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Article

Variance-constrained H_∞ State Estimation For Time-varying Delayed Neural Networks With Random Access Protocol and Sensor Failures

Yan Gao [1,2], Jun Hu [1,2,3,] *, Hui Yu [2,3], Chaoqing Jia [2,3], and Cai Chen [2,3]

[1] Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, China
[2] Heilongjiang Provincial Key Laboratory of Optimization Control and Intelligent Analysis for Complex Systems, Harbin University of Science and Technology, Harbin 150080, China
[3] School of Automation, Harbin University of Science and Technology, Harbin 150080, China
* Correspondence: jhu@hrbust.edu.cn

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Abstract: In this paper, a recursive state estimation method is proposed for delayed recurrent neural networks subject to random sensor failures and random access protocol, where time-varying characteristic and two performance evaluation indices are taken into account. The phenomenon of random sensor failures is characterized by introducing a random variable with certain occurrence probability. In order to prevent the data from collisions and save the resources, the random access protocol is introduced into the transmission channel, in which it is assumed that only one sensor node is allowed to access the network. Our aim is to propose the H_∞ state estimation strategy without utilizing the augmentation method, where the estimator has the same order of the original neural state. In particular, we provide some sufficient conditions, which can guarantee two performance requirements from the noise attenuation and estimation accuracy perspectives. Finally, we use a simulation example to illustrate the feasibility of proposed H_∞ state estimation method handling the concerned communication constraints.

Keywords: time-varying delayed recurrent neural networks; H_∞ performance; random sensor failures; random access protocol; error variance

1. Introduction

In the past two decades, the neural networks (NNs) have long been concerned by researchers because of their strong learning ability and adaptability to the environment changes [1–3]. Accordingly, the NNs have been well discussed with wide applications including combinatorial optimization, pattern recognition and associative memory [4–6]. In the process of analyzing the problems related to NNs, we need to propose suitable estimation algorithms to estimate the system state [7–9]. Up to now, many achievements have been published on various state estimation (SE) problems of NNs [10–12]. Nevertheless, note that the results on time-varying delayed NNs (TVDNNs) have not been fully studied, where the time-varying characteristics should be discussed [13–15]. For instance, in [14], the authors have discuss the dynamical behaviour analysis of multi-rate systems with time-varying parameters and some sufficient criteria have been obtained to guarantee the H_∞ performance requirement and upper bound of error covariance (UBOEC). Different from the minimum variance estimation method, it can be recognized that some variance-constrained SE methods can meet additional requirement and propose more loose technique. Recently, the H_∞ SE issues have been investigated in [16, 17] under variance constraint, the sufficient conditions have been given to guarantee the H_∞ performance requirement. Based on the existing results, we aim to discuss the dynamical behaviours of TVDNNs with stochastic noises and propose the corresponding SE method guaranteeing the noise attenuation level and certain estimation accuracy.

In the actual engineering system, the phenomenon of sensor failures or fault is commonly encountered due to sensor aging, component failures and false data injection attacks [18–24]. So far, some preliminary results have been released to solve the SE issue for NNs with sensor failures [25–27]. Specifically, new SE strategy with additional robustness of the estimator gain perturbation has been presented in [26] for NNs with sensor failures and Markov jump parameters, and the sufficient condition has been obtained to guarantee the mean-square exponential stability by selecting suitable Lyapunov functions. Recently, the SE strategy handling the sensor failures has been proposed in [27] for NNs, and the finite-time state estimator has been designed and implemented to guarantee the stochastic stability within finite-time framework of resultant singular error dynamical system. However, it is commonly assumed that the sensor failures are deterministic in some existing results, the sensor failures may be random due to variations in the operating environment. When the SE problem subject to certain sensor failure becomes a concern, the estimation performance may be degraded if the influences of random sensor failure is not handled properly. According to the existing results, the SE problem for TVDNNs with random sensor failures has not been studied in depth, which is one main motivation of current research.

For most existing results, all sensor nodes can simultaneously access the communication channels and transmit the data, which need more network resources [28–30]. Unfortunately, it cannot be realized in some engineering practices, because simultaneous access via shared communication channels could lead to inevitable data conflicts [31–34]. In order to avoid the data conflict, many scholars have proposed a variety of network communication protocols to schedule the transmission sequence of nodes [35–38]. Accordingly, the network communication protocols widely used in practice include the try-once-discard protocol, the random access protocol (RAP) and the round-robin protocol [39–41]. So far, the SE problem for NNs under some communication protocols has aroused primary research interest [42–44]. Specifically, in [44], the H_∞ SE problem has been studied for NNs under stochastic communication protocol, and the sufficient conditions have been given to ensure the H_∞ performance requirement. As far as the authors know, although there has explicit engineering significance in the field of communication, the SE problem for TVDNNs with RAP has not been thoroughly studied, which deserves further investigations.

Based on the above discussions, the dynamical behaviour of TVDNNs with random sensor failures and RAP is analyzed and the desirable SE scheme is given by considering the needs of noise attenuation as well as estimation accuracy. The RAP is introduced into the transmission channel to save communication costs. Our purpose is to develop new H_∞ SE method with guaranteed estimation performance without utilizing augmentation method, and the sufficient criteria are given to ensure the two desirable requirements involving the desirable H_∞ performance index and the UBOEC. In addition, the main challenges or difficulties are listed as follows: (i) How to tackle the influences from random sensor failures and nonlinear activation function for addressed TVDNNs? (ii) How to evaluate the estimation method when both the noise attenuation ability and estimation accuracy requirement are considered? (iii) How to further analyze the behaviour of TVDNNs subject to random sensor failures under RAP schedule and propose an effective SE method? Our main innovations can be listed as follows: 1) the SE issue is investigated for discrete delayed TVDNNs with random sensor failures and RAP by combining the H_∞ performance over finite-horizon with error variance constraint simultaneously; 2) both the random sensor failures and RAP are well discussed, where a new SE method meeting the H_∞ performance and guaranteed estimation error constraint is proposed accordingly; and 3) the newly developed SE approach has the recursive format suitable for online application, which may reduce the computational burden due to the fact that there is no need to resort the augmentation technique.

Notations: The symbols used throughout the paper are fairly standard. $\mathbb{R}^r$, A^T, $\mathrm{tr}(A)$, I, $\mathrm{Prob}\{x\}$ and $\mathbb{E}\{x\}$ denote the r-dimensional Euclidean space, the transpose of matrix A, the trace of matrix A, the identity matrix with proper dimension, the probability of x and the mathematical expectation of x, respectively. The $\mathrm{diag}\{\cdots\}$ is defined as the block diagonal matrix. Let $X > 0$ be a positive definite symmetric matrix.

2. Problem Formulation

As shown in Figure 1, in the actual networked environment, the RAP is introduced into TVDNNs to save bandwidth and network resources and determine which component can access the network channel and has the right to transmit data.

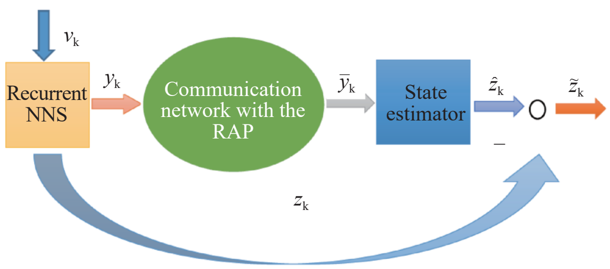

Figure 1. The SE Framework Scheduled by RAP.

Consider the following n-neurons TVDNNs subject to random sensor failures:

$$\begin{aligned} x_{k+1} =& A_k x_k + A_{dk} x_{k-d} + B_k \mathbf{f}(x_k) + C_k v_{1k}, \\ y_k =& \zeta_k D_k x_k + E_k v_{2k}, \\ z_k =& H_k x_k, \\ x_k =& \phi_k, \quad \forall k \in \{-d, -d+1, \cdots, 0\}, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ stands for the state vector of NNs, $y_k \in \mathbb{R}^m$ depicts the measurement output and $z_k \in \mathbb{R}^r$ denotes the controlled output, respectively. $A_k = \mathrm{diag}\{a_{i,k}\}$ $(i = 1, 2, \cdots, n)$ denotes state coefficient matrix, B_k represents the connection weight matrix and A_{dk} depicts delayed connection weight matrix. $\mathbf{f}(x_k)$ stands for the neuron activation function, ϕ_k depicts the initial sequence, and d stands for the time-delay. C_k, D_k, E_k and H_k are proper compatible matrices. v_{1k} and v_{2k} denote zero-mean white noises with covariances $V_{1k} > 0$ and $V_{2k} > 0$. The phenomenon of random sensor failures is characterized by random variable ζ_k over the interval $[0, 1]$ based on the certain probability mass function $P(s)$, $\bar{\zeta}_k$ stands for the mean value and $\tilde{\zeta}_k$ represents the variance. In what follows, assume that ζ_k, v_{1k} and v_{2k} are mutually independent.

The nonlinear activation function $\mathbf{f}(\cdot)$: $\mathbb{R}^n \mapsto \mathbb{R}^n$ satisfies $\mathbf{f}(0) = 0$ and the following sector-bounded condition

$$[\mathbf{f}(x) - \mathbf{f}(x) - \mathfrak{W}_1(x-y)]^T [\mathbf{f}(x) - \mathbf{f}(x) - \mathfrak{W}_2(x-y)] \leqslant 0, \quad \forall x, y \in \mathbb{R}^n, \tag{2}$$

where $\mathfrak{W}_1$ and $\mathfrak{W}_2$ stand for known matrices of proper dimensions and $\mathfrak{W} = \mathfrak{W}_2 - \mathfrak{W}_1$ stands for the symmetric positive definite matrix (PDM).

