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Article

Recursive Filtering for Nonlinear Systems with Relay Communication, Energy Harvesting and Correlated Noises

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Abstract: In this paper, the recursive filtering problem is investigated for stochastic nonlinear systems with relay communication, energy harvesting and correlated noises. The relay node receives signals transmitted from the sensor, then amplifies and forwards these signals to the filter. The transmission power of the sensor and the relay node is characterized by random variables obeying certain probability distribution. The energy harvesting technique is also employed to sustain the operation of both the sensor and the relay node, where the corresponding energy harvesting models are established to describe evolution of their energy levels. The process and the measurement noises are one-step autocorrelated and two-step cross-correlated, respectively. Furthermore, the channel noises are considered as one-step autocorrelated due to influence of the communication environment. The objective of this paper is to establish a recursive filter to address the state estimation problem in the presence of relay communication, energy harvesting, and correlated noises within a unified framework. By means of recursive computation and stochastic analysis, a certain upper bound is guaranteed on the second moment matrix of the filtering error, and then minimized by appropriately designed filter gain at each time step. In addition, the boundedness issue is also discussed to assess the filtering performance. Finally, two examples are presented to illustrate the effectiveness and practicability of the designed filtering strategy.

Keywords: nonlinear system; recursive filtering; relay communication; energy harvesting; correlated noises

1. Introduction

Over the past several decades, the filtering problem has emerged as a fundamental topic in the field of signal processing and has garnered extensive attention. The filter designed aims to estimate the unobservable system state based on the system model and measured output signal, which has found widespread application in fields such as navigation, target tracking, communication systems and automatic control [1–4]. To date, a multitude of filtering methods have been developed, catering to diverse engineering application requirements [5–9]. In particular, the H_∞ filtering algorithm aims to guarantee a certain level of disturbance attenuation described by the L_2 gain from external noises to the estimation error [10]. The Kalman filtering focuses on designing a suitable filter gain to minimize the trace of the filtering error variance matrix. In its core, the Kalman filtering is a recursive algorithm that combines the system model with the real-time measurements to continuously predict/update the estimation for the system state. The classic Kalman filter is indeed an optimal estimator when dealing with linear systems. However, due to modeling errors within the system and the external environmental disturbances, the engineering systems often exhibit nonlinear behaviors. Up to now, there has been a substantial amount of literature on the estimation problem for nonlinear systems [11, 12]. Particularly, the stochastic nonlinearity arising probably from the high maneuverability of the tracked target, the intermittent network congestion or the random failures, has garnered special attention [13–16].

Most existing filtering results implicitly assume that the sensors can transmit their measurements directly to the filter. However, this assumption is often unrealistic in engineering practice because of the restricted transmission capacity of the practical communication channels, primarily stemming from the physical or material constraints. Especially when deploying the low-cost sensors and requiring the data transmission from sensors via wireless links, the long-distance communication is often unattainable [17]. To address such signal transmission challenges, relay communication strategies have been widely adopted in industrial systems. Relays, as critical components, receive and transmit the signals from transmitter to the receiver, enhancing signal transmission and ensuring reliable communication. As a result, relay communication strategies have attracted initial research attention with some results emerging in [18–21]. Notably, the amplify-and-forward (AF) relay has attracted significant attention due to its simplicity and practicality in industrial systems. In this approach, the signal received from the source is amplified and then retransmitted [22].

To date, the filtering problem involving relays has garnered increasing research attention, with several filter design strategies being proposed in the literature [17, 23, 24]. However, the above mentioned literature does not take into account the energy supply issues of the sensors and relays. It is well known that, in scenarios where the relay communication strategies are applied, the sensors and relay nodes are often deployed in complex external environments such as the remote geographical locations. In this case, ensuring the energy required for information transmission/processing becomes a crucial issue. The energy harvesting technology, considered as an ideal solution for sustainable energy provisioning, has recently gained substantial interest from both the academic and industrial sectors [25–29]. It is worth noting that limited research methods are available to deal with the filtering problems subject to both relay communication and energy harvesting, which is one of the research motivations of this paper.

On another research front, the correlation of noises is an important yet complex phenomenon. For the target tracking problem, when both the process noise and the measurement noise depend on the system state, a correlation is present between the process and the measurement noises [30]. Specifically, when the process noise depends on the system state and the system state exhibits continuity (i.e., the system state at time z is related to those at adjacent time points), the process noise becomes autocorrelated. In addition, when the target deploys an electronic countermeasure, such as noise jamming, the measurement noises are almost never white [31–33]. Neglecting this correlation in state estimation can significantly degrade the filter performance, and even lead to divergence of the filtering error system. Therefore, to date, some research has focused on the estimation problems with correlated noises [30, 34–36]. Furthermore, it is worth mentioning that the correlation of channel noises is also taken into account in this paper, undoubtedly increasing the complexity and difficulty of the analysis/synthesis of the systems. As far as the authors are aware, the filtering problem for stochastic nonlinear systems with relay communication, energy harvesting and correlated noises remains under-explored, and the primary goal of this paper is to bridge this gap.

Summarizing the above discussions, we aim to address the recursive filtering problem for stochastic nonlinear systems with relay communication, which is a challenging task due to the following challenges: 1) How to establish appropriate energy harvesting models to describe the evolution of energy levels in the sensor and the relay node? 2) How to describe the correlations between noises and correctly handle these correlations in subsequent calculations? 3) How a filter can be recursively determined in the relay communication, ensuring the upper bound is optimized at each step? 4) How to evaluate the effectiveness of the proposed filter? The main objective of this paper is to offer comprehensive solutions to these issues, and the key contributions can be summarized as follows:

1) This paper presents initial investigations on the recursive filtering problem for stochastic systems with relay communication, energy harvesting and correlated noises.

2) A recursive filter is designed, where the upper bound of its error second moment matrix is minimized at each time step.

3) Boundedness of the filtering performance is also analyzed by mathematical induction.

4) The recursive methodology is employed in designing the filter gain, rendering it well-suited for online applications.

The structure of this paper is as follows. Section 2 introduces the stochastic nonlinear system model, and presents the recursive filtering problem to be addressed. In Section 3, an appropriate filter gain is designed to minimize the obtained upper bound at each sampling instant. In Section 4, the boundedness performance is analyzed for the obtained upper bound. Section 5 provides two examples to illustrate the effectiveness and practicability of the proposed filtering algorithm. Conclusions are drawn in Section 6.

Notations: The notations employed in this paper are standard. The identity matrix is denoted by I with appropriate dimensions. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively. P^{-1} and P^T represent the inverse and the transpose of matrix P, respectively. $\mathbb{E}\{x\}$ denotes the expectation of a stochastic variable x, and $\mathrm{tr}(\cdot)$ is the trace of some matrix. $\delta_{i,j}$ is the Kronecker delta function, which is equal to one for $i=j$ and zero otherwise. For a symmetric matrix Q, $Q>0$ and $Q\geq 0$ mean that Q is positive definite and positive semidefinite, respectively. For nonnegative integers a and b with $a\leqslant b$, $[a\ b]$

represents a finite set $\{a, a+1, \cdots, b-1, b\}$.

2. Problem Formulation and Some Preliminaries

Consider the following time-varying stochastic system:

$$\begin{cases} x_{z+1} = A_z x_z + B_z \omega_z + f(x_z, \xi_z), \\ y_z = C_z x_z + D_z v_z, \end{cases} \tag{1}$$

where the time $z \in \mathbb{N} := \{0, 1, 2, \cdots\}$; $x_z \in \mathbb{R}^n$ is the system state with initial value x_0 being a random vector that has a mean $\bar{x}_0$ and a variance P_0, $y_z \in \mathbb{R}^m$ is the measurement output. $\omega_z \in \mathbb{R}^{n_\omega}$ and $v_z \in \mathbb{R}^{m_v}$ stand for the process noise and the measurement noise, respectively, which are characterized by the following statistical properties:

$$\mathbb{E}\{\omega_s\} = 0, \qquad \mathbb{E}\{v_s\} = 0, \tag{2a}$$

$$\mathbb{E}\{\omega_s \omega_t^T\} = Q_s \delta_{s,t} + Q_{s,t} \delta_{s,t-1} + Q_{s,t} \delta_{s,t+1}, \tag{2b}$$

$$\mathbb{E}\{v_s v_t^T\} = R_s \delta_{s,t} + R_{s,t} \delta_{s,t-1} + R_{s,t} \delta_{s,t+1}, \tag{2c}$$

$$\mathbb{E}\{\omega_s v_t^T\} = S_s \delta_{s,t} + S_{s,t} \delta_{s,t-1} + S_{s,t} \delta_{s,t-2}, \tag{2d}$$

where $s, t \in \mathbb{N}$; Q_s, $Q_{s,t}$, R_s, $R_{s,t}$, S_s and $S_{s,t}$ are known correlation matrices. Matrices A_z, B_z, C_z and D_z are known with appropriate dimensions.

The stochastic nonlinear function $f(x_z, \xi_z)$ in (1) has the following properties:

$$f(0, \xi_z) = 0, \tag{3a}$$

$$\mathbb{E}\{f(x_z, \xi_z) | x_z\} = 0, \tag{3b}$$

$$\mathbb{E}\{f(x_z, \xi_z) f^T(x_j, \xi_j) | x_z\} = \sum_{i=1}^{o} \Pi_i x_z^T \Gamma_i x_z \delta_{z,j}, \qquad j \geqslant z, \tag{3c}$$

where ξ_z is a zero mean stochastic noise sequence with variance $\sigma^2 I$, o is a given positive integer, Π_i and Γ_i are known positive semidefinite matrices.

Remark 1. From (2a)–(2d), it can be observed that the process noise and the measurement noise considered in this paper exhibit the characteristics of one-step autocorrelation and two-step cross-correlation, respectively. More specifically, ω_z is correlated with ω_{z-1} and ω_{z+1}, v_z is correlated with v_{z-1} and v_{z+1}, v_z is correlated with ω_z, ω_{z-1} and ω_{z-2}. As both the system and the filter to be designed inherently involve recursions, such correlations persist and propagate throughout the entire recursive process, which will greatly complicate the analysis and synthesis of the addressed system, as manifested in the subsequent design of the filter gains. Notably, when the correlation matrices $Q_{s,t}$, $R_{s,t}$, S_s and $S_{s,t}$ are all zero, the noise sequences considered here degenerate to the standard uncorrelated case, which has been extensively studied in the literature.

