2 = \frac{1}{2}; \alpha_1 = \frac{5}{6}; \alpha_0 = \frac{2}{3}; y = \frac{5}{3}$.



(a) Real part of $\{u_{1,1}\}$ in 3-dimensions **(b)** Real part of $\{u_{1,1}\}$ with contour surface

Figure 1. Three-dimension and contour surface plot of (24), where $-20 \le x \le 20,\ 0 \le t \le 40$.



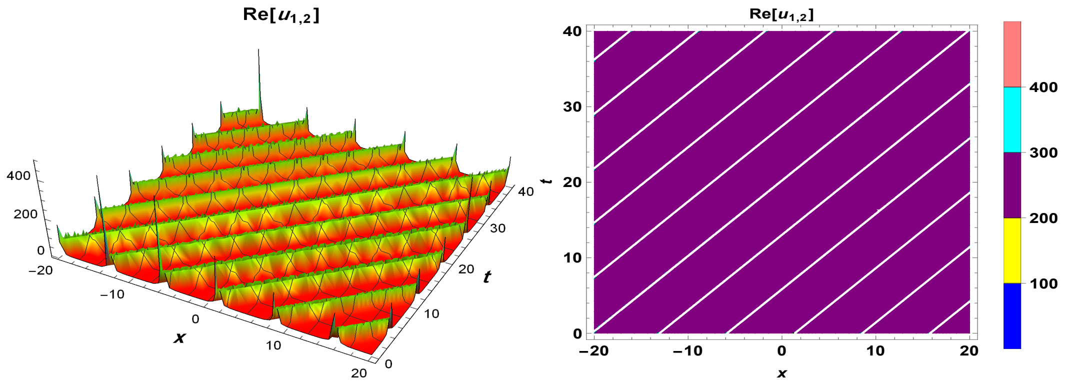
(a) Three dimensional revolving plot of $\{u_{1,1}\}$ **(b)** Two dimensional time-evolution graphs of $\{u_{1,1}\}$

Figure 2. Three dimensional revolving plot and 2D time evolution graph of (24) where $-20 \le x \le 20$, time values are given in the legend.

Case 1.2. The following solution to Equation (1) is obtained using Equations (23) and (8) together.

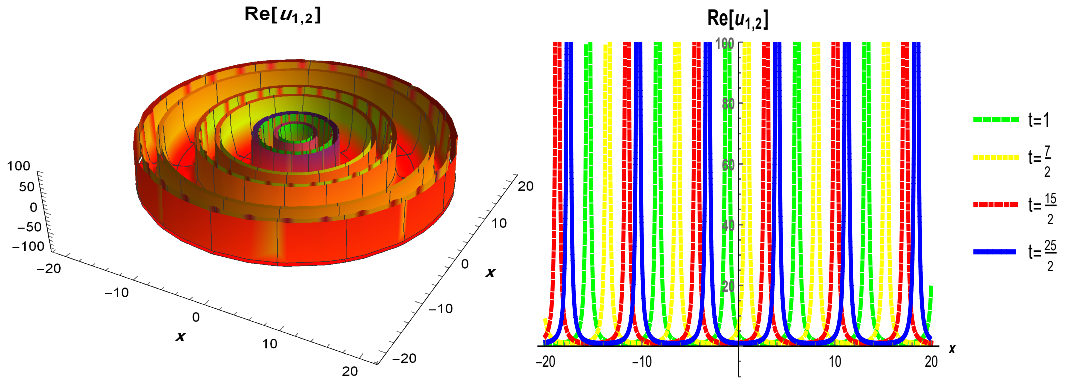
$$u_{1,2} = \frac{A_3\mathscr{H}\left(4\mathscr{H} - \mathscr{G}^2\right)}{B_1\left(\mathscr{G}\cos\left(\frac{1}{2}(\wp + \mathscr{Q})\mathcal{A}\right) - \mathcal{A}\sin\left(\frac{1}{2}(\wp + \mathscr{Q})\mathcal{A}\right)\right)^2}, \tag{25}$$

herein $\mathcal{A} = \sqrt{4\mathscr{H} - \mathscr{G}^2}$, $\mathscr{Q} = \mathscr{I}_1 x + \mathscr{I}_2 y - \mathscr{I}_3 t$. Solution (25), has been figured out as follows for the specific parameters $\wp = \frac{1}{2}; \mathscr{G} = \frac{1}{2}; \mathscr{H} = \frac{5}{4}; A_3 = -\frac{1}{2}; B_1 = -\frac{2}{3}; \mathscr{I}_1 = -\frac{2}{5}; \mathscr{I}_2 = \frac{1}{2}; \alpha_1 = \frac{1}{3}; \alpha_0 = -\frac{2}{3}; y = \frac{1}{2}$. The time-space evolution of the real part, where highly localized oscillations are represented by Figures 3 and 4.