Let ξ_k indicate the sensor with privileged access to the channel at time k. According to [45], $\{\xi_k\}_{k \geqslant 0}$ depicts a sequence of random variables. It is assumed that all random variables are independent of each other. Furthermore, the occurrence probability is described by

$$\mathrm{Prob}\{\xi_k = i\} = p_i, \tag{3}$$

where $p_i > 0$ depicts the occurrence probability of data transmission through the communication network and $\sum_{i=1}^{m} p_i = 1$. Furthermore, let $y_k^i \in \mathbb{R}$ depict the measurement output of ith sensor and $\bar{y}_k^i$ is defined as the measurement output sent by the ith sensor under the RAP, then the update rule satisfies the following form:

$$\bar{y}_k^i = \begin{cases} y_k^i, & \text{if } i = \xi_k, \ k \geqslant 0, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

Based on the updating rule (4), the RAP is described as follows:

$$\bar{y}_k = \Psi_{\xi_k} y_k, \tag{5}$$

where $\Psi_{\xi_k} = \mathrm{diag}\{\delta(\xi_k - 1), \delta(\xi_k - 2), \cdots, \delta(\xi_k - m)\}$ and $\delta(\cdot) \in \{0, 1\}$ depicts the Kronecker delta function.

Furthermore, it can be concluded from the definition of Ψ_{ξ_k} that

$$\Psi_{\xi_k} = \sum_{i=1}^{m} \delta(\xi_k - i) \Psi_i. \tag{6}$$

We can get

$$\begin{aligned} \mathbb{E}\{\delta(\xi_k - i)\} &= \sum_{j=1}^{N} p_j \delta(j - i) = p_i, \\ \delta(\xi_k - i)\delta(\xi_k - j) &= \begin{cases} \delta(\xi_k - i), & i = j, \\ 0, & i \neq j. \end{cases} \end{aligned} \tag{7}$$

Remark 1: As it is well known, the RAP plays an important role in the resource-constrained environments, the data conflicts are a major problem especially when multiple transmissions of data are involved. The RAP effectively reduces the probability of such collisions, which can ensure the timeliness of data transmission. On the other hand, the RAP contributes to optimizing the utilization of network resources. In a network with limited bandwidth and time slots for data transmission, it enables efficient allocation of these resources. This ensures that the data has a fair chance to transmit without over-saturating the network. In addition, when the data packet cannot be updated in the context of the RAP with partial data transmission, we use a zero-input strategy to handle the case that the packet cannot be updated, which implies that no external inputs are introduced to the system.

In this paper, we design the following finite-horizon state estimator:

$$\begin{aligned}\hat{x}_{k+1} =& A_k\hat{x}_k + A_{dk}\hat{x}_{k-d} + B_k\mathbf{f}(\hat{x}_k) + K_k(\bar{y}_k - \bar{\zeta}_k\Psi_{\xi_k}D_k\hat{x}_k),\\ \hat{z}_k =& H_k\hat{x}_k,\end{aligned} \tag{8}$$

where $\hat{x}_k$ represents SE of x_k and K_k denotes the estimator gain matrix.

To proceed, we denote controlled output EE as $\tilde{z}_k = z_k - \hat{z}_k$ and estimation error (EE) as $e_k = x_k - \hat{x}_k$. Based on (1) and (8), the EE dynamics system is obtained through simple calculations

$$\begin{aligned}e_{k+1} =& (A_k - \bar{\zeta}_kK_k\Psi_{\xi_k}D_k)e_k + A_{dk}e_{k-d} + B_k\bar{\mathbf{f}}(e_k) - (\zeta_k - \bar{\zeta}_k)K_k\Psi_{\xi_k}D_k\hat{x}_k\\ &- (\zeta_k - \bar{\zeta}_k)K_k\Psi_{\xi_k}D_ke_k - K_k\Psi_{\xi_k}E_kv_{2k} + C_kv_{1k},\\ \tilde{z}_k =& H_ke_k,\end{aligned} \tag{9}$$

where $\bar{\mathbf{f}}(e_k) = \mathbf{f}(x_k) - \mathbf{f}(\hat{x}_k)$ and $e_{k-d} = x_{k-d} - \hat{x}_{k-d}$.

The EE covariance matrix P_k is defined as follows:

$$P_k = \mathbb{E}\{e_ke_k^T\}. \tag{10}$$

The main purpose is to propose the H_∞ SE method guaranteeing the following two performance requirements.

(R1) Let the scalar $\gamma > 0$, the PDMs U_ϕ, U_φ and U_ψ be given. For the initial state e_l $(l = -d, -d+1, \cdots, 0)$, the following inequality holds

$$J_1 := \mathbb{E}\left\{\sum_{k=0}^{N-1}(\|\tilde{z}_k\|^2 - \gamma^2\|v_k\|^2_{U_\varphi})\right\} - \gamma^2\mathbb{E}\left\{e_0^TU_\phi e_0 + \sum_{l=-d}^{-1}e_l^TU_\psi e_l\right\} < 0, \tag{11}$$

with

$$\|v_k\|^2_{U_\varphi} = v_k^TU_\varphi v_k,\ \ v_k = \begin{bmatrix}v_{1k}^T & v_{2k}^T\end{bmatrix}^T.$$

(R2) The EE covariance has following upper bound

$$J_2 := P_k \leqslant \mathfrak{Y}_k \tag{12}$$

with $\mathfrak{Y}_k > 0$ $(0 \leqslant k \leqslant N)$ being a known matrix, which reflects the admissible estimation precision demand corresponding to the actual situation.

Remark 2: The traditional H_∞ performance index within the infinite horizon situation is applicable for handling the time-invariant system with steady performance attention. Unlike the traditional infinite-horizon performance index, the index in (11) is designed with a finite-horizon consideration and can better handle the transient behavior requirement. In addition, in real systems, especially those with time-varying characteristics or short-term operational requirements, the H_∞ performance index within the finite horizon may be more practical.

3. Main Results

In this section, the sufficient criteria are derived to guarantee the H_∞ performance requirement and the UBOEC.

3.1. H_∞ Performance Analysis

Firstly, a sufficient condition is given to ensure the desirable H_∞ performance index by using the recursive linear matrix inequalities (RLMIs) technique.

Theorem 1: Consider the TVDNNs subject to random sensor failures and RAP. Suppose that PDMs U_φ, U_ϕ and U_ψ, the positive scalar γ and the estimator gain matrix K_k in (8) are given. Under initial conditions $\mathbf{R}_0 \leqslant \gamma^2U_\phi$ and $\mathfrak{P}_l \leqslant \gamma^2U_\psi$ $(l = -d, -d+1, \cdots, -1)$, if there are a series of PDMs $\{\mathbf{R}_k\}_{1\leqslant k\leqslant N+1}$ and $\{\mathfrak{P}_k\}_{0\leqslant k\leqslant N}$ satisfying the following inequality

$$\Phi = \begin{bmatrix}\Phi_{11} & \Phi_{12} & -R_{3k}^T & 0 & 0 & 0\\ * & \Phi_{22} & \mathcal{F}_k(\hat{x}_k) & 0 & 0 & 0\\ * & * & \Phi_{33} & 0 & 0 & 0\\ * & * & * & \Phi_{44} & 0 & 0\\ * & * & * & * & \Phi_{55} & 0\\ * & * & * & * & * & \Phi_{66}\end{bmatrix} < 0 \tag{13}$$

with

$$\begin{aligned}
\Phi_{11} =&(3+\bar{\zeta}_k)A_k^T\mathbf{R}_{k+1}A_k+(4\bar{\zeta}_k+2\tilde{\zeta}_k+\bar{\zeta}_k^2)\sum_{i=1}^{m}p_iD_k^T\Psi_iK_k^T\mathbf{R}_{k+1}K_k\Psi_iD_k\\
&+\mathfrak{P}_k+H_k^TH_k-\mathbf{R}_k-R_{1k},\\
\Phi_{12} =&A_k^T\mathbf{R}_{k+1}B_k-R_{2k},\\
\Phi_{22} =&(3+\bar{\zeta}_k)B_k^T\mathbf{R}_{k+1}B_k-I,\\
\Phi_{33} =&2\tilde{\zeta}_k\sum_{i=1}^{m}p_i\hat{x}_k^TD_k^T\Psi_iK_k^T\mathbf{R}_{k+1}K_k\Psi_iD_k\hat{x}_k+(4+\bar{\zeta}_k)\mathcal{F}_k^T(\hat{x}_k)B_k^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(\hat{x}_k)\\
&-\mathcal{F}_k^T(\hat{x}_k)\mathcal{F}_k(\hat{x}_k),\\
\Phi_{44} =&(4+\bar{\zeta}_k)A_{dk}^T\mathbf{R}_{k+1}A_{dk}-\mathfrak{P}_{k-d},\\
\Phi_{55} =&C_k^T\mathbf{R}_{k+1}C_k-\gamma^2U_{\varphi},\\
\Phi_{66} =&\sum_{i=1}^{m}p_iE_k^T\Psi_iK_k^T\mathbf{R}_{k+1}K_k\Psi_iE_k-\gamma^2U_{\varphi},
\end{aligned}\tag{14}$$

then the addressed H_∞ performance constraint can be ensured.

