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Nonovershooting Prescribed Finite-Time Control for Nonlinear Pure-Feedback Systems

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Abstract: The paper investigates the the problem of non-overshooting tracking prescribed finite-time control for nonlinear pure-feedback systems is studied.Currently, most existing results focus on nonlinear strict-feedback systems, while studies on the more general pure-feedback systems are scarce. Moreover, the available conditions for ensuring non-overshooting performance are conservative, making it difficult to select appropriate constraints for diverse engineering applications. In this paper, we design a prescribed finite-time controller by proposing a new prescribed finite-time lemma and the backstepping technique. At the same time, a relatively simple closed-loop system is obtained. Furthermore, an analytical expression for the tracking error is derived, which facilitates the analysis of a wider range of non-overshooting conditions and thereby reduces conservatism. The proposed approach not only solves the problem of prescribed finite-time control, but also solves the problem of non-overshooting control. Finally, the simulation examples demonstrate the effectiveness of the proposed algorithm.

Keywords: strict-feedback systems; pure-feedback systems; non-overshooting control; backstepping

1. Introduction

There are many problems in the control theory of systems, among which the research on tracking problems is of great significance in both control theory and practical application [1–3]. In numerous engineering applications, the system is often required to achive small overshoot or even non-overshooting tracking of the reference signal. The term 'non-overshooting' means that the sign of the tracking error remains unchanged [4]. For many control systems, it is actually required to achieve non-overshooting, such as missile launch , attitude control of aircraft [5], current control during welding, etc. Therefore, the research of non-overshooting control is highly valuable for industry, national defense and science and technology. This significance further underscores the fundamental importance of studying non-overshooting tracking control.

The results on non-overshooting tracking control for nonlinear systems are very scarce. At present, the main contributions can be found in references [4–9]. These works achieve a simple closed-loop system by designing a controller for strict-feedback nonlinear systems with all-knowing signal information, and then analyze the closed-loop system to obtain the tracking conditions with non-overshooting. However, if the closed-loop system is too complex, it is difficult to analyze the conditions of non-overshooting. Moreover, the conditions obtained in these existing results are often very conservative. The conditions of non-overshooting obtained in reference [4] were to ensure that the initial values of all closed-loop system states are negative by selecting design parameters, thereby ensuring that all states remain negative during the convergence process. This approach, which ensures the first state remains negative by maintaining all states in negativity, is quite conservative. For example, at present, due to the inherent nature of positive systems, the initial values of the system states need to be greater than or equal to zero, so the non-overshooting control algorithm in [4] is difficult to apply. Another example is temperature control of the sintering furnace requires a constant temperature of 1000 °C for sintering [10]. In this case, it is necessary to design a system that raises the temperature without exceeding 1000 °C. To achieve non-overshooting control, the tracking

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error must satisfy $e(t) = y(t) - r(t) \leq 0$, where $y(t)$ is the actual temperature and $r(t) = 1000$ °C. Therefore, it is essential to study this kind of problem with the goal of deriving less conservative and more general non-overshooting. Only in this way can the controller be designed according to the practical requirements.

Nonlinear pure-feedback systems are more general than strict-feedback systems [11–20]. However, little research on the non-overshooting tracking control of these systems. Therefore, studying the non-overshooting control of nonlinear pure-feedback systems is of great significance for broadening their applicability.

Because the control variables and virtual control variables in nonlinear pure-feedback systems are often implied in non-affine functions [11–20], most control methods suitable for strict-feedback systems cannot be directly used in pure-feedback nonlinear systems. At present, the common strategy for controlling pure-feedback nonlinear systems is to transform pure-feedback systems into affine nonlinear systems by using approximate linearization, Taylor series, mean value theorem and inverse systems, and then design the controller according to the control method for affine nonlinear system. However, these methods can only ensure the local stability of the control system, and cannot achive the large-scale control of the system. Furthermore, the closed-loop system obtained by these methods is often very complex. As mentioned above, if the closed-loop system is too complex, an analytical expression for the tracking error, and it is difficult to analyze a wide range of conditions with non-overshooting.

This paper addresses the problem of non-overshooting tracking control for pure-feedback nonlinear systems. By integrating prescribed finite-time control techniques, we derive a relatively simple closed-loop system. This approach allows us to establish less conservative sufficient conditions for achieving non-overshooting tracking performance. In this paper, there are the following innovations:

(1) In recent years, the prescribed finite-time control has attracted significant attention [21–27]. The concept was put forward in 2016 [22]. The core idea is to construct a time-varying function that tends to infinity as time approaches a preset finite time and to use this function to transform the original system into a new one. If it is proved that the new system is input-state-stable, then the states of the original system are guaranteed to converge within the prescribed time. The general real variable function is $(T/(T-t))^{m+n}$, where T represents the expected convergence time, and n and m are the system dimension and a constant, respectively. This method has been applied to a variety of nonlinear systems, demonstrating its effectiveness and practicality.

However, it is difficult to obtain a simple closed-loop system by the existing prescribed finite-time criterion lemma [26]. Therefore, this paper proposes a new criterion lemma of prescribed finite-time control. This contribution enriches the toolbox of prescribed finite-time control algorithms. As a result, the obtained closed-loop system is relatively simple, which facilitates the analysis of less conservative non-overshooting tracking conditions.

(2) As mentioned in 1), this paper also addresses the prescribed finite-time control problem of nonlinear pure-feedback systems. At present, this issue has been reported less. The corresponding conditions of non-overshooting can be selected according to the requirements of initial states and control gain parameters in engineering practice.

(3) Compared with the existing results [4–9], the non-overshooting conditions proposed in this paper are less conservative. Compared with the algorithm in [4–6], the algorithm in this paper can make the choice of control gain c_i $(i = 1, \cdots, n+1)$ wider because it does not need to consider the control term z_i.This reduction in conservatism offers greater flexibility.

The remaining parts of this paper are organized as follows. Section 2 focuses on non-overshooting control for nonlinear strict-feedback systems. Section 3 extends the proposed method to non-overshooting control for nonlinear pure-feedback systems. Three examples are given in Section 4, and the conclusions are obtained in Section 5.

2. Problem Formulation

Consider the nonlinear strict-feedback system as

$$\begin{cases} \dot{x}_i = f_i(x_1, \cdots, x_{i+1}), \quad i = 1, \cdots, n-1, \\ \dot{x}_n = f_n(u, x_1, \cdots, x_n), \\ y = x_1, \\ e = y - r, \end{cases} \tag{1}$$

where $u \in \mathbb{R}$ denotes the control input; $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$ represents the system states; The function $f_i(\bar{x}_i)$ $(i = 1, \cdots, n)$ represents a nonlinear non-affine smooth known function with $f_i(0, \cdots, 0) = 0$; $y \in \mathbb{R}$ represents the output of the system; r represents a tracking signal; e is the tracking error.

We deal with the non-affine function $f_n(u, \bar{x}_n)$ by introducing the following first-order system

$$\dot{u}(t) = \omega(t), \tag{2}$$

where $\omega(t)$ is the auxiliary control input.

Assumption 1. *The tracking signal r is assumed to be a smooth and bounded signal.*

Assumption 2. *[16] The non-affine functions $f_i(\bar{x}_{i+1})$ $(i = 1, \cdots, n-1)$ and $f_n(x, u)$ in (1) satisfy the following inequality*

$$0 < \eta_{(i+1)1} < |\frac{\partial f_i(\bar{x}_{i+1})}{\partial x_{i+1}}| < \eta_{(i+1)2} < \infty, \tag{3}$$

$$0 < \eta_{(n+1)1} < |\frac{\partial f_n(x, u)}{\partial u}| < \eta_{(n+1)2} < \infty. \tag{4}$$

where η_{i1} $(i = 1, \cdots, n+1)$ and η_i are positive constants.

According to reference [1], the non-overshooting control means that the sign of the tracking error $e(t)$ in $t \in [0, +\infty]$ remains unchanged, that is

$$e(t) \leq 0 \tag{5}$$

This paper aims to complete the control goal as follows: For the nonlinear pure-feedback system (1), we want to solve the prescribed finite-time control problem of the controlled system by designing the prescribed finite-time controller, so that all signals of the closed-loop system are stable at the finite-time T and the tracking error satisfies $\lim_{t\to T} e(t) = 0$. At the same time, it is necessary to get a relatively simple closed-loop system to facilitate further analysis and obtain the tracking control conditions with non-overshooting.

Similar to the approach in [12], in order to solve the problem of tracking control with non-overshooting, we need to get a relatively simple closed-loop system. The closed-loop systems obtained by the existing prescribed finite-time control criterion lemma are very complicated [1]. This complexity makes it difficult to derive an analytical expression of tracking error, which in turn hinders the establishment of less conservative non-overshooting conditions. Therefore, we propose a new prescribed finite-time criterion lemma as shown below.

Lemma 1. *Consider the monotonically increasing function*

$$\varphi(t) = \frac{T}{T-t}, \quad t \in [0, T) \tag{6}$$

where $T > 0$, $t \in \mathbb{R}_+$. If a continuously differentiable function $V(t) : [0, T] \to [0, +\infty)$ satisfies

$$\dot{V}(t) \leq -2\lambda_1 \varphi(t) V(t) + \lambda_2 V(t) \tag{7}$$

where λ_1 and λ_2 are positive constants.

Then, $\lim_{t\to T} V(t) = 0$.