The energy-constrained sensor broadcasts the measurement to the remote filter via an energy-constrained relay. Let s_z represent the measurement received by the relay node, which is expressed as follows:

$$s_z = \gamma_{1,z} \sqrt{p_{1,z}} C_{1,z} y_z + v_{1,z}, \tag{4}$$

where $\gamma_{1,z}$ is determined by

$$\gamma_{1,z} = \begin{cases} 1, & h_{1,z} > 0, \\ 0, & h_{1,z} = 0, \end{cases} \tag{5}$$

in which $h_{1,z} \in \mathbb{N} := \{0, 1, 2, \cdots\}$ denoting the sensor's energy level at time z. Assume $\bar{S}_1$ is the sensor's maximum storage capacity, and dynamics of the sensor energy are given by

$$h_{1,z+1} = \min\{h_{1,z} + \overleftarrow{z}_{1,z} - \gamma_{1,z}, \bar{S}_1\} \tag{6}$$

with $h_{1,0} \leqslant \bar{S}_1$. $\overleftarrow{z}_{1,z}$ is the energy harvested by the sensor at time z, which is a random variable with the following distribution:

$$\mathrm{P}(\overleftarrow{z}_{1,z}=m)=\bar{z}_1^{(m)},\quad m=0,1,2,\cdots \tag{7}$$

where ($\mathrm{P}(\cdot)$) means the probability operator, $0\leqslant\bar{z}_1^{(m)}\leqslant 1$ is a known scalar and $\sum_{m=0}^{+\infty}\bar{z}_1^{(m)}=1$. $p_{1,z}$ is a random variable representing the transmission power from the sensor to the relay, and following the distribution:

$$\mathrm{P}(p_{1,z}=p_{1,z}^{(i)})=\bar{p}_{1,z}^{(i)},\quad i=0,1,\cdots,u_1, \tag{8}$$

where $p_{1,z}^{(i)}\geqslant 0$, $0\leqslant\bar{p}_{1,z}^{(i)}\leqslant 1$ are known scalars and $\sum_{i=0}^{u_1}\bar{p}_{1,z}^{(i)}=1$. $C_{1,z}$ is a known matrix. $v_{1,z}$ is the sensor-to-relay channel noise with zero mean, and has the following property:

$$\mathbb{E}\{v_{1,s}v_{1,t}^T\}=T_s^{(1)}\delta_{s,t}+T_{s,t}^{(1)}\delta_{s,t-1}+T_{s,t}^{(1)}\delta_{s,t+1}, \tag{9}$$

where the correlation matrices $T_s^{(1)}$ and $T_{s,t}^{(1)}$ are known.

After receiving the measurement, the relay node serves to amplify and forward the signal to the remote filter via wireless channel. The measurement received by the filter is given by

$$r_z=a_z\gamma_{2,z}\sqrt{p_{2,z}}C_{2,z}s_z+v_{2,z}, \tag{10}$$

where a_z is the amplification factor, $\gamma_{2,z}$ is defined by

$$\gamma_{2,z}=\begin{cases}1, & h_{2,z}>0\\ 0, & h_{2,z}=0\end{cases} \tag{11}$$

with $h_{2,z}\in\mathbb{N}$ being the energy level of the relay, and the energy evolution concerning the relay can be characterized as follows:

$$h_{2,z+1}=\min\{h_{2,z}+\overleftarrow{z}_{2,z}-\gamma_{2,z},\bar{S}_2\}, \tag{12}$$

in which $\bar{S}_2$ is the maximum energy unit that the relay can store, the initial condition $h_{2,0}\leqslant\bar{S}_2$. $\overleftarrow{z}_{2,z}$ is the energy harvested by the relay node at time z, and the probability distribution of $\overleftarrow{z}_{2,z}$ is given as follows:

$$\mathrm{P}(\overleftarrow{z}_{2,z}=m)=\bar{z}_2^{(m)},\quad m=0,1,2,\cdots \tag{13}$$

where $0\leqslant\bar{z}_2^{(m)}\leqslant 1$ is a known scalar and $\sum_{m=0}^{+\infty}\bar{z}_2^{(m)}=1$. $p_{2,z}$ is the transmission power from the relay to the filter, which has the following distribution:

$$\mathrm{P}(p_{2,z}=p_{2,z}^{(i)})=\bar{p}_{2,z}^{(i)},\quad i=0,1,\cdots,u_2, \tag{14}$$

where $p_{2,z}^{(i)}\geqslant 0$, $0\leqslant\bar{p}_{2,z}^{(i)}\leqslant 1$ are known scalars with $\sum_{i=0}^{u_2}\bar{p}_{2,z}^{(i)}=1$. $C_{2,z}$ is a known matrix. $v_{2,z}$ is the channel noise with zero mean, and it has the following property:

$$\mathbb{E}\{v_{2,s}v_{2,t}^T\}=T_s^{(2)}\delta_{s,t}+T_{s,t}^{(2)}\delta_{s,t-1}+T_{s,t}^{(2)}\delta_{s,t+1}, \tag{15}$$

in which $T_s^{(2)}$ and $T_{s,t}^{(2)}$ are known correlation matrices.

Remark 2. In many engineering applications, the signals cannot be transmitted directly from sensors to the signal center because of the long communication distance. Thus, an AF relay is usually employed to tackle this communication challenge. The transmission powers of the sensor and the relay are considered to be random, characterized by random variables with certain probability distributions. Furthermore, in application of the relay strategy, sensors and the relay nodes are frequently deployed in the remote location. Therefore, it is imperative to have effective schemes for energy collection. The estimation problem to be addressed fully considers the energy harvesting and relay communication schemes, thus has substantial practical significance. On the other hand, the channel noises $v_{1,z}$ and $v_{2,z}$ are both characterized to be autocorrelated, which constitute a significant distinction from the relay channel model presented in [23], not to mention that the energy harvesting strategies are simultaneously considered here.

Assumption 1. The random variable x_0 and the random sequences ξ_z, $p_{1,z}$, $p_{2,z}$, $\overleftarrow{z}_{1,z}$, $\overleftarrow{z}_{2,z}$, $v_{1,z}$, and $v_{2,z}$ are mutually uncorrelated. Furthermore, the noise sequences ω_z and v_z are uncorrelated with the aforementioned random variables.

Assumption 2. The random sequences $p_{1,z}, p_{2,z}, \overleftarrow{z}_{1,z}, \overleftarrow{z}_{2,z}$ are white.

This paper aims to design a recursive filter for system (1) given by

$$\begin{cases}\hat{x}_{z+1|z}=A_z\hat{x}_{z|z},\\ \hat{x}_{z+1|z+1}=\hat{x}_{z+1|z}+K_{z+1}(r_{z+1}-E_{z+1}\hat{x}_{z+1|z}),\end{cases} \tag{16}$$

where

$$E_{z+1} := a_{z+1}\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1}C_{2,z+1}C_{1,z+1}C_{z+1},$$

$\hat{x}_{z+1|z}$ is the one-step prediction of state x_{z+1}, $\hat{x}_{z+1|z+1}$ is an updated estimate of state x_{z+1} with $\hat{x}_{0|0} = \bar{x}_0$, K_{z+1} is the filter gain to be designed; in which

$$\bar{p}_{1,z+1} := \mathbb{E}\{\sqrt{p_{1,z+1}}\} = \sum_{i=0}^{u_1} \sqrt{p^{(i)}_{1,z+1}\bar{p}^{(i)}_{1,z+1}},$$

$$\bar{p}_{2,z+1} := \mathbb{E}\{\sqrt{p_{2,z+1}}\} = \sum_{i=0}^{u_2} \sqrt{p^{(i)}_{2,z+1}\bar{p}^{(i)}_{2,z+1}};$$

$\bar{\gamma}_{1,z+1}$ and $\bar{\gamma}_{2,z+1}$ are the expectation of $\gamma_{1,z+1}$ and $\gamma_{2,z+1}$ respectively, that will be obtained in subsequent calculations.

Denote the prediction error as $\tilde{x}_{z+1|z} := x_{z+1} - \hat{x}_{z+1|z}$ and the filtering error as $\tilde{x}_{z+1|z+1} := x_{z+1} - \hat{x}_{z+1|z+1}$. Then, considering (1), (4), (10) and (16), we have

$$\tilde{x}_{z+1|z} = A_z\tilde{x}_{z|z} + B_z\omega_z + f(x_z,\xi_z), \tag{17a}$$

$$\begin{aligned}\tilde{x}_{z+1|z+1} =&(I - K_{z+1}E_{z+1})\tilde{x}_{z+1|z} - a_{z+1}\left(\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}} - \bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1}\right)\\ &\times K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}x_{z+1} - a_{z+1}\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}}K_{z+1}C_{2,z+1}C_{1,z+1}\\ &\times D_{z+1}v_{z+1} - a_{z+1}\gamma_{2,z+1}\sqrt{p_{2,z+1}}K_{z+1}C_{2,z+1}v_{1,z+1} - K_{z+1}v_{2,z+1}.\end{aligned} \tag{17b}$$

The objective of this paper is mainly to establish a unified framework for addressing the state estimation problem for the stochastic nonlinear system (1), considering the correlated noises under the simultaneous relay communication and energy harvesting schemes.

3. Main Result

This section focuses on solving the estimation problem for system (1) with relay communication, correlated noises and energy harvesting mechanisms. Firstly, the upper bounds for the second moment matrices of the filtering errors are provided. Then, such bounds are minimized in the sense of trace by appropriately designing the filter gains.

To arrive at our main results, the following lemmas are introduced.

Lemma 1. Let a and b be vectors of compatible dimensions. For any positive scalar ϵ, the following inequality holds:

$$ab^T + ba^T \leqslant \epsilon aa^T + \epsilon^{-1}bb^T.$$

Lemma 2. Denote the probability distribution laws of the energy levels $h_{1,z}$ and $h_{2,z}$ by $\bar{h}_{1,z} := [\bar{h}^{(0)}_{1,z}\ \bar{h}^{(1)}_{1,z}\ \cdots\ \bar{h}^{(\bar{S}_1)}_{1,z}]^T$ and $\bar{h}_{2,z} := [\bar{h}^{(0)}_{2,z}\ \bar{h}^{(1)}_{2,z}\ \cdots\ \bar{h}^{(\bar{S}_2)}_{2,z}]^T$ respectively, where $\bar{h}^{(i)}_{j,z} = \mathrm{P}(h_{j,z} = i)$ for each $i = 0,1,\cdots,S_j$ $(j = 1,2)$. Then, the probability distributions of random variables $\gamma_{1,z+1}$ and $\gamma_{2,z+1}$ are provided as follows:

$$\mathrm{P}(\gamma_{1,z+1} = 1) = \bar{\gamma}_{1,z+1}, \quad \mathrm{P}(\gamma_{2,z+1} = 1) = \bar{\gamma}_{2,z+1}, \tag{18}$$

where

$$\bar{\gamma}_{1,z+1} = 1 - R_1\bar{h}_{1,z+1}, \quad R_1 = [1\underbrace{0\cdots0}_{\bar{S}_1}], \quad \bar{\gamma}_{2,z+1} = 1 - R_2\bar{h}_{2,z+1}, \quad R_2 = [1\underbrace{0\cdots0}_{\bar{S}_2}],$$
$$\bar{h}_{1,z+1} = b_1 + G_1\bar{h}_{1,z}, \quad b_1 = [\underbrace{0\cdots0}_{\bar{S}_1}1]^T, \quad \bar{h}_{2,z+1} = b_2 + G_2\bar{h}_{2,z}, \quad b_2 = [\underbrace{0\cdots0}_{\bar{S}_2}1]^T,$$

$$G_1 = \begin{bmatrix} \bar{z}^{(0)}_1 & \bar{z}^{(0)}_1 & 0 & \cdots & 0 \\ \bar{z}^{(1)}_1 & \bar{z}^{(1)}_1 & \bar{z}^{(0)}_1 & \cdots & 0 \\ \bar{z}^{(2)}_1 & \bar{z}^{(2)}_1 & \bar{z}^{(1)}_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{z}^{(\bar{S}_1-1)}_1 & \bar{z}^{(\bar{S}_1-1)}_1 & \bar{z}^{(\bar{S}_1-2)}_1 & \cdots & \bar{z}^{(0)}_1 \\ -\sum_{i=0}^{\bar{S}_1-1}\bar{z}^{(i)}_1 & -\sum_{i=0}^{\bar{S}_1-1}\bar{z}^{(i)}_1 & -\sum_{i=1}^{\bar{S}_1-1}\bar{z}^{(i-1)}_1 & \cdots & -\bar{z}^{(0)}_1 \end{bmatrix},$$

$$G_2=\begin{bmatrix} \bar{z}_2^{(0)} & \bar{z}_2^{(0)} & 0 & \cdots & 0 \\ \bar{z}_2^{(1)} & \bar{z}_2^{(1)} & \bar{z}_2^{(0)} & \cdots & 0 \\ \bar{z}_2^{(2)} & \bar{z}_2^{(2)} & \bar{z}_2^{(1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{z}_2^{(\bar{S}_2-1)} & \bar{z}_2^{(\bar{S}_2-1)} & \bar{z}_2^{(\bar{S}_2-2)} & \cdots & \bar{z}_2^{(0)} \\ -\sum_{i=0}^{\bar{S}_2-1}\bar{z}_2^{(i)} & -\sum_{i=0}^{\bar{S}_2-1}\bar{z}_2^{(i)} & -\sum_{i=1}^{\bar{S}_2-1}\bar{z}_2^{(i-1)} & \cdots & -\bar{z}_2^{(0)} \end{bmatrix}.$$