Proof: Firstly, we introduce:

$$M(e_k)=e_k^T\mathbf{R}_ke_k+\sum_{l=k-d}^{k-1}e_l^T\mathfrak{P}_le_l.\tag{15}$$

Calculate the $M(e_k)$ along the EE dynamics system (9) as follows:

$$\begin{aligned}
\mathbb{E}\{\Delta M(e_k)\} =&\mathbb{E}\left\{e_k^TA_k^T\mathbf{R}_{k+1}A_ke_k+\bar{\zeta}_k^2e_k^TD_k^T\Psi_{\xi_k}K_k^T\mathbf{R}_{k+1}K_k\Psi_{\xi_k}D_ke_k+e_{k-d}^TA_{dk}^T\mathbf{R}_{k+1}A_{dk}e_{k-d}\right.\\
&+\mathcal{F}_k^T(e_k,\hat{x}_k)B_k^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(e_k,\hat{x}_k)+\mathcal{F}_k^T(\hat{x}_k)B_k^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(\hat{x}_k)\\
&+v_{1k}^TC_k^T\mathbf{R}_{k+1}C_kv_{1k}+\tilde{\zeta}_k\hat{x}_k^TD_k^T\Psi_{\xi_k}K_k^T\mathbf{R}_{k+1}K_k\Psi_{\xi_k}D_k\hat{x}_k\\
&+\tilde{\zeta}_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\mathbf{R}_{k+1}K_k\Psi_{\xi_k}D_ke_k+v_{2k}^TE_k^T\Psi_{\xi_k}K_k^T\mathbf{R}_{k+1}K_k\Psi_{\xi_k}E_kv_{2k}\\
&-2\bar{\zeta}_ke_k^TA_k^T\mathbf{R}_{k+1}K_k\Psi_{\xi_k}D_ke_k+2e_k^TA_k^T\mathbf{R}_{k+1}A_{dk}e_{k-d}+2e_k^TA_k^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(e_k,\hat{x}_k)\\
&-2\bar{\zeta}_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\mathbf{R}_{k+1}A_{dk}e_{k-d}-2\bar{\zeta}_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(e_k,\hat{x}_k)\\
&+2e_{k-d}^TA_{dk}^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(e_k,\hat{x}_k)+2\tilde{\zeta}_k\hat{x}_k^TD_k^T\Psi_{\xi_k}K_k^T\mathbf{R}_{k+1}K_k\Psi_{\xi_k}D_ke_k\\
&-2e_k^TA_k^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(\hat{x}_k)+2\bar{\zeta}_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(\hat{x}_k)-2e_{k-d}^TA_{dk}^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(\hat{x}_k)\\
&\left.-2\mathcal{F}_k^T(e_k,\hat{x}_k)B_k^T\mathbf{R}_{k+1}B_k\mathcal{F}_k(\hat{x}_k)-e_k^T\mathbf{R}_ke_k+e_k^T\mathfrak{P}_ke_k-e_{k-d}^T\mathfrak{P}_{k-d}e_{k-d}\right\},
\end{aligned}\tag{16}$$

where $\mathcal{F}_k(e_k,\hat{x}_k)=\mathbf{f}(\hat{x}_k+e_k)$ and $\mathcal{F}_k(\hat{x}_k)=\mathbf{f}(\hat{x}_k)$.

Using the inequality $2x^T\mathcal{P}y\leqslant x^T\mathcal{P}x+y^T\mathcal{P}y$ $(\mathcal{P}>0)$, we can get

$$
\begin{aligned}
&\mathbb{E}\{-2\bar{\zeta}_k e_k^T A_k^T \mathbf{R}_{k+1} K_k \Psi_{\xi_k} D_k e_k\}\\
\leqslant&\mathbb{E}\{\bar{\zeta}_k e_k^T A_k^T \mathbf{R}_{k+1} A_k e_k + \bar{\zeta}_k e_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} K_k \Psi_{\xi_k} D_k e_k\},\\
&\mathbb{E}\{2e_{k-d}^T A_{dk}^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(e_k,\hat{x}_k)\}\\
\leqslant&\mathbb{E}\{e_{k-d}^T A_{dk}^T \mathbf{R}_{k+1} A_{dk} e_{k-d} + \mathcal{F}_k^T(e_k,\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(e_k,\hat{x}_k)\},\\
&\mathbb{E}\{-2\bar{\zeta}_k e_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(e_k,\hat{x}_k)\}\\
\leqslant&\mathbb{E}\{\bar{\zeta}_k e_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} K_k \Psi_{\xi_k} D_k e_k + \bar{\zeta}_k \mathcal{F}_k^T(e_k,\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(e_k,\hat{x}_k)\},\\
&\mathbb{E}\{2e_k^T A_k^T \mathbf{R}_{k+1} A_{dk} e_{k-d}\}\\
\leqslant&\mathbb{E}\{e_k^T A_k^T \mathbf{R}_{k+1} A_k e_k + e_{k-d}^T A_{dk}^T \mathbf{R}_{k+1} A_{dk} e_{k-d}\},\\
&\mathbb{E}\{-2\bar{\zeta}_k e_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} A_{dk} e_{k-d}\}\\
\leqslant&\mathbb{E}\{\bar{\zeta}_k e_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} K_k \Psi_{\xi_k} D_k e_k + \bar{\zeta}_k e_{k-d}^T A_{dk}^T \mathbf{R}_{k+1} A_{dk} e_{k-d}\},\\
&\mathbb{E}\{2\tilde{\zeta}_k \hat{x}_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} K_k \Psi_{\xi_k} D_k e_k\}\\
\leqslant&\mathbb{E}\{\tilde{\zeta}_k \hat{x}_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} K_k \Psi_{\xi_k} D_k \hat{x}_k + \tilde{\zeta}_k e_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} K_k \Psi_{\xi_k} D_k e_k\},\\
&\mathbb{E}\{-2e_k^T A_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k)\}\\
\leqslant&\mathbb{E}\{e_k^T A_k^T \mathbf{R}_{k+1} A_k e_k + \mathcal{F}_k^T(\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k)\},\\
&\mathbb{E}\{2\bar{\zeta}_k e_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k)\}\\
\leqslant&\mathbb{E}\{\bar{\zeta}_k e_k^T D_k^T \Psi_{\xi_k} K_k^T \mathbf{R}_{k+1} K_k \Psi_{\xi_k} D_k e_k + \bar{\zeta}_k \mathcal{F}_k^T(\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k)\},\\
&\mathbb{E}\{-2e_{k-d}^T A_{dk}^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k)\}\\
\leqslant&\mathbb{E}\{e_{k-d}^T A_{dk}^T \mathbf{R}_{k+1} A_{dk} e_{k-d} + \mathcal{F}_k^T(\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k)\},\\
&\mathbb{E}\{-2\mathcal{F}_k^T(e_k,\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k)\}\\
\leqslant&\mathbb{E}\{\mathcal{F}_k^T(e_k,\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(e_k,\hat{x}_k) + \mathcal{F}_k^T(\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k)\}.
\end{aligned}
\tag{17}
$$

To proceed, it can be derived from (6), (7), (16) and (17) that

$$
\begin{aligned}
\mathbb{E}\{\Delta M(e_k)\}\leqslant&\mathbb{E}\Big\{(3+\bar{\zeta}_k)e_k^T A_k^T \mathbf{R}_{k+1} A_k e_k + (4+\bar{\zeta}_k)e_{k-d}^T A_{dk}^T \mathbf{R}_{k+1} A_{dk} e_{k-d}\\
&+(4+\bar{\zeta}_k)\mathcal{F}_k^T(\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k) + (3+\bar{\zeta}_k)\mathcal{F}_k^T(e_k,\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(e_k,\hat{x}_k)\\
&+(4\bar{\zeta}_k+2\tilde{\zeta}_k+\bar{\zeta}_k^2)e_k^T D_k^T \sum_{i=1}^{m}\delta(\xi_k-i)\Psi_i K_k^T \mathbf{R}_{k+1} K_k \sum_{i=1}^{m}\delta(\xi_k-i)\Psi_i D_k e_k\\
&+2\tilde{\zeta}_k \hat{x}_k^T D_k^T \sum_{i=1}^{m}\delta(\xi_k-i)\Psi_i K_k^T \mathbf{R}_{k+1} K_k \sum_{i=1}^{m}\delta(\xi_k-i)\Psi_i D_k \hat{x}_k\\
&+v_{2k}^T E_k^T \sum_{i=1}^{m}\delta(\xi_k-i)\Psi_i K_k^T \mathbf{R}_{k+1} K_k \sum_{i=1}^{m}\delta(\xi_k-i)\Psi_i E_k v_{2k} + v_{1k}^T C_k^T \mathbf{R}_{k+1} C_k v_{1k}\\
&+2e_k^T A_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(e_k,\hat{x}_k) + e_k^T \mathfrak{P}_k e_k - e_{k-d}^T \mathfrak{P}_{k-d} e_{k-d} - e_k^T \mathbf{R}_k e_k\Big\}\\
=&\mathbb{E}\Big\{(3+\bar{\zeta}_k)e_k^T A_k^T \mathbf{R}_{k+1} A_k e_k + (4+\bar{\zeta}_k)e_{k-d}^T A_{k-d}^T \mathbf{R}_{k+1} A_{k-d} e_{k-d}\\
&+(3+\bar{\zeta}_k)\mathcal{F}_k^T(e_k,\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(e_k,\hat{x}_k) + (4+\bar{\zeta}_k)\mathcal{F}_k^T(\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k)\\
&+v_{1k}^T C_k^T \mathbf{R}_{k+1} C_k v_{1k} + (4\bar{\zeta}_k+2\tilde{\zeta}_k+\bar{\zeta}_k^2)\sum_{i=1}^{m} p_i e_k^T D_k^T \Psi_i K_k^T \mathbf{R}_{k+1} K_k \Psi_i D_k e_k\\
&+2\tilde{\zeta}_k \sum_{i=1}^{m} p_i \hat{x}_k^T D_k^T \Psi_i K_k^T \mathbf{R}_{k+1} K_k \Psi_i D_k \hat{x}_k\\
&+\sum_{i=1}^{m} p_i v_{2k}^T E_k^T \Psi_i K_k^T \mathbf{R}_{k+1} K_k \Psi_i E_k v_{2k}\\
&+2e_k^T A_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(e_k,\hat{x}_k) + e_k^T \mathfrak{P}_k e_k - e_{k-d}^T \mathfrak{P}_{k-d} e_{k-d} - e_k^T \mathbf{R}_k e_k\Big\}.
\end{aligned}
\tag{18}
$$