(a) Real part of $\{u_{1,1}\}$ in 3-dimensions **(b)** Real part of $\{u_{1,1}\}$ with contour surface

Figure 3. Three dimension and contour surface plot of (25), where $-20 \leq x \leq 20,\ 0 \leq t \leq 40$.



(a) Three-dimensional revolving plot of $\{u_{1,2}\}$ **(b)** Two-dimensional time-evolution graphs of $\{u_{1,2}\}$

Figure 4. Three-dimensional revolving plot and 2D time evolution graph of (25) where $-20 \leq x \leq 20$, time values are given in the legend.

Case 1.3. The following solution to Equation (1) is obtained using Equations (23) and (9) together.

$$u_{1,3} = \frac{A_3\mathscr{G}^2\operatorname{csch}^2\left(\frac{1}{2}\mathscr{G}\left(\wp + \mathscr{I}_1 x + \mathscr{I}_2 y - \mathscr{G}^2\left(\alpha_0\mathscr{I}_1^3 + \alpha_1\mathscr{I}_2^3\right)t\right)\right)}{4B_1}. \tag{26}$$

Case 1.4. The following solution to Equation (1) is obtained using Equations (23) and (10) together.

$$u_{1,4} = \frac{A_3\mathscr{G}^2}{B_1(\mathscr{G}(\wp + \mathscr{I}_1 x + \mathscr{I}_2 y) + 2)^2}. \tag{27}$$

CASE 2. The following is a list of the obtained coefficients:

$$
\begin{aligned}
A_0 &= A_2\mathscr{H} - \frac{\sqrt{2}\mathscr{H}\sqrt{A_3^3B_1^2(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)(2A_3\mathscr{H}(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)-3B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3)}}{A_3B_1(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)};\\
A_1 &= \frac{\sqrt{2}A_2\sqrt{A_3^3B_1^2(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)(2A_3\mathscr{H}(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)-3B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3)}}{A_3^2B_1(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)}\\
&-3A_3\mathscr{H}+\frac{6B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3}{\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3};\alpha_1=-\frac{6\alpha_0\mathscr{I}_2\mathscr{I}_1^4+\frac{A_3\left(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3\right)}{B_1}}{6\mathscr{I}_1\mathscr{I}_2^4};\\
\mathscr{G} &= \frac{\sqrt{2}\sqrt{A_3^3B_1^2(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)(2A_3\mathscr{H}(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)-3B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3)}}{A_3^2B_1(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)};\\
B_0 &= \frac{A_2B_1}{A_3}-\frac{\sqrt{2}\sqrt{A_3^3B_1^2(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)(2A_3\mathscr{H}(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)-3B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3)}}{A_3^2(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)}.
\end{aligned} \tag{28}
$$

The collected parameters Equation (28), based on the provided solutions Equations (7)–(11) for the specified Equation (6) are producing the following sub-cases for the solution of the given model Equation (1):

Case 2.1. The following solution to Equation (1) is obtained using Equations (28) and (7) together.

$$
u_{2,1}=-\frac{3A_3^3B_1^3\mathscr{I}_1^2\mathscr{I}_2^2\mathscr{I}_3^2\mathscr{H}\operatorname{sech}^2\left(\sqrt{3}(\wp+\mathscr{Q})\sqrt{-\frac{B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3}{2\alpha_2A_3\mathscr{I}_1^3+2\alpha_3A_3\mathscr{I}_2^3}}\right)}{\left(\mathcal{A}-\sqrt{3}A_3B_1^2\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3\tanh\left(\sqrt{3}(\wp+\mathscr{Q})\sqrt{-\frac{B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3}{2\alpha_2A_3\mathscr{I}_1^3+2\alpha_3A_3\mathscr{I}_2^3}}\right)\right)^2}, \tag{29}
$$

here $\mathcal{A}=\sqrt{-\frac{B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3}{A_3\left(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3\right)}}\sqrt{A_3^3B_1^2(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)(2A_3\mathscr{H}(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)-3B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3)}$, and $\mathscr{Q}=\mathscr{I}_1x+\mathscr{I}_2y-\mathscr{I}_3t$.

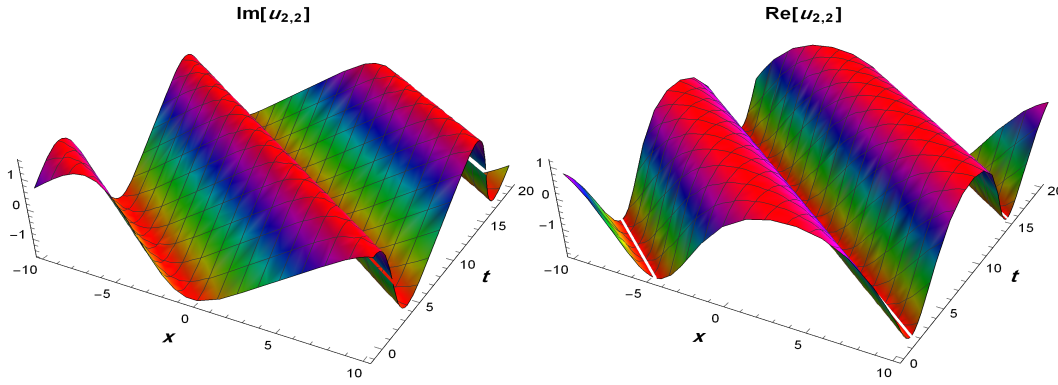
Case 2.2. The following solution to Equation (1) is obtained using Equations (28) and (8) together.