Proof. Without loss of generality, set $j=1$. It is straightforward to observe that the random variable $h_{1,z+1}$ is independent of $\overleftarrow{z}_{1,z+1}$, and hence, for each $i=0,1,\cdots,\bar{S}_1-1$, we have

$$\begin{aligned}\bar{h}_{1,z+1}^{(i)}=&\mathrm{P}(h_{1,z+1}=i)=\mathrm{P}(\min\{h_{1,z}+\overleftarrow{z}_{1,z}-\gamma_{1,z},\bar{S}_1\}=i)\\ =&\mathrm{P}(h_{1,z}+\overleftarrow{z}_{1,z}-\gamma_{1,z}=i)\\ =&\mathrm{P}(h_{1,z}=0,\overleftarrow{z}_{1,z}=i)+\sum_{l=1}^{i+1}\mathrm{P}(h_{1,z}=l,\overleftarrow{z}_{1,z}=i+1-l)\\ =&\mathrm{P}(h_{1,z}=0)\mathrm{P}(\overleftarrow{z}_{1,z}=i)+\sum_{l=1}^{i+1}\mathrm{P}(h_{1,z}=l)\mathrm{P}(\overleftarrow{z}_{1,z}=i+1-l)\\ =&\bar{h}_{1,z}^{(0)}\bar{z}_1^{(i)}+\sum_{l=1}^{i+1}\bar{h}_{1,z}^{(l)}\bar{z}_1^{(i+1-l)}.\end{aligned}\tag{19}$$

When $i=\bar{S}_1$, $\bar{h}_{1,z+1}^{(\bar{S}_1)}$ can be given by

$$\begin{aligned}\bar{h}_{1,z+1}^{(\bar{S}_1)}=&1-\sum_{i=0}^{\bar{S}_1-1}\bar{h}_{1,z+1}^{(i)}\\ =&1-\bar{h}_{1,z}^{(0)}\sum_{i=0}^{\bar{S}_1-1}\bar{z}_1^{(i)}-\sum_{i=0}^{\bar{S}_1-1}\sum_{l=1}^{i+1}\bar{h}_{1,z}^{(l)}\bar{z}_1^{(i+1-l)}\\ =&1-\bar{h}_{1,z}^{(0)}\sum_{i=0}^{\bar{S}_1-1}\bar{z}_1^{(i)}-\sum_{l=1}^{\bar{S}_1}\bar{h}_{1,z}^{(l)}\sum_{i=l-1}^{\bar{S}_1-1}\bar{z}_1^{(i+1-l)}.\end{aligned}\tag{20}$$

Considering (19) and (20), the recursive form of $\bar{h}_{1,z+1}$ can be obtained as follow:

$$\bar{h}_{1,z+1}=b_1+G_1\bar{h}_{1,z}.\tag{21}$$

From (5) and (19), the probability distribution of $\gamma_{1,z}$ can be easily obtained. The same method can be used to access the probability distribution of $\gamma_{2,z}$, and the proof is now complete. □

Lemma 3. For the random variables $\gamma_{1,z}$ and $\gamma_{2,z}$ defined respectively in (5) and (11), the following relationships hold:

$$\mathrm{P}(\gamma_{1,z+1}\gamma_{1,z}=0)=\bar{q}_{1,z},\qquad \mathrm{P}(\gamma_{2,z+1}\gamma_{2,z}=0)=\bar{q}_{2,z},\tag{22}$$

where

$$\bar{q}_{1,z}=\bar{h}_{1,z}^{(0)}+\bar{h}_{1,z}^{(1)}\bar{z}_1^{(0)},\qquad \bar{q}_{2,z}=\bar{h}_{2,z}^{(0)}+\bar{h}_{2,z}^{(1)}\bar{z}_2^{(0)}.$$

Proof. It follows from (5)–(7) that

$$\begin{aligned}\mathrm{P}(\gamma_{1,z+1}\gamma_{1,z}=0)=&\mathrm{P}(\gamma_{1,z+1}\gamma_{1,z}=0,\gamma_{1,z}=0)+\mathrm{P}(\gamma_{1,z+1}\gamma_{1,z}=0,\gamma_{1,z}=1)\\ =&\mathrm{P}(\gamma_{1,z}=0)+\mathrm{P}(\gamma_{1,z+1}=0,\gamma_{1,z}=1)\\ =&\mathrm{P}(h_{1,z}=0)+\mathrm{P}(h_{1,z}+\overleftarrow{z}_{1,z}-1=0,h_{1,z}>0)\\ =&\mathrm{P}(h_{1,z}=0)+\mathrm{P}(h_{1,z}=1,\overleftarrow{z}_{1,z}=0)\\ =&\bar{h}_{1,z}^{(0)}+\bar{h}_{1,z}^{(1)}\bar{z}_1^{(0)},\end{aligned}\tag{23}$$

from which and (5), it is directly obtained that

$$\mathrm{P}(\gamma_{1,z+1}\gamma_{1,z}=1)=1-\mathrm{P}(\gamma_{1,z+1}\gamma_{1,z}=0)=1-\bar{q}_{1,z}. \tag{24}$$

It follows from (23) and (24) that the first equality in (22) holds. Subsequently, the other equality in (22) can also be established through similar calculations, thereby completing the proof of this lemma. □

Lemma 4. The second moment matrix of the state is denoted as $X_{z+1}:=\mathbb{E}\{x_{z+1}x_{z+1}^T\}$, which satisfies

$$X_{z+1}=A_zX_zA_z^T+B_zQ_zB_z^T+\sum_{i=1}^{o}\Pi_i\mathrm{tr}(X_z\Gamma_i)+A_zB_{z-1}Q_{z-1,z}B_z^T+B_zQ_{z-1,z}^TB_{z-1}^TA_z^T \tag{25}$$

with initial value $X_0=P_0+\bar{x}_0\bar{x}_0^T$.

Proof. From (1), one has

$$\begin{aligned}X_{z+1}=&A_z\mathbb{E}\{x_zx_z^T\}A_z^T+B_z\mathbb{E}\{\omega_z\omega_z^T\}B_z^T+\mathbb{E}\{f(x_z,\xi_z)f^T(x_z,\xi_z)\}+A_z\mathbb{E}\{x_z\omega_z^T\}B_z^T\\&+B_z\mathbb{E}\{\omega_zx_z^T\}A_z^T+A_z\mathbb{E}\{x_zf^T(x_z,\xi_z)\}+\mathbb{E}\{f(x_z,\xi_z)x_z^T\}A_z^T+B_z\mathbb{E}\{\omega_zf^T(x_z,\xi_z)\}\\&+\mathbb{E}\{f(x_z,\xi_z)\omega_z^T\}B_z^T.\end{aligned}$$

Considering (3b) and the fact that the random variables ω_z and ξ_z are independent, we have

$$\mathbb{E}\{f(x_z,\xi_z)\omega_z^T\}=0. \tag{26}$$

According to the property of the conditional expectation, it results from (3b)–(3c) that

$$\mathbb{E}\{x_zf^T(x_z,\xi_z)\}=\mathbb{E}\{\mathbb{E}\{x_zf^T(x_z,\xi_z)|x_z\}\}=\mathbb{E}\{x_z\mathbb{E}\{f^T(x_z,\xi_z)|x_z\}\}=0 \tag{27}$$

and

$$\begin{aligned}\mathbb{E}\{f(x_z,\xi_z)f^T(x_z,\xi_z)\}=&\mathbb{E}\{\mathbb{E}\{f(x_z,\xi_z)f^T(x_z,\xi_z)|x_z\}\}\\=&\sum_{i=1}^{o}\Pi_i\mathbb{E}\{x_z^T\Gamma_ix_z\}=\sum_{i=1}^{o}\Pi_i\mathrm{tr}(X_z\Gamma_i).\end{aligned} \tag{28}$$

Noting (1) and (2a)–(2b), the term $\mathbb{E}\{x_z\omega_z^T\}$ can be calculated as follows:

$$\mathbb{E}\{x_z\omega_z^T\}=\mathbb{E}\{[A_{z-1}x_{z-1}+B_{z-1}\omega_{z-1}+f(x_{z-1},\xi_{z-1})]\omega_z^T\}=B_{z-1}Q_{z-1,z}. \tag{29}$$

Then, (25) can be concluded. □

Lemma 5. For system (1), the recursive form of the second moment matrix for the one-step prediction error denoted by $P_{z+1|z}:=\mathbb{E}\{\tilde{x}_{z+1|z}\tilde{x}_{z+1|z}^T\}$ is given by

$$P_{z+1|z}=A_zP_{z|z}A_z^T+B_zQ_zB_z^T+\sum_{i=1}^{o}\Pi_i\mathrm{tr}(X_z\Gamma_i)+A_zM_zB_z^T+B_zM_z^TA_z^T, \tag{30}$$

where $P_{z|z}:=\mathbb{E}\{\tilde{x}_{z|z}\tilde{x}_{z|z}^T\}$ is the second moment matrix of the filtering error $\tilde{x}_{z|z}$, and

$$M_z=\left(I-a_z\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z}K_zC_{2,z}C_{1,z}C_z\right)B_{z-1}Q_{z-1,z}-a_z\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z}K_zC_{2,z}C_{1,z}D_zS_z^T.$$

Proof. Since variable ξ_z is uncorrelated with $\tilde{x}_{z|z}$, one has from (3b) that

$$\begin{aligned}\mathbb{E}\{\tilde{x}_{z|z}f^T(x_z,\xi_z)\}=&\mathbb{E}\{\mathbb{E}\{\tilde{x}_{z|z}f^T(x_z,\xi_z)|\tilde{x}_{z|z}\}\}\\=&\mathbb{E}\{\tilde{x}_{z|z}\mathbb{E}\{f^T(x_z,\xi_z)|\tilde{x}_{z|z}\}\}\\=&\mathbb{E}\{\tilde{x}_{z|z}\mathbb{E}\{f^T(x_z,\xi_z)|x_z\}\}=0.\end{aligned} \tag{31}$$

