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Distributed Constrained Optimization for Nonlinear Stochastic Multi-Agent Systems: Application to Resource Allocation

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Abstract: This paper investigates the resource allocation problem (RAP) for nonlinear stochastic systems subject to random disturbances. The communication network is modeled as a weight-balanced digraph, where each agent can only access its own differentiable and strongly convex local cost function. A fully distributed adaptive state-feedback algorithm is proposed, and rigorous analysis shows that the decision variables converge almost surely to the optimal solution. Unlike existing studies on deterministic RAPs, the system considered here is affected by two types of stochastic factors-Brownian motion and unknown nonlinear dynamics—which significantly increase the difficulty of algorithm design. Finally, numerical simulations on a resource allocation example are provided to demonstrate the effectiveness of the proposed approach.

Keywords: resource allocation; nonlinear stochastic system; adaptive control

1. Introduction

Distributed coordination control has rapidly emerged as a prominent research direction in recent years, driven by its extensive applications in cutting-edge fields such as wind energy generation, smart grid management, and the coordinated control of complex physical systems. The core focus in this area lies in achieving efficient cooperation and collaborative optimization among multiple independent agents. In multi-agent systems (MASs), each agent possesses its own cost function, which is not accessible to other agents. Thus, a key research challenge is to design distributed algorithms that enable all agents to reach the optimal solution of the global cost function, while only exchanging limited information with neighboring agents, which means that the coordinated decision-making even under constrained information sharing conditions are achieved. For example, Weng et al. [1] and Lian et al. [2] have studied the distributed coordination control problem for both homogeneous and heterogeneous MASs, respectively. Additionally, numerous researchers have explored the distributed constrained optimization problem, addressing aspects such as DoS attacks [3], event-triggered strategies [4], and online optimization [5,6].

Resource allocation is a fundamental optimization problem that has drawn considerable attention due to its broad applications in smart grids [7,8], supply chain management [9], and transportation systems [10]. Unlike centralized frameworks, distributed strategies solve global optimization problems cooperatively, where each agent exchanges information only with its neighbors. Such decentralized approaches reduce communication dependence, improve scalability, and enhance robustness against communication failures and uncertainties. To exploit these advantages, various distributed RAP algorithms have been developed. For example, Binetti et al. [11] investigated discrete-time systems and extended the results to cyber-physical systems [12], achieving predefined-time allocation. This work was further generalized to switching networks with cyber attacks [13] and second-order MASs [14]. To handle nonsmooth and coupled cost functions, Huang et al. [15] and Deng et al. [16] proposed proximal-based and fully distributed schemes for general nonlinear systems. Moreover, Chai et al. [17] explored event-triggered resource allocation over switching topologies, improving communication efficiency and convergence performance.

The study of distributed systems has traditionally focused on deterministic models, where agents are assumed to operate in perfectly controlled environments. However, real-world applications often involve uncertainties that cannot be ignored if the performance and adaptability of agents are to be improved. A significant milestone in this direction was made by [18], who introduced a beam search method to optimize resource allocation under random conditions. Building on this, Yi et al. [19] expanded the approach to distributed systems, tackling the complexities of resource allocation in random networks while accounting for additive communication noise. As research in this area progressed, Doan et al. [20] addressed the challenge of managing unknown resources in distributed settings, further advancing the field. Additionally, Zhou et al. [21] examined the EMPC problem under stochastic disturbances, where the cost function is not required to be either positive definite or quadratic. Mai et al. [22] introduced a novel algorithm to solve globally constrained optimization problems, specifically targeting separable cost functions in noisy environments.

The aforementioned studies primarily focus on computational failures or communication disturbances between agents, without considering the inherent randomness of renewable energy generation. In recent years, researchers have introduced the Itô process into the power sector, exploring its application in modeling the uncertainties of wind and solar power output, as well as in the stochastic analysis and control of power systems [23,24]. While few studies have applied the Itô process to distributed resource allocation, its formulation as a stochastic differential equation provides a natural framework for modeling the continuous uncertainties in renewable energy output. Furthermore, this approach facilitates the integration of these uncertainties into the distributed optimization problems faced by multi-agent systems, enabling more effective coordination and control strategies that account for the variability in renewable energy sources.

Building on the preceding discussion, this paper primarily investigates the distributed RAP for nonlinear stochastic MASs. The proposed algorithm, based on state-feedback control, ensures that the decision variables converge to the optimal solution, allowing for the selection of an optimal scheme. The main contributions of this paper are as follows:

- The coordination optimization problem for stochastic MASs is established to how a group of agents with Itô dynamics can collaboratively accomplish resource allocation task, serving as an extension of the works presented in [18–22]. Compare with them, where the gradient or communication noises was only considered, this paper considers a more general situation, namely, the stochastic system with Itô process is studied. It can offer a continuous stochastic approach, providing higher precision in simulating renewable energy fluctuations (see [23,24]). In addition, the drift and diffusion coefficients are nonlinear and unknown, which can lead to difficulties in designing and analyzing distributed algorithms.
- Adaptive technology is introduced to design the optimization algorithm for Itô MASs, which not only enhances the efficiency of the resource allocation process but also makes the system more robust and flexibility than other algorithms.
- Inspired by works [14–16], a adaptive optimization algorithm with feedback control law is developed. By introducing a new state feedback, the designed protocol can be more easily adapted to nonlinear systems than [12–17]. Furthermore, energy generation is first explored in the context of nonlinear stochastic MASs.

Notation: 0_r and 1_r are zero and ones column vector with r-dimension respectively. $\mathbb{R}^n$ is the Euclidean space with n-dimension. For a vector β, β^T is the transpose, $\|\beta\|$ is its Euclidean norm. $\mathbb{L}^p([0,\infty];\mathbb{R}^d)$ is the family of $\mathbb{R}^n$-value processes $\{g(k)\}_{k\geq 0}$ such that for every $k>0$, $\int_0^K \|g(k)\|^p < \infty$, a.s.. $col\{\cdot\}$ represents the column vector. $diag\{\cdot\}$ represents the diagonal matrix. $\otimes$ is the Kronecker product.