Next, consider $\mathbb{E}\{\Delta M(e_k)\}$ by adding $\tilde{z}_k^T \tilde{z}_k - \gamma^2 v_k^T U_\varphi v_k - \tilde{z}_k^T \tilde{z}_k + \gamma^2 v_k^T U_\varphi v_k$, and one has

$$
\mathbb{E}\{\Delta M(e_k)\}\leqslant\mathbb{E}\left\{\begin{bmatrix}\Theta_k^T & v_k^T\end{bmatrix}\tilde{\Phi}\begin{bmatrix}\Theta_k\\ v_k\end{bmatrix} - \tilde{z}_k^T \tilde{z}_k + \gamma^2 v_k^T U_\varphi v_k\right\},
\tag{19}
$$

where

$$\Theta_k = \begin{bmatrix} e_k^T & \mathcal{F}_k^T(e_k, \hat{x}_k) & 1 & e_{k-d}^T \end{bmatrix}^T,$$

$$\tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_{11} & A_k^T \mathbf{R}_{k+1} B_k & 0 & 0 & 0 & 0 \\ * & (3+\bar{\zeta}_k) B_k^T \mathbf{R}_{k+1} B_k & 0 & 0 & 0 & 0 \\ * & * & \tilde{\Phi}_{33} & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 \\ * & * & * & * & * & \Phi_{66} \end{bmatrix},$$

$$\begin{aligned} \tilde{\Phi}_{11} =& (3+\bar{\zeta}_k) A_k^T \mathbf{R}_{k+1} A_k + (4\bar{\zeta}_k + 2\tilde{\zeta}_k + \bar{\zeta}_k^2) \sum_{i=1}^{m} p_i D_k^T \Psi_i K_k^T \mathbf{R}_{k+1} K_k \Psi_i D_k \\ & + \mathfrak{P}_k + H_k^T H_k - \mathbf{R}_k, \\ \tilde{\Phi}_{33} =& 2\tilde{\zeta}_k \sum_{i=1}^{m} p_i \hat{x}_k^T D_k^T \Psi_i K_k^T \mathbf{R}_{k+1} K_k \Psi_i D_k \hat{x}_k + (4+\bar{\zeta}_k) \mathcal{F}_k^T(\hat{x}_k) B_k^T \mathbf{R}_{k+1} B_k \mathcal{F}_k(\hat{x}_k) \end{aligned}$$

with Φ_{44}, Φ_{55} and Φ_{66} defined below (14).

Notice that the activation function obeys (2), and we can easily derive

$$\begin{bmatrix} e_k \\ \mathcal{F}_k(e_k, \hat{x}_k) \\ 1 \end{bmatrix}^T \begin{bmatrix} R_{1k} & R_{2k} & R_{3k}^T \\ R_{2k}^T & I & -\mathcal{F}_k(\hat{x}_k) \\ R_{3k} & -\mathcal{F}_k^T(\hat{x}_k) & \mathcal{F}_k^T(\hat{x}_k)\mathcal{F}_k(\hat{x}_k) \end{bmatrix} \begin{bmatrix} e_k \\ \mathcal{F}_k(e_k, \hat{x}_k) \\ 1 \end{bmatrix} \leqslant 0, \tag{20}$$

where

$$R_{1k} = \frac{\mathfrak{W}_1^T \mathfrak{W}_2 + \mathfrak{W}_2^T \mathfrak{W}_1}{2}, \quad R_{2k} = -\frac{\mathfrak{W}_1^T + \mathfrak{W}_2}{2}, \quad R_{3k} = \frac{\mathcal{F}_k^T(\hat{x}_k)\mathfrak{W}_1 + \mathcal{F}_k^T(\hat{x}_k)\mathfrak{W}_2}{2}.$$

According to (20), we obtain

$$\begin{aligned} \mathbb{E}\{\Delta M(e_k)\} \leqslant& \mathbb{E}\left\{ \begin{bmatrix} \Theta_k^T & v_k^T \end{bmatrix} \tilde{\Phi} \begin{bmatrix} \Theta_k \\ v_k \end{bmatrix} - \tilde{z}_k^T \tilde{z}_k + \gamma^2 v_k^T U_\varphi v_k - [e_k^T R_{1k} e_k + 2e_k^T R_{2k} \mathcal{F}_k(e_k, \hat{x}_k) + 2e_k^T R_{3k}^T \right. \\ & \left. + \mathcal{F}_k^T(e_k, \hat{x}_k) \mathcal{F}_k(e_k, \hat{x}_k) - 2\mathcal{F}_k^T(e_k, \hat{x}_k) \mathcal{F}_k(\hat{x}_k) + \mathcal{F}_k^T(\hat{x}_k) \mathcal{F}_k(\hat{x}_k)] \right\} \\ =& \mathbb{E}\left\{ \begin{bmatrix} \Theta_k^T & v_k^T \end{bmatrix} \Phi \begin{bmatrix} \Theta_k \\ v_k \end{bmatrix} - \tilde{z}_k^T \tilde{z}_k + \gamma^2 v_k^T U_\varphi v_k \right\}, \end{aligned} \tag{21}$$

where Φ is denoted in (13). The sum of equation (21) from 0 to $N-1$ is given by

$$\begin{aligned} \sum_{k=0}^{N-1} \mathbb{E}\{\Delta M(e_k)\} =& \mathbb{E}\left\{ e_N^T \mathbf{R}_N e_N - e_0^T \mathbf{R}_0 e_0 + \sum_{l=N-d}^{N-1} e_l^T \mathfrak{P}_l e_l - \sum_{l=-d}^{-1} e_l^T \mathfrak{P}_l e_l \right\} \\ \leqslant& \mathbb{E}\left\{ \sum_{k=0}^{N-1} \begin{bmatrix} \Theta_k^T & v_k^T \end{bmatrix} \Phi \begin{bmatrix} \Theta_k \\ v_k \end{bmatrix} \right\} - \mathbb{E}\left\{ \sum_{k=0}^{N-1} \left(\tilde{z}_k^T \tilde{z}_k - \gamma^2 v_k^T U_\varphi v_k \right) \right\}. \end{aligned} \tag{22}$$

Furthermore, we obtain

$$\begin{aligned} J_1 \leqslant& \mathbb{E}\left\{ \sum_{k=0}^{N-1} \begin{bmatrix} \Theta_k^T & v_k^T \end{bmatrix} \Phi \begin{bmatrix} \Theta_k \\ v_k \end{bmatrix} + e_0^T \left(\mathbf{R}_0 - \gamma^2 U_\phi \right) e_0 + \sum_{l=-d}^{-1} e_l^T \left(\mathfrak{P}_l - \gamma^2 U_\psi \right) e_l \right\} \\ & - \mathbb{E}\left\{ e_N^T \mathbf{R}_N e_N + \sum_{l=N-d}^{N-1} e_l^T \mathfrak{P}_l e_l \right\}. \end{aligned} \tag{23}$$

Notice that $\mathbf{R}_N > 0$, $\mathfrak{P}_l > 0$, $\Phi < 0$, $\mathbf{R}_0 \leqslant \gamma^2 U_\phi$ and $\mathfrak{P}_l \leqslant \gamma^2 U_\psi$ $(l = -d, -d+1, \cdots, -1)$, we can get $J_1 < 0$.

□

Remark 3: The primary contribution of Theorem 1 is to ensure that the proposed SE strategy satisfies the H_∞ performance requirements and is capable of attenuating noise interference. Specifically, (20) is derived from the literature based on the neuron activation function satisfying (2). During the H_∞ performance analysis, we utilize (20) to handle the neuron activation function. The derivation of (20) plays a crucial role in the proof of the Theorem 1. This innovative approach provides a solid theoretical foundation for guaranteeing the H_∞ performance of the SE strategy. It offers a new perspective and a practical solution for researchers who aim to design SE strategies for NNs.

3.2. Variance Constraint Analysis

According to the stochastic analysis technique, a sufficient criterion is obtained to guarantee the UBOEC.

Theorem 2: Consider the TVDNNs with random sensor failures and RAP and suppose that the estimator gain matrix K_k is given. For the initial condition $Q_0 = P_0$, if there are a series of PDMs $\{Q_k\}_{1\leqslant k\leqslant N+1}$ satisfying the following inequality

$$Q_{k+1}\geqslant\mathfrak{X}(Q_k) \tag{24}$$

with

$$\begin{aligned}\mathfrak{X}(Q_k) =&(3+\bar{\zeta}_k)A_kQ_kA_k^T+(3\bar{\zeta}_k+\bar{\zeta}_k^2+2\tilde{\zeta}_k)\sum_{i=1}^{m}p_iK_k\Psi_iD_kQ_kD_k^T\Psi_iK_k^T\\&+(3+\bar{\zeta}_k)A_{dk}Q_{k-d}A_{dk}^T+(3+\bar{\zeta}_k)\mathcal{Y}\mathrm{tr}(Q_k)B_kB_k^T\\&+2\tilde{\zeta}_k\sum_{i=1}^{m}p_iK_k\Psi_iD_k\hat{x}_k\hat{x}_k^TD_k^T\Psi_iK_k^T+C_kV_{1k}C_k^T\\&+\sum_{i=1}^{m}p_iK_k\Psi_iE_kV_{2k}E_k^T\Psi_iK_k^T,\\ \mathcal{Y}=&\sum_{t=1}^{2}\mu_t\mathrm{tr}(\mathfrak{W}_t^T\mathfrak{W}_t),\ \ \rho\in(0,1),\end{aligned} \tag{25}$$

where $\mu_1=\dfrac{\rho+\frac{1}{\rho}}{2(1-\rho)}\mathrm{tr}(\mathfrak{W}_1^T\mathfrak{W}_1)$ and $\mu_2=\dfrac{1}{\rho(1-\rho)}\mathrm{tr}(\mathfrak{W}_2^T\mathfrak{W}_2)$ with $\mathfrak{W}_t$ being proper compatible matrices, then it is obvious to obtain $Q_k\geqslant P_k$ $(k\in[1,N+1])$.