Then, from the expression of $\tilde{x}_{z+1|z}$, we have

$$\begin{aligned}P_{z+1|z}=&A_z\mathbb{E}\{\tilde{x}_{z|z}\tilde{x}_{z|z}^T\}A_z^T+B_z\mathbb{E}\{\omega_z\omega_z^T\}B_z^T+\mathbb{E}\{f(x_z,\xi_z)f^T(x_z,\xi_z)\}+A_z\mathbb{E}\{\tilde{x}_{z|z}\omega_z^T\}B_z^T\\&+B_z\mathbb{E}\{\omega_z\tilde{x}_{z|z}^T\}A_z^T+A_z\mathbb{E}\{\tilde{x}_{z|z}f^T(x_z,\xi_z)\}+\mathbb{E}\{f(x_z,\xi_z)\tilde{x}_{z|z}^T\}A_z^T+B_z\mathbb{E}\{\omega_zf^T(x_z,\xi_z)\}\\&+\mathbb{E}\{f(x_z,\xi_z)\omega_z^T\}B_z^T.\end{aligned} \tag{32}$$

Based on (2b), (2d) and Assumption 1, the following relationship for term $\mathbb{E}\{\tilde{x}_{z|z}\omega_z^T\}$ can be established:

$$
\begin{aligned}
\mathbb{E}\{\tilde{x}_{z|z}\omega_z^T\} =&\mathbb{E}\Big\{\big[(I-a_z\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z}K_zC_{2,z}C_{1,z}C_z)\big(A_{z-1}\tilde{x}_{z-1|z-1}+B_{z-1}\omega_{z-1}\\
&+f(x_{z-1},\xi_{z-1})\big)-a_z(\gamma_{1,z}\gamma_{2,z}\sqrt{p_{1,z}p_{2,z}}-\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z})K_zC_{2,z}C_{1,z}C_zx_z\\
&-a_z\gamma_{1,z}\gamma_{2,z}\sqrt{p_{1,z}p_{2,z}}K_zC_{2,z}C_{1,z}D_zv_z-a_z\gamma_{2,z}\sqrt{p_{2,z}}K_zC_{2,z}v_{1,z}-K_zv_{2,z}\big]\omega_z^T\Big\}\\
=&(I-a_z\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z}K_zC_{2,z}C_{1,z}C_z)B_{z-1}\mathbb{E}\{\omega_{z-1}\omega_z^T\}\\
&-a_z\mathbb{E}\{(\gamma_{1,z}\gamma_{2,z}\sqrt{p_{1,z}p_{2,z}}-\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z})K_zC_{2,z}C_{1,z}C_zx_z\omega_z^T\}\\
&-a_z\mathbb{E}\{\gamma_{1,z}\gamma_{2,z}\sqrt{p_{1,z}p_{2,z}}K_zC_{2,z}C_{1,z}D_zv_z\omega_z^T\}\\
=&(I-a_z\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z}K_zC_{2,z}C_{1,z}C_z)B_{z-1}Q_{z-1,z}\\
&-a_z\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z}K_zC_{2,z}C_{1,z}D_zS_z^T.
\end{aligned}\tag{33}
$$

It follows from (26), (28) and (31)–(33) that (30) holds, which completes the proof of this lemma. □

Theorem 1. Let ϵ_1 be a given positive scalar and X_{z+1} satisfy (25). Considering system (1) with recursive filter (16), the second moment matrices $P_{z+1|z}$ and $P_{z+1|z}$ are bounded by

$$
P_{z+1|z}\leqslant\vec{P}_{z+1|z},\qquad P_{z+1|z+1}\leqslant\vec{P}_{z+1|z+1},\tag{34}
$$

where matrix sequences $\vec{P}_{z+1|z}$ and $\vec{P}_{z+1|z+1}$ with initial value $\vec{P}_{0|0}=P_0$ are derived as solutions to the following recursive equations:

$$
\vec{P}_{z+1|z}=A_z\vec{P}_{z|z}A_z^T+B_zQ_zB_z^T+\sum_{i=1}^{o}\Pi_i\mathrm{tr}(X_z\Gamma_i)+A_zM_zB_z^T+B_zM_z^TA_z^T,\tag{35a}
$$

$$
\begin{aligned}
\vec{P}_{z+1|z+1}=&(1+\epsilon_1)(I-K_{z+1}E_{z+1})\vec{P}_{z+1|z}(I-K_{z+1}E_{z+1})^T+K_{z+1}L_{z+1}K_{z+1}^T\\
&-(I-K_{z+1}E_{z+1})H_{z+1}K_{z+1}^T-K_{z+1}H_{z+1}^T(I-K_{z+1}E_{z+1})^T,
\end{aligned}\tag{35b}
$$

in which

$$
\begin{aligned}
H_{z+1}=&a_{z+1}\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1}\big[A_z(I-K_zE_z)B_{z-1}S_{z-1,z+1}+B_zS_{z,z+1}\big]D_{z+1}^TC_{1,z+1}^TC_{2,z+1}^T\\
&-a_{z+1}a_z(1-\bar{q}_{2,z})\bar{p}_{2,z+1}\bar{p}_{2,z}A_zK_zC_{2,z}T_{z,z+1}^{(1)}C_{2,z+1}^T-A_zK_zT_{z,z+1}^{(2)}\\
&-a_{z+1}a_z\big[(1-\bar{q}_{1,z})(1-\bar{q}_{2,z})\bar{p}_{1,z+1}\bar{p}_{2,z+1}\bar{p}_{1,z}\bar{p}_{2,z}\\
&-\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1}\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z}\big]A_zK_zC_{2,z}C_{1,z}C_zB_{z-1}S_{z-1,z+1}\\
&\times D_{z+1}^TC_{1,z+1}^TC_{2,z+1}^T-a_{z+1}a_z(1-\bar{q}_{1,z})(1-\bar{q}_{2,z})\bar{p}_{1,z+1}\bar{p}_{2,z+1}\bar{p}_{1,z}\bar{p}_{2,z}\\
&\times A_zK_zC_{2,z}C_{1,z}D_zR_{z,z+1}D_{z+1}^TC_{1,z+1}^TC_{2,z+1}^T,
\end{aligned}\tag{36}
$$

$$
\begin{aligned}
L_{z+1}=&(1+\epsilon_1^{-1})a_{z+1}^2(\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}-\bar{\gamma}_{1,z+1}^2\bar{\gamma}_{2,z+1}^2\bar{p}_{1,z+1}^2\bar{p}_{2,z+1}^2)C_{2,z+1}C_{1,z+1}\\
&\times C_{z+1}X_{z+1}C_{z+1}^TC_{1,z+1}^TC_{2,z+1}^T+a_{z+1}^2\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}C_{2,z+1}C_{1,z+1}D_{z+1}\\
&\times R_{z+1}D_{z+1}^TC_{1,z+1}^TC_{2,z+1}^T+a_{z+1}^2\bar{\gamma}_{2,z+1}\breve{p}_{2,z+1}C_{2,z+1}T_{z+1}^{(1)}C_{2,z+1}^T+T_{z+1}^{(2)}\\
&+a_{z+1}^2(\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}-\bar{\gamma}_{1,z+1}^2\bar{\gamma}_{2,z+1}^2\bar{p}_{1,z+1}^2\bar{p}_{2,z+1}^2)C_{2,z+1}C_{1,z+1}\\
&\times\big[C_{z+1}(A_zB_{z-1}S_{z-1,z+1}+B_zS_{z,z+1})D_{z+1}^T\\
&+D_{z+1}(S_{z,z+1}^TB_z^T+S_{z-1,z+1}^TB_{z-1}^TA_z^T)C_{z+1}^T\big]C_{1,z+1}^TC_{2,z+1}^T,
\end{aligned}\tag{37}
$$

$$
\breve{p}_{1,z}=\sum_{i=0}^{u_1}p_{1,z}^{(i)}\bar{p}_{1,z}^{(i)},\qquad \breve{p}_{2,z}=\sum_{i=0}^{u_2}p_{2,z}^{(i)}\bar{p}_{2,z}^{(i)}.\tag{38}
$$

Furthermore, the trace of the upper bound $\vec{P}_{z+1|z+1}$ is minimized when the filter gain is designed as follows:

$$
\begin{aligned}
K_{z+1}=&[(1+\epsilon_1)\vec{P}_{z+1|z}E_{z+1}^T+H_{z+1}]\big((1+\epsilon_1)E_{z+1}\vec{P}_{z+1|z}E_{z+1|z}^T\\
&+L_{z+1}+E_{z+1}H_{z+1}+H_{z+1}^TE_{z+1}^T\big)^{-1}.
\end{aligned}\tag{39}
$$

Proof. By using the definition of $P_{z+1|z+1}$ and the expression of $\tilde{x}_{z+1}$, one has

$$
\begin{aligned}
P_{z+1|z+1} = & \mathbb{E}\{(I-K_{z+1}E_{z+1})\tilde{x}_{z+1|z}\tilde{x}_{z+1|z}^T(I-K_{z+1}E_{z+1})^T\} + \mathbb{E}\{K_{z+1}v_{2,z+1}v_{2,z+1}^T K_{z+1}^T\} \\
& + \mathcal{Z}_{1,z+1} + \mathcal{Z}_{2,z+1} + \mathcal{Z}_{3,z+1} - \mathcal{Z}_{4,z+1} - \mathcal{Z}_{4,z+1}^T - \mathcal{Z}_{5,z+1} - \mathcal{Z}_{5,z+1}^T - \mathcal{Z}_{6,z+1} \\
& - \mathcal{Z}_{6,z+1}^T - \mathcal{Z}_{7,z+1} - \mathcal{Z}_{7,z+1}^T + \mathcal{Z}_{8,z+1} + \mathcal{Z}_{8,z+1}^T + \mathcal{Z}_{9,z+1} + \mathcal{Z}_{9,z+1}^T + \mathcal{Z}_{10,z+1} \\
& + \mathcal{Z}_{10,z+1}^T + \mathcal{Z}_{11,z+1} + \mathcal{Z}_{11,z+1}^T + \mathcal{Z}_{12,z+1} + \mathcal{Z}_{12,z+1}^T + \mathcal{Z}_{13,z+1} + \mathcal{Z}_{13,z+1}^T,
\end{aligned} \tag{40}
$$

where

$$
\begin{aligned}
\mathcal{Z}_{1,z+1} = & \mathbb{E}\{a_{z+1}^2(\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}} - \bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1})^2 \\
& \times K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}x_{z+1}x_{z+1}^T C_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{2,z+1} = & \mathbb{E}\{a_{z+1}^2\gamma_{1,z+1}^2\gamma_{2,z+1}^2 p_{1,z+1}p_{2,z+1}K_{z+1}C_{2,z+1}C_{1,z+1}D_{z+1}v_{z+1}v_{z+1}^T \\
& \times D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{3,z+1} = & \mathbb{E}\{a_{z+1}^2\gamma_{2,z+1}^2 p_{2,z+1}K_{z+1}C_{2,z+1}v_{1,z+1}v_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{4,z+1} = & \mathbb{E}\{a_{z+1}(\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}} - \bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1})(I-K_{z+1}E_{z+1}) \\
& \times \tilde{x}_{z+1|z}x_{z+1}^T C_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{5,z+1} = & \mathbb{E}\{a_{z+1}\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}}(I-K_{z+1}E_{z+1})\tilde{x}_{z+1|z}v_{z+1}^T D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{6,z+1} = & \mathbb{E}\{a_{z+1}\gamma_{2,z+1}\sqrt{p_{2,z+1}}(I-K_{z+1}E_{z+1})\tilde{x}_{z+1|z}v_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{7,z+1} = & \mathbb{E}\{(I-K_{z+1}E_{z+1})\tilde{x}_{z+1|z}v_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{8,z+1} = & \mathbb{E}\{a_{z+1}^2(\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}} - \bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1})\gamma_{1,z+1}\gamma_{2,z+1} \\
& \times \sqrt{p_{1,z+1}p_{2,z+1}}K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}x_{z+1}v_{z+1}^T D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{9,z+1} = & \mathbb{E}\{a_{z+1}^2(\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}} - \bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1}) \\
& \times \gamma_{2,z+1}\sqrt{p_{2,z+1}}K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}x_{z+1}v_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{10,z+1} = & \mathbb{E}\{a_{z+1}(\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}} - \bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1}) \\
& \times K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}x_{z+1}v_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{11,z+1} = & \mathbb{E}\{a_{z+1}^2\gamma_{1,z+1}\gamma_{2,z+1}^2\sqrt{p_{1,z+1}}p_{2,z+1}K_{z+1}C_{2,z+1}C_{1,z+1}D_{z+1}v_{z+1}v_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{12,z+1} = & \mathbb{E}\{a_{z+1}\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}}K_{z+1}C_{2,z+1}C_{1,z+1}D_{z+1}v_{z+1}v_{2,z+1}^T K_{z+1}^T\}, \\
\mathcal{Z}_{13,z+1} = & \mathbb{E}\{a_{z+1}\gamma_{2,z+1}\sqrt{p_{2,z+1}}K_{z+1}C_{2,z+1}v_{1,z+1}v_{2,z+1}^T K_{z+1}^T\}.
\end{aligned}
$$