2. Preliminaries and Problem Formulation

2.1. Preliminaries

Let $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{A}\}$ be a directed graph with node set $\mathscr{V} = \{1, \cdots, m\}$, edge set $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$. $\mathscr{A} = [a_{ij}]_{m\times m} \in \mathbb{R}^{m\times m}$ is the adjacency matrix, where $a_{ij} > 0$ if node i can receive information from node j; otherwise, $a_{ij} = 0$. The Laplacian matrix of the directed graph is defined as: $\mathscr{L} = [l]_{ij} \in \mathcal{R}^{m\times m}$, where $l_{ii} = \sum_{j=1,j\neq i}^{m} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$. The graph is strongly connected if there exists a path between any two nodes, where the path between nodes v_i and v_1 can be represented by a sequence $(v_{i-1}, v_i), (v_{i-2}, v_{i-1}), \cdots, (v_1, v_2)$. If the row sum and column sum of the matrix $\mathscr{L}$ are 0, then the graph is weight-balanced.

2.2. Problem Formulation

Consider a network with m agents over a directed graph, where the dynamical behavior of the ith agent can be formulated by the following stochastic differential equation of type Itô:

$$dz_i(k) = [g(z_i(k), k) + u_i(k)]dk + h(z_i(k), k)d\omega(k), \tag{1}$$

where $z_i(k) \in \mathbb{R}^n$ and $u_i(k) \in \mathbb{R}^n$ are the decision variable and control input, respectively; $g \in \mathbb{L}^1([0, \infty), \mathbb{R}^n)$ and $h \in \mathbb{L}^2([0, \infty), \mathbb{R}^{n\times d})$ are the drift and diffusion coefficients, which are the Lipschitz continuous. $\omega(k)$ is a d-dimensional Brownian motion with $E(d\omega^2(k)) = dk$. The goal of MASs is mainly to solve the following RAP:

$$\min_{z_i \in \mathcal{R}^n} \phi(z) = \textstyle\sum_{i=1}^m \phi_i(z_i), \sum_{i=1}^m z_i = \sum_{i=1}^m c_i, \tag{2}$$

where $\phi_i(z_i)$ is convex and differential function. Our task is to design a reasonable scheme for dynamics (1) such that the decision variable z_i can not only satisfy the network constraints condition $\sum_{i=1}^m z_i = \sum_{i=1}^m c_i$, but also minimize the global function $\phi(z)$.

Remark 1. *Different from the traditional resource allocation algorithms [12–17], this study examines the impact of stochastic disturbances and unknown nonlinear diffusion terms induced by Brownian motion. Based on the analysis, these results cannot be directly applied to system (1). Moreover, system (1) includes unknown nonlinearities in both the diffusion and drift terms, complicating the convergence analysis of the algorithm.*

The subsequent analysis requires several lemmas and specific assumptions regarding the graphs and objective functions.

Lemma 1. *[25] Consider a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_k\}_{k\geq 0}$. Let $g \in \mathbb{L}^1([k_0, K], \mathbb{R}^{n\times r})$ and $h \in \mathbb{L}^2([k_0, K], \mathbb{R}^{n\times r})$, both of which are Borel measurable functions. The n-dimensional Itô-type stochastic differential equation is given by:*

$$ds(k) = g(s(k), k)dk + h(s(k), k)d\omega(k), k \leq k_0, \tag{3}$$

with the initial condition $s(k_0) = s_0$. Suppose there exists a twice continuously differentiable function $V(s(k), k)$ and positive constants α_1 and α_2 such that for all $s(k) \neq 0$, the following conditions hold:

(i) $\alpha_1\|s(k)\|^2 \leq V(s(k), k) \leq \alpha_2\|s(k)\|^2$,

(ii) $\mathcal{L}V(s(k), k) \leq 0$,

where $\mathcal{L}V(s(k), k)$ is defined as:

$$\begin{aligned}\mathcal{L}V(s(k), k) &= \tfrac{\partial V(s(k),k)}{\partial t} + \tfrac{\partial V(s(k),k)}{\partial s(k)} g(s(k), k) \\ &+ \tfrac{1}{2} trace(h(s(k), k)^T \tfrac{\partial V^2(s(k),k)}{\partial s(k)^2} h(s(k), k)),\end{aligned}$$

and $trace(\cdot)$ represents the trace of a square matrix. Under these conditions, the trivial solution of equation (3) is said to be stochastically asymptotically stable.

Lemma 2. *[14] If $z_i^* \in \mathbb{R}^n (i = 1, \cdots, m)$ is the optimal solution of problem (2), then one has*

$$\nabla\phi_i(z_i^*) = \nabla\phi_j(z_j^*), \textstyle\sum_{i=1}^m z_i^* = \sum_{i=1}^m c_i, \forall i, j \in \mathscr{V}.$$

Conversely, if the above equations can be satisfied for point $z_i^* \in \mathbb{R}^n$, $z_i^*(i = 1, \cdots, m)$ is the global optimum of problem (2).