Proof: On the basis of (10), we calculate the EE covariance matrix P_{k+1} as:

$$\begin{aligned}P_{k+1}=&\mathbb{E}\{\bar{\zeta}_k^2K_k\Psi_{\xi_k}D_ke_ke_k^TD_k^T\Psi_{\xi_k}K_k^T+A_ke_ke_k^TA_k^T+A_{dk}e_{k-d}e_{k-d}^TA_{dk}^T\\&+B_k\mathcal{F}_k(e_k)\mathcal{F}_k^T(e_k)B_k^T+C_kv_{1k}v_{1k}^TC_k^T+\tilde{\zeta}_kK_k\Psi_{\xi_k}D_k\hat{x}_k\hat{x}_k^TD_k^T\Psi_{\xi_k}K_k^T\\&+\tilde{\zeta}_kK_k\Psi_{\xi_k}D_ke_ke_k^TD_k^T\Psi_{\xi_k}K_k^T+K_k\Psi_{\xi_k}E_kv_{2k}v_{2k}^TE_k^T\Psi_{\xi_k}K_k^T\\&-\bar{\zeta}_kA_ke_ke_k^TD_k^T\Psi_{\xi_k}K_k^T-\bar{\zeta}_kK_k\Psi_{\xi_k}D_ke_ke_k^TA_k^T\\&+A_{dk}e_{k-d}e_k^TA_k^T+B_k\mathcal{F}_k(e_k)e_k^TA_k^T+A_ke_ke_{k-d}^TA_{dk}^T+A_ke_k\mathcal{F}_k^T(e_k)B_k^T\\&-\bar{\zeta}_kA_{dk}e_{k-d}e_k^TD_k^T\Psi_{\xi_k}K_k^T-\bar{\zeta}_kK_k\Psi_{\xi_k}D_ke_ke_{k-d}^TA_{dk}^T\\&-\bar{\zeta}_kB_k\mathcal{F}_k(e_k)e_k^TD_k^T\Psi_{\xi_k}K_k^T-\bar{\zeta}_kK_k\Psi_{\xi_k}D_ke_k\mathcal{F}_k^T(e_k)B_k^T\\&+B_k\mathcal{F}_k(e_k)e_{k-d}^TA_{dk}^T+A_{dk}e_{k-d}\mathcal{F}_k^T(e_k)B_k^T\\&+\tilde{\zeta}_kK_k\Psi_{\xi_k}D_ke_k\hat{x}_k^TD_k^T\Psi_{\xi_k}K_k^T+\tilde{\zeta}_kK_k\Psi_{\xi_k}D_k\hat{x}_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\},\end{aligned}$$

where $\mathcal{F}_k(e_k)=\bar{\mathbf{f}}(e_k)$.

By using the inequality $xy^T+yx^T\leqslant xx^T+yy^T$, we can easily get the following inequalities:

$$\begin{aligned}&\mathbb{E}\{-\bar{\zeta}_kK_k\Psi_{\xi_k}D_ke_ke_k^TA_k^T-\bar{\zeta}_kA_ke_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\}\\\leqslant&\mathbb{E}\{\bar{\zeta}_kA_ke_ke_k^TA_k^T+\bar{\zeta}_kK_k\Psi_{\xi_k}D_ke_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\},\\&\mathbb{E}\{A_{dk}e_{k-d}e_k^TA_k^T+A_ke_ke_{k-d}^TA_{dk}^T\}\\\leqslant&\mathbb{E}\{A_ke_ke_k^TA_k^T+A_{dk}e_{k-d}e_{k-d}^TA_{dk}^T\},\\&\mathbb{E}\{B_k\mathcal{F}_k(e_k)e_k^TA_k^T+A_ke_k\mathcal{F}_k^T(e_k)B_k^T\}\\\leqslant&\mathbb{E}\{A_ke_ke_k^TA_k^T+B_k\mathcal{F}_k(e_k)\mathcal{F}_k^T(e_k)B_k^T\},\\&\mathbb{E}\{B_k\mathcal{F}_k(e_k)e_{k-d}^TA_{dk}^T+A_{dk}e_{k-d}\mathcal{F}_k^T(e_k)B_k^T\}\\\leqslant&\mathbb{E}\{A_{dk}e_{k-d}e_{k-d}^TA_{dk}^T+B_k\mathcal{F}_k(e_k)\mathcal{F}_k^T(e_k)B_k^T\},\\&\mathbb{E}\{-\bar{\zeta}_kA_{dk}e_{k-d}e_k^TD_k^T\Psi_{\xi_k}K_k^T-\bar{\zeta}_kK_k\Psi_{\xi_k}D_ke_ke_{k-d}^TA_{dk}^T\}\\\leqslant&\mathbb{E}\{\bar{\zeta}_kA_{dk}e_{k-d}e_{k-d}^TA_{dk}^T+\bar{\zeta}_kK_k\Psi_{\xi_k}D_ke_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\},\\&\mathbb{E}\{-\bar{\zeta}_kB_k\mathcal{F}_k(e_k)e_k^TD_k^T\Psi_{\xi_k}K_k^T-\bar{\zeta}_kK_k\Psi_{\xi_k}D_ke_k\mathcal{F}_k^T(e_k)B_k^T\}\\\leqslant&\mathbb{E}\{\bar{\zeta}_kB_k\mathcal{F}_k(e_k)\mathcal{F}_k^T(e_k)B_k^T+\bar{\zeta}_kK_k\Psi_{\xi_k}D_ke_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\},\\&\mathbb{E}\{\tilde{\zeta}_kK_k\Psi_{\xi_k}D_ke_k\hat{x}_k^TD_k^T\Psi_{\xi_k}K_k^T+\tilde{\zeta}_kK_k\Psi_{\xi_k}D_k\hat{x}_ke_k^TD_k^T\Psi_{\xi_k}K_k^T\}\\\leqslant&\mathbb{E}\{\tilde{\zeta}_kK_k\Psi_{\xi_k}D_ke_ke_k^TD_k^T\Psi_{\xi_k}K_k^T+\tilde{\zeta}_kK_k\Psi_{\xi_k}D_k\hat{x}_k\hat{x}_k^TD_k^T\Psi_{\xi_k}K_k^T\}.\end{aligned}$$

To proceed, together with (6), (7) and the above formulas, it can be calculated that

$$\begin{aligned}P_{k+1}\leqslant\mathbb{E}\Bigg\{&(3+\bar{\zeta}_k)A_ke_ke_k^TA_k^T+(3\bar{\zeta}_k+\bar{\zeta}_k^2+2\tilde{\zeta}_k)K_k\sum_{i=1}^{m}\delta(\xi_k-i)\Psi_iD_ke_ke_k^TD_k^T\sum_{i=1}^{m}\delta(\xi_k-i)\Psi_iK_k^T\\&+(3+\bar{\zeta}_k)A_{dk}e_{k-d}e_{k-d}^TA_{dk}^T+(3+\bar{\zeta}_k)B_k\mathcal{F}_k(e_k)\mathcal{F}_k^T(e_k)B_k^T\\&+2\tilde{\zeta}_kK_k\sum_{i=1}^{m}\delta(\xi_k-i)\Psi_iD_k\hat{x}_k\hat{x}_k^TD_k^T\sum_{i=1}^{m}\delta(\xi_k-i)\Psi_iK_k^T+C_kv_{1k}v_{1k}^TC_k^T\\&+K_k\sum_{i=1}^{m}\delta(\xi_k-i)\Psi_iE_kv_{2k}v_{2k}^TE_k^T\sum_{i=1}^{m}\delta(\xi_k-i)\Psi_iK_k^T\Bigg\}\\=\mathbb{E}\Bigg\{&(3+\bar{\zeta}_k)A_ke_ke_k^TA_k^T+(3\bar{\zeta}_k+\bar{\zeta}_k^2+2\tilde{\zeta}_k)\sum_{i=1}^{m}p_iK_k\Psi_iD_ke_ke_k^TD_k^T\Psi_iK_k^T\\&+(3+\bar{\zeta}_k)A_{dk}e_{k-d}e_{k-d}^TA_{dk}^T+(3+\bar{\zeta}_k)B_k\mathcal{F}_k(e_k)\mathcal{F}_k^T(e_k)B_k^T\\&+2\tilde{\zeta}_k\sum_{i=1}^{m}p_iK_k\Psi_iD_k\hat{x}_k\hat{x}_k^TD_k^T\Psi_iK_k^T+C_kv_{1k}v_{1k}^TC_k^T\\&+\sum_{i=1}^{m}p_iK_k\Psi_iE_kv_{2k}v_{2k}^TE_k^T\Psi_iK_k^T\Bigg\}.\end{aligned}$$

The activation function obeys (2), and we can easily derive

$$\mathcal{F}_k^T(e_k)\mathcal{F}_k(e_k)\leqslant\sum_{t=1}^{2}\mu_t\mathrm{tr}(\mathfrak{W}_t^T\mathfrak{W}_t)\|e_k\|^2, \tag{26}$$

where μ_1, μ_2 and $\mathfrak{W}_t$ are defined in (25).