Observing that the terms $\mathcal{Z}_{9,z+1}$, $\mathcal{Z}_{10,z+1}$, $\mathcal{Z}_{11,z+1}$, $\mathcal{Z}_{12,z+1}$ and $\mathcal{Z}_{13,z+1}$ in (40) are equal to zero, matrix $P_{z+1|z+1}$ can be calculated as follows:

$$
\begin{aligned}
P_{z+1|z+1} = & (I-K_{z+1}E_{z+1})P_{z+1|z}(I-K_{z+1}E_{z+1})^T + K_{z+1}T_{z+1}^{(2)}K_{z+1}^T + \mathcal{Z}_{1,z+1} + \mathcal{Z}_{2,z+1} \\
& + \mathcal{Z}_{3,z+1} - \mathcal{Z}_{4,z+1} - \mathcal{Z}_{4,z+1}^T - \mathcal{Z}_{5,z+1} - \mathcal{Z}_{5,z+1}^T - \mathcal{Z}_{6,z+1} - \mathcal{Z}_{6,z+1}^T - \mathcal{Z}_{7,z+1} \\
& - \mathcal{Z}_{7,z+1}^T + \mathcal{Z}_{8,z+1} + \mathcal{Z}_{8,z+1}^T.
\end{aligned} \tag{41}
$$

By utilizing the properties of the expectation operator, we can readily derive from Assumption 1 and Lemma 2 that

$$
\begin{aligned}
\mathcal{Z}_{1,z+1} = & a_{z+1}^2\mathbb{E}\{(\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}} - \bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1})^2\} \\
& \times K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}X_{z+1}C_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T \\
= & a_{z+1}^2(\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1} - \bar{\gamma}_{1,z+1}^2\bar{\gamma}_{2,z+1}^2\bar{p}_{1,z+1}^2\bar{p}_{2,z+1}^2) \\
& \times K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}X_{z+1}C_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T,
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\mathcal{Z}_{2,z+1} = & a_{z+1}^2\mathbb{E}\{\gamma_{1,z+1}^2\gamma_{2,z+1}^2 p_{1,z+1}p_{2,z+1}\}K_{z+1}C_{2,z+1}C_{1,z+1}D_{z+1}R_{z+1}D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T \\
= & a_{z+1}^2\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}K_{z+1}C_{2,z+1}C_{1,z+1}D_{z+1}R_{z+1}D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T,
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\mathcal{Z}_{3,z+1} = & a_{z+1}^2\mathbb{E}\{\gamma_{2,z+1}^2 p_{2,z+1}\}K_{z+1}C_{2,z+1}T_{z+1}^{(1)}C_{2,z+1}^T K_{z+1}^T \\
= & a_{z+1}^2\bar{\gamma}_{2,z+1}\breve{p}_{2,z+1}K_{z+1}C_{2,z+1}T_{z+1}^{(1)}C_{2,z+1}^T K_{z+1}^T.
\end{aligned} \tag{44}
$$

Now, we are ready to calculate the cross-terms in (41). By Assumptions 1–2, it follows from (1), (17a), (17b) and Lemmas 2–3 that

$$\begin{aligned}\mathcal{Z}_{5,z+1} =& \mathbb{E}\{a_{z+1}\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}}(I-K_{z+1}E_{z+1})A_z(I-K_zE_z)B_{z-1}\\&\times\omega_{z-1}v_{z+1}^TD_{z+1}^TC_{1,z+1}^TC_{2,z+1}^TK_{z+1}^T\}-\mathbb{E}\{a_{z+1}a_z\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}}\\&\times(I-K_{z+1}E_{z+1})(\gamma_{1,z}\gamma_{2,z}\sqrt{p_{1,z}p_{2,z}}-\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z})A_zK_zC_{2,z}C_{1,z}C_zx_zv_{z+1}^T\\&\times D_{z+1}^TC_{1,z+1}^TC_{2,z+1}^TK_{z+1}^T\}-\mathbb{E}\{a_{z+1}a_z\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}}\\&\times(I-K_{z+1}E_{z+1})\gamma_{1,z}\gamma_{2,z}\sqrt{p_{1,z}p_{2,z}}A_zK_zC_{2,z}C_{1,z}D_zv_zv_{z+1}^TD_{z+1}^TC_{1,z+1}^TC_{2,z+1}^TK_{z+1}^T\}\\&+\mathbb{E}\{a_{z+1}\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}}(I-K_{z+1}E_{z+1})B_z\omega_zv_{z+1}^TD_{z+1}^TC_{1,z+1}^TC_{2,z+1}^TK_{z+1}^T\}\\=&(I-K_{z+1}E_{z+1})\Big(a_{z+1}\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1}\big[A_z(I-K_zE_z)B_{z-1}S_{z-1,z+1}\\&+B_zS_{z,z+1}\big]D_{z+1}^TC_{1,z+1}^TC_{2,z+1}^T-a_{z+1}a_z\big[(1-\bar{q}_{1,z})(1-\bar{q}_{2,z})\bar{p}_{1,z+1}\bar{p}_{2,z+1}\bar{p}_{1,z}\bar{p}_{2,z}\\&-\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1}\bar{\gamma}_{1,z}\bar{\gamma}_{2,z}\bar{p}_{1,z}\bar{p}_{2,z}\big]A_zK_zC_{2,z}C_{1,z}C_zB_{z-1}S_{z-1,z+1}D_{z+1}^TC_{1,z+1}^T\\&\times C_{2,z+1}^T-a_{z+1}a_z(1-\bar{q}_{1,z})(1-\bar{q}_{2,z})\bar{p}_{1,z+1}\bar{p}_{2,z+1}\bar{p}_{1,z}\bar{p}_{2,z}A_zK_zC_{2,z}C_{1,z}D_zR_{z,z+1}\\&\times D_{z+1}^TC_{1,z+1}^TC_{2,z+1}^T\Big)K_{z+1}^T,\end{aligned}\tag{45}$$

$$\begin{aligned}\mathcal{Z}_{6,z+1} =& -\mathbb{E}\{a_{z+1}\gamma_{2,z+1}\sqrt{p_{2,z+1}}(I-K_{z+1}E_{z+1})A_za_z\gamma_{2,z}\sqrt{p_{2,z}}K_zC_{2,z}v_{1,z}v_{1,z+1}^TC_{2,z+1}^TK_{z+1}^T\}\\=&(I-K_{z+1}E_{z+1})\big(-a_{z+1}a_z(1-\bar{q}_{2,z})\bar{p}_{2,z+1}\bar{p}_{2,z}A_zK_zC_{2,z}T_{z,z+1}^{(1)}C_{2,z+1}^T\big)K_{z+1}^T,\end{aligned}\tag{46}$$

$$\begin{aligned}\mathcal{Z}_{7,z+1} =& -\mathbb{E}\{(I-K_{z+1}E_{z+1})A_zK_zv_{2,z}v_{2,z+1}^TK_{z+1}^T\}\\=&(I-K_{z+1}E_{z+1})\big(-A_zK_zT_{z,z+1}^{(2)}\big)K_{z+1}^T,\end{aligned}\tag{47}$$

$$\begin{aligned}\mathcal{Z}_{8,z+1} =& a_{z+1}^2\mathbb{E}\big\{(\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}}-\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1})\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}}\\&\times\sqrt{p_{2,z+1}}\big\}K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}\mathbb{E}\{x_{z+1}v_{z+1}^T\}D_{z+1}^TC_{1,z+1}^TC_{2,z+1}^TK_{z+1}^T\\=&a_{z+1}^2(\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\check{p}_{1,z+1}\check{p}_{2,z+1}-\bar{\gamma}_{1,z+1}^2\bar{\gamma}_{2,z+1}^2\bar{p}_{1,z+1}^2\bar{p}_{2,z+1}^2)K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}\\&\times(A_zB_{z-1}S_{z-1,z+1}+B_zS_{z,z+1})D_{z+1}^TC_{1,z+1}^TC_{2,z+1}^TK_{z+1}^T.\end{aligned}\tag{48}$$

On the other hand, due to the simultaneous presence of stochastic nonlinearity defined in (3a)–(3c) and the autocorrelation of random sequences ω_z, $\gamma_{1,z}$ and $\gamma_{2,z}$, the exact value of term $-\mathcal{Z}_{4,z+1}-\mathcal{Z}_{4,z+1}^T$ cannot be determined. Therefore, a feasible choice is to obtain an upper bound for such a term. It follows from the definition of $\mathcal{Z}_{4,z+1}$ and Lemma 1 that

$$\begin{aligned}&-\mathcal{Z}_{4,z+1}-\mathcal{Z}_{4,z+1}^T\\\leqslant&\epsilon_1\mathbb{E}\{(I-K_{z+1}E_{z+1})\tilde{x}_{z+1|z}\tilde{x}_{z+1|z}^T(I-K_{z+1}E_{z+1})^T\}+\epsilon_1^{-1}\mathbb{E}\{a_{z+1}^2(\gamma_{1,z+1}\gamma_{2,z+1}\\&\times\sqrt{p_{1,z+1}p_{2,z+1}}-\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1})^2K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}x_{z+1}x_{z+1}^T\\&\times C_{z+1}^TC_{1,z+1}^TC_{2,z+1}^TK_{z+1}^T\}\\=&\epsilon_1(I-K_{z+1}E_{z+1})P_{z+1|z}(I-K_{z+1}E_{z+1})^T+\epsilon_1^{-1}a_{z+1}^2(\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\check{p}_{1,z+1}\check{p}_{2,z+1}\\&-\bar{\gamma}_{1,z+1}^2\bar{\gamma}_{2,z+1}^2\bar{p}_{1,z+1}^2\bar{p}_{2,z+1}^2)K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}X_{z+1}C_{z+1}^TC_{1,z+1}^TC_{2,z+1}^TK_{z+1}^T.\end{aligned}\tag{49}$$

Substituting (42)–(49) into (41), it can be obtained directly that

$$\begin{aligned}P_{z+1|z+1}\leqslant&(1+\epsilon_1)(I-K_{z+1}E_{z+1})P_{z+1|z}(I-K_{z+1}E_{z+1})^T+K_{z+1}L_{z+1}K_{z+1}^T\\&-(I-K_{z+1}E_{z+1})H_{z+1}K_{z+1}^T-K_{z+1}H_{z+1}^T(I-K_{z+1}E_{z+1})^T\end{aligned}\tag{50}$$

where L_{z+1} and H_{z+1} are defined in (36) and (37).