Assumption 1. *Assume that the functions g and h satisfy the Lipschitz condition, i.e., there exist two positive constants μ_1 and μ_2 such that for any $\xi_1, \xi_2 \in \mathbb{R}^n$, the following inequalities hold:*

$$\|g(\xi_1, k) - g(\xi_2, k)\| \leq \mu_1\|\xi_1 - \xi_2\|, \|h(\xi_1, k) - h(\xi_2, k)\| \leq \mu_2\|\xi_1 - \xi_2\|.$$

Assumption 2. *The cost function $\phi_i(z_i)$ is τ_i-strongly convex and differentiable with respect to z_i.*

Assumption 3. *The communication topology is strongly connected and weight-balanced.*

Remark 2. *Assumption 1 is a widely adopted condition, particularly for systems governed by Itô dynamics. Under Assumption 2, using the properties of strongly convex functions, the inequality* $(\nabla\varphi(\xi_1)-\nabla\varphi(\xi_2))^T(\xi_1-\xi_2)\geq \tau(\xi_1-\xi_2)^T(\xi_1-\xi_2)$ *holds, where* $\tau=\max_{1\leq i\leq m}\tau_i$*. Furthermore, Assumption 3 guarantees that the decision states reach consensus.*

3. Design of Control Protocol and Convergence Analysis

To tackle problem (2), the following fully distributed algorithm based on state feedback is designed for the ith agent:

$$\begin{aligned} u_i &= -\alpha_i\nabla\phi_i(z_i)-\alpha_i\rho_i-\beta_i\theta_i \\ \dot{\theta}_i &= \alpha_i\nabla\phi_i(z_i)+\alpha_i\rho_i \\ \dot{\rho}_i &= -\textstyle\sum_{j=1}^m a_{ij}[(\rho_i-\rho_j)-(\varpi_i-\varpi_j)]+z_i-c_i \\ \dot{\varpi}_i &= -\textstyle\sum_{j=1}^m a_{ij}(\rho_i-\rho_j) \end{aligned} \tag{4}$$

where $\alpha_i>\frac{2\mu_1+\frac{1}{2}\mu_1^2+\mu_2^2+\frac{\beta^2}{\beta-\frac{1}{2}}}{\tau}$, $\beta_i>\frac{1}{2}$ are the control parameters. The selection of the parameters α_i and β_i ensure the convergence of the algorithm, which has been proved in detail later. This approach is similar to the choice of control parameters in some optimization algorithms discussed in [16,17,19].

Remark 3. *In Algorithm (4), the term* $-\alpha_i\nabla\phi_i(z_i)$ *serves as the driving force that guides each agent toward the optimal solution. To facilitate cooperative behavior, the term* $\sum_{j=1}^m a_{ij}(\varpi_i-\varpi_j)$ *enables information exchange among neighboring agents, ensuring that local decisions incorporate shared knowledge from the network. Meanwhile,* ρ_i *is introduced to enforce local constraints, preventing infeasible updates. Additionally,* θ_i *plays a crucial role in maintaining consistency across agents, ensuring that the gradient condition* $\nabla\phi_i(z_i^*)=\nabla\phi_j(z_j^*)$ *holds, which is essential for achieving a globally coordinated solution.*

For simplicity, the time variable k is first omitted. Defined $z=[z_1^T,\cdots,z_m^T]^T$, $\rho=[\rho_1^T,\cdots,\rho_m^T]^T$, $\theta=[\theta_1^T,\cdots,\theta_m^T]^T$, $c=col\{c_1,\cdots,c_m\}$, $\nabla\phi(z)=[\nabla\phi_1(z_1)^T,\cdots,\nabla\phi_m(z_m)^T]^T$, $\varpi=[\varpi_1^T,\cdots,\varpi_m^T]^T$. Then, the system (1) can be rewritten the following form:

$$dz=[g(z,k)-\alpha\nabla\phi(z)-\alpha\rho-\beta\theta]dk+h(z,k)d\omega, \tag{5a}$$

$$\dot{\theta}=\alpha\nabla\phi(z)+\alpha\rho, \tag{5b}$$

$$\dot{\rho}=-(\mathscr{L}\otimes I_n)\rho+(\mathscr{L}\otimes I_n)\varpi+z-c, \tag{5c}$$

$$\dot{\varpi}=-(\mathscr{L}\otimes I_n)\rho, \tag{5d}$$

where $\alpha=diag\{\alpha_1,\cdots,\alpha_m\}$, $\beta=diag\{\beta_1,\cdots,\beta_m\}$.

Before the convergence analysis can be provided, the following lemma regarding the equilibrium point is needed.

Lemma 3. *If Assumptions 1, 2, 3 are satisfied, then* z^* *is the optimal solution to* (2) *if and only if there exist* θ^*, ρ^*, ϖ^*, z^* *such that* $(\theta^*,\rho^*,\varpi^*,z^*)$ *is the equilibrium point* (5b), (5c), (5d)*, while ensuring that* (5a) *also holds.*

Proof. $Step1$: If $(\theta^*,\rho^*,\varpi^*)$ be the equilibrium point of (5b), (5c), (5d), and (5a) holds, then

$$dz^*=[g(z^*,k)-\alpha\nabla\phi(z^*)-\alpha\rho^*-\beta\theta^*]dk+h(z^*,k)d\omega \tag{6a}$$

$$0=\alpha\nabla\phi(z^*)+\alpha\rho^* \tag{6b}$$

$$0=-(\mathscr{L}\otimes I_n)\rho^*+(\mathscr{L}\otimes I_n)\varpi^*+z^*-c \tag{6c}$$

$$0=-(\mathscr{L}\otimes I_n)\rho^* \tag{6d}$$

Multiplying the left-hand side of (6c) by 1_m^T,

$$\textstyle\sum_{i=1}^m z_i^*=\sum_{i=1}^m c_i$$

Because the graph is weight-balanced, i.e., $(\mathscr{L} \otimes I_n)1_m = 0_m$ and $1_m^T(\mathscr{L} \otimes I_n) = 0_m^T$, (6d) yields that $\rho_1^* = \rho_2^* = \cdots = \rho_m^*$, hence,

$$\nabla\phi_1(z_1^*) = \nabla\phi_2(z_2^*) = \cdots = \nabla\phi_m(z_m^*)$$

Therefore, z^* is the optimal point of problem (2).