Proof. From (6), we have

$$\dot{V}(t) \leq (-2\lambda_1 \varphi(t) + \lambda_2) V(t) \tag{8}$$

The above differential equation is solved to obtain

$$V(t) \leq V(0) e^{\int_0^t (-2\lambda_1 \varphi(\tau) + \lambda_2) d\tau} \tag{9}$$

From (4), we get

$$\begin{aligned} V(t) \leq & V(0) e^{\int_0^t (-2\lambda_1 \frac{T}{T-\tau} + \lambda_2) d\tau} \\ = & V(0) e^{(2\lambda_1 T \ln(T-t) - 2\lambda_1 T \ln(T) + \lambda_2 t} \end{aligned} \tag{10}$$

According to the properties of the function $\ln(T-t)$, we have

$$\lim_{t\to T} V(t) = 0 \tag{11}$$

□

Remark 1. *This lemma is different from existing lemmas of prescribed finite-time control criterion [5]. The primary aim of this paper mainly proposed to obtain a relatively simple closed-loop system (23) in the following.*

3. Control Design and Stability Analysis

Next, the controller will be designed using the backstepping method. The following coordinate transformation is introduced as

$$\begin{cases} z_1 = x_1 - r, \\ z_i = f_{i-1}(\bar{x}_i) - \alpha_{i-1}, \quad i = 2, \cdots, n, \\ z_{n+1} = f_n(u, \bar{x}_n) - \alpha_n, \\ \phi_i(t) = \varphi(t) z_i, \quad i = 1, \cdots, n+1, \end{cases} \tag{12}$$

where $\alpha_1, \cdots, \alpha_n$ are the virtual controllers. And it is designed as

$$\begin{cases} \alpha_1 = -c_1\phi_1 + \dot{r} - \frac{1}{T}\varphi(t) z_1 \\ \alpha_2 = \frac{-c_2\phi_2 - \frac{\partial f_1(\bar{x}_2)}{\partial x_1} f_1(\bar{x}_2) + \dot{\alpha}_1 - \frac{1}{T}\varphi(t) z_2}{\frac{\partial f_1(\bar{x}_2)}{\partial x_2}} \\ \alpha_i = \frac{-c_i\phi_i - \sum_{j=1}^{i-1} \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_j} f_j(\bar{x}_{j+1}) + \dot{\alpha}_{i-1} - \frac{1}{T}\varphi(t) z_i}{\frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i}} \\ \omega = \frac{-c_{n+1}\phi_{n+1} - \sum_{j=1}^{n} \frac{\partial f_n(u, \bar{x}_n)}{\partial x_j} f_j(\bar{x}_{j+1}) + \dot{\alpha}_n - \frac{1}{T}\varphi(t) z_{n+1}}{\frac{\partial f_n(u, \bar{x}_n)}{\partial u}} \end{cases} \tag{13}$$

where $i = 3, \cdots, n$; $c_i > 0$ is a designable parameter, and

$$\begin{cases} \dot{\alpha}_{i-1} = \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} f_j(x_{j+1}) + \frac{\partial \alpha_{i-1}}{\partial \varphi(t)} \dot{\varphi}(t) + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial r^j} r^{j+1} \\ \dot{\varphi}(t) = \frac{1}{T}\varphi^2(t). \end{cases} \tag{14}$$

According to the controller (13) designed above, the following theorem can be obtained.

Theorem 1. *The controller and the virtual controllers (13) are designed for the system (1). Then, the following conclusions can be obtained in* $t \in [0, T)$*.*

(1) The state $z_i(t)$ *of the closed-loop system satisfies the prescribed finite-time* $\lim_{t \to T} z_i(t) = 0$*;*

(2) The tracking error $e(t)$ *satisfies* $\lim_{t \to T} e(t) = 0$*;*

(3) All states of the system are bounded.

Proof. Next, define the total Lyapunov function $V(t)$ as

$$V(t) = \frac{1}{2}\phi_1^2(t) + \cdots + \frac{1}{2}\phi_{n+1}^2(t) = \sum_{j=1}^{n+1} \phi_j^2(t). \tag{15}$$

The derivative of $V(t)$ over time t is

$$\dot{V}(t) = \phi_1(t)\dot{\phi}_1(t) + \cdots + \phi_{n+1}(t)\dot{\phi}_{n+1}(t). \tag{16}$$

According to (12), we have

$$\begin{cases} \dot{\phi}_1 = \varphi(t)[z_2 + \alpha_1 - \dot{r} + \frac{1}{T}\varphi(t) z_1] \\ \dot{\phi}_i = \varphi(t)[\sum_{j=1}^{i-1} \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_j} f_j(\bar{x}_{j+1}) + \frac{\partial f_{i-1}(\bar{x}_{i-1})}{\partial x_i}(z_{i+1} + \alpha_i) + \dot{\alpha}_{i-1} + \frac{1}{T}\varphi(t) z_i], \quad i = 2, \cdots, n \\ \dot{\phi}_{n+1} = \varphi(t)[\sum_{j=1}^{n} \frac{\partial f_n(\bar{x}_i)}{\partial x_j} f_j(\bar{x}_{j+1}) + \frac{\partial f_n(u, \bar{x}_n)}{\partial u}\omega + \dot{\alpha}_n + \frac{1}{T}\varphi(t) z_n] \end{cases} \tag{17}$$

Substituting (13) and (17) into (16), the following equation can be obtained

$$\dot{V}(t) = -\sum_{j=1}^{n+1} c_j \varphi(t)\phi_j^2 + \phi_1\phi_2 + \sum_{j=1}^{n} \frac{\partial f_{j-1}(\bar{x}_j)}{\partial x_j} \phi_j\phi_{j+1}. \tag{18}$$

According to Assumption 2, we can get

$$\begin{aligned}
&\phi_1\phi_2+\sum_{j=1}^{n}\frac{\partial f_{j-1}(\bar{x}_j)}{\partial x_j}\phi_j\phi_{j+1}\\
&\leq \frac{1}{2}\phi_1^2+\frac{1}{2}\phi_2^2+\frac{1}{2}\phi_2^2+\frac{1}{2}(\frac{\partial f_1}{\partial x_2})^2\phi_3^2+\frac{1}{2}\phi_3^2+\frac{1}{2}(\frac{\partial f_1}{\partial x_2})^2\phi_4^2+\cdots+\frac{1}{2}\phi_n^2+\frac{1}{2}(\frac{\partial f_{n-1}}{\partial x_n})^2\phi_{n+1}^2\\
&\leq V+\frac{1}{2}\phi_2^2+\frac{1}{2}\eta_{22}^2\phi_3^2+\frac{1}{2}\eta_{32}^2\phi_4^2+\cdots+\frac{1}{2}\eta_{n2}^2\phi_{n+1}^2\\
&\leq V+\frac{1}{2}\kappa\phi_2^2+\frac{1}{2}\kappa\phi_3^2+\frac{1}{2}\kappa\phi_4^2+\cdots+\frac{1}{2}\kappa\phi_{n+1}^2\\
&=(1+\kappa)V,
\end{aligned} \tag{19}$$

where $\kappa \geq \max\{1,\eta_{22}^2,\eta_{32}^2,\cdots,\eta_{n2}^2\}$.

According to the above formula, Formula (18) can be rewritten as

$$\begin{aligned}
\dot{V}(t)\leq& -\sum_{j=1}^{n+1}c_j\varphi(t)\phi_j^2+(1+\kappa)V\\
\leq& -2c\varphi(t)V+(1+\kappa)V,
\end{aligned} \tag{20}$$

where $c\leq\min\{c_1,\cdots,c_{n+1}\}$.

From Lemma 1, we can get $\lim_{t\to T}V(t)=0$. From this result, the following deduction process can be obtained

$$\begin{aligned}
\lim_{t\to T}V(t)=0 &\Rightarrow \lim_{t\to T}\phi_i(t)=0\\
&\Rightarrow \lim_{t\to T}\varphi(t)z_i(t)=0\\
&\Rightarrow \lim_{t\to T}T\frac{z_i(t)}{T-t}=0\\
&\Rightarrow \lim_{t\to T}z_i(t)=0,
\end{aligned} \tag{21}$$

where $i=1,\cdots,n+1$.

Therefore, we can get that $z_i(t)$ is stable in T. If $z_1(t)=x_1(t)-r(t)$, then $x_1(t)$ is bounded. If $z_2(t)=f_1(x_1(t),x_2(t))-\alpha_1$, then $f_1(x_1(t),x_2(t))$ is bounded. There is at least one point μ ($\mu\in(\min(0,x_2),\max(0,x_2))$) to satisfy

$$f_1(x_1,x_2)-f_1(x_1,0)=\frac{\partial f_1(x_1,x_2)}{\partial x_2}|_{x_2=\mu}(x_2-0). \tag{22}$$

According to Assumption 2, we know that $x_2(t)$ is bounded. Similarly, the system states $x_1(t),x_2(t),\cdots,x_n(t)$ are bounded. □

4. Non-Overshooting Control

The section focuses on the non-overshooting control design of the system. According to Equation (17), the control condition with non-overshooting can be designed through the closed-loop system (12). In this regard, by substituting Equation (13) into Equation (17), we can obtain

$$\begin{cases}
\dot{\phi}_1=-c_1\varphi(t)\phi_1+\phi_2,\\
\dot{\phi}_i=-c_i\varphi(t)\phi_i+\frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i}\phi_{i+1}, \quad i=2,\cdots,n,\\
\dot{\phi}_{n+1}=-c_{n+1}\varphi(t)\phi_{n+1}.
\end{cases} \tag{23}$$

In the following, the tracking conditions without overshoot will be discussed in turn according to the system order $n=1$, $n=2$, and $n\geq 3$.

(1) The situation when $n=1$.

In this situation, the system (1) is a first-order system, so the closed-loop system can be obtained as $\dot{\phi}_1=-c_1\varphi(t)\phi_1+\phi_2$ and $\dot{\phi}_2=-c_2\varphi(t)\phi_2$. We discuss the tracking control with non-overshooting in the following two cases.