$$
\begin{aligned}
u_{2,2} = &\frac{\mathscr{H}\left(A_3^2B_1\sec^2\left(\frac{1}{2}\mathcal{A}(\mathscr{Q}+\wp)\right)\left(\mathscr{G}^2\cos(\mathcal{A}(\mathscr{Q}+\wp))-\mathcal{A}\mathscr{G}\sin(\mathcal{A}(\mathscr{Q}+\wp))+4\mathscr{H}\right)\right)}{A_3B_1^2\left(\mathscr{G}-\mathcal{A}\tan\left(\frac{1}{2}\mathcal{A}(\mathscr{Q}+\wp)\right)\right)^2}\\
&-\frac{\mathcal{B}\mathscr{H}\left(\mathscr{G}-\mathcal{A}\tan\left(\frac{1}{2}\mathcal{A}(\mathscr{Q}+\wp)\right)\right)}{A_3B_1^2\left(\mathscr{G}-\mathcal{A}\tan\left(\frac{1}{2}\mathcal{A}(\mathscr{Q}+\wp)\right)\right)^2},
\end{aligned} \tag{30}
$$

here

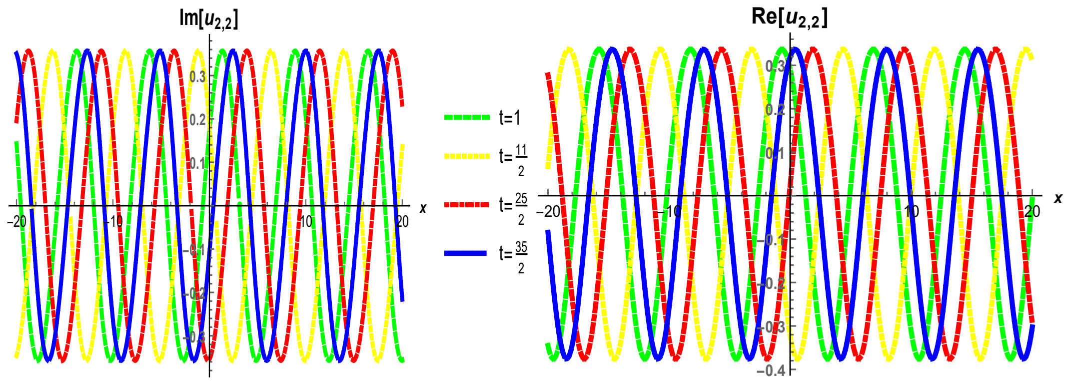
$$
\mathcal{B}=\frac{2\sqrt{2}\sqrt{A_3^3B_1^2(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)(2A_3\mathscr{H}(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)-3B_1\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3)}}{\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3},\ \mathcal{A}=\sqrt{4\mathscr{H}-\mathscr{G}^2}
$$

and $\mathscr{Q}=\mathscr{I}_1x+\mathscr{I}_2y-\mathscr{I}_3t$. Given solution in (30) are plotted in the follows where $\wp=\frac{2}{3};\mathscr{H}=-\frac{1}{2};A_3=-\frac{1}{2};B_1=-\frac{2}{3};\mathscr{I}_1=-\frac{1}{5};\mathscr{I}_2=\frac{1}{6};\mathscr{I}_3=\frac{1}{4};\alpha_2=\frac{1}{4};\alpha_3=-\frac{2}{5};y=\frac{1}{2}$. Here, we obtained the periodic traveling wave solutions represented by Figures 5 and 6.



(a) Imaginary part of $\{u_{2,2}\}$ in 3-dimensions **(b)** Real part of $\{u_{2,2}\}$ in 3-dimensions

Figure 5. Three-dimensional figures of (30), where $-20 \leq x \leq 20,\ 0 \leq t \leq 20$.



(a) Imaginary part of $\{u_{2,2}\}$ in 2-dimensions **(b)** Real part of $\{u_{2,2}\}$ in 2-dimensions

Figure 6. Two-dimensional time evolution graphs of (30) where $-20 \leq x \leq 20$, and time values are given in the legend.