$Step2$: If z^* is the optimal point of problem (2), we have

$$\sum_{i=1}^m z_i^* = \sum_{i=1}^m c_i, \nabla\phi_i(z_i^*) = \nabla\phi_j(z_j^*),$$

for any $i, j \in \mathscr{V}$. Besides, there exists ρ^* and ϖ^* such that $\rho^* = -\nabla\phi(z^*)$ with $\rho_i^* = \rho_j^*, \forall i, j \in \mathscr{V}$ and $\varpi^* = 1_m\varpi$, then, (5b), (5c) and (5d) can be satisfied. Besides, there exists θ^* such that (5a) holds.

Next, it is proved that $z^*(k)$ is stable. It results from (5b),

$$dz^*(k) = [g(z^*(k), k) - \beta\theta^*(k)]dk + h(z^*(k), k)d\omega(k). \tag{7}$$

Let $\theta^*(k) = \frac{g(z^*(k),k) - Pz^*(k) - \iota(k)}{\beta}$. If there exist some positive constants ϱ_1, ϱ_2, ϱ_3, ϱ_4, ϱ_5, a matrix $P \in \mathbb{R}^{mn\times mn}$ with eigenvalues of negative real parts, and $\iota(k) : \mathbb{R}_+ \to \mathbb{R}^{mn}$, such that $\|e^{Pk}\|^2 \leq \varrho_1 e^{-\varrho_2 k}$, and $\max\{E[\|\iota(k)\|^2], E[\|h(z^*(k), k)\|^2]\} \leq \varrho_3 e^{-\varrho_4 k}$, $k > 0$. Then, by the definition of stochastic $It\hat{o}$ integrals, (7) is equivalent to the following integral form:

$$z^*(k) = \quad e^{P(k-k_0)}s^*(k_0) + \int_{k_0}^k e^{P(k-s)}\iota(s)ds + \int_{k_0}^k e^{P(k-s)}h(z^*(s), s)d\omega(s).$$

Taking the expectation on $\|z_i^*(k)\|^2$ yields that

$$\begin{aligned} &E[\|z^*(k)\|^2] \\ &\quad\leq 3\varrho_1\varrho_3(k - k_0 + 1)\int_{k_0}^k e^{-\varrho_2(t-k)-\varrho_4 k}ds + 3\varrho_1\|z^*(k_0)\|^2 e^{-\varrho_2(k-k_0)} \\ &\quad\leq 3\varrho_1\varrho_3(k - k_0 + 1)(k - k_0)e^{-\min\{\varrho_2,\varrho_4\}k} + 3\varrho_1\|z^*(k_0)\|^2 e^{-\varrho_2(k-k_0)}. \end{aligned}$$

Let $\vartheta = 3\varrho_1\|z^*(k_0)\|^2 + 3\varrho_1\varrho_3\sup_{k\geq k_0}[(k - k_0 + 1)(k - k_0)e^{-\epsilon k}]$, where $0 < \epsilon < \min\{\varrho_2, \varrho_4\}/2$ is arbitrary, which yields that

$$E[\|z^*(t)\|^2] \leq \vartheta e^{-(\min\{\varrho_2,\varrho_4\}-\epsilon)(k-k_0)}, \; k \geq k_0.$$

Let $p = 1, 2, \cdots$. Note that

$$z^*(k) = \quad z^*(k_0 + p - 1) + \int_{k_0+p-1}^t [Pz^*(s) + \iota(s)]ds + \int_{k_0+p-1}^k h(z^*(s), s)d\omega(s).$$

By Hölder's inequality and Doob's martingale inequality, one has that

$$\begin{aligned} &E[\sup_{k_0+p-1\leq k\leq k_0+v}\|z^*(k)\|^2] \\ &\quad\leq 6\int_{k_0+p-1}^{k_0+p}\vartheta\|P\|^2 e^{-(\min\{\varrho_2,\varrho_4\}-\epsilon)(s-k_0)}ds \\ &\qquad+3\varrho_1 e^{-(\min\{\varrho_2,\varrho_4\}-\epsilon)(p-1)} + 6\varrho_3\int_{k_0+p-1}^{t_0+v} e^{-\varrho_4 s}ds. \end{aligned}$$

Since ϵ is arbitrary, $z^*(k)$ is stable in mean square as $p \to \infty$, which means that there θ^* such that (5a) holds. This proof is completed. □

Theorem 1. *Under Assumptions 1–3, if* $\alpha_i > \frac{2\mu_1 + \frac{1}{2}\mu_1^2 + \mu_2^2 + \frac{\beta^2}{\beta - \frac{1}{2}}}{\tau}$ *and* $\beta_i > \frac{1}{2}$*, the RAP (2) can be minimized for Itô MASs, that is, each decision variable can effectively converge to the optimal value.*

Proof. Let $\bar{z} = z - z^*$, $\bar{\theta} = \theta - \theta^*$, $\bar{\rho} = \rho - \rho^*$, $\bar{\varpi} = \varpi - \varpi^*$, (5a)–(5d) can be equivalently replaced by the following form:

$$\begin{aligned} d\bar{z} &= [\bar{g}(\cdot, k) - \alpha\nabla\bar{\phi}(\cdot) - \alpha\bar{\rho} - \beta\bar{\theta}]dk + \bar{h}(\cdot, k)d\omega \\ \dot{\bar{\theta}} &= \alpha\nabla\bar{\phi}(\cdot) + \alpha\bar{\rho} \\ \dot{\bar{\rho}} &= -(\mathscr{L} \otimes I_n)\bar{\rho} + (\mathscr{L} \otimes I_n)\bar{\varpi} + \bar{z} \\ \dot{\bar{\varpi}} &= -(\mathscr{L} \otimes I_n)\bar{\rho}, \end{aligned}$$

where $\nabla\bar{\phi}(\cdot) = \nabla\bar{\phi}(z) - \nabla\bar{\phi}(z^*)$, $\bar{g}(\cdot,k) = \bar{g}(z,k) - \bar{g}(z^*,k)$, and $\bar{h}(\cdot,k) = \bar{h}(z,k) - \bar{h}(z^*,k)$. Taking the candidate Lyapunov function as