Next, we will prove that $\vec{P}_{z+1|z+1}$ is an upper bound for $P_{z+1|z+1}$ by mathematical induction. Firstly, we have $\vec{P}_{0|0}=P_{0|0}$ from the initial conditions. Then, assume that $P_{z|z}\leqslant\vec{P}_{z|z}$ holds for $z=z_0\geqslant 0$. From (30) and (35a), the following relationship can be obtained:

$$P_{z_0+1|z_0}\leqslant A_{z_0}\vec{P}_{z_0|z_0}A_{z_0}^T+B_{z_0}Q_{z_0}B_{z_0}^T+\sum_{i=1}^{o}\Pi_i\mathrm{tr}(X_{z_0}\Gamma_i)+A_{z_0}M_{z_0}B_{z_0}^T+B_{z_0}M_{z_0}^TA_{z_0}^T=\vec{P}_{z_0+1|z_0},$$

from which, together with (35b) and (50), one has

$$P_{z_0+1|z_0+1}-\vec{P}_{z_0+1|z_0+1}\leqslant(1+\epsilon_1)(I-K_{z_0+1}E_{z_0+1})(P_{z_0+1|z_0}-\vec{P}_{z_0+1|z_0})(I-K_{z_0+1}E_{z_0+1})^T\leqslant 0.$$

Therefore, the inequality $P_{z+1|z+1}\leqslant\vec{P}_{z+1|z+1}$ holds for all $z\in\mathbb{N}$.

Finally, we proceed to design an appropriate filter gain K_{z+1} which minimizes such an upper bound $\vec{P}_{z+1|z+1}$.

Taking the partial derivative of $\mathrm{tr}(\vec{P}_{z+1|z+1})$ with respect to K_{z+1}, we have

$$\begin{aligned}\frac{\partial \mathrm{tr}(\vec{P}_{z+1|z+1})}{\partial K_{z+1}} =& -2(1+\epsilon_1)(I-K_{z+1}E_{z+1})\vec{P}_{z+1|z}E_{z+1}^T+2K_{z+1}L_{z+1}\\ &-2H_{z+1}+2K_{z+1}E_{z+1}H_{z+1}+2K_{z+1}H_{z+1}^TE_{z+1}^T.\end{aligned} \tag{51}$$

Let (51) be zero and then the filter gain K_{z+1} is given as (39) shown. The proof of this theorem is complete. □

Remark 3. Based on Theorem 1, we have derived a novel solution to the filtering problem for the stochastic nonlinear system (1). The system considered in this paper is comprehensive, encompassing various phenomena, namely, relay communication, energy harvesting, stochastic nonlinearity, autocorrelated noises and cross-correlated noises. These phenomena, which are commonly encountered in engineering applications, have been successfully addressed within a unified framework. Moreover, all these phenomena are reflected in the main result of this study. Specifically, $\bar{p}_{1,z}$ and $\bar{p}_{2,z}$ describe the average power levels of the transmission power in the sensor-to-relay communication channel and the relay-to-filter communication channel respectively. Constants $\bar{\gamma}_{1,z}$ and $\bar{\gamma}_{2,z}$ account for the effects of energy harvesting on the communication capabilities. Matrices Π_i and Γ_i (for $i=1,2,\cdots,o$) quantify the impact of stochastic nonlinearity. Matrices $Q_s, Q_{s,t}, R_s, R_{s,t}, T_s^{(1)}, T_{s,t}^{(1)}, T_s^{(2)}$ and $T_{s,t}^{(2)}$ reflect the autocorrelated noises, while S_s and $S_{s,t}$ represent the cross-correlated phenomenon of the noises.

4. Performance Analysis

After deriving the locally minimal upper bound for the second moment matrix $P_{z+1|z+1}$ of the filtering error at each time step, it is valuable to evaluate the performance of the designed filter. This section further analyzes the bound with respect to $P_{z+1|z+1}$. To proceed, the following assumption is presented.

Assumption 3. Positive scalars $\bar{a},\bar{b},\bar{m},\bar{\pi}_i,\bar{\gamma}_i$ $(i\in[1\ \ o])$ and $\bar{q}$ exist such that the following relationships are satisfied for all $z\in\mathbb{N}$:

$$\begin{aligned}&\mathrm{tr}(A_zA_z^T)\leqslant\bar{a}, \quad \mathrm{tr}(B_zB_z^T)\leqslant\bar{b}, \quad \mathrm{tr}(X_z)\leqslant\bar{m},\\ &\mathrm{tr}(\Pi_i)\leqslant\bar{\pi}_i, \quad\ \ \mathrm{tr}(\Gamma_i)\leqslant\bar{\gamma}_i, \quad\ \ \mathrm{tr}(Q_z)\leqslant\bar{q}.\end{aligned}$$

Theorem 2. Let ϵ_i $(i=2,3,4,5,6)$ denote a set of positive scalars with ϵ_1 being given in Theorem 1. Consider the time-varying system (1) with filter (16), in which the gain parameter K_{z+1} is presented in (39). It follows from Assumption 3 that matrix $\vec{P}_{z+1|z+1}$ is bounded by

$$\vec{P}_{z+1|z+1}\leqslant\eta_{z+1}I \tag{52}$$

with initial constraint $\vec{P}_{0|0}\leqslant\eta_0 I$, where

$$\begin{aligned}&\eta_{z+1}=\lambda_{z+1}\eta_0+\sum_{i=0}^{z}\lambda_i\beta_2, \qquad \lambda_z=\beta_1\lambda_{z-1}, \quad \lambda_0=1,\\ &\beta_1=(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)(1+\epsilon_2)\bar{a},\\ &\beta_2=(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)\left[(1+\epsilon_2^{-1})\bar{b}\bar{q}+\bar{m}\sum_{i=1}^{o}\bar{\pi}_i\bar{\gamma}_i\right].\end{aligned}$$

Proof. By applying Lemma 1 and Assumption 3, it can be obtained from (33), (34), (35a) and the definition of M_z that

$$\begin{aligned}\vec{P}_{z+1|z}\leqslant& A_z\vec{P}_{z|z}A_z^T+B_zQ_zB_z^T+\sum_{i=1}^{o}\Pi_i\mathrm{tr}(X_z\Gamma_i)+\epsilon_2A_z\mathbb{E}\{\tilde{x}_{z|z}\tilde{x}_{z|z}^T\}A_z^T+\epsilon_2^{-1}B_z\mathbb{E}\{\omega_z\omega_z^T\}B_z^T\\ \leqslant&(1+\epsilon_2)A_z\vec{P}_{z|z}A_z^T+(1+\epsilon_2^{-1})B_zQ_zB_z^T+\sum_{i=1}^{o}\Pi_i\mathrm{tr}(X_z\Gamma_i)\\ :=&\Sigma_{z+1|z}.\end{aligned} \tag{53}$$

For the positive scalars ϵ_i $(i=3,...,6)$, one has directly from Lemma 1, (34) and (40) that

$$
\begin{aligned}
-\mathcal{Z}_{5,z+1}-\mathcal{Z}_{5,z+1}^T \leqslant & \epsilon_3\mathbb{E}\{(I-K_{z+1}E_{z+1})\tilde{x}_{z+1|z}\tilde{x}_{z+1|z}^T(I-K_{z+1}E_{z+1})^T\} \\
& +\epsilon_3^{-1}\mathbb{E}\{a_{z+1}^2\gamma_{1,z+1}^2\gamma_{2,z+1}^2 p_{1,z+1}p_{2,z+1}K_{z+1}C_{2,z+1}C_{1,z+1} \\
& \times D_{z+1}v_{z+1}v_{z+1}^T D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\} \\
\leqslant & \epsilon_3(I-K_{z+1}E_{z+1})\vec{P}_{z+1|z}(I-K_{z+1}E_{z+1})^T \\
& +\epsilon_3^{-1}a_{z+1}^2\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}K_{z+1}C_{2,z+1}C_{1,z+1} \\
& \times D_{z+1}R_{z+1}D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T,
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
-\mathcal{Z}_{6,z+1}-\mathcal{Z}_{6,z+1}^T \leqslant & \epsilon_4\mathbb{E}\{(I-K_{z+1}E_{z+1})\tilde{x}_{z+1|z}\tilde{x}_{z+1|z}^T(I-K_{z+1}E_{z+1})^T\} \\
& +\epsilon_4^{-1}\mathbb{E}\{a_{z+1}^2\gamma_{2,z+1}^2 p_{2,z+1}K_{z+1}C_{2,z+1}v_{1,z+1}v_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\} \\
\leqslant & \epsilon_4(I-K_{z+1}E_{z+1})\vec{P}_{z+1|z}(I-K_{z+1}E_{z+1})^T \\
& +\epsilon_4^{-1}a_{z+1}^2\bar{\gamma}_{2,z+1}\breve{p}_{2,z+1}K_{z+1}C_{2,z+1}T_{z+1}^{(1)}C_{2,z+1}^T K_{z+1}^T,
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
-\mathcal{Z}_{7,z+1}-\mathcal{Z}_{7,z+1}^T \leqslant & \epsilon_5\mathbb{E}\{(I-K_{z+1}E_{z+1})\tilde{x}_{z+1|z}\tilde{x}_{z+1|z}^T(I-K_{z+1}E_{z+1})^T\} \\
& +\epsilon_5^{-1}\mathbb{E}\{K_{z+1}v_{2,z+1}v_{2,z+1}^T K_{z+1}^T\} \\
\leqslant & \epsilon_5(I-K_{z+1}E_{z+1})\vec{P}_{z+1|z}(I-K_{z+1}E_{z+1})^T+\epsilon_5^{-1}K_{z+1}T_{z+1}^{(2)}K_{z+1}^T,
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
\mathcal{Z}_{8,z+1}+\mathcal{Z}_{8,z+1}^T \leqslant & \epsilon_6\mathbb{E}\{a_{z+1}^2(\gamma_{1,z+1}\gamma_{2,z+1}\sqrt{p_{1,z+1}p_{2,z+1}}-\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\bar{p}_{1,z+1}\bar{p}_{2,z+1})^2K_{z+1}C_{2,z+1} \\
& \times C_{1,z+1}C_{z+1}x_{z+1}x_{z+1}^T C_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\}+\epsilon_6^{-1}\mathbb{E}\{a_{z+1}^2\gamma_{1,z+1}^2\gamma_{2,z+1}^2 \\
& \times p_{1,z+1}p_{2,z+1}K_{z+1}C_{2,z+1}C_{1,z+1}D_{z+1}v_{z+1}v_{z+1}^T D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T\} \\
= & \epsilon_6 a_{z+1}^2(\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}-\bar{\gamma}_{1,z+1}^2\bar{\gamma}_{2,z+1}^2\bar{p}_{1,z+1}^2\bar{p}_{2,z+1}^2) \\
& \times K_{z+1}C_{2,z+1}C_{1,z+1}C_{z+1}X_{z+1}C_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T \\
& +\epsilon_6^{-1}a_{z+1}^2\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}K_{z+1}C_{2,z+1}C_{1,z+1}D_{z+1} \\
& \times R_{z+1}D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T.
\end{aligned} \tag{57}
$$