Case 2.3. The following solution to Equation (1) is obtained using Equations (28) and (9) together.

$$u_{2,3} = \frac{3\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3 \csc^2\left(\frac{\sqrt{\frac{3}{2}}\sqrt{B_1}\sqrt{\mathscr{I}_1}\sqrt{\mathscr{I}_2}\sqrt{\mathscr{I}_3}(\wp-\mathscr{I}_3 t+\mathscr{I}_1 x+\mathscr{I}_2 y)}{\sqrt{A_3}\sqrt{\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3}}\right)}{2(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3)}. \tag{31}$$

CASE 3. The following is a list of the obtained coefficients:

$$\begin{aligned} A_1 &= \frac{A_3\left(B_1\mathscr{G}^2+6B_0\mathscr{G}+2B_1\mathscr{H}\right)}{6B_1}; A_2 = \frac{A_3(B_1\mathscr{G}+B_0)}{B_1}; \\ A_0 &= \frac{A_3B_0\left(\mathscr{G}^2+2\mathscr{H}\right)}{6B_1}; \mathscr{I}_3 = \frac{A_3\left(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3\right)\left(\mathscr{G}^2-4\mathscr{H}\right)}{6B_1\mathscr{I}_1\mathscr{I}_2}; \\ \alpha_0 &= \frac{-\alpha_3A_3\mathscr{I}_2^3-\alpha_2A_3\mathscr{I}_1^3-6\alpha_1B_1\mathscr{I}_1\mathscr{I}_2^4}{6B_1\mathscr{I}_1^4\mathscr{I}_2}. \end{aligned} \tag{32}$$

Specified coefficients Equation (32) furthermore to utilizing provided solutions Equations (7)–(11) for the known Equation (6) produce the following sub-cases, which includes the solution of the given model Equation (1).

Case 3.1. When performing Equation (32) furthermore with Equation (7) the result is the following solution for Equation (1).

$$u_{3,1} = \frac{\mathcal{A}^2 A_3 \operatorname{sech}^2\left(\frac{1}{2}\mathcal{A}(\wp+\mathscr{Q})\right)\left(\left(\mathscr{G}^2-2\mathscr{H}\right)\cosh(\mathcal{A}(\wp+\mathscr{Q}))-4\mathscr{H}\right)}{6B_1\left(\mathcal{A}\tanh\left(\frac{1}{2}\mathcal{A}(\wp+\mathscr{Q})\right)+\mathscr{G}\right)^2} + \frac{\mathcal{A}^3 A_3 \mathscr{G}\sinh(\mathcal{A}(\wp+\mathscr{Q}))\operatorname{sech}^2\left(\frac{1}{2}\mathcal{A}(\wp+\mathscr{Q})\right)}{6B_1\left(\mathcal{A}\tanh\left(\frac{1}{2}\mathcal{A}(\wp+\mathscr{Q})\right)+\mathscr{G}\right)^2}. \tag{33}$$

here $\mathcal{A}=\sqrt{\mathscr{G}^2-4\mathscr{H}}$, and $\mathscr{Q}=\mathscr{I}_1 x+\mathscr{I}_2 y-\mathscr{I}_3 t.$

Case 3.2. When performing Equation (32) furthermore with Equation (8) the result is the following solution for Equation (1).

$$u_{3,2} = \frac{A_3\left(\mathscr{G}^2-4\mathscr{H}\right)\sec^2\left(\frac{1}{2}\mathcal{A}(\mathscr{Q}+\wp)\right)\left(\left(\mathscr{G}^2-2\mathscr{H}\right)\cos(\mathcal{A}(\mathscr{Q}+\wp))-4\mathscr{H}\right)}{6B_1\left(\mathscr{G}-\mathcal{A}\tan\left(\frac{1}{2}\mathcal{A}(\mathscr{Q}+\wp)\right)\right)^2} - \frac{\mathcal{A}A_3\mathscr{G}\left(\mathscr{G}^2-4\mathscr{H}\right)\sin(\mathcal{A}(\mathscr{Q}+\wp))\sec^2\left(\frac{1}{2}\mathcal{A}(\mathscr{Q}+\wp)\right)}{6B_1\left(\mathscr{G}-\mathcal{A}\tan\left(\frac{1}{2}\mathcal{A}(\mathscr{Q}+\wp)\right)\right)^2}, \tag{34}$$

where $\mathcal{A}=\sqrt{4\mathscr{H}-\mathscr{G}^2}$, $\mathscr{Q}=\mathscr{I}_1 x+\mathscr{I}_2 y-\mathscr{I}_3 t.$

Case 3.3. When performing Equation (32) furthermore with Equation (9) the result is the following solution for Equation (1).

$$u_{3,3} = \frac{A_3\mathscr{G}^2\left(3\operatorname{csch}^2\left(\frac{1}{2}\mathscr{G}\left(-\frac{A_3\mathscr{G}^2 t\left(\alpha_2\mathscr{I}_1^3+\alpha_3\mathscr{I}_2^3\right)}{6B_1\mathscr{I}_1\mathscr{I}_2}+\wp+\mathscr{I}_1 x+\mathscr{I}_2 y\right)\right)+2\right)}{12B_1}. \tag{35}$$