$$V = V_1 + V_2 + V_3,$$

where $V_1 = \frac{1}{2}\bar{z}^T\bar{z} + \frac{1}{2}\Theta^T\Theta$, $V_2 = \frac{\alpha}{2}\bar{\rho}^T\bar{\rho}$, $V_3 = \frac{\alpha}{2}\bar{\varpi}^T\bar{\varpi}$, and $\Theta = \bar{z} + \bar{\theta}$. When nonlinear stochastic systems (1) executes the algorithm (4), one derives

$$\begin{aligned} dV_1 = \ & \bar{z}^T[\bar{g}(\cdot,k) - \alpha\nabla\bar{\phi}(\cdot) - \alpha\bar{\rho} - \beta\bar{\theta}]dk + \bar{z}^T\bar{h}(\cdot,k)d\omega + \tfrac{1}{2}\bar{h}(\cdot,k)^T\bar{h}(\cdot,k)dk \\ & + \Theta^T\bar{h}(\cdot,k)d\omega + \Theta^T(\bar{g}(\cdot,k) - \beta\bar{\theta})dk + \tfrac{1}{2}\bar{h}(\cdot,k)^T\bar{h}(\cdot,k)dk. \end{aligned}$$

By the $It\hat{o}$ formula and the definition of $\mathcal{L}V$ in Lemma 1, we have

$$\mathcal{L}V_1 = \ \bar{z}^T[\bar{g}(\cdot,k) - \alpha\nabla\bar{\phi}(\cdot) - \alpha\bar{\rho} - \beta\bar{\theta}] + \bar{h}(\cdot,k)^T\bar{h}(\cdot,k) + \Theta^T(\bar{g}(\cdot,k) - \beta\bar{\theta}).$$

Applying to the Lipschitz conditions of nonlinear functions g and h in Assumption 1, one obtains

$$\bar{z}^T\bar{g}(\cdot,k) \le \mu_1\bar{z}^T\bar{z},\ \bar{h}(\cdot,k)^T\bar{h}(\cdot,k) \le \mu_2^2\bar{z}^T\bar{z}.$$

According to the property of strongly convex in Assumption 2, one gets

$$-\bar{z}^T\nabla\bar{\phi}(\cdot) \le -\tau\bar{z}^T\bar{z}.$$

Using Young's inequality, one has

$$\bar{\theta}^T\bar{g}(\cdot,k) \le \frac{1}{2}\bar{\theta}^T\bar{\theta} + \frac{1}{2}\mu_1^2\bar{z}^T\bar{z}.$$

Thus,

$$\begin{aligned} \mathcal{L}V_1 \le \ & 2\mu_1\bar{z}^T\bar{z} + \mu_2^2\bar{z}^T\bar{z} - \alpha\tau\bar{z}^T\bar{z} + \bar{z}^T[-\alpha\bar{\rho} - \beta\bar{\theta}] \\ & + \tfrac{1}{2}\mu_1^2\bar{z}^T\bar{z} - \beta\bar{z}^T\bar{\theta} - (\beta - \tfrac{1}{2})\bar{\theta}^T\bar{\theta}. \end{aligned} \tag{8}$$

Note that

$$\mathcal{L}V_2 = -\alpha\bar{\rho}^T(\mathscr{L}\otimes I_n)\bar{\rho} + \alpha\bar{\rho}^T(\mathscr{L}\otimes I_n)\bar{\varpi} + \alpha\bar{\rho}^T\bar{z}, \tag{9}$$

and

$$\mathcal{L}V_3 = -\alpha\bar{\varpi}^T(\mathscr{L}\otimes I_n)\bar{\rho}. \tag{10}$$

Concluding (8), (9) with (10), we have

$$\begin{aligned} \mathcal{L}V \le \ & (2\mu_1 + \tfrac{1}{2}\mu_1^2 + \mu_2^2 + \tfrac{\beta^2}{\beta - \frac{1}{2}} - \alpha\tau)\bar{z}^T\bar{z} \\ & - \|\tfrac{\beta}{\sqrt{\beta - \frac{1}{2}}}\bar{z} + \sqrt{\beta - \tfrac{1}{2}}\bar{\theta}\|^2 - \alpha\bar{\rho}^T(\mathscr{L}\otimes I_n)\bar{\rho}. \end{aligned}$$

Since α_i and β_i are selected such that $\alpha_i > \frac{2\mu_1 + \frac{1}{2}\mu_1^2 + \mu_2^2 + \frac{\beta^2}{\beta - \frac{1}{2}}}{\tau}$ and $\beta_i > \frac{1}{2}$. From Lemma 1, we can get that $E[\|\bar{z}\|^2] \to 0$. Therefore, for any $i = 1, 2, \cdots, m$, one deduces that

$$E[\lim_{k\to\infty} \|z_i(k) - z^*(k)\|^2] = 0,$$

namely, the energy allocation can be optimized to minimize losses. The proof is achieved. □

Remark 4. *According to Theorem 1, the proposed algorithm effectively solves the RAP for stochastic system (1) driven by a diffusion process. Compared with several representative works [11–17], this study explicitly addresses random disturbances induced by Brownian motion, which significantly enhances the robustness and practical applicability of the proposed method in real-world stochastic environments. The adaptive feedback design enables the system to maintain stable performance under model uncertainties and communication perturbations, aligning with the principles of robust control theory discussed in [26]. In particular, when the variance of the Brownian motion approaches zero, the system reduces to a deterministic first-order case, and the proposed algorithm becomes simpler than the second-order schemes in [14], while preserving convergence and robustness properties.*