According to the above formula and (26), one has

$$\begin{aligned}P_{k+1}\leqslant\mathbb{E}\Bigg\{&(3+\bar{\zeta}_k)A_ke_ke_k^TA_k^T+(3\bar{\zeta}_k+\bar{\zeta}_k^2+2\tilde{\zeta}_k)\sum_{i=1}^{m}p_iK_k\Psi_iD_ke_ke_k^TD_k^T\Psi_iK_k^T\\&+(3+\bar{\zeta}_k)A_{dk}e_{k-d}e_{k-d}^TA_{dk}^T+(3+\bar{\zeta}_k)\mathcal{Y}B_ke_k^Te_kB_k^T\\&+2\tilde{\zeta}_k\sum_{i=1}^{m}p_iK_k\Psi_iD_k\hat{x}_k\hat{x}_k^TD_k^T\Psi_iK_k^T+C_kv_{1k}v_{1k}^TC_k^T\\&+\sum_{i=1}^{m}p_iK_k\Psi_iE_kv_{2k}v_{2k}^TE_k^T\Psi_iK_k^T\Bigg\},\end{aligned} \tag{27}$$

where $\mathcal{Y}$ is defined in (25). In line with the feature of the trace, it is obvious to verify that

$$\mathbb{E}\{e_k^Te_k\}=\mathbb{E}\{\mathrm{tr}(e_ke_k^T)\}=\mathrm{tr}(P_k). \tag{28}$$

Combining (27) with (28) results in

$$\begin{aligned}P_{k+1}\leqslant&(3+\bar{\zeta}_k)A_kP_kA_k^T+(3\bar{\zeta}_k+\bar{\zeta}_k^2+2\tilde{\zeta}_k)\sum_{i=1}^{m}p_iK_k\Psi_iD_kP_kD_k^T\Psi_iK_k^T\\&+(3+\bar{\zeta}_k)A_{dk}P_{k-d}A_{dk}^T+(3+\bar{\zeta}_k)\mathrm{tr}(P_k)\mathcal{Y}B_kB_k^T\\&+2\tilde{\zeta}_k\sum_{i=1}^{m}p_iK_k\Psi_iD_k\hat{x}_k\hat{x}_k^TD_k^T\Psi_iK_k^T+C_kV_{1k}C_k^T\\&+\sum_{i=1}^{m}p_iK_k\Psi_iE_kV_{2k}E_k^T\Psi_iK_k^T\\=&\mathfrak{X}(P_k).\end{aligned}$$

Noting $Q_0\geqslant P_0$ and letting $Q_k\geqslant P_k$, we can get

$$\mathfrak{X}(Q_k)\geqslant\mathfrak{X}(P_k)\geqslant P_{k+1}. \tag{29}$$

Furthermore, the following inequality can be deduced as follows:

$$Q_{k+1} \geqslant \mathfrak{X}(Q_k) \geqslant \mathfrak{X}(P_k) \geqslant P_{k+1}. \tag{30}$$

The proof is complete. □

Remark 4: The main contribution of Theorem 2 is to ensure that the proposed SE strategy satisfies the UBOEC, thereby enhancing the estimation accuracy. To facilitate the analysis of the UBOEC, note that (26) is derived from the literature based on the neuron activation function satisfying (2), which contributes significantly to Theorem 2. Compared with existing results, we consider two performance indices by the consideration of H_∞ performance requirement and the UBOEC, which can provide more flexible sufficient criteria and cater application needs.

Next, through the analysis of Theorem 1 and Theorem 2 above, some sufficient criteria can be obtained to guarantee two desirable constraints.

Theorem 3: Consider the TVDNNs with random sensor failures and RAP and assume that the estimator gain matrix K_k is given. For given scalar $\gamma > 0$, PDMs U_φ, U_ϕ and U_ψ, under the initial criteria $Q_0 = P_0$, $\mathbf{R}_0 \leqslant \gamma^2 U_\phi$ and $\mathfrak{P}_l \leqslant \gamma^2 U_\psi$ $(l = -d, -d+1, \cdots, -1)$, if there are three PDMs $\{Q_k\}_{1\leqslant k\leqslant N+1}$, $\{\mathbf{R}_k\}_{1\leqslant k\leqslant N+1}$ and $\{\mathfrak{P}_k\}_{0\leqslant k\leqslant N}$ satisfying inequalities

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & 0 & 0 \\ * & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & 0 \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36} \\ * & * & * & \Xi_{44} & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 \\ * & * & * & * & * & \Xi_{66} \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ * & \Gamma_{22} \end{bmatrix} < 0 \tag{32}$$

with

$$\begin{aligned}
\Xi_{11} &= -R_{1k} - \mathbf{R}_k + \mathfrak{P}_k + H_k^T H_k, \\
\Xi_{12} &= \begin{bmatrix} -R_{2k} & -R_{3k}^T \end{bmatrix}, \\
\Xi_{13} &= \begin{bmatrix} 0 & 0 & 0 & A_k^T \end{bmatrix}, \\
\Xi_{14} &= \begin{bmatrix} \sqrt{2+\bar{\zeta}_k} A_k^T & \sqrt{\left(4\bar{\zeta}_k + 2\tilde{\zeta}_k + \bar{\zeta}_k^2\right)} \sum_{i=1}^m p_i D_k^T \Psi_i K_k^T \end{bmatrix}, \\
\Xi_{22} &= \begin{bmatrix} -I & \mathcal{F}_k(\hat{x}_k) \\ * & -\mathcal{F}_k^T(\hat{x}_k)\mathcal{F}_k(\hat{x}_k) \end{bmatrix}, \\
\Xi_{23} &= \begin{bmatrix} 0 & 0 & 0 & B_k^T \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Xi_{25} &= \begin{bmatrix} \sqrt{2+\bar{\zeta}_k} B_k^T & 0 & 0 \\ 0 & \sqrt{2\tilde{\zeta}_k} \sum_{i=1}^m p_i \hat{x}_k^T D_k^T \Psi_i K_k^T & \sqrt{4+\bar{\zeta}_k} \mathcal{F}_k^T(\hat{x}_k) B_k^T \end{bmatrix}, \\
\Xi_{36} &= \begin{bmatrix} \sqrt{4+\bar{\zeta}_k} A_{dk}^T & 0 & 0 \\ 0 & C_k^T & 0 \\ 0 & 0 & \sum_{i=1}^m p_i E_k^T \Psi_i K_k^T \\ 0 & 0 & 0 \end{bmatrix}, \\
\Xi_{33} &= \mathrm{diag}\{-\mathfrak{P}_{k-d}, -\gamma^2 U_\varphi, -\gamma^2 U_\varphi, -\mathbf{R}_{k+1}^{-1}\}, \\
\Xi_{44} &= \mathrm{diag}\{-\mathbf{R}_{k+1}^{-1}, -\mathbf{R}_{k+1}^{-1}\}, \\
\Xi_{55} &= \mathrm{diag}\{-\mathbf{R}_{k+1}^{-1}, -\mathbf{R}_{k+1}^{-1}, -\mathbf{R}_{k+1}^{-1}\}, \\
\Xi_{66} &= \mathrm{diag}\{-\mathbf{R}_{k+1}^{-1}, -\mathbf{R}_{k+1}^{-1}, -\mathbf{R}_{k+1}^{-1}\}, \\
\Gamma_{11} &= -Q_{k+1} + (3+\bar{\zeta}_k) A_k Q_k A_k^T + (3+\bar{\zeta}_k) A_{dk} Q_{k-d} A_{dk}^T + (3+\bar{\zeta}_k)\mathcal{Y}\mathrm{tr}(Q_k) B_k B_k^T + C_k V_{1k} C_k^T, \\
\Gamma_{12} &= \begin{bmatrix} \sqrt{2\tilde{\zeta}_k} \sum_{i=1}^m p_i K_k \Psi_i D_k \hat{x}_k & \sqrt{\left(3\bar{\zeta}_k + \bar{\zeta}_k^2 + 2\tilde{\zeta}_k\right)} \sum_{i=1}^m p_i K_k \Psi_i D_k Q_k & \sum_{i=1}^m p_i K_k \Psi_i E_k V_{2k} \end{bmatrix}, \\
\Gamma_{22} &= \mathrm{diag}\{-I, -Q_k, -V_{2k}\},
\end{aligned}$$

then the EE dynamics system achieves the boundedness of error covariance and the prescribed H_∞ performances.

Proof: According to Theorem 1 and Theorem 2 and under the initial conditions, the H_∞ performance index in (11) satisfies $J_1 < 0$ and the boundedness of error covariance obeys $J_2 := P_k \leqslant \mathfrak{Y}_k$. Thus, we obtain sufficient conditions to ensure the boundedness of error covariance and the H_∞ performance constraints simultaneously. □

4. Design of The Finite-Horizon State Estimator

In this part, sufficient conditions are obtained to solve the design issue of finite-horizon state estimator, which give the solution of the finite-horizon state estimator gain.