Case 3.4. When performing Equation (32) furthermore with Equation (10) the result is the following solution for Equation (1).

$$u_{3,4} = \frac{A_3\mathscr{G}^2}{B_1(\mathscr{G}(\wp+\mathscr{I}_1 x+\mathscr{I}_2 y)+2)^2}. \tag{36}$$

Case 3.5. When performing Equation (32) furthermore with Equation (11) the result is the following solution for Equation (1).

$$u_{3,5} = \frac{A_3}{B_1(\wp+\mathscr{I}_1 x+\mathscr{I}_2 y)^2}. \tag{37}$$

Remark 3. *We omitted the graphs of the offered solutions in Equations (33)–(37) as they have the same appearance as the graphs in the preceding sub-cases.*

3.2. Application of ERSCM to the (2+1)-Dimensional Novikov-Veselov System

In this subsection, the extended rational $\sin-\cos$ method is applied to the nonlinear (2+1)-dimensional Novikov-Veselov system. The various forms of this model have been explored using numerous analytic approaches; we have listed several in the literature. To start solving Equation (17) , make the assumption that solutions to the researched model have the following form:

$$U(\mathscr{Q}) = \frac{\gamma_0\sin(\mathscr{H}\mathscr{Q})}{\gamma_1\cos(\mathscr{H}\mathscr{Q})+\gamma_2},\ \cos(\mathscr{H}\mathscr{Q})\neq-\frac{\gamma_2}{\gamma_1}, \tag{38}$$

where $\gamma_0, \gamma_1, \gamma_2$ are parameters to be determined and $\mathscr{H}$ is a non-zero wave number. Here should be $\gamma_0\neq 0$, and $\gamma_1^2+\gamma_2^2\neq 0$. Take the derivatives of the assumed solution Equation (38), one immediately gets

$$U'(\mathscr{Q}) = \frac{\gamma_0\mathscr{H}(\gamma_2\cos(\mathscr{H}\mathscr{Q})+\gamma_1)}{(\gamma_1\cos(\mathscr{H}\mathscr{Q})+\gamma_2)^2}, \tag{39}$$

and

$$U''(\mathscr{Q}) = \frac{\gamma_0 \mathscr{H}^2 \sin(\mathscr{H}\mathscr{Q})\left(\gamma_2\gamma_1 \cos(\mathscr{H}\mathscr{Q}) + 2\gamma_1^2 - \gamma_2^2\right)}{\left(\gamma_1 \cos(\mathscr{H}\mathscr{Q}) + \gamma_2\right)^3}. \tag{40}$$

Installing Equations (38)–(40) into Equation (17), one gets the following:

$$\begin{aligned}
&2\alpha_0\gamma_1^2\mathscr{I}_2\mathscr{I}_1^4\mathscr{H}^2 + \alpha_2\gamma_0\gamma_2\mathscr{I}_1^3\sin(\mathscr{H}\mathscr{Q}) + \alpha_2\gamma_0\gamma_1\mathscr{I}_1^3\sin(\mathscr{H}\mathscr{Q})\cos(\mathscr{H}\mathscr{Q}) - \gamma_2^2\mathscr{I}_1\mathscr{I}_3\mathscr{I}_2 \\
&+ \alpha_0\gamma_1\gamma_2\mathscr{I}_2\mathscr{I}_1^4\mathscr{H}^2\cos(\mathscr{H}\mathscr{Q}) - \mathscr{H}^2\alpha_0\gamma_2^2\mathscr{I}_2\mathscr{I}_1^4 + \alpha_3\gamma_0\gamma_1\mathscr{I}_2^3\sin(\mathscr{H}\mathscr{Q})\cos(\mathscr{H}\mathscr{Q}) \\
&+ \alpha_1\gamma_1\gamma_2\mathscr{I}_1\mathscr{I}_2^4\mathscr{H}^2\cos(\mathscr{H}\mathscr{Q}) + 2\alpha_1\gamma_1^2\mathscr{I}_1\mathscr{I}_2^4\mathscr{H}^2 + \alpha_3\gamma_0\gamma_2\mathscr{I}_2^3\sin(\mathscr{H}\mathscr{Q}) \\
&- \mathscr{H}^2\alpha_1\gamma_2^2\mathscr{I}_1\mathscr{I}_2^4 - \gamma_1^2\mathscr{I}_1\mathscr{I}_3\mathscr{I}_2\cos^2(\mathscr{H}\mathscr{Q}) - 2\gamma_1\gamma_2\mathscr{I}_1\mathscr{I}_3\mathscr{I}_2\cos(\mathscr{H}\mathscr{Q}) = 0.
\end{aligned} \tag{41}$$