4. Simulations

In this section, the practical examples about the RAP of smart grids are considered to support theoretical results. Specifically, in the IEEE-39 bus system, such nonlinearities stem from the complex interactions between generators, loads, and transmission lines, which are subject to stochastic disturbances caused by fluctuations in renewable energy output and demand variations. In the smart grid, there exist m machines to generate energy to supply to the user, and each machine has its own cost function attributed to the different working efficiency of the machine. The energy dispatch problem of the smart grids is formulated as follows:

$$\min_{\tilde{P}_i\in\mathcal{R}^n} \phi(\tilde{P}) = \sum_{i=1}^m \phi_i(\tilde{P}_i), \sum_{i=1}^m \tilde{P}_i = \sum_{i=1}^m c_i, \tag{11}$$

where $\tilde{P}_i \in \mathbb{R}^n$, in MW, $\phi_i(\tilde{P}_i) : \mathbb{R}^n \to \mathbb{R}^n$, in $M\$$, and $c_i \in \mathbb{R}^n$, in MW, are the output power, the cost function and the local energy demand of the ith machine, respectively. Moreover, the energy power output is also called as a continuous random process, which satisfies some characteristics of time sequence. It has been shown in [23,24] that the process of energy renewal can be expressed by Itô stochastic differential system, described by the following equation:

$$d\tilde{P}_i(k) = [g(\tilde{P}_i(k), k) + u_i(k)]dk + h(\tilde{P}_i(k), k)d\omega(k),$$

where $g(\tilde{P}_i(k), k) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ and $h(\tilde{P}_i(k), k) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear functions, and $u_i(k)$ is the input mechanical power of a component in energy system.

The proposed uncertainty scheme is first verified on the IEEE 39-bus system with 10-generators in Figure 1. Each machine has a local generation function given by

$$\phi_i(\tilde{P}_i) = \tilde{\chi}_i\tilde{P}_i^2 + \tilde{\gamma}_i\tilde{P}_i + \tilde{\zeta}_i, \tag{12}$$

where the output power $P_i = [P_{i1}, P_{i2}] \in \mathbb{R}^2$ represents the power distribution in a two-dimensional Euclidean space and the parameters are presented in Table 1 (see [11]).

Table 1. Generator parameters of the 39-bus system.

Bus	$\tilde{\chi}_i[\$] \in \mathbb{R}^2$	$\tilde{\gamma}_i[\$/\text{MW}] \in \mathbb{R}^2$	$\tilde{\zeta}_i[\$/\text{MW}^2] \in \mathbb{R}^2$
30	[0.325, 0.325]	[15.00, 20.00]	[103.15, 104.25]
31	[0.210, 0.210]	[13.55, 13.55]	[110.35, 121.36]
32	[0.365, 0.365]	[12.00, 16.00]	[123.01, 137.65]
33	[0.175, 0.310]	[22.88, 26.00]	[201.15, 189.75]
34	[0.210, 0.265]	[20.00, 26.88]	[164.35, 179.68]
35	[0.255, 0.460]	[14.00, 27.00]	[191.01, 127.02]
36	[0.215, 0.275]	[16.00, 26.00]	[256.05, 279.07]
37	[0.430, 0.225]	[18.00, 32.00]	[136.97, 147.01]
38	[0.450, 0.325]	[18.00, 28.00]	[196.75, 221.02]
39	[0.130, 0.140]	[12.80, 30.00]	[98.650, 82.210]

The generators can exchange information on a weight-balanced digraph, whose Laplacian matrix can be described as follows:

$$\begin{pmatrix} 4 & -3 & 0 & 0 & 0 & 0 & 0 & -1 \\ -3 & 5 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 6 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 5 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 5 & -2 \\ -1 & 0 & 0 & 0 & 0 & 0 & -2 & 3 \end{pmatrix}.$$

The simulations have been conducted in MATLAB, where we completed the experimental analysis.

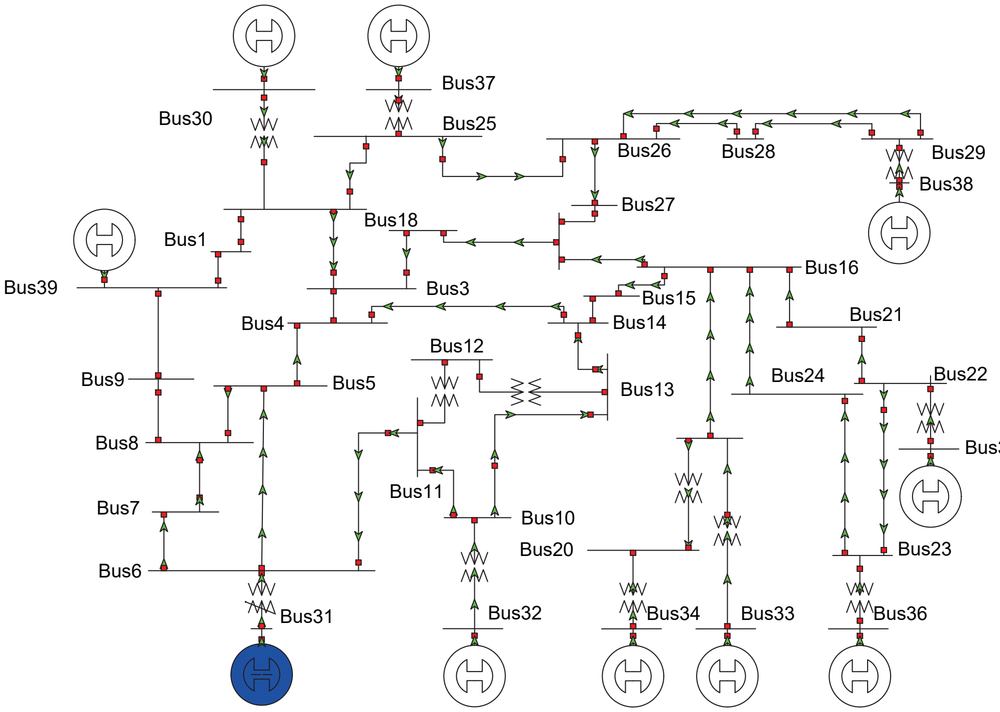

Figure 1. IEEE 39 bus system. (https://icseg.iti.illinois.edu/ieee-39-bus-system/, accessed on 25 August 2025).