Case 1: $c_1\neq\frac{1}{T}+c_2$

In this case, the system (1) is a first-order system, so the closed-loop system can be obtained as $\dot{\phi}_1=$

$-c_1\varphi(t)\phi_1 + \phi_2$ and $\dot{\phi}_2 = -c_2\varphi(t)\phi_2$. We integrate this subsystem along $[0, T)$. Then, $\phi_2(t)$ is obtained as

$$\phi_2(t) = \phi_2(0)e^{-c_2T\int_0^t \frac{1}{T-\tau}d\tau} = \phi_2(0)\varphi^{-c_2T}, \tag{24}$$

where $\phi_2(0)$ is the initial value of $\phi_2(t)$.

Substituting (24) into $\dot{\phi}_1 = -c_1\varphi(t)\phi_1 + \phi_2$. We have

$$\dot{\phi}_1 = -c_1\varphi(t)\phi_1 + \phi_2(0)\varphi^{-c_2T}. \tag{25}$$

We integrate this subsystem along $[0, T)$. Then, $\phi_1(t)$ is obtained as

$$\begin{aligned}\phi_1(t) &= \phi_1(0)e^{-\int_0^t c_1\varphi(\tau)d\tau} + e^{-\int_0^t c_1\varphi(\tau)d\tau}\int_0^t e^{\int_0^\tau c_1\varphi(\sigma)d\sigma}\phi_2(0)\varphi^{-c_2T}(\tau)d\tau \\ &= \phi_1(0)\varphi^{-c_1T}(t) + \phi_2(0)\varphi^{-c_1T}(t)\int_0^t \varphi^{(c_1-c_2)T}(t)d\tau \\ &= [\phi_1(0) - \phi_2(0)\frac{T}{(c_1-c_2)T-1}](\varphi(t))^{-c_1T} + \phi_2(0)\frac{T}{(c_1-c_2)T-1}(\varphi(t))^{-c_2T-1}.\end{aligned} \tag{26}$$

Case 2: $c_1 = \frac{1}{T} + c_2$

Substituting (24) into $\dot{\phi}_1 = -(\frac{1}{T} + c_2)\varphi(t)\phi_1 + \phi_2$. We have

$$\dot{\phi}_1 = -(\frac{1}{T} + c_2)\varphi(t)\phi_1 + \phi_2(0)\varphi^{-c_2T}. \tag{27}$$

We integrate this subsystem along $[0, T)$. Then, $\phi_1(t)$ is obtained as

$$\begin{aligned}\phi_1(t) &= \phi_1(0)e^{-\int_0^t(\frac{1}{T}+c_2)\varphi(\tau)d\tau} + e^{-\int_0^t(\frac{1}{T}+c_2)\varphi(\tau)d\tau} \times \int_0^t \phi_2(0)\varphi^{-c_2T}(\tau)e^{\int_0^\tau(\frac{1}{T}+c_2)\varphi(\sigma)d\sigma}d\tau \\ &= \phi_1(0)\varphi^{-(1+c_2T)}(t) + \phi_2(0)\varphi^{-(1+c_2T)}(t)T\int_0^t \frac{1}{T-\tau}d\tau \\ &= [\phi_1(0) + \phi_2(0)T\ln\varphi](\varphi(t))^{-(1+c_2T)}.\end{aligned} \tag{28}$$

Combining Equations (12), (26) and (28), the expression of tracking error is

$$\begin{cases} z_1(t) = [z_1(0) - z_2(0)\frac{T}{(c_1-c_2)T-1}](\varphi(t))^{-c_1T-1} + z_2(0)\frac{T}{(c_1-c_2)T-1}(\varphi(t))^{-c_2T-2}, c_1 \neq \frac{1}{T} + c_2, \\ z_1(t) = [z_1(0) + z_2(0)T\ln\varphi](\varphi(t))^{-(1+c_2T)}, c_1 = \frac{1}{T} + c_2. \end{cases} \tag{29}$$

where $\varphi(0) = 1$.

Noting that $\varphi \in [1, +\infty)$, so $0 \leq \ln\varphi < +\infty$. Therefore, by analyzing the above expression, the conditions of non-overshooting tracking are obtained as follows

$$\begin{cases} z_1(0) \geq 0 \\ z_2(0) \geq 0 \end{cases} \Rightarrow z_1(t) \geq 0, \tag{30}$$

$$\begin{cases} z_1(0) \geq z_2(0)\frac{T}{(c_1-c_2)T-1} \\ z_2(0) \geq 0 \\ c_1 > \frac{1}{T} + c_2 \end{cases} \Rightarrow z_1(t) \geq 0 \tag{31}$$

$$\begin{cases} z_1(0) \geq z_2(0)\frac{T}{(c_1-c_2)T-1} \\ z_2(0) \leq 0 \\ c_1 < \frac{1}{T} + c_2 \end{cases} \Rightarrow z_1(t) \geq 0 \tag{32}$$

$$\begin{cases} z_1(0) \le 0 \\ z_2(0) \le 0 \end{cases} \Rightarrow z_1(t) \le 0, \tag{33}$$

$$\begin{cases} z_1(0) \le z_2(0)\dfrac{T}{(c_1 - c_2)T - 1}, \\ z_2(0) \le 0 \\ c_1 > \dfrac{1}{T} + c_2 \end{cases} \Rightarrow z_1(t) \le 0, \tag{34}$$

$$\begin{cases} z_1(0) \le z_2(0)\dfrac{T}{(c_1 - c_2)T - 1} \\ z_2(0) \ge 0 \\ c_1 < \dfrac{1}{T} + c_2 \end{cases} \Rightarrow z_1(t) \le 0 \tag{35}$$

(2) The situation when $n = 2$.

The system (1) is a 2-order system. According to (23), $\dot{\phi}_1 = -c_1\varphi(t)\phi_1 + \phi_2$, $\dot{\phi}_2 = -c_2\varphi(t)\phi_2 + \frac{\partial f_1}{\partial x_2}\phi_3$, and $\dot{\phi}_3 = -c_3\varphi(t)\phi_3$. From (24), we obtain

$$\phi_3(t) = \phi_3(0)\varphi^{-c_3T} \tag{36}$$

The following content examines the non-overshooting control design under various control gain conditions.

Case 1: $c_1 \ne \frac{1}{T} + c_2 \ne \frac{2}{T} + c_3$, and $z_3(0) \le 0$.

Then, $\phi_3(t) \le 0$. Substituting (36) into $\dot{\phi}_2 = -c_2\varphi(t)\phi_2 + \frac{\partial f_1}{\partial x_2}\phi_3$, we have

$$\dot{\phi}_2 \le -c_2\varphi(t)\phi_2 + |\frac{\partial f_1}{\partial x_2}||\phi_3|, \tag{37}$$

The discussion starts with the situation of $\phi_2(t)$. The case of $\phi_2(t)$ can be divided into $\phi_2(t) \le 0$ and $\phi_2(t) \ge 0$. Let's discuss $\phi_2(t) \le 0$ first. According to Assumption 1, the following formula can be obtained

$$\dot{\phi}_2 \le -c_2\varphi(t)\phi_2 - \eta_{21}\phi_3(0)\varphi^{-c_3T} \tag{38}$$

Integrating this subsystem over the interval $[0, T]$. We arrive at the expression for $\phi_2(t)$:

$$\begin{aligned} \phi_2(t) &\le \phi_2(0)e^{-\int_0^t c_2\varphi(\tau)d\tau} - e^{-\int_0^t c_2\varphi(\tau)d\tau} \times \int_0^t e^{\int_0^\tau c_2\varphi(\sigma)d\sigma}\eta_{21}\phi_3(0)\varphi^{-c_3T}(\tau)d\tau \\ &= \phi_2(0)\varphi^{-c_2T}(t) - \eta_{21}\phi_3(0)\varphi^{-c_2T}(t)\int_0^t \varphi^{(c_2-c_3)T}(t)d\tau \\ &= [\phi_2(0) + \phi_3(0)\frac{\eta_{21}T}{(c_2 - c_3)T - 1}](\varphi(t))^{-c_2T} - \phi_3(0)\frac{\eta_{21}T}{(c_2 - c_3)T - 1}(\varphi(t))^{-c_3T-1}. \end{aligned} \tag{39}$$

Based on the above discussion, the condition for $\phi_2(t) \le 0$ are as follows

$$\begin{cases} \phi_2(0) \le -\phi_3(0)\dfrac{\eta_{21}T}{(c_2 - c_3)T - 1} \\ \phi_3(0) \le 0, c_2 < \dfrac{1}{T} + c_3 \end{cases} \tag{40}$$

Substituting (39) into $\dot{\phi}_1 = -c_1\varphi(t)\phi_1 + \phi_2$. This differential equation can be solved as follows

$$\begin{aligned}
\phi_1(t) \leq\, & \phi_1(0)e^{-\int_0^t c_1\varphi(\tau)d\tau} - e^{-\int_0^t c_1\varphi(\tau)d\tau} \\
& \times \int_0^t e^{\int_0^\tau c_1\varphi(\sigma)d\sigma}\{[\phi_2(0) + \frac{\phi_3(0)\eta_{21}T}{(c_2-c_3)T-1}] \\
& \times \varphi(\tau)^{-c_2T} - \frac{\phi_3(0)\eta_{21}T}{(c_2-c_3)T-1}\varphi(\tau)^{-c_3T-1}\}d\tau \\
=\, & [\phi_1(0) - \phi_2(0)\frac{T}{(c_1-c_2)T-1} \\
& - \phi_3(0)\frac{\eta_{21}T^2}{[(c_1-c_2)T-1][(c_2-c_3)T-1]} \\
& + \phi_3(0)\frac{\eta_{21}T^2}{[(c_1-c_3)T-2][(c_2-c_3)T-1]}](\varphi(t))^{-c_1T} \\
& + [\phi_2(0)\frac{T}{(c_1-c_2)T-1} \\
& + \phi_3(0)\frac{\eta_{21}T^2}{[(c_1-c_2)T-1][(c_2-c_3)T-1]}](\varphi(t))^{-c_2T-1} \\
& - \phi_3(0)\frac{\eta_{21}T^2}{[(c_1-c_3)T-2][(c_2-c_3)T-1]}(\varphi(t))^{-c_3T-2}.
\end{aligned} \tag{41}$$