In Equation (41) collecting all terms with the same powers of $\cos^{\tau_1}(\mathscr{H}\mathscr{Q})\sin^{\tau_2}(\mathscr{H}\mathscr{Q})$ where $\tau_1, \tau_2 = 0, 1, 2$ and equating coefficients of the same powers to zero gives a system of algebraic equations as follows:

$$\left.\begin{aligned}
&\sin^0(\mathscr{H}\mathscr{Q})\cos^2(\mathscr{H}\mathscr{Q}) : -\gamma_1^2\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3 = 0, \\
&\sin^0(\mathscr{H}\mathscr{Q})\cos^1(\mathscr{H}\mathscr{Q}) : \alpha_0\gamma_1\gamma_2\mathscr{I}_2\mathscr{I}_1^4\mathscr{H}^2 + \alpha_1\gamma_1\gamma_2\mathscr{I}_2^4\mathscr{I}_1\mathscr{H}^2 \\
&\qquad\qquad\qquad - 2\gamma_1\gamma_2\mathscr{I}_2\mathscr{I}_3\mathscr{I}_1 = 0, \\
&\sin^1(\mathscr{H}\mathscr{Q})\cos^0(\mathscr{H}\mathscr{Q}) : \alpha_2\gamma_0\gamma_1\mathscr{I}_1^3 + \alpha_3\gamma_0\gamma_1\mathscr{I}_2^3 = 0, \\
&\sin^0(\mathscr{H}\mathscr{Q})\cos^0(\mathscr{H}\mathscr{Q}) : -\mathscr{H}^2\alpha_0\gamma_2^2\mathscr{I}_2\mathscr{I}_1^4 - \alpha_1\gamma_2^2\mathscr{I}_2^4\mathscr{I}_1\mathscr{H}^2 \\
&\qquad\qquad\qquad - \gamma_2^2\mathscr{I}_2\mathscr{I}_3\mathscr{I}_1 = 0, \\
&\sin^1(\mathscr{H}\mathscr{Q})\cos^1(\mathscr{H}\mathscr{Q}) : \alpha_2\gamma_0\gamma_2\mathscr{I}_1^3 + \alpha_3\gamma_0\gamma_2\mathscr{I}_2^3 = 0.
\end{aligned}\right\} \tag{42}$$

Solving the system in Equation (42) by using some computer software packages, the following cases are obtained:

Case 1. The acquired coefficients in solving the related system Equation (42) are presented in the following:

$$\alpha_0 = 0; \alpha_2 = -\frac{\alpha_3\mathscr{I}_2^3}{\mathscr{I}_1^3}; \mathscr{H} = -\frac{i\sqrt{\mathscr{I}_3}}{\sqrt{\alpha_1}\mathscr{I}_2^{3/2}}; \gamma_1 = 0. \tag{43}$$

By substituting (43) into Equation (38), the solution for Equation (1) is obtained as follows:

$$u_1 = -\frac{i\gamma_0}{\gamma_2}\sinh\left(\frac{\sqrt{\mathscr{I}_3}(-\mathscr{I}_3 t + \mathscr{I}_1 x + \mathscr{I}_2 y)}{\sqrt{\alpha_1}\mathscr{I}_2^{3/2}}\right). \tag{44}$$

Case 2. The acquired coefficients in solving the related system Equation (42) are presented in the following:

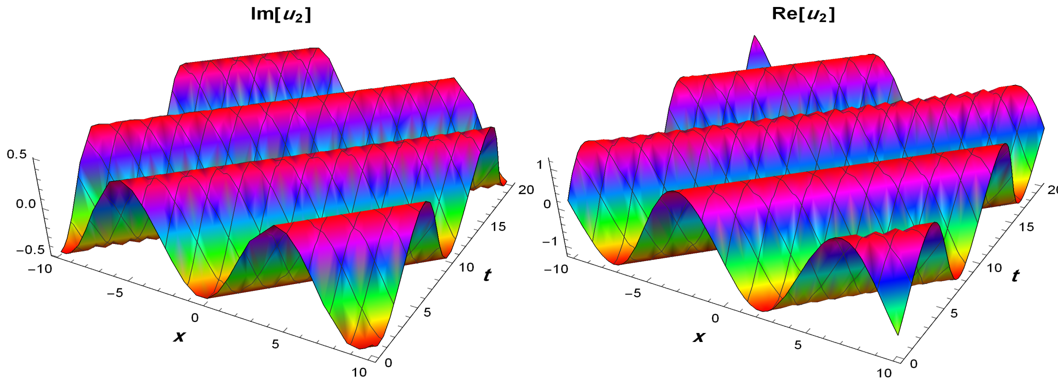
$$\mathscr{I}_2 = \frac{\sqrt[3]{-1}\sqrt[3]{\alpha_2}\mathscr{I}_1}{\sqrt[3]{\alpha_3}}; \mathscr{I}_3 = -\frac{(\alpha_0\alpha_3 - \alpha_1\alpha_2)\mathscr{I}_1^3\mathscr{H}^2}{\alpha_3}; \gamma_1 = 0. \tag{45}$$