4.1. Example: Fixed Demands

In this context, the method designed for continuous-time systems without stochastic disturbances, as detailed in [14], was applied to the same problem for comparative analysis. The local demands of each generator are shown in Table 2. Taking initial values $\tilde{P}_1(0) = 25$, $\tilde{P}_2(0) = 29$, $\tilde{P}_3(0) = 24$, $\tilde{P}_4(0) = 23$, $\tilde{P}_5(0) = 38$, $\tilde{P}_6(0) = 36$, $\tilde{P}_7(0) = 31$, $\tilde{P}_8(0) = 32$, $\tilde{P}_9(0) = 41.5$, $\tilde{P}_10(0) = 43.5$. $\theta_i(0)$, $\rho_i(0)$ and $\varpi_i(0)$ are all ones. The nonlinear functions are defined as $g(\tilde{P}_i(k), k) = -0.045 * \tilde{P}_i(k) + 0.025 * sin(\tilde{P}_i(k))$ and $h(\tilde{P}_i(k), k) = 0.005 * sin(\tilde{P}_i(k))$. The control parameters α_i, β_i of the algorithm (4) are 26 and 1, satisfying the conditions $\alpha_i > \frac{2\mu_1+\frac{1}{2}\mu_1^2+\mu_2^2+\frac{\beta^2}{\beta-\frac{1}{2}}}{\tau}$ and $\beta_i > \frac{1}{2}$. Sampling time $T = 0.01s$. Evolutions of output power of each generator are presented in Figure 2, illustrating that the power generator can converge to the local demand. In order to show the superiority of this algorithm, it is compared with the asymptotic convergence algorithm (4) in [14]. As shown in Figure 2, the algorithm in this paper only needs about 1700 iterations to complete the RAP, while the algorithm in [14] requires 11,000 iterations. This greatly improves the convergence rate and robustness of the algorithm.

Table 2. IEEE 39-bus system: local demands.

Bus	**Demand** $(c_i, i = 1, \cdots, 5)$	**Bus**	**Demand** $(c_i, i = 1, \cdots, 5)$
30	50	35	70
31	55	36	68
32	58	37	62
33	60	38	65
34	65	39	62

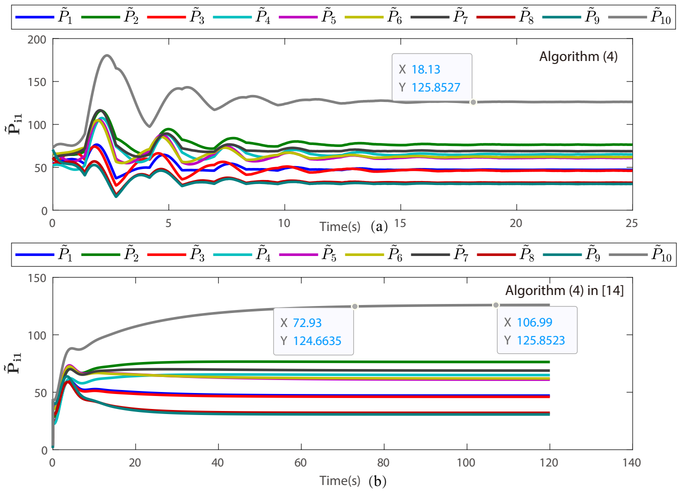

Figure 2. Output power evolution of generator$_i$ under different algorithms.

4.2. Example: Time-Varying Demands

In this scenario, all parameters are set to align with the fixed requirements specified in Table 1. It is important to note that, in real-world applications, both external conditions and internal dynamics frequently change, resulting in shifting requirements. As a result, local requirements cannot remain static over time. The algorithm proposed in this paper can effectively address the challenges posed by these changing demands. Let the initial values $\tilde{P}(0)$ be randomly selected in $[1, 10](MW)$, and other initials $\theta(0)$, $\rho(0)$ and $\varpi(0)$ are one. For time $k \in [0, 2800](s)$, the capacity $c_i = [c_{i1}, c_{i2}]$ is constrained to the ranges $[50, 70] \bigcup [30, 60](MW)$; for time $k \in (2800, 5000](s)$, $c_i = [c_{i1}, c_{i2}]$ falls within the ranges $[30, 60] \bigcup [10, 30](MW)$; lastly, while time $k \in (5000, 7000](s)$, $c_i = [c_{i1}, c_{i2}] \in [70, 90] \bigcup [60, 90](MW)$. Moreover, other parameters α_i, β_i of algorithm (4) are 2 and 3. The simulation results are presented in Figures 3–6, where the evolutions of output energy of generators$_{i1,i2}$, and the comparison between total output power and demand are depicted, which means that the proposed scheme ensures the expected performance. Figures 4 and 5 show that the outputs of 8 generators can converge to the value of the energy allocation which satisfies the load demand. Figure 6 presents that the total output power can be consistent with the load demand, even if the initial values does not satisfy the load. A residual errors are drawn in Figure 3 to represent the convergence rate of the algorithms. It is shown that the convergence rate of the two types of algorithms is similar, but the algorithm in this paper can handle the disturbed MASs.