Theorem 4: Consider the TVDNNs with random sensor failures and RAP. For given $\gamma > 0$, a series of desired upper bounds matrices $\{\mathfrak{Y}_k\}_{0\leqslant k\leqslant N+1}$, the PDMs U_ϕ, U_φ and U_ψ, with the initial conditions

$$\begin{cases} \mathbf{R}_0 \leqslant \gamma^2 U_\phi, \\ \mathfrak{P}_l \leqslant \gamma^2 U_\psi,\ (l=-d,-d+1,\cdots,-1), \\ \mathbb{E}\{e_0 e_0^T\} = Q_0 \leqslant \mathfrak{Y}_0, \end{cases} \tag{33}$$

suppose that there are several PDMs $\{Q_k\}_{1\leqslant k\leqslant N+1}$, $\{\mathbf{R}_k\}_{1\leqslant k\leqslant N+1}$, $\{\mathfrak{P}_k\}_{0\leqslant k\leqslant N}$ and the estimator gain matrix $\{K_k\}_{0\leqslant k\leqslant N}$ satisfying

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & 0 & 0 \\ * & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & 0 \\ * & * & \Upsilon_{33} & 0 & 0 & \Xi_{36} \\ * & * & * & \Upsilon_{44} & 0 & 0 \\ * & * & * & * & \Upsilon_{55} & 0 \\ * & * & * & * & * & \Upsilon_{66} \end{bmatrix} < 0, \tag{34}$$

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ * & \Gamma_{22} \end{bmatrix} < 0, \tag{35}$$

$$Q_{k+1} - \mathfrak{Y}_{k+1} \leqslant 0 \tag{36}$$

with the following updating rule

$$\bar{\mathbf{R}}_k = \mathbf{R}_k^{-1}, \tag{37}$$

where

$$\begin{aligned}
\Xi_{11} &= -R_{1k} - \mathbf{R}_k + \mathfrak{P}_k + H_k^T H_k, \\
\Xi_{12} &= \begin{bmatrix} -R_{2k} & -R_{3k}^T \end{bmatrix}, \\
\Xi_{13} &= \begin{bmatrix} 0 & 0 & 0 & A_k^T \end{bmatrix}, \\
\Xi_{14} &= \begin{bmatrix} \sqrt{2+\bar{\zeta}_k} A_k^T & \sqrt{\left(4\bar{\zeta}_k + 2\tilde{\zeta}_k + \tilde{\zeta}_k^2\right)} \sum_{i=1}^{m} p_i D_k^T \Psi_i K_k^T \end{bmatrix}, \\
\Xi_{22} &= \begin{bmatrix} -I & \mathcal{F}_k(\hat{x}_k) \\ * & -\mathcal{F}_k^T(\hat{x}_k)\mathcal{F}_k(\hat{x}_k) \end{bmatrix}, \\
\Xi_{23} &= \begin{bmatrix} 0 & 0 & 0 & B_k^T \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\Xi_{25} &= \begin{bmatrix} \sqrt{2+\bar{\zeta}_k} B_k^T & 0 & 0 \\ 0 & \sqrt{2\tilde{\zeta}_k} \sum_{i=1}^{m} p_i \hat{x}_k^T D_k^T \Psi_i K_k^T & \sqrt{4+\bar{\zeta}_k} \mathcal{F}_k^T(\hat{x}_k) B_k^T \end{bmatrix}, \\
\Xi_{36} &= \begin{bmatrix} \sqrt{4+\bar{\zeta}_k} A_{dk}^T & 0 & 0 \\ 0 & C_k^T & 0 \\ 0 & 0 & \sum_{i=1}^{m} p_i E_k^T \Psi_i K_k^T \\ 0 & 0 & 0 \end{bmatrix}, \\
\Upsilon_{33} &= \mathrm{diag}\{-\mathfrak{P}_{k-d}, -\gamma^2 U_\varphi, -\gamma^2 U_\varphi, -\bar{\mathbf{R}}_{k+1}\}, \\
\Upsilon_{44} &= \mathrm{diag}\{-\bar{\mathbf{R}}_{k+1}, -\bar{\mathbf{R}}_{k+1}\}, \\
\Upsilon_{55} &= \mathrm{diag}\{-\bar{\mathbf{R}}_{k+1}, -\bar{\mathbf{R}}_{k+1}, -\bar{\mathbf{R}}_{k+1}\}, \\
\Upsilon_{66} &= \mathrm{diag}\{-\bar{\mathbf{R}}_{k+1}, -\bar{\mathbf{R}}_{k+1}, -\bar{\mathbf{R}}_{k+1}\}, \\
\Gamma_{11} &= -Q_{k+1} + (3+\bar{\zeta}_k) A_k Q_k A_k^T + (3+\bar{\zeta}_k) A_{dk} Q_{k-d} A_{dk}^T + (3+\bar{\zeta}_k)\mathcal{Y}\mathrm{tr}(Q_k) B_k B_k^T + C_k V_{1k} C_k^T, \\
\Gamma_{12} &= \begin{bmatrix} \sqrt{2\tilde{\zeta}_k} \sum_{i=1}^{m} p_i K_k \Psi_i D_k \hat{x}_k & \sqrt{\left(3\bar{\zeta}_k + \tilde{\zeta}_k^2 + 2\tilde{\zeta}_k\right)} \sum_{i=1}^{m} p_i K_k \Psi_i D_k Q_k & \sum_{i=1}^{m} p_i K_k \Psi_i E_k V_{2k} \end{bmatrix}, \\
\Gamma_{22} &= \mathrm{diag}\{-I, -Q_k, -V_{2k}\},
\end{aligned}$$

it is easy to get the value of the finite-horizon state estimator, then the problem of estimator design can be solved.

Proof: The proof of Theorem 4 is not difficult to be obtained and then it is omitted. ■

According to Theorem 4, the H_∞ SE algorithm with random sensor failures is proposed as follows.

The H_∞ SE Algorithm with Random Sensor Failures
Step 1. Let the PDMs U_φ, U_ϕ, U_ψ and $\mathfrak{Y}_k$ $(0\leqslant k\leqslant N+1)$, the H_∞ performance constraint γ be given. Select matrices $\left\{\mathbf{R}_0, Q_0, \mathfrak{P}_l\leqslant\gamma^2 U_\psi\ (l=-d,-d+1,\cdots,-1)\right\}$ obeying (33).
Step 2. By solving the RLMIs (34)-(36), the PDMs and the estimator gain matrix K_k can be obtained.
Step 3. Set $k=k+1$, if $k<N$, then go back to Step 2, otherwise go to Step 4.
Step 4. Over.

Remark 5: In this paper, the H_∞ SE algorithm with random sensor failures is proposed. Specifically, the PDMs U_φ, U_ϕ and U_ψ are defined in step 1, and these matrices play a fundamental role in the subsequent calculations. The H_∞ performance constraint γ is also given at this stage, and it represents a key parameter that determines the level of algorithm performance in terms of noise attenuation and estimation accuracy. Then, the matrix initialization iteration procedure that conforms to equation (33) is selected. In step 2, the calculation is carried out by solving the RLMIs (34)-(36), which yield the updated PDMs and the estimator gain matrix K_k. Step 3 is a control step for the iterative process, where the index k is incremented by 1 and then checked against N; if $k<N$, the process returns to step 2, and if not, it proceeds to step 4. Finally, step 4 is the termination step, indicating the completion of the algorithm's execution and the end of the SE process.

Remark 6: In fact, most of the existing results focus on the H_∞ SE issue with steady performance analysis for time-invariant recurrent NNs. Here, we have proposed the H_∞ SE approach for delayed TVDNNs with random sensor failures and RAP under variance constraint. Compared with the existing literature, the main differences or advantages can be listed as follows: a) the variance-constrained H_∞ SE issue is investigated for TVDNNs with RAP and sensor failures, thereby reflecting engineering practice; b) the H_∞ SE algorithm with estimation error accuracy consideration is designed within finite-horizon case, which has time-varying recursive feature with online application potential; and c) a new non-augmented SE method is proposed with guaranteed noise attenuation ability and certain estimation accuracy, which might reduce the computational burden as same order estimation method is proposed.

5. A Simulation Example

In this section, we present a simulation example to illustrate the effectiveness of H_∞ SE method.

The following relevant parameters can be given

$$A_k=\begin{bmatrix}-0.1 & 0\\ 0 & 0.2\sin(2k)\end{bmatrix},\ A_{dk}=\begin{bmatrix}-0.45 & 0.2\\ 0.2\sin(2k) & -0.1\end{bmatrix},\ B_k=\begin{bmatrix}-0.12\sin(2k) & -0.1\\ -0.2 & 0.04\end{bmatrix},$$
$$D_k=\begin{bmatrix}-0.13\sin(k) & -1.8\end{bmatrix},\ C_k=\begin{bmatrix}-0.1 & -0.1\sin(2k)\end{bmatrix}^T,\ E_k=\begin{bmatrix}-0.15\sin(k) & 0.1\end{bmatrix},$$
$$H_k=\begin{bmatrix}-0.06 & -0.12\sin(2k)\end{bmatrix},\ \rho=0.2,\ d=2.$$

The occurrence probability of ξ_k is described

$$\begin{cases}\text{Prob}\{\xi_k=1\}=0.4,\\ \text{Prob}\{\xi_k=2\}=0.6.\end{cases}$$

The activation function is depicted by

$$\mathbf{f}(x_k)=\begin{bmatrix}0.59x_{1,k}+\tanh(0.05x_{1,k})-0.11x_{2,k}\\ 0.2x_{1,k}+0.6x_{2,k}+\tanh(0.02x_{2,k})\end{bmatrix},$$

where the state variable depicts $x_k=\begin{bmatrix}x_{1,k} & x_{2,k}\end{bmatrix}^T$, the initial value of $\hat{x}_k$ is $\hat{x}_0=\begin{bmatrix}0.4 & 0.32\end{bmatrix}^T$, the mean of the initial value is $\phi_0=\begin{bmatrix}0.5 & 0.2\end{bmatrix}^T$ and initial values of time-delay are $\phi_{-1}=\begin{bmatrix}-1 & 1\end{bmatrix}^T$, $\phi_{-2}=\begin{bmatrix}1 & -1\end{bmatrix}^T$, $\hat{x}_{-1}=\begin{bmatrix}1 & -2\end{bmatrix}^T$ and $\hat{x}_{-2}=\begin{bmatrix}-1 & -1\end{bmatrix}^T$. Let the covariances be $V_{1k}=1$ and $V_{2k}=I$, the correlation matrices of the activation function are $\mathfrak{W}_1=\begin{bmatrix}0.59 & 0.11\\ 0.2 & 0.6\end{bmatrix}$ and $\mathfrak{W}_2=\begin{bmatrix}0.64 & 0.11\\ 0.2 & 0.62\end{bmatrix}$, the disturbance attenuation level be $\gamma=0.8$ and $N=80$, the weighted matrices be $U_\varphi=I$, $U_\phi=2I$ and $U_\psi=3I$, the upper bounds be $\{\mathfrak{Y}_k\}_{0\leqslant k\leqslant N+1}=\text{diag}\{0.4,0.4\}$. For comparison purpose, two different probability values are considered.