Subbing Equation (45) into Equation (38), the solution of Equation (1) is constructed as follows

$$u_2 = \frac{\gamma_0}{\gamma_2}\sin\left(\mathscr{H}\left(\frac{(\alpha_0\alpha_3 - \alpha_1\alpha_2)\mathscr{I}_1^3\mathscr{H}^2 t}{\alpha_3} + \mathscr{I}_1 x + \frac{\sqrt[3]{-1}\sqrt[3]{\alpha_2}\mathscr{I}_1 y}{\sqrt[3]{\alpha_3}}\right)\right). \tag{46}$$

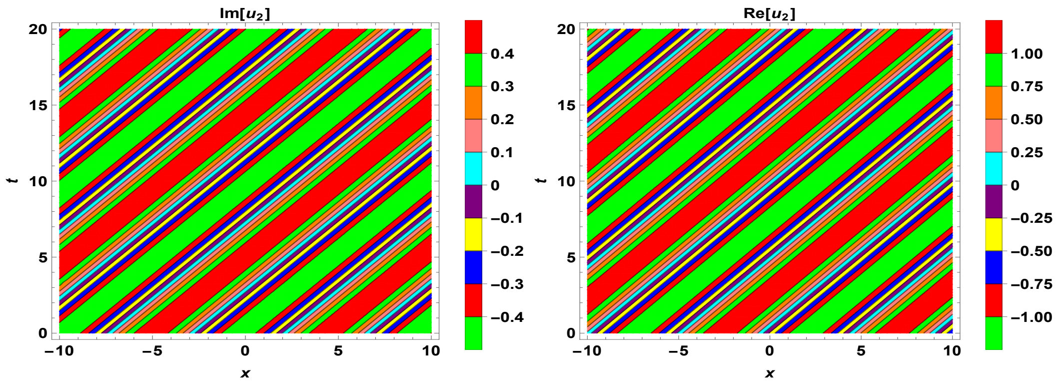
The solution (44) has been plotted in the following where $\mathscr{I}_1 = -\frac{2}{5}; \alpha_0 = \frac{2}{5}; \alpha_1 = \frac{5}{3}; \alpha_2 = -\frac{3}{5}; \alpha_3 = -\frac{3}{8}; \gamma_0 = -\frac{3}{4}; \gamma_2 = \frac{2}{3}; \mathscr{H} = -\frac{5}{3}; y = \frac{5}{8}$.

Here, we present the periodic traveling wave solutions, which have been represented by Figures 7–9.



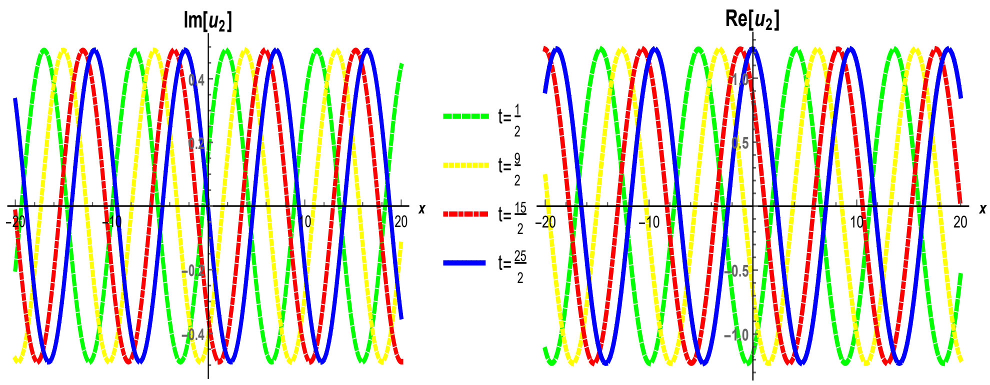
(a) Im part of $\{u_2\}$ in 3-dimensions **(b)** Re part of $\{u_2\}$ in 3-dimensions

Figure 7. Three dimension plots of (46), where $-10 \leq x \leq 10,\ 0 \leq t \leq 20$.



(a) Im part of $\{u_2\}$ **(b)** Re part of $\{u_2\}$

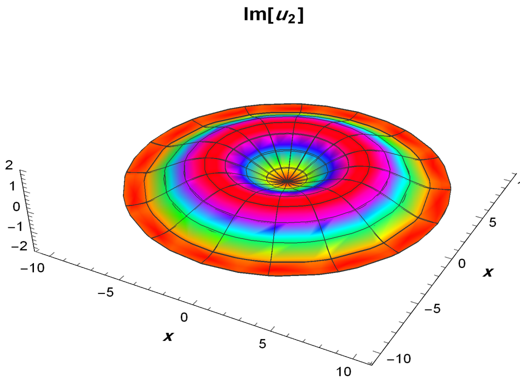
Figure 8. Contour surface graphs of (46) where $-10 \leq x \leq 10,\ 0 \leq t \leq 20$.