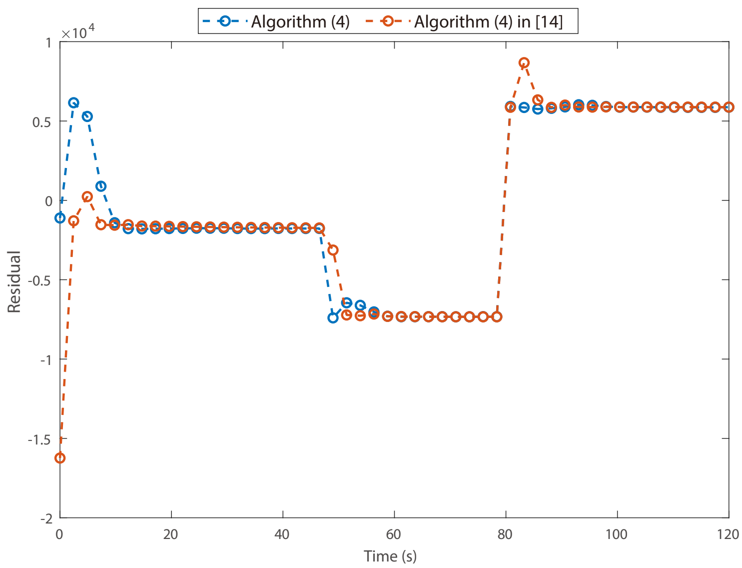

Figure 3. Residual errors: comparing the performance of the two algorithms (Algorithm (4) and Algorithm (4) Adapted from [14]) in solving RAP.

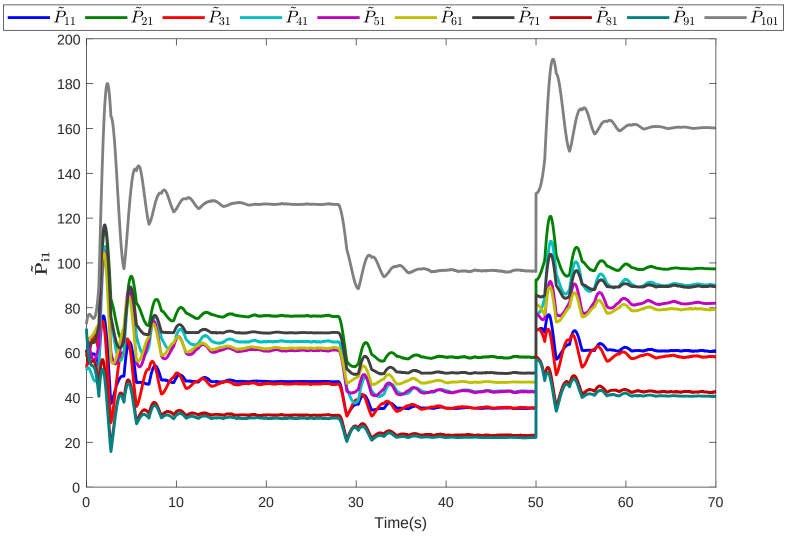

Figure 4. Evolutions of output power $\tilde{P}_{i1}$.

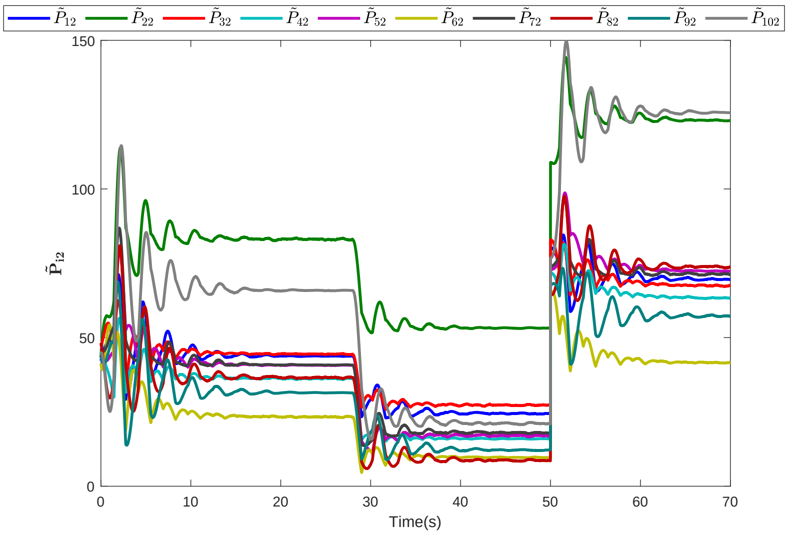

Figure 5. Evolutions of output power $\tilde{P}_{i2}$.

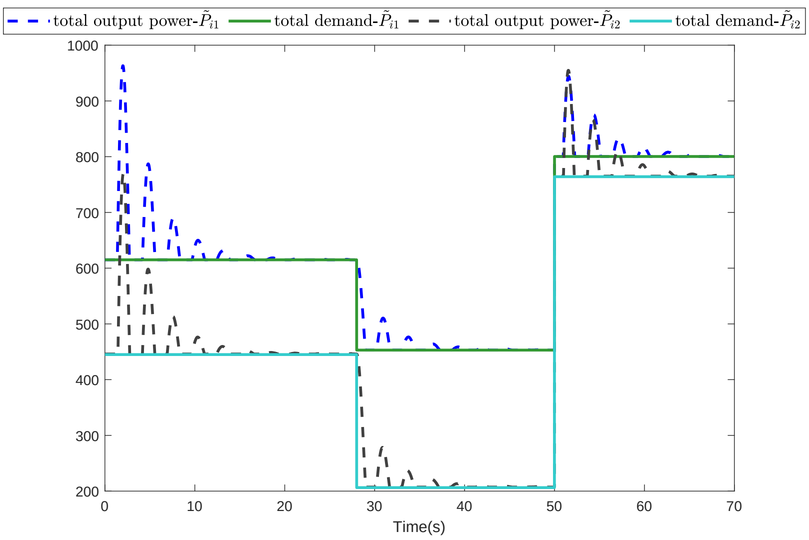

Figure 6. Mispairing of total output power and demand of generators$_{i1}$ and generators$_{i2}$.

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H.H.: writing—original draft preparation, conceptualization, methodology, software; Q.Z.: visualization, supervision. All authors have read and agreed to the published version of the manuscript.

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