Case Ⅰ: The random variable ζ_k satisfies the probability mass function as follows:

$$P(s) = \begin{cases} 0.8, & s = 0, \\ 0.2, & s = 0.5, \\ 0, & s = 1, \end{cases}$$

where $\bar{\zeta}_k = 0.1$ and $\tilde{\zeta}_k = 0.04$.

Case Ⅱ: The random variable ζ_k satisfies the probability mass function as follows:

$$P(s) = \begin{cases} 0, & s = 0, \\ 0.2, & s = 0.5, \\ 0.8, & s = 1, \end{cases}$$

where $\bar{\zeta}_k = 0.9$ and $\tilde{\zeta}_k = 0.04$. Furthermore, based on (34)-(36), Table 1 and Table 2 give the partial parameter matrices in the two cases.

Table 1 Parameter matrices (Case Ⅰ: $\bar{\zeta}_k = 0.1$)

k	1	2	3	…
K_k	$K_1 = \begin{bmatrix} 1.2595 & -0.1230 \end{bmatrix}^T$	$K_2 = \begin{bmatrix} 0.8933 & 0.0535 \end{bmatrix}^T$	$K_3 = \begin{bmatrix} 1.2687 & -0.0400 \end{bmatrix}^T$	…
$\mathbf{R}_k$	$\mathbf{R}_1 = \begin{bmatrix} 0.4822 & 0.4706 \\ 0.4526 & 0.4371 \end{bmatrix}$	$\mathbf{R}_2 = \begin{bmatrix} 0.4287 & 0.4192 \\ 0.4083 & 0.3984 \end{bmatrix}$	$\mathbf{R}_3 = \begin{bmatrix} 0.3905 & 0.3800 \\ -0.3711 & 0.3663 \end{bmatrix}$	…
Q_k	$Q_1 = \begin{bmatrix} 0.4197 & 0.3001 \\ 0.2392 & 0.2093 \end{bmatrix}$	$Q_2 = \begin{bmatrix} 0.1776 & 0.1594 \\ 0.1396 & 0.1264 \end{bmatrix}$	$Q_3 = \begin{bmatrix} 0.1271 & 0.1359 \\ 0.1287 & -0.1366 \end{bmatrix}$	…

Table 2 Parameter matrices (Case Ⅱ: $\bar{\zeta}_k = 0.9$)

k	1	2	3	…
K_k	$K_1 = \begin{bmatrix} 0.3525 & -0.7586 \end{bmatrix}^T$	$K_2 = \begin{bmatrix} 0.1521 & -0.1137 \end{bmatrix}^T$	$K_3 = \begin{bmatrix} 0.3446 & 0.1180 \end{bmatrix}^T$	…
$\mathbf{R}_k$	$\mathbf{R}_1 = \begin{bmatrix} 0.1240 & 0.4628 \\ 0.2165 & 0.2826 \end{bmatrix}$	$\mathbf{R}_2 = \begin{bmatrix} 0.3505 & 0.2677 \\ 0.2771 & 0.2669 \end{bmatrix}$	$\mathbf{R}_3 = \begin{bmatrix} 0.2664 & 0.2115 \\ -0.2166 & 0.2643 \end{bmatrix}$	…
Q_k	$Q_1 = \begin{bmatrix} 0.1116 & 0.2957 \\ 0.0874 & 0.2943 \end{bmatrix}$	$Q_2 = \begin{bmatrix} -0.0757 & 0.0855 \\ 0.2966 & 0.1453 \end{bmatrix}$	$Q_3 = \begin{bmatrix} 0.1844 & 0.0818 \\ 0.0943 & 0.0945 \end{bmatrix}$	…

The simulation results are shown in Figures 2–5 and the controlled output EEs are provided in Table 3. In particular, Figure 2 describes z_k and its estimation, and Figure 3 describes $\tilde{z}_k$. Figures 4–5 describe the UBOEC and actual error variance. In addition, we make a simple comparison of the controlled output EE under different probabilities. According to Table 3, we can see that the bigger mathematical expectation, the better estimation output performs. At the same time, it also validates the effectiveness of newly designed H_∞ SE result.

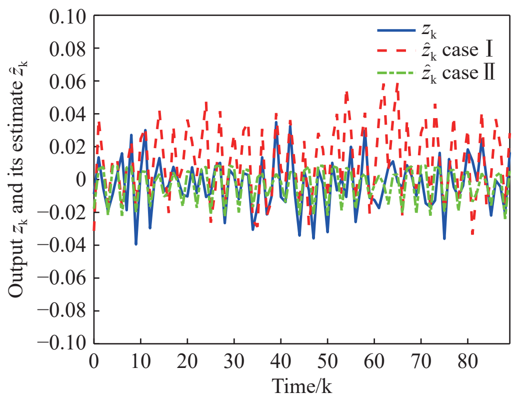

Figure 2. z_k and its estimation $\hat{z}_k$.

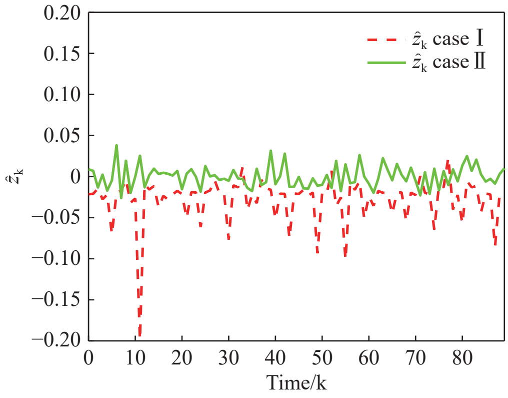

Figure 3. EEs $\tilde{z}_k$.

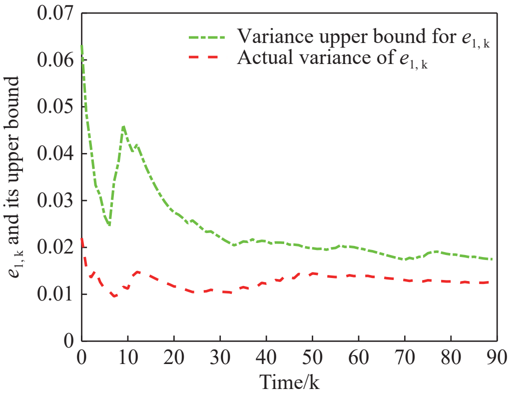

Figure 4. The UBOEC $e_{1,k}$ and actual error variance.

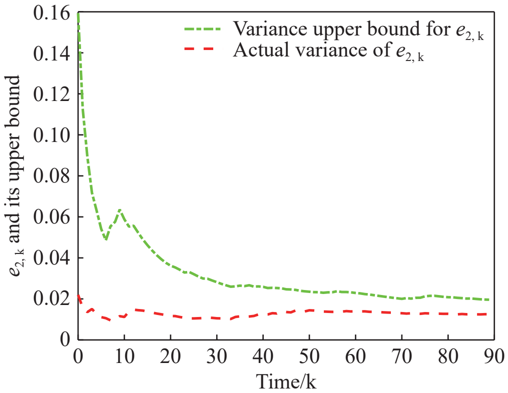

Figure 5. The UBOEC $e_{2,k}$ and actual error variance.

Table 3 The controlled output EEs in two cases

	$\sum_{k=0}^{N-1} \|\tilde{z}_k\|^2$
Case I : $\bar{\zeta}_k = 0.1$	0.0590
Case II : $\bar{\zeta}_k = 0.9$	0.0321

6. Conclusion

This paper has presented the H_∞ SE method for TVDNNs with random sensor failures and RAP under variance constraint. In addition, the RAP has been characterized by the independent and identically distributed random variables sequence with known occurrence probabilities. The finite-horizon state estimator has been designed for TVDNNs subject to random sensor failures, and the sufficient conditions have been obtained to ensure both two constraints. Furthermore, a novel H_∞ SE approach has been presented for TVDNNs, and the solution of the estimator gain matrix has been obtained by using the RLMIs method. Finally, in order to further verify the effectiveness of the presented H_∞ SE approach, we have made a simple comparison of the controlled output EE under different probabilities, and the effectiveness of presented H_∞ SE strategy has been verified by a simulation example. In future research, the H_∞ SE problem by considering the estimation error bound for memristive NNs can be discussed based on the more flexible protocols with mixed regulation idea relying on the probabilistic attacks.

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