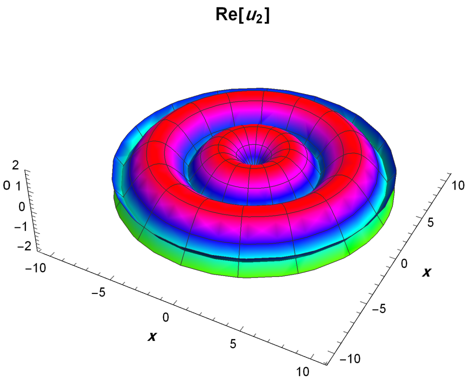
(a) Im part of (46) **(b)** Re part of (46)

Figure 9. Two-dimensional temporal-evolution graphs of (46) where $-10 \leq x \leq 10$, and given t values in the legend.

Figure 10 illustrates the complex structure of the wave field in one dimension, highlighting its circular symmetry and revealing the amplitude and phase distributions at a specific time.



(a) Im part of (46)



(b) Re part of (46)

Figure 10. Three dimensional revolution plots of (46) for $t = 1,\ -10 \leq x \leq 10$.

Case 3. The acquired coefficients in solving the related system Equation (42) are presented in the following:

$$\gamma_1 = 0; \alpha_3 = -\frac{\alpha_2 \mathscr{I}_1^3}{\mathscr{I}_2^3}; \mathscr{H} = \frac{\sqrt{\mathscr{I}_3}}{\sqrt{-\alpha_0 \mathscr{I}_1^3 - \alpha_1 \mathscr{I}_2^3}}. \tag{47}$$

Plugging Equation (47) into Equation (38), the solution of Equation (1) is obtained as follows:

$$u_3 = \frac{\gamma_0}{\gamma_2} \sin\left(\frac{\sqrt{\mathscr{I}_3}(-\mathscr{I}_3 t + \mathscr{I}_1 x + \mathscr{I}_2 y)}{\sqrt{-\alpha_0 \mathscr{I}_1^3 - \alpha_1 \mathscr{I}_2^3}}\right). \tag{48}$$

Remark 4. *Through similar previous arguments and by supposing the following formulation of the solution:*

$$U(\mathscr{Q}) = \frac{\gamma_0 \cos(\mathscr{H}\mathscr{Q})}{\gamma_2 + \gamma_1 \sin(\mathscr{H}\mathscr{Q})},\ \sin(\mathscr{H}\mathscr{Q}) \neq -\frac{\gamma_2}{\gamma_1},$$

where γ_i *for* $i = 0, 1, 2$ *are parameters and* $\mathscr{H}$ *is a non-zero wave number, someone else obtains other solutions for Equation (1).*

4. Conclusions

In this work, the third-order nonlinear (2+1)-dimensional Novikov-Veselov system of equations with constant coefficients has been explored utilizing the extended rational $\sin - \cos$ approach and the modified exponential function method as two reliable and trustworthy techniques for solving nonlinear systems analytically. The major purpose is to acquire particular, absolutely exact traveling waves, periodic waves, and soliton solutions. In that they shed light on the important features of the physical phenomena, the provided solutions are both new and substantial. The resultant solutions are written as a variety of trigonometric functions, including hyperbolic trigonometric functions, exponential functions, and rational functions. The qualities of the solutions have been shown in a diversity of forms, including two- and three-dimensional ones. All of the generated solutions have been confirmed by substituting them back into relevant equations using efficient technologies in software programming packages. The specified values for the free parameters have a major impact on molding the mechanical and physical properties of the produced solutions.

Bright soliton solutions are achieved in Figures 1 and 2. The time-space evolution of the real part, where highly localized oscillations are represented by Figures 3 and 4. Using two distinct methodologies, we found periodic traveling wave solutions by comparing Figures 5 and 6 with Figures 7–9. Figure 10 illustrates the complex structure of a wave field along one spatial dimension, with circular symmetry in the cross-section, offering insight into its amplitude and phase distribution at a fixed time.

The outcomes indicate that the employed methodologies are straightforward and appropriate for application to other mathematical, physical, and engineering models. Moreover, for the best optical observations of the physical characteristics of the generated solutions, the efficacy of the temporal evolution has been illustrated by two-dimensional graphs.

Author Contributions

The authors investigated the research model, developed applications, and performed calculations. All authors contributed equally to the writing of the paper and equally to the assessment of the results. A.A.M.: Conceptualization, Writing—original Draft, Writing—Review and Editing, Software. K.A.M.: Investigation, Methodology, Resources, Validation, Formal Analysis, Visualization. T.T.; Supervision, Formal Analysis, Conceptualization, Investigation, Resources. All authors have read and agreed to the published version of the manuscript.

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The data produced and/or evaluated during the present study are not publicly accessible; however, they are available from the relevant author upon a justifiable request.

Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

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