Then, According to the above formula and $\varphi(0) = 1$, the inequality expression of $z_1(t)$ is

$$\begin{aligned}
z_1(t) \leq\, & [z_1(0) - z_2(0)\frac{T}{(c_1-c_2)T-1} \\
& - z_3(0)\frac{\eta_{21}T^2}{[(c_1-c_2)T-1][(c_2-c_3)T-1]} \\
& + z_3(0)\frac{\eta_{21}T^2}{[(c_1-c_3)T-2][(c_2-c_3)T-1]}](\varphi(t))^{-c_1T-1} \\
& + [z_2(0)\frac{T}{(c_1-c_2)T-1} \\
& + z_3(0)\frac{\eta_{21}T^2}{[(c_1-c_2)T-1][(c_2-c_3)T-1]}](\varphi(t))^{-c_2T-2} \\
& - z_3(0)\frac{\eta_{21}T^2}{[(c_1-c_3)T-2][(c_2-c_3)T-1]}(\varphi(t))^{-c_3T-3}.
\end{aligned} \tag{42}$$

Therefore, the following non-overshooting control condition of $z_1(t) \leq 0$ can be obtained

$$\begin{cases}
z_1(0) \leq \dfrac{Tz_2(0)}{(c_1-c_2)-1} + \dfrac{\eta_{21}T^2z_3(0)}{[(c_1-c_3)T-2][(c_1-c_2)T-1]} \\
z_2(0) \leq -\dfrac{\eta_{21}Tz_3(0)}{(c_2-c_3)T-1} \\
z_3(0) \leq 0 \\
c_2 < \dfrac{1}{T} + c_3 < c_1 - \dfrac{1}{T}
\end{cases} \tag{43}$$

Similarly, if $\phi_2(0) \geq 0$, then we can get the following condition

$$\begin{cases}
z_1(0) \leq \dfrac{Tz_2(0)}{(c_1-c_2)-1} - \dfrac{\eta_{21}T^2z_3(0)}{[(c_1-c_3)T-2][(c_1-c_2)T-1]} \\
z_2(0) \leq \dfrac{\eta_{21}Tz_3(0)}{(c_2-c_3)T-1} \\
z_3(0) \geq 0 \\
c_2 < \dfrac{1}{T} + c_3 < c_1 - \dfrac{1}{T}
\end{cases} \tag{44}$$

Thus, when $\phi_3(0) \le 0$, we can get the following conditions

$$\begin{cases} z_1(0) \ge \dfrac{Tz_2(0)}{(c_1 - c_2) - 1} - \dfrac{\eta_{21}T^2 z_3(0)}{[(c_1 - c_3)T - 2][(c_1 - c_2)T - 1]} \\ z_2(0) \ge \dfrac{\eta_{21}Tz_3(0)}{(c_2 - c_3)T - 1} \\ z_3(0) \le 0 \\ c_2 < \dfrac{1}{T} + c_3 < c_1 - \dfrac{1}{T} \end{cases} \tag{45}$$

$$\begin{cases} z_1(0) \ge \dfrac{Tz_2(0)}{(c_1 - c_2) - 1} + \dfrac{\eta_{21}T^2 z_3(0)}{[(c_1 - c_3)T - 2][(c_1 - c_2)T - 1]} \\ z_2(0) \ge -\dfrac{\eta_{21}Tz_3(0)}{(c_2 - c_3)T - 1} \\ z_3(0) \ge 0 \\ c_2 < \dfrac{1}{T} + c_3 < c_1 - \dfrac{1}{T} \end{cases} \tag{46}$$

Case 2: $c_1 = \frac{1}{T} + c_2 \ne \frac{2}{T} + c_3$, and $z_3(0) \le 0$.

Since $c_2 \ne \frac{1}{T} + c_3$, the algorithm of $\phi_2(t)$ remains the same as (37)–(40) in Case 1. Therefore, substituting (39) into $\dot{\phi}_1 = -c_1\varphi(t)\phi_1 + \phi_2$. This differential equation can be solved as follows

$$\begin{aligned} \phi_1(t) &\le \phi_1(0)e^{-\int_0^t (c_2+\frac{1}{T})\varphi(\tau)d\tau} - e^{-\int_0^t (c_2+\frac{1}{T})\varphi(\tau)d\tau} \\ &\quad \times \int_0^t e^{\int_0^\tau (c_2+\frac{1}{T})\varphi(\sigma)d\sigma}\{[\phi_2(0) + \frac{\phi_3(0)\eta_{21}T}{(c_2 - c_3)T - 1}] \\ &\quad \times \varphi(\tau)^{-c_2T} - \frac{\phi_3(0)\eta_{21}T}{(c_2 - c_3)T - 1}\varphi(\tau)^{-c_3T-1}\}d\tau \\ &= [\phi_1(0) + T(\phi_2(0) + \frac{\phi_3(0)\eta_{21}T}{(c_2 - c_3)T - 1})\ln\varphi \\ &\quad + \frac{\phi_3(0)\eta_{21}T^2}{((c_2 - c_3)T - 1)^2}]\varphi(t)^{-c_2T-1} \\ &\quad - \frac{\phi_3(0)\eta_{21}T^2}{((c_2 - c_3)T - 1)^2}\varphi(t)^{-c_3T-1}. \end{aligned} \tag{47}$$

According to (47), we have

$$\begin{aligned} z_1(t) \le [z_1(0) &+ T(z_2(0) + \frac{z_3(0)\eta_{21}T}{(c_2 - c_3)T - 1})\ln\varphi \\ &+ \frac{z_3(0)\eta_{21}T^2}{((c_2 - c_3)T - 1)^2}]\varphi(t)^{-c_2T-2} \\ &- \frac{z_3(0)\eta_{21}T^2}{((c_2 - c_3)T - 1)^2}\varphi(t)^{-c_3T-2}. \end{aligned} \tag{48}$$

Therefore, we can obtain the following condition

$$\begin{cases} z_1(0) \le -z_3(0)\dfrac{\eta_{21}T^2}{[(c_1 - c_3)T - 1]^2} \\ z_2(0) \le -z_3(0)\dfrac{\eta_{21}T}{(c_2 - c_3)T - 1} \\ z_3(0) \le 0, c_2 \le c_3 \end{cases} \Rightarrow z_1(t) \le 0 \tag{49}$$

The cases of $\phi_2(t) \ge 0$ and $z_3(t) \ge 0$ are similar to Case 1, so this article will not discuss them again.

Next, we discuss the case where $c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3$. In this case, if we discuss the non-overshooting condition of $c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3$ according to the algorithm of Case 1, it is difficult to obtain the non-overshooting condition. However, according to Assumption 1, we can consider two subcases for $\frac{\partial f_1(\bar{x}_2)}{\partial x_2}$ into two cases: $\frac{\partial f_1(\bar{x}_2)}{\partial x_2} \ge 0$ and $\frac{\partial f_1(\bar{x}_2)}{\partial x_2} \le 0$. Then, we discuss the conditions of non-overshooting in the following cases.

Case 3: $c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3$, $\frac{\partial f_1}{\partial x_2} > 0$ and $z_3(0) \le 0$.
Then, $\phi_3(t) \le 0$. Substituting (36) into $\dot{\phi}_2 = -(c_1 - \frac{1}{T})\varphi(t)\phi_2 + \frac{\partial f_1}{\partial x_2}\phi_3$, we have

$$\dot{\phi}_2 \le -(c_1 - \frac{1}{T})\varphi(t)\phi_2 + \eta_{11}\phi_3(0)\varphi^{-c_1T+2} \tag{50}$$

We integrate this subsystem along $[0, T]$. Then, $\phi_2(t)$ is obtained as

$$\begin{aligned}\phi_2(t) \le\ & \phi_2(0)e^{-\int_0^t (c_1 - \frac{1}{T})\varphi(\tau)d\tau} + e^{-\int_0^t (c_1 - \frac{1}{T})\varphi(\tau)d\tau} \\ & \times \int_0^t e^{\int_0^\tau (c_1 - \frac{1}{T})\varphi(\sigma)d\sigma}\eta_{11}\phi_3(0)\varphi^{-c_1T+2}(\tau)d\tau \\ =\ & [\phi_2(0) + T\eta_{11}\phi_3(0)\ln\varphi](\varphi(t))^{-c_1T+1}\end{aligned} \tag{51}$$

Based on the above discussion, the condition for $\phi_2(t) \le 0$ are as follows

$$\phi_2(0) \le 0, \phi_3(0) \le 0 \tag{52}$$

Substituting (51) into $\dot{\phi}_1 = -c_1\varphi(t)\phi_1 + \phi_2$. This differential equation can be solved as follows

$$\begin{aligned}\phi_1(t) \le\ & \phi_1(0)e^{-\int_0^t c_1\varphi(\tau)d\tau} + e^{-\int_0^t c_1\varphi(\tau)d\tau}\int_0^t e^{\int_0^\tau c_1\varphi(\sigma)d\sigma} \times [\phi_2(0) + T\eta_{11}\phi_3(0)\ln\varphi(\tau)](\varphi(\tau))^{-c_1T+1}d\tau \\ =\ & [\phi_1(0) + T\phi_2(0)\ln\varphi + \frac{1}{2}T^2\eta_{11}\phi_3(0)(\ln\varphi(\tau))^2](\varphi(t))^{-c_1T}.\end{aligned} \tag{53}$$