Considering (54)–(56) and the definition of H_{z+1}, we have

$$
\begin{aligned}
& -(I-K_{z+1}E_{z+1})H_{z+1}K_{z+1}^T-K_{z+1}H_{z+1}^T(I-K_{z+1}E_{z+1})^T \\
\leqslant & (\epsilon_3+\epsilon_4+\epsilon_5)(I-K_{z+1}E_{z+1})\vec{P}_{z+1|z}(I-K_{z+1}E_{z+1})^T \\
& +\epsilon_3^{-1}a_{z+1}^2\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}K_{z+1}C_{2,z+1}C_{1,z+1}D_{z+1}R_{z+1}D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T K_{z+1}^T \\
& +\epsilon_4^{-1}a_{z+1}^2\bar{\gamma}_{2,z+1}\breve{p}_{2,z+1}K_{z+1}C_{2,z+1}T_{z+1}^{(1)}C_{2,z+1}^T K_{z+1}^T+\epsilon_5^{-1}K_{z+1}T_{z+1}^{(2)}K_{z+1}^T,
\end{aligned}
$$

from which and (35b), (37) as well as (57), one obtains that

$$
\begin{aligned}
\vec{P}_{z+1|z+1} \leqslant & (1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)(I-K_{z+1}E_{z+1})\vec{P}_{z+1|z}(I-K_{z+1}E_{z+1})^T+K_{z+1}F_{z+1}K_{z+1}^T \\
:= & \Sigma_{z+1|z+1}
\end{aligned} \tag{58}
$$

holds for any matrix K_{z+1} with compatible dimensions, where

$$
\begin{aligned}
F_{z+1}= & (1+\epsilon_1^{-1}+\epsilon_6)a_{z+1}^2(\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}-\bar{\gamma}_{1,z+1}^2\bar{\gamma}_{2,z+1}^2\bar{p}_{1,z+1}^2\bar{p}_{2,z+1}^2) \\
& \times C_{2,z+1}C_{1,z+1}C_{z+1}X_{z+1}C_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T+(1+\epsilon_3^{-1}+\epsilon_6^{-1}) \\
& \times a_{z+1}^2\bar{\gamma}_{1,z+1}\bar{\gamma}_{2,z+1}\breve{p}_{1,z+1}\breve{p}_{2,z+1}C_{2,z+1}C_{1,z+1}D_{z+1}R_{z+1}D_{z+1}^T C_{1,z+1}^T C_{2,z+1}^T \\
& +(1+\epsilon_4^{-1})a_{z+1}^2\bar{\gamma}_{2,z+1}\breve{p}_{2,z+1}C_{2,z+1}T_{z+1}^{(1)}C_{2,z+1}^T+(1+\epsilon_5^{-1})T_{z+1}^{(2)}.
\end{aligned}
$$

View $\vec{P}_{z+1|z+1}$ and $\Sigma_{z+1|z+1}$ as functions with respect to K_{z+1}, which have the following expressions:

$$
\begin{aligned}
\vec{P}_{z+1|z+1}(K_{z+1}) = & (1+\epsilon_1)(I-K_{z+1}E_{z+1})\vec{P}_{z+1|z}(I-K_{z+1}E_{z+1})^T+K_{z+1}L_{z+1}K_{z+1}^T \\
& -(I-K_{z+1}E_{z+1})H_{z+1}K_{z+1}^T-K_{z+1}H_{z+1}^T(I-K_{z+1}E_{z+1})^T, \\
\Sigma_{z+1|z+1}(K_{z+1}) = & (1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)(I-K_{z+1}E_{z+1})\vec{P}_{z+1|z}(I-K_{z+1}E_{z+1})^T \\
& +K_{z+1}F_{z+1}K_{z+1}^T.
\end{aligned}
$$

Then, define matrices $K_{z+1}^{(1)}$ and $K_{z+1}^{(2)}$ as follows:

$$\begin{aligned}K^{(1)}_{z+1} =&\left[(1+\epsilon_1)\vec{P}_{z+1|z}E^T_{z+1}+H_{z+1}\right]\left((1+\epsilon_1)E_{z+1}\vec{P}_{z+1|z}E^T_{z+1|z}+L_{z+1}\right.\\&\left.+E_{z+1}H_{z+1}+H^T_{z+1}E^T_{z+1}\right)^{-1},\\K^{(2)}_{z+1} =&(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)\vec{P}_{z+1|z}E^T_{z+1}\left((1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)E_{z+1}\vec{P}_{z+1|z}E^T_{z+1}+F_{z+1}\right)^{-1}.\end{aligned}$$

In this case, $\Sigma_{z+1|z+1}(K^{(2)}_{z+1})$ can be expressed as

$$\begin{aligned}\Sigma_{z+1|z+1}(K^{(2)}_{z+1}) =&(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)\vec{P}_{z+1|z}-(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)^2\vec{P}_{z+1|z}E^T_{z+1}\\&\times\left((1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)E_{z+1}\vec{P}_{z+1|z}E^T_{z+1}+F_{z+1}\right)^{-1}E_{z+1}\vec{P}_{z+1|z}.\end{aligned}\tag{59}$$

Based on the theory of matrices, when the gain parameter K_{z+1} is presented in (39), one has from (53), (58) and (59) that

$$\begin{aligned}\vec{P}_{z+1|z+1}\leqslant&\mathrm{tr}(\vec{P}_{z+1|z+1}(K^{(1)}_{z+1}))I\\\leqslant&\mathrm{tr}(\vec{P}_{z+1|z+1}(K^{(2)}_{z+1}))I\leqslant\mathrm{tr}(\Sigma_{z+1|z+1}(K^{(2)}_{z+1}))I\\\leqslant&(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)\mathrm{tr}(\Sigma_{z+1|z})I.\end{aligned}\tag{60}$$

In the subsequent discussion, we establish the proof for assertion (52) by induction. The initial step is dedicated to demonstrating the validity of (52) for $z=1$. By setting $z=1$, we derive from Assumption 3 and (60) that

$$\begin{aligned}\vec{P}_{1|1}\leqslant&(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)\mathrm{tr}(\Sigma_{1|0})I\\\leqslant&(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)\left[(1+\epsilon_2)\bar{a}\eta_0+(1+\epsilon_2^{-1})\bar{b}\bar{q}+\bar{m}\sum_{i=1}^{o}\bar{\pi}_i\bar{\gamma}_i\right]I\\=&\beta_1\eta_0I+\beta_2I=\eta_1I.\end{aligned}$$

Furthermore, we assume that $\vec{P}_{z|z}\leqslant\eta_zI$ holds for all $z\in\{0,1,2,...,z_0\}$. The identical assertion can be deduced for $z=z_0+1$ as follows:

$$\begin{aligned}\vec{P}_{z_0+1|z_0+1}\leqslant&(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)\mathrm{tr}(\Sigma_{z_0+1|z_0})I\\\leqslant&(1+\epsilon_1+\epsilon_3+\epsilon_4+\epsilon_5)\left[(1+\epsilon_2)\bar{a}\eta_{z_0}+(1+\epsilon_2^{-1})\bar{b}\bar{q}+\bar{m}\sum_{i=1}^{o}\bar{\pi}_i\bar{\gamma}_i\right]I\\=&\beta_1\eta_{z_0}I+\beta_2I=\beta_1(\lambda_{z_0}\eta_0+\sum_{i=0}^{z_0-1}\lambda_i\beta_2)I+\beta_2I\\=&\lambda_{z_0+1}\eta_0I+\sum_{i=0}^{z_0}\lambda_i\beta_2I=\eta_{z_0+1}I.\end{aligned}$$

Based on the inductive method, the proof of this theorem is completed. □

5. Numerical Examples

In this section, two simulation examples are provided to validate the effectiveness and practicability of the designed filtering algorithm.

Example 1. Considering the following discrete time-varying system:

$$\begin{cases}x_{z+1}=A_zx_z+B_z\omega_z+f(x_z,\xi_z),\\y_z=C_zx_z+D_zv_z,\\\omega_z=\overleftarrow{u}_{z+2}+\overleftarrow{u}_{z+1},\\v_z=\overleftarrow{u}_{z+1}+\overleftarrow{u}_z,\end{cases}$$

where $x_z=[x^{(1)}_zx^{(2)}_zx^{(3)}_z]^T$, $\overleftarrow{u}_z=[\overleftarrow{u}^{(1)}_z\ \ \overleftarrow{u}^{(2)}_z]^T$ with $\overleftarrow{u}^{(1)}_z$ and $\overleftarrow{u}^{(2)}_z$ being 5×10^{-1} and 4×10^{-1} times of the unit Gaussian white noise respectively,

$$A_z=\begin{bmatrix}0.95+\sin(z)&0&-0.5\sin(z)\\0.21&0.97+\sin(z)&-0.5\sin(z)\\0.22&0&0.97+\sin(2z)\end{bmatrix},\quad B_z=\begin{bmatrix}0&0.3\\0.4&0\\0.5&0\end{bmatrix},$$

$$C_z=\begin{bmatrix}0.65&-0.5&-0.2\\-0.2&0.4&1+\cos(z)\end{bmatrix},\quad D_z=\begin{bmatrix}0&1\\0&1+\cos(z)\end{bmatrix}.$$

The stochastic nonlinear function $f(x_z,\xi_z)$ has the following expression:

$$f(x_z,\xi_z)=[0.1\ \ 0.15\ \ 0.2]^T[0.1\mathrm{sign}(x_z^{(1)})x_z^{(1)}\xi_z^{(1)}+0.2\mathrm{sign}(x_z^{(2)})x_z^{(2)}\xi_z^{(2)}+0.3\mathrm{sign}(x_z^{(3)})x_z^{(3)}\xi_z^{(3)}]$$

where ξ_z is a zero mean Gaussian white noise with variance I, and $\xi_z^{(i)}$ $(i=1,2,3)$ represents the i-th element of ξ_z. Moreover, the initial conditions are chosen as $\bar{x}_0=[1\ 0.5\ 0.1]^T$ and $P_0=4I$.

For the communication channel, we set the amplification factor as $a_z=1$, and the channel parameters are configured as follows:

$$C_{1,z}=\begin{bmatrix}0.35 & 0\\ 0 & 0.5\end{bmatrix},\quad C_{2,z}=\begin{bmatrix}0.6 & 0\\ 0 & 0.4\end{bmatrix}.$$

The probability distributions of the transmission power $p_{1,z},p_{2,z}$ are described as follows:

$$\begin{aligned}&\mathrm{P}(p_{1,z}=1)=0.1,\quad \mathrm{P}(p_{1,z}=1.5)=0.6,\quad \mathrm{P}(p_{1,z}=2)=0.3,\\ &\mathrm{P}(p_{2,z}=1)=0.2,\quad \mathrm{P}(p_{2,z}=1.5)=0.5,\quad \mathrm{P}(p_{2,z}=2)=0.3.\end{aligned}$$

Variances of the channel noises $v_{1,z}$ and $v_{2,z}$ are selected as

$$T_s^{(1)}=T_{s,t}^{(1)}=\begin{bmatrix}0.03 & 0\\ 0 & 0.04\end{bmatrix},\quad T_s^{(2)}=T_{s,t}^{(2)}=\begin{bmatrix}0.03 & 0\\ 0 & 0.04\end{bmatrix}$$

for all $s,t\in\mathbb{N}$.