Combining the above formula with the initial condition $\varphi(0) = 1$, we obtain the following inequality for $z_1(t)$:

$$z_1(t) \le [z_1(0) + Tz_2(0)\ln\varphi + \frac{1}{2}T^2\eta_{11}z_3(0)(\ln\varphi(\tau))^2](\varphi(t))^{-c_1T-1} \tag{54}$$

Therefore, the following non-overshooting control conditions of $z_1(t) \le 0$ can be obtained

$$\begin{cases} z_1(0) \le 0, z_2(0) \le 0, z_3(0) \le 0 \\ c_1 = \dfrac{1}{T} + c_2 = \dfrac{2}{T} + c_3 \end{cases} \Rightarrow z_1(t) \le 0. \tag{55}$$

$$\begin{cases} z_2(0) \le 0, z_3(0) \le 0 \\ 2\eta_{11}z_1(0)z_3(0) - z_2^2(0) \le 0 \\ c_1 = \dfrac{1}{T} + c_2 = \dfrac{2}{T} + c_3 \end{cases} \Rightarrow z_1(t) \le 0. \tag{56}$$

Case 4: $c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3$, $\frac{\partial f_1}{\partial x_2} > 0$ and $z_3(0) \ge 0$.
Then, $\phi_3(t) \ge 0$. According to $\dot{\phi}_2 = -(c_1 - \frac{1}{T})\varphi(t)\phi_2 + \frac{\partial f_1}{\partial x_2}\phi_3$, we have

$$\dot{\phi}_2 \ge -(c_1 - \frac{1}{T})\varphi(t)\phi_2 + \eta_{11}\phi_3(0)\varphi^{-c_1T+2}. \tag{57}$$

Integrating this subsystem over the interval $[0, T]$. Then, $\phi_2(t)$ is obtained as

$$\phi_2(t) \ge [\phi_2(0) + T\eta_{11}\phi_3(0)\ln\varphi](\varphi(t))^{-c_1T+1}. \tag{58}$$

Based on the above discussion, the condition for $\phi_2(t) \ge 0$ is summarized as follows

$$\phi_2(0) \ge 0, \phi_3(0) \ge 0, \tag{59}$$

Substituting (58) into $\dot{\phi}_1 = -c_1\varphi(t)\phi_1 + \phi_2$. This differential equation can be solved as follows

$$\phi_1(t) \ge [\phi_1(0) + T\phi_2(0)\ln\varphi + \frac{1}{2}T^2\eta_{11}\phi_3(0)(\ln\varphi(\tau))^2](\varphi(t))^{-c_1T}. \tag{60}$$

Then, According to the above formula and $\varphi(0) = 1$, the inequality expression of $z_1(t)$ is

$$z_1(t) \geq [z_1(0) + Tz_2(0)\ln\varphi + \frac{1}{2}T^2\eta_{11}z_3(0)(\ln\varphi(\tau))^2](\varphi(t))^{-c_1T-1}. \tag{61}$$

Therefore, the following non-overshooting control conditions for $z_1(t) \leq 0$ can be obtained

$$\begin{cases} z_1(0) \geq 0, z_2(0) \geq 0, z_3(0) \geq 0 \\ c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3 \end{cases} \Rightarrow z_1(t) \geq 0 \tag{62}$$

$$\begin{cases} z_2(0) \geq 0, z_3(0) \geq 0 \\ 2\eta_{11}z_1(0)z_3(0) - z_2^2(0) \leq 0 \\ c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3 \end{cases} \Rightarrow z_1(t) \geq 0. \tag{63}$$

Case 5: $c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3$, $\frac{\partial f_1}{\partial x_2} < 0$ and $z_3(0) \leq 0$.
Then, $\phi_3(t) \leq 0$. According to $\dot{\phi}_2 = -(c_1 - \frac{1}{T})\varphi(t)\phi_2 + \frac{\partial f_1}{\partial x_2}\phi_3$, we have

$$\dot{\phi}_2 \geq -(c_1 - \frac{1}{T})\varphi(t)\phi_2 - \eta_{12}\phi_3(0)\varphi^{-c_1T+2}. \tag{64}$$

We integrate this subsystem along $[0, T]$. Then, $\phi_2(t)$ is obtained as

$$\begin{aligned} \phi_2(t) &\geq \phi_2(0)e^{-\int_0^t (c_1-\frac{1}{T})\varphi(\tau)d\tau} - e^{-\int_0^t (c_1-\frac{1}{T})\varphi(\tau)d\tau} \times \int_0^t e^{\int_0^\tau (c_1-\frac{1}{T})\varphi(\sigma)d\sigma}\eta_{12}\phi_3(0)\varphi^{-c_1T+2}(\tau)d\tau \\ &= [\phi_2(0) - T\eta_{12}\phi_3(0)\ln\varphi](\varphi(t))^{-c_1T+1}. \end{aligned} \tag{65}$$

Based on the above discussion, the condition for $\phi_2(t) \geq 0$ are as follows

$$\phi_2(0) \geq 0, \phi_3(0) \leq 0, \tag{66}$$

Substituting (65) into $\dot{\phi}_1 = -c_1\varphi(t)\phi_1 + \phi_2$. This differential equation can be solved as follows

$$\begin{aligned} \phi_1(t) &\geq \phi_1(0)e^{-\int_0^t c_1\varphi(\tau)d\tau} + e^{-\int_0^t c_1\varphi(\tau)d\tau}\int_0^t e^{\int_0^\tau c_1\varphi(\sigma)d\sigma} \times [\phi_2(0) - T\eta_{12}\phi_3(0)\ln\varphi(\tau)](\varphi(\tau))^{-c_1T+1}d\tau \\ &= [\phi_1(0) + T\phi_2(0)\ln\varphi - \frac{1}{2}T^2\eta_{12}\phi_3(0)(\ln\varphi(\tau))^2](\varphi(t))^{-c_1T}. \end{aligned} \tag{67}$$

Then, According to the above formula and $\varphi(0) = 1$, the inequality expression of $z_1(t)$ is

$$z_1(t) \geq [z_1(0) + Tz_2(0)\ln\varphi - \frac{1}{2}T^2\eta_{12}z_3(0)(\ln\varphi(\tau))^2](\varphi(t))^{-c_1T-1}. \tag{68}$$

Therefore, the following non-overshooting control conditions of $z_1(t) \leq 0$ can be obtained

$$\begin{cases} z_1(0) \geq 0, z_2(0) \geq 0, z_3(0) \leq 0 \\ c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3 \end{cases} \Rightarrow z_1(t) \geq 0 \tag{69}$$

$$\begin{cases} z_2(0) \geq 0, z_3(0) \leq 0 \\ 2\eta_{12}z_1(0)z_3(0) + z_2^2(0) \leq 0 \\ c_1 = \dfrac{1}{T} + c_2 = \dfrac{2}{T} + c_3 \end{cases} \Rightarrow z_1(t) \geq 0. \tag{70}$$

Case 6: $c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3$, $\frac{\partial f_1}{\partial x_2} < 0$ and $z_3(0) \geq 0$.
Then, $\phi_3(t) \geq 0$. According to $\dot{\phi}_2 = -(c_1 - \frac{1}{T})\varphi(t)\phi_2 + \frac{\partial f_1}{\partial x_2}\phi_3$, we have

$$\dot{\phi}_2 \leq -(c_1 - \frac{1}{T})\varphi(t)\phi_2 - \eta_{12}\phi_3(0)\varphi^{-c_1T+2} \tag{71}$$

We integrate this subsystem along $[0, T]$. Then, $\phi_2(t)$ is obtained as

$$\phi_2(t) \leq [\phi_2(0) - T\eta_{12}\phi_3(0)\ln\varphi](\varphi(t))^{-c_1T+1} \tag{72}$$

Based on the above discussion, the condition for $\phi_2(t) \geq 0$ are as follows

$$\phi_2(0) \leq 0, \phi_3(0) \geq 0 \tag{73}$$

Substituting (65) into $\dot{\phi}_1 = -c_1\varphi(t)\phi_1 + \phi_2$. This differential equation can be solved as follows

$$\phi_1(t) \leq [\phi_1(0) + T\phi_2(0)\ln\varphi - \frac{1}{2}T^2\eta_{12}\phi_3(0)(\ln\varphi(\tau))^2](\varphi(t))^{-c_1T} \tag{74}$$

Then, According to the above formula and $\varphi(0) = 1$, the inequality expression of $z_1(t)$ is

$$z_1(t) \leq [z_1(0) + Tz_2(0)\ln\varphi - \frac{1}{2}T^2\eta_{12}z_3(0)(\ln\varphi(\tau))^2](\varphi(t))^{-c_1T-1} \tag{75}$$

Therefore, the following non-overshooting control conditions of $z_1(t) \leq 0$ can be obtained

$$\begin{cases} z_1(0) \leq 0, z_2(0) \leq 0, z_3(0) \geq 0 \\ c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3 \end{cases} \Rightarrow z_1(t) \leq 0 \tag{76}$$

$$\begin{cases} z_2(0) \leq 0, z_3(0) \geq 0 \\ 2\eta_{12}z_1(0)z_3(0) + z_2^2(0) \leq 0 \\ c_1 = \frac{1}{T} + c_2 = \frac{2}{T} + c_3 \end{cases} \Rightarrow z_1(t) \leq 0 \tag{77}$$

Then, we can get the following theorem

Theorem 2. *For the nonlinear system (1) with $n = 2$, the virtual controllers α_1, α_2 and auxiliary controller ω are designed to make the closed-loop system stable for a prescribed finite-time. Then, the condition of non-overshooting tracking control needs to satisfy one of (43)–(46), (49), (55), (56), (62), (63), (69), (70), (76) and (77).*

(3) The situation when $n \geq 3$.

In order to obtain more extensive tracking conditions with non-overshooting, we need to get the expression of tracking error $e(t)$. However, with the increase of system order n, the expression of tracking error is very complicated. In addition, some conditions with non-overshooting also need to rely on parameters η_{i1} and η_{i2}, which also increases the computational burden. Therefore, next, we introduce the following methods to solve these problems.

According to Assumption 1, the following inequality is obtained

$$\eta_{i1} < \frac{\partial f_i(\bar{x}_i)}{\partial x_i} < \eta_{i2} \quad or \quad -\eta_{i2} < \frac{\partial f_i(\bar{x}_i)}{\partial x_i} < -\eta_{i1} \tag{78}$$

where $i = 1, \cdots, n+1$.

Therefore, the function $\frac{\partial f_i(\bar{x}_i)}{\partial x_i}$ can be divided into two cases: $\frac{\partial f_i(\bar{x}_i)}{\partial x_i} > 0$ and $\frac{\partial f_i(\bar{x}_i)}{\partial x_i} < 0$. As follows:

Case 1: $\frac{\partial f_i(\bar{x}_i)}{\partial x_i} > 0$.

From (23), we obtain $\dot{\phi}_{n+1} = -c_{n+1}\varphi(t)\phi_{n+1}$. Integrating this subsystem along $[0, t]$. Then,

$$\begin{aligned} \phi_{n+1}(t) &= \phi_{n+1}(0)e^{-\int_0^t c_{n+1}\varphi(\tau)d\tau} \\ &= \phi_{n+1}(0)\varphi^{-c_{n+1}t} \end{aligned} \tag{79}$$

The conditions that satisfy $\phi_{n+1}(t) \leq 0$ are as follows

$$\phi_{n+1}(0) \leq 0 \Rightarrow \phi_{n+1}(t) \leq 0 \tag{80}$$

Since $\frac{\partial f_{n-1}(\bar{x}_n)}{\partial x_n}\phi_{n+1}(t) \leq 0$, we can get

$$\begin{aligned}\dot{\phi}_n =& - c_n\varphi(t)\phi_n + \frac{\partial f_{n-1}(\bar{x}_n)}{\partial x_n}\phi_{n+1} \\ \leq& - c_n\varphi(t)\phi_n\end{aligned} \tag{81}$$

We integrate (81) along $[0, t]$. Then,

$$\begin{aligned}\phi_n(t) &\leq \phi_n(0)e^{-\int_0^t c_n\varphi(\tau)d\tau} \\ &= \phi_n(0)\varphi^{-c_n t}\end{aligned} \tag{82}$$

The conditions for $\phi_n(t) \leq 0$ are as follows.

$$\phi_n(0) \leq 0 \Rightarrow \phi_n(t) \leq 0 \tag{83}$$

Since $\frac{\partial f_{n-2}(\bar{x}_{n-1})}{\partial x_{n-1}}\phi_n(t) \leq 0$, we can get

$$\begin{aligned}\dot{\phi}_{n-1} =& - c_{n-1}\varphi(t)\phi_{n-1} + \frac{\partial f_{n-2}(\bar{x}_{n-1})}{\partial x_{n-1}}\phi_n \\ \leq& - c_{n-1}\varphi(t)\phi_{n-1}\end{aligned} \tag{84}$$

We integrate (84) along $[0, t]$. Then,

$$\begin{aligned}\phi_{n-1}(t) &\leq \phi_{n-1}(0)e^{-\int_0^t c_{n-1}\varphi(\tau)d\tau} \\ &= \phi_{n-1}(0)\varphi^{-c_{n-1} t}\end{aligned} \tag{85}$$

The conditions for $\phi_{n-1}(t) \leq 0$ are as follows

$$\phi_{n-1}(0) \leq 0 \Rightarrow \phi_{n-1}(t) \leq 0. \tag{86}$$

In this way, we can obtain

$$\phi_1(t) \leq \phi_1(0)\varphi^{-c_1 t} \tag{87}$$

To sum up, the tracking control with non-overshooting satisfies the following condition

$$\begin{cases} \phi_i(0) \leq 0, i = 1, \cdots, n+1 \\ \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i} > 0, i = 1, \cdots, n+1, \end{cases} \Rightarrow \phi_1(t) \leq 0 \tag{88}$$

According to (12), We deduced that

$$\begin{aligned}z_i(0) \leq 0 \Rightarrow \phi_i(0) \leq 0 \\ \phi_1(t) \leq 0 \Rightarrow z_i(t) \leq 0\end{aligned} \tag{89}$$

Therefore, we have

$$\begin{cases} z_i(0) \leq 0, i = 1, \cdots, n+1 \\ \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i} > 0, i = 1, \cdots, n+1 \end{cases} \Rightarrow z_1(t) \leq 0 \tag{90}$$

Similarly, another tracking condition with non-overshooting can be obtained as follows

$$\begin{cases} z_i(0) \geq 0, i = 1, \cdots, n+1 \\ \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i} > 0, i = 1, \cdots, n+1 \end{cases} \Rightarrow z_1(t) \geq 0 \tag{91}$$

Case 2: $\frac{\partial f_i(\bar{x}_i)}{\partial x_i} < 0$, and n is an odd number.
From (79), the conditionss for $\phi_{n+1}(t) \geq 0$ are as follows

$$\phi_{n+1}(0) \leq 0 \Rightarrow \phi_{n+1}(t) \leq 0 \tag{92}$$

Since $\frac{\partial f_{n-1}(\bar{x}_n)}{\partial x_n}\phi_{n+1}(t) \geq 0$, we can obtain

$$\begin{aligned}\dot{\phi}_n =& - c_n\varphi(t)\phi_n + \frac{\partial f_{n-1}(\bar{x}_n)}{\partial x_n}\phi_{n+1} \\ \geq& - c_n\varphi(t)\phi_n\end{aligned} \tag{93}$$

We integrate (81) along $[0, t]$. Then,

$$\begin{aligned}\phi_n(t) \geq& \phi_n(0)e^{-\int_0^t c_n\varphi(\tau)d\tau} \\ =& \phi_n(0)\varphi^{-c_n t}\end{aligned} \tag{94}$$

The conditions for $\phi_n(t) \geq 0$ are as follows

$$\left\{\begin{array}{l}\phi_{n+1}(0) \leq 0 \\ \phi_n(0) \geq 0\end{array}\right. \Rightarrow \phi_n(t) \geq 0 \tag{95}$$

Since $\frac{\partial f_{n-2}(\bar{x}_{n-1})}{\partial x_{n-1}}\phi_n(t) \leq 0$, we can get

$$\begin{aligned}\dot{\phi}_{n-1} =& - c_{n-1}\varphi(t)\phi_{n-1} + \frac{\partial f_{n-2}(\bar{x}_{n-1})}{\partial x_{n-1}}\phi_n \\ \leq& - c_{n-1}\varphi(t)\phi_{n-1}.\end{aligned} \tag{96}$$

We integrate (84) along $[0, t]$. Then,

$$\begin{aligned}\phi_{n-1}(t) \leq& \phi_{n-1}(0)e^{-\int_0^t c_{n-1}\varphi(\tau)d\tau} \\ =& \phi_{n-1}(0)\varphi^{-c_{n-1} t}\end{aligned} \tag{97}$$

The condition that satisfy $\phi_{n-1}(t) \leq 0$ are as follows

$$\left\{\begin{array}{l}\phi_{n+1}(0) \leq 0 \\ \phi_n(0) \geq 0 \\ \phi_{n-1}(0) \leq 0\end{array}\right. \Rightarrow \phi_{n-1}(t) \leq 0 \tag{98}$$

In this way, we can obtain

$$\phi_1(t) \leq \phi_1(0)\varphi^{-c_1 t} \tag{99}$$

To sum up, the tracking control with non-overshooting satisfies the following condition

$$\left\{\begin{array}{l}\phi_i(0) \geq 0, i = 3, 5, 7, \cdots, n \\ \phi_i(0) \leq 0, i = 2, 4, 6, \cdots, n+1 \\ \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i} < 0, i = 1, 2, 3, \cdots, n+1 \\ \phi_1(0) \leq 0\end{array}\right. \Rightarrow \phi_1(t) \leq 0 \tag{100}$$

Therefore, we have

$$\left\{\begin{array}{l}z_i(0) \geq 0, i = 3, 5, 7, \cdots, n \\ z_i(0) \leq 0, i = 2, 4, 6, \cdots, n+1 \\ \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i} < 0, i = 1, 2, 3, \cdots, n+1 \\ z_1(0) \leq 0\end{array}\right. \Rightarrow z_1(t) \leq 0 \tag{101}$$

Similarly, another tracking condition with non-overshooting can be obtained as follows

$$\left\{\begin{array}{l}z_i(0) \leq 0, i = 3, 5, 7, \cdots, n \\ z_i(0) \geq 0, i = 2, 4, 6, \cdots, n+1 \\ \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i} < 0, i = 1, 2, 3, \cdots, n+1 \\ z_1(0) \geq 0\end{array}\right. \Rightarrow z_1(t) \geq 0 \tag{102}$$

In the same way, if n is an even number, then we can get the following tracking control conditions with non-overshooting

$$\begin{cases} z_i(0) \leq 0, i = 3, 5, 7, \cdots, n+1 \\ z_i(0) \geq 0, i = 2, 4, 6, \cdots, n \\ \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i} < 0, i = 1, 2, 3, \cdots, n+1 \\ z_1(0) \geq 0 \end{cases} \Rightarrow z_1(t) \geq 0 \tag{103}$$

$$\begin{cases} z_i(0) \geq 0, i = 3, 5, 7, \cdots, n+1 \\ z_i(0) \leq 0, i = 2, 4, 6, \cdots, n \\ \frac{\partial f_{i-1}(\bar{x}_i)}{\partial x_i} < 0, i = 1, 2, 3, \cdots, n+1 \\ z_1(0) \leq 0 \end{cases} \Rightarrow z_1(t) \leq 0 \tag{104}$$

Theorem 3. *For the system (1) of $n \geq 3$, by designing the virtual controllers α_i ($i = 1, \cdots, n$) and the auxiliary controller $\omega(t)$, the closed-loop system is a non-overshooting tracking control result, which needs to satisfy one of inequations (101)–(104).*

5. Simulation Example

In this section, we give an example to illustrate the effectiveness of this algorithm.