The probability distributions of energy harvesting for the sensor and relay node are given as follows:

$$\mathrm{P}(\overleftarrow{z}_{j,z}=0)=0.2,\quad \mathrm{P}(\overleftarrow{z}_{j,z}=1)=0.6,\quad \mathrm{P}(\overleftarrow{z}_{j,z}=2)=0.2\ \ (j=1,2).$$

The other parameters are selected as $h_{1,0}=h_{2,0}=1,\bar{S}_1=\bar{S}_2=3$ and $\epsilon_1=0.5$.

The corresponding simulation results are shown in Figures 1–6. Specifically, Figure 1 shows the evolution of the energy stored in the sensor and the relay node. Figures 2–4 present the trajectories of the system state $x_z^{(i)}$ and its estimation $\hat{x}_{z|z}^{(i)}$. It can be seen from Figures 2–4 that the filtering algorithm proposed in this paper performs quite well.

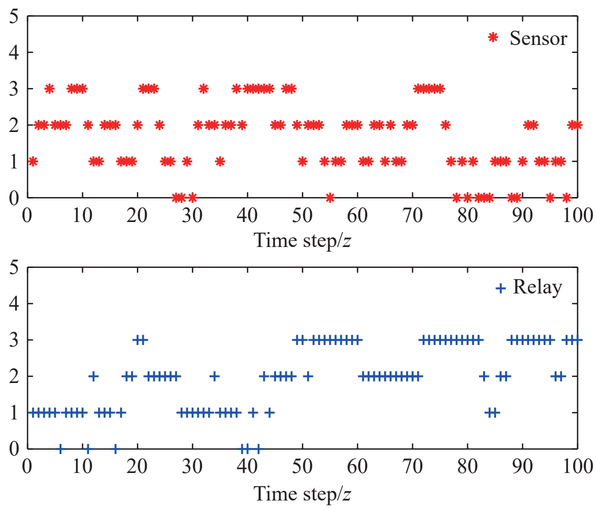

Figure 1. Evolution of the energy levels of the sensor and the relay node.

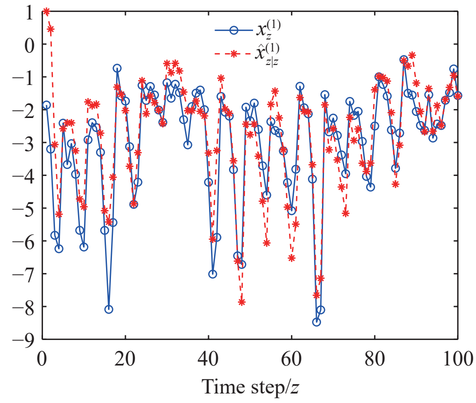

Figure 2. Evolution of $x_z^{(1)}$ and its estimation $\hat{x}_{z|z}^{(1)}$.

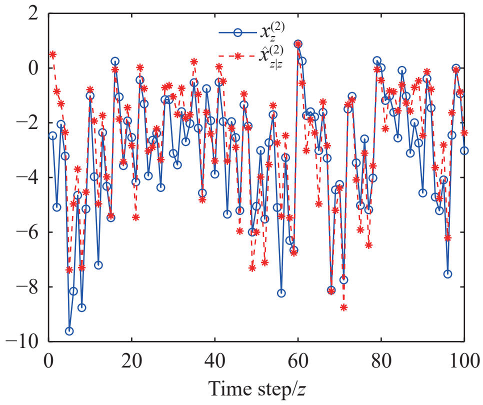

Figure 3. Evolution of $x_z^{(2)}$ and its estimation $\hat{x}_{z|z}^{(2)}$.

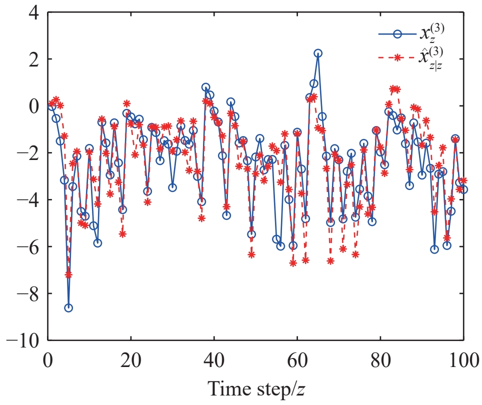

Figure 4. Evolution of $x_z^{(3)}$ and its estimation $\hat{x}_{z|z}^{(3)}$.

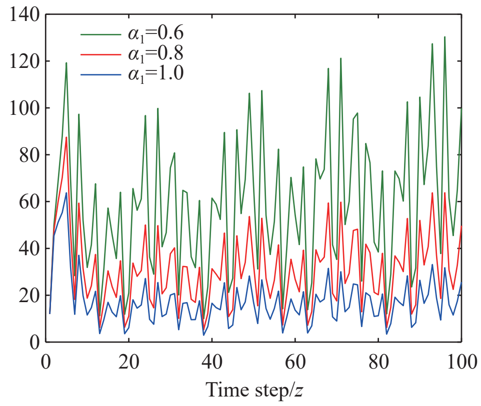

Figure 5. Trajectories of the trace of $\vec{P}_{z+1|z+1}$ under different values of α_1.

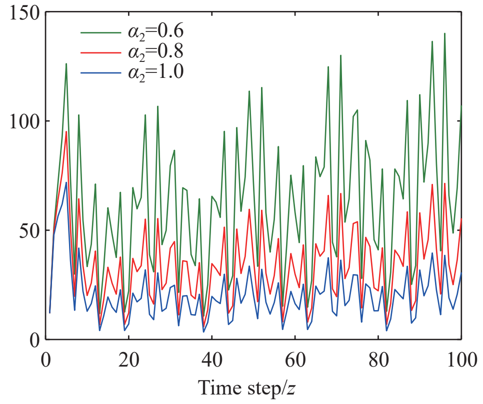

Figure 6. Trajectories of the trace of $\vec{P}_{z+1|z+1}$ under different values of α_2.

To analyze the impact of the energy harvesting scheme on the filtering performance, we denote $\alpha_1 := \sum_{m=0}^{3}(\overleftarrow{z}_{1,z} = m)\bar{z}_1^{(m)}$ and $\alpha_2 := \sum_{m=0}^{3}(\overleftarrow{z}_{2,z} = m)\bar{z}_2^{(m)}$, which represent respectively the mean energy units collected by the sensor and the relay node at time instant z. We reset $\alpha_1 = 0.6, 0.8$ and 1 while remaining all the other parameters. In this case, Figure 5 displays the trace of the minimum upper bound $\vec{P}_{z+1|z+1}$ for the second moment matrix $P_{z+1|z+1}$ of the filtering error. Additionally, Figure 6 plots the trace trajectories of $\vec{P}_{z+1|z+1}$ when α_2 is set to 0.6, 0.8 and 1, while keeping all the other parameters invariant. From Figures 5–6, we confirm the monotonicity of the filtering performance with regard to α_1 and α_2.

Example 2. In order to validate the applicability of the filtering algorithm proposed in this paper for practical systems, we examine a network-based testbed consisting of a plant (DC servo system) and a remote controller [37]. After discretization, the corresponding system parameters can be obtained:

$$A = \begin{bmatrix} 1.12 & 0.213 & -0.333 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.8 \\ 0 \\ 0 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_z = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}.$$

In this simulation, we set $\overleftarrow{u}_z$ as 5×10^{-1} times of the unit Gaussian white noise, and variances of the channel noises $v_{1,z}$ and $v_{2,z}$ are both determined as $0.025I$. In addition, the initial condition is chosen as $\bar{x}_0 = [0.1\ 0.2\ 1]^T$ and $P_0 = 10I$. In view of the presence of unmodeled dynamics and environmental perturbations, we consider the following time-varying system parameters:

$$A_z = A + \sin(10/z) \times \begin{bmatrix} -0.1 & 0 & 0 \\ -0.01 & 0.01 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_z = B + \sin(10/z) \times \begin{bmatrix} 0.01 \\ 0.01 \\ 0 \end{bmatrix}.$$

Selection of the other parameters remains consistent with Example 1.

Based on the provided parameters, the simulation results are depicted in Figures 7–9 that present the trajectories of the system state and its estimation, which further demonstrate the practicability of our filtering algorithm.

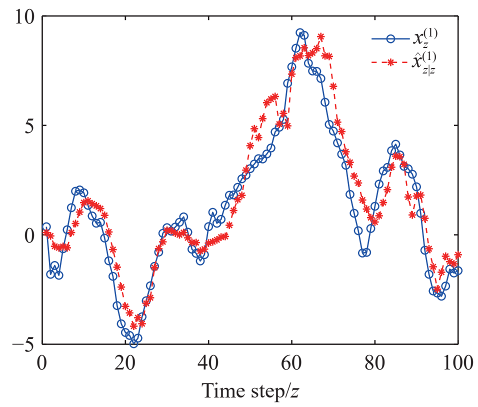

Figure 7. Evolution of $x_z^{(1)}$ and its estimation $\hat{x}_{z|z}^{(1)}$.

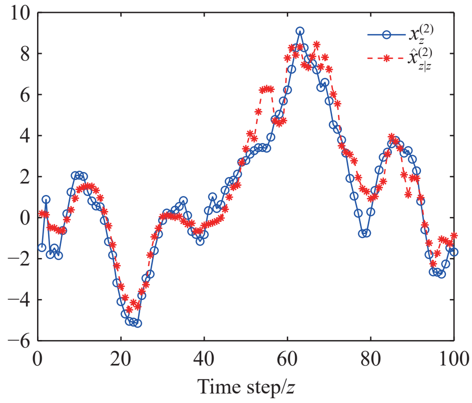

Figure 8. Evolution of $x_z^{(2)}$ and its estimation $\hat{x}_{z|z}^{(2)}$.

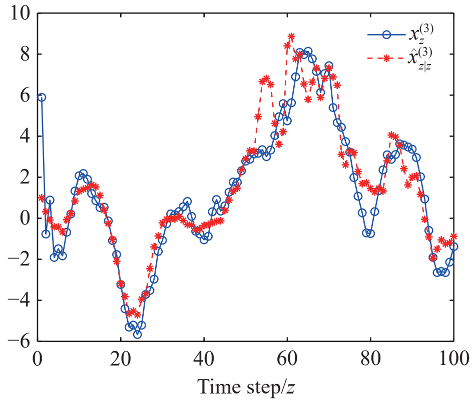

Figure 9. Evolution of $x_z^{(3)}$ and its estimation $\hat{x}_{z|z}^{(3)}$.

6. Conclusion

This paper has tackled the filtering problem for systems with stochastic nonlinearity, relay communication, energy harvesting and correlated noises, where the signal has been received by the relay node and then amplified and forwarded to the filter. The operational energy of the sensor and the relay node is supplied by the energy harvesting techniques. An upper bound for the second moment matrix of the estimation error has been constructed by mathematical induction, which has been locally minimized by an appropriately designed filter. In addition, boundedness of this upper bound has been analyzed so as to reflect the filtering performance. Simulation results have been presented to validate the efficiency and practicality of the filtering algorithm.

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