Example 1. *Consider a nonlinear system of order $n = 2$, where the functions $f_1(\bar{x}_2) = x_2 + 1/3x_2^3$, $f_2(u, \bar{x}_2) = u + 0.5\sin(u) + x_2$ and the reference signal is $r = 1 - \sin(t)$.*

The control parameters are selected as $c_1 = 10/3$, $c_2 = 3$, $c_3 = 4$, $x_1(0) = 0$, $x_2(0) = 1$, and $x_3(0) = -1$. The prescribed finite-time T is selected as $T = 1$.

Then, we can know that $z_1(0) = -1$, $z_2(0) = -2$ and $z_3(0) = 0.5\sin(-1) - 5/3$. According to the algorithm in this paper, we have

$$\begin{cases} z_3(0) = (0.5\sin(-1) - 5/3)\varphi^{-4t-1} \leq 0 \\ z_2(0) \leq -2\varphi^{-3t-1} \leq 0 \\ z_1(0) \leq -\varphi^{-t-1} \leq 0 \end{cases} \tag{105}$$

Therefore, the non-overshooting condition (90) is satisfied. The selected parameters in (105) meet the requirements for non-overshooting tracking control, as confirmed by the simulation results shown in Figures 1 and 2.

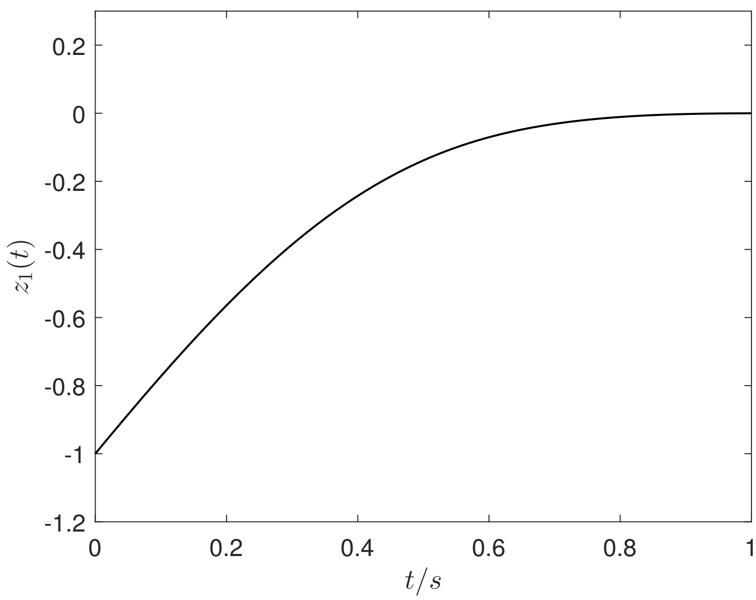

Figure 1. The time trajectories of $z_1(t)$.

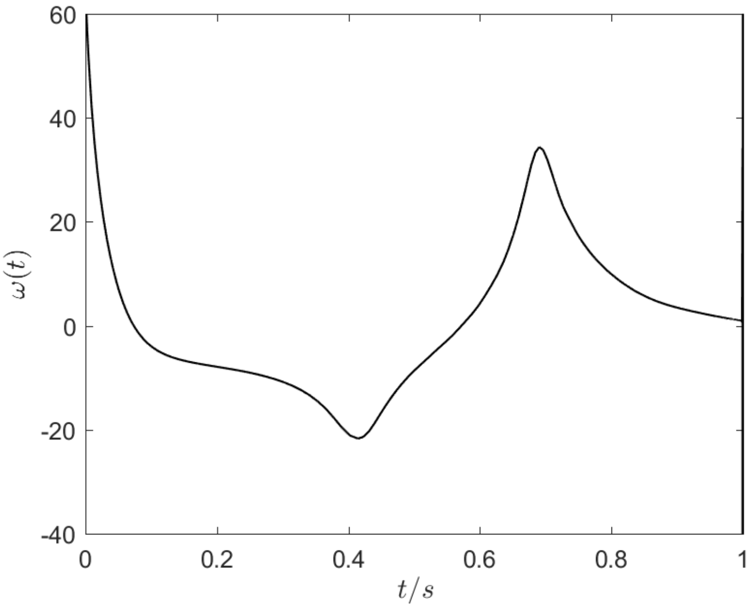

Figure 2. The time trajectories of auxiliary control input $\omega(t)$.

Figure 1 shows that the tracking error $e(t)$ *can converge to the origin in the moment of* $T = 1$*, and it is always not greater than* 0 *in the process of convergence, which satisfies the tracking control with non-overshooting; Figure 2 shows that the auxiliary controller* $\omega(t)$ *is also bounded. Therefore, the algorithm in this paper is effective.*

Example 2. *Consider the following Chua's circuit system in [22]*

$$\begin{cases} C_a\dot{V}_a = \frac{V_b - V_a}{R_0} - I_a, \\ C_b\dot{V}_b = -\frac{V_b - V_a}{R_0} + I_b, \\ L_0\dot{I}_b = -V_b - R_1 I_b + u, \end{cases} \tag{106}$$

where V_a *and* V_b *are the voltages on capacitors* C_a *and* C_b*, respectively.* I_a *is the current through Chus's diode* D*.* I_b *is the current on inductance* L_0*. Let* $x_1 = V_a$*,* $x_2 = V_b$*, and* $x_3 = I_b$*,* $I_a = x_1^2$*. Then, equation (106) can be written as*

$$\begin{cases} \dot{x}_1 = \varepsilon_1 x_2 + f_1(x_1), \\ \dot{x}_2 = \varepsilon_2 x_3 + f_2(\bar{x}_2), \\ \dot{x}_3 = \varepsilon_3 u + f_3(\bar{x}_3), \end{cases} \tag{107}$$

where $\varepsilon_1 = \frac{1}{R_0 C_a}$*,* $\varepsilon_2 = \frac{1}{C_b}$*,* $\varepsilon_3 = \frac{1}{L_0}$*,* $f_1(x_1) = -\frac{1}{R_0 C_a}x_1 - \frac{1}{C_a}I_a$*,* $f_2(\bar{x}_2) = -\frac{1}{RC_b}x_2 + \frac{1}{RC_b}x_1$*,* $f_3(\bar{x}_3) = -\frac{1}{L_0}x_2 - \frac{R_1}{L_0}x_3$*,* $L_0 = 0.25H$*,* $C_b = 0.1F$*,* $R_1 = 0.01\Omega$*,* $C_a = 1F$*, and* $R_0 = 0.2\Omega$*.*

Define $z_1 = x_1 - r$*,* $z_2 = \varepsilon_1 x_2 - \alpha_1$ *and* $z_3 = \varepsilon_1\varepsilon_2 x_3 - \alpha_2$*, where* $r = 1 - \sin(t)$*. Following the algorithm in this paper, the controllers are designed as follows:*

$$\begin{cases} \alpha_1 = -c_1\phi_1 - f_1(x_1) + \dot{r} - 1/T\varphi z_1 \\ \alpha_2 = -c_2\phi_2 - 5f_2(\bar{x}_2) + \dot{\alpha}_1 - 1/T\varphi z_2 \\ u = \dfrac{1}{\varepsilon_1\varepsilon_2\varepsilon_3}[-c_3\phi_3 - f_3(\bar{x}_3) + \dot{\alpha}_2 - 1/T\varphi z_3] \end{cases} \tag{108}$$

Then, we can obtain the closed-loop system (8). The design parameters and initial values are selected as $c_1 = 6$*,* $c_2 = 5$*,* $c_3 = 4$*,* $x_1(0) = 0$*,* $x_2(0) = 1$*,* $x_3(0) = -5.3$*. The selection of these parameters can satisfy the non-overshooting condition (59). Then,* $z_1(0) = -1$*,* $z_2(0) = -1$*. Since* $\frac{\partial \dot{x}_1}{\partial x_2} = 5$*, then we choose* $\eta_{21} = 1$ *according to Assumption 1. To sum up,* $z_3(0) = -5$*. In this way, the selected parameters also meet the condition of non-overshooting (56).*

The simulation result are given in Figures 3 and 4.

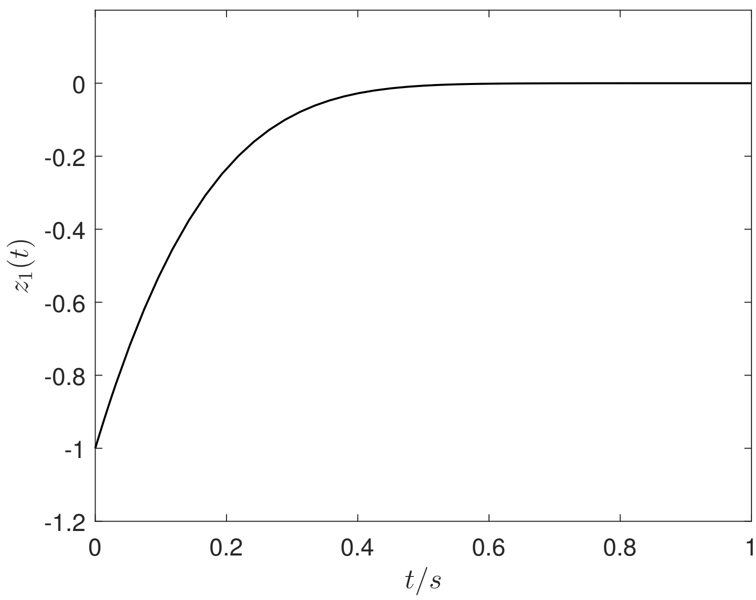

Figure 3. The time trajectories of $z_1(t)$.

According to Figure 3, it can be seen that $z_1(t)$ converge to the origin asymptotically, and $z_1(t)$ is always negative. This shows that the algorithm in this paper is effective.

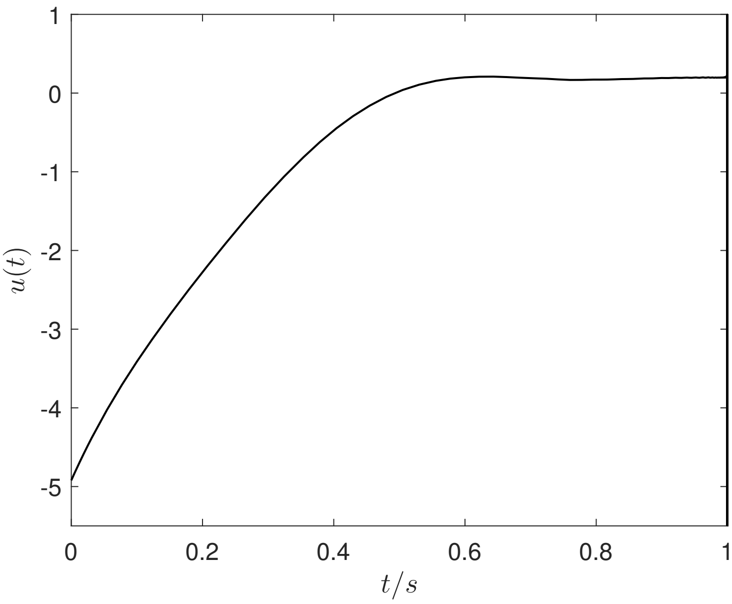

Figure 4. The time trajectories of auxiliary control input $u(t)$.

6. Conclusions

The problem of non-overshooting control of nonlinear systems is challenging. Based on the backstepping method, the controller and auxiliary controller are designed for nonlinear strict-feedback systems and nonlinear pure-feedback systems, and the closed-loop systems are obtained, respectively. Then, the expressions for the tracking error are deduced according to the closed-loop systems. By analyzing the expressions, the more widely applicable conditions of non-overshooting are obtained. Finally, the simulation results of three examples have demonstrate the effectiveness and superiority of the proposed algorithm.

A limitation of this work, which is inherent to the backstepping approach itself, is the requirement for the nonlinear system to have a specific structure. The backstepping method necessitates that the controlled system be in a strict-feedback form, as the lower-triangular structure of such systems allows for the recursive control design intrinsic to backstepping. Future research will focus on this very aspect, aiming to develop more effective control strategies that are not constrained by specific system structures and to explore control methods with better generalization capabilities.

Author Contributions

Z.Z.: writing—reviewing and editing; conceptualization, methodology; S.L.: writing—original draft preparation; visualization, investigation; F.J.: supervision. All authors have read and agreed to the published version of the manuscript.

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The authors declare no conflict of interest.

Use of AI and AI-assisted Technologies

No AI tools were utilized for this paper.

References

1. Min, H.; Xu, S.; Zhang, Z. Adaptive finite-time stabilization of stochastic nonlinear systems subject to full-state constraints and input saturation. *IEEE Trans. Autom. Control* **2021**, *66*, 1306–1313.
2. Zhao, X.; Wang, X.; Zong, G.; et al. Fuzzy-approximation-based adaptive output-feedback control for uncertain nonsmooth nonlinear systems. *IEEE Trans. Fuzzy Syst.* **2018**, *26*, 3847–3859.
3. Xia, J.; Zhang, J.; Sun, W.; et al. Finite-time adaptive fuzzy control for nonlinear systems with full state constraints. *IEEE Trans. Syst. Man Cybern. Syst.* **2019**, *49*, 1541–1548.
4. Krstic, M.; Bement, M. Nonovershooting control of strict-feedback nonlinear systems. *IEEE Trans. Autom. Control* **2006**, *51*, 1938–1943.
5. Liang, S. Design and Implementation of Overshoot-Free Control Algorithm for Aircraft Pitch Angle. Master's Thesis, University of Electronic Science and Technology of China, Chengdu, China, 2012.
6. Chen, X.P.; Xu, H.B. Non-overshooting control for a class of nonlinear systems. *Control. Decis.* **2013**, *28*, 627–631.
7. Zhu, C.; Zhao, C.C. Non-overshooting output tracking of feedback linearisable nonlinear systems. *Int. J. Control* **2013**, *86*, 821–832.
8. Li, W.Q.; Krstic, M. Mean-nonovershooting control of stochastic nonlinear systems. *IEEE Trans. Autom. Control* **2021**, *66*, 6756–5771.
9. Song, L.; Tong, S. Finite-time mean-nonovershooting control for stochastic nonlinear systems. *Int. J. Syst. Sci.* **2023**, *54*, 1047–1055.
10. Zhan, J.Z.; Liu, Y.L. The design of a sinter furnace hearting control system for quick temperature rise without overshooting. *Ind. Instrum. Autom.* **2006**, *3*, 64–65.
11. Cui, G.Z.; Xu, S.Y.; Chen, X.k.; et al. Distributed containment control for nonlinear multiagent systems in pure-feedback form. *Int. J. Robust Nonlinear Control* **2018**, *28*, 2742–2758.
12. Liu, W.; Ma, Q.; Lu, J.W.; et al. A neural composite dynamic surface control for pure-feedback systems with unknown control gain signs and full state constraints. *Int. J. Robust Nonlinear Control* **2019**, *29*, 5720–5743.
13. Liu, W.H.; Lu, J.W.; Zhang, Z.Q.; et al. Observer-based neural control for MIMO pure-feedback non-linear systems with input saturation and disturbances. *IET Control. Theory Appl.* **2016**, *10*, 2314–2324.

14. Gao, T.T.; Liu, Y.J.; Liu, L.; et al. Adaptive Neural Network-Based Control for a Class of Nonlinear Pure-Feedback Systems With Time-Varying Full State Constraints. *IEEE-Caa J. Autom. Sin.* **2018**, *5*, 923–933.
15. Wang, C.X.; Wu, Y.Q.; Yu, J.B. Barrier Lyapunov functions-based dynamic surface control for pure-feedback systems with full state constraints. *Iet Control. Theory Appl.* **2017**, *11*, 524–530.
16. Chen, L.S.; Wang, Q. Prescribed performance-barrier Lyapunov function for the adaptive control of unknown pure-feedback systems with full-state constraints. *Nonlinear Dyn.* **2019**, *95*, 2443–2459.
17. Shi, W.J.; Dong, X.D.; Yang, F.F. Nussbaum gain adaptive neural control for stochastic pure-feedback nonlinear time-delay systems with full-state constraints. *Neurocomputing* **2018**, *292*, 130–141.
18. Liu, Y.J.; Tong, S.C. Barrier Lyapunov Functions-based adaptive control for a class of nonlinear pure-feedback systems with full state constraints. *Automatica* **2016**, *64*, 70–75.
19. Liu, W.; Ma, Q.; Zhang, G.M.; et al. An improved adaptive neural dynamic surface control for pure-feedback systems with full state constraints and disturbance. *Appl. Math. Comput.* **2019**, *358*, 37–50.
20. Zhang, T.P.; Xia, M.Z.; Yang, Y.; et al. Adaptive Neural Dynamic Surface Control of Pure-Feedback Nonlinear Systems With Full State Constraints and Dynamic Uncertainties. *IEEE Trans. Syst. Man-Cybern.-Syst.* **2017**, *47*, 2378–2387.
21. Ye, H.F.; Song, Y.D. Prescribed-time control of uncertain strict-feedback-like systems. *Int. J. Robust Nonlinear Control* **2021**, *31*, 5281–5297.
22. Song, Y.D.; Wang, Y.J.; Holloway, J.; et al. Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. *Automatica* **2017**, *83*, 243–251.
23. Song, Y.D.; Su, J. A unified Lyapunov characterization for finite time control and prescribed time control. *Int. J. Robust Nonlinear Control.* **2023**, *33*, 2930–2949.
24. Zhao, K.; Song, Y.D.; Ma, T.D.; et al. Prescribed performance control of uncertain Euler-Lagrange systems subject to full-state constraints. *IEEE Trans. Neural Netw. Learn. Syst.* **2018**, *29*, 3478–3489.
25. Orlov, Y.; Kairuz, R.I.V. Autonomous Output Feedback Stabilization With Prescribed Settling-Time Bound. *IEEE Trans. Autom. Control.* **2023**, *68*, 2452–2459.
26. Song, Y.D.; Ye, H.F.; Lewis, F.L. Prescribed-time control and its latest developments. *IEEE Trans. Syst. Man, Cybern. Syst.* **2023**, *53*, 4102–4116.
27. Wang, Y.J.; Song, Y.D. Leader-following control of high-order multi-agent systems under directed graphs. *Automatica* **2018**, *87*, 113–120.