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Robust H_∞ Estimation for 2D FMLSS Systems Under Asynchronous Multi-channel Delays: A Partition Reconstruction Approach

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Abstract: This paper investigates the robust H_∞ estimation problem for a class of 2D FMLSS systems under asynchronous multi-channel delays. We overcame the substantial challenges caused by incomplete information due to external random perturbation and multi-channel delay. First, by employing partition reconstruction approach, the original delayed systems is transformed into an equivalent multi-channel observation delay-free systems. Then, by making two modifications to the quadratic performance function, an equivalence relation is established between the design of the H_∞ filter and a minimization problem of an indefinite quadratic form. i) the first modification involves adding a random perturbation matrix term to the quadratic performance function, ensuring that the perturbation is simultaneously considered when the original system is dualized in line with the quadratic performance function; ii) the second modification replaces the initial observation and noise sequences in the quadratic performance function with reconstructed observation and noise sequences. This ensures the completeness of information in the estimation algorithm during real-time recursive computation. Then, an N-step robust H_∞ filter is designed in the 2D Krein space stochastic system, and the necessary and sufficient conditions for its existence are provided. Finally, the effectiveness of the proposed recursive filtering algorithm is verified through a numerical example.

Keywords: robust H_∞ estimation; 2D FMLSS system; multi-channel delay; Krein space stochastic system

1. Introduction

As modern industrialization progresses, the demand for multivariate analysis has been increasingly growing. Two-dimensional (2D) systems have drawn widespread attention due to their propagation along two independent directions, with bidirectional propagation being a significant feature that distinguishes them from classical one-dimensional (1D) systems. The inherent complexity and richness of 2D systems have presented challenging and novel problems for control theory and its applications. Up to now, substantial work has been conducted in areas such as system stabilization, estimation and fault diagnosis, and controller design within the 2D framework, as referenced in [1–10].

State estimation, with its widespread applications in fields such as image processing [11], surface analysis [12], environmental monitoring [13], and biomedical signal processing [14], has become a research topic of high interest in both the academic and industrial communities. To date, a series of efficient estimation approach have been developed for different types of noise environments [15–21]. The introduction of these approach is based on a deep understanding of the noise characteristics in specific application scenarios, aimed at enhancing the accuracy and reliability of state estimation, thereby optimizing system performance and improving system robustness.

In practical engineering applications, systems often operate within complex and variable environments. These environmental factors include, but are not limited to, uncertainty in parameter, external perturbation, and internal dynamic change. Such uncertainty and perturbation pose significant challenges to system control/estimation. Especially in engineering design, obtaining an accurate mathematical model of the system is difficult, leading to a situation where existing models cannot perfectly reflect the actual behavior of the system. Against this backdrop, the H_∞ state estimation strategy becomes particularly important. H_∞ state estimation achieves robust resistance to uncertainty and external perturbation by optimizing the performance indicator in the worst-case scenario during the estimation process. This means that even when the system model is not fully clear or is subjected to unknown perturbation, the H_∞ approach can ensure that the performance of the estimator meets predefined robustness standards.

Review recent research related to H_∞ estimation. Ref. [22] have investigated the design problem of asynchronous deconvolution filter for a class of 2D Markov jump systems with random packet dropout. The asynchronous phenomenon between system modes and filter modes has been described using a hidden Markov model. An asynchronous 2D deconvolution filter has been designed, taking into account the random packet dropout phenomenon that may be caused by limited bandwidth. An improved 2D single exponential smoothing scheme has been proposed to generate predictions for the missing information and perform compensation. Furthermore, based on 2D Lyapunov stability theory, sufficient conditions have been derived for the system to achieve asymptotic mean square stability and satisfy H_∞ disturbance attenuation performance. Ref. [23] have investigated the dissipative deconvolution filtering problem for a class of 2D digital systems. Initially, in a 2D setting, they have developed a new multinode round-robin protocol to reduce communication overhead and proposed a new compensation strategy to enhance the performance of the deconvolution filter. Subsequently, by using coupled matrix inequalities, they have established sufficient conditions to ensure the dissipativity of the deconvolution filter. Furthermore, they have designed filter gains with certain optimized performance using the particle swarm optimization algorithm. Ref. [24] have investigated the robust H_∞ fault estimation problem for a class of 2D uncertain systems with norm-bounded unknown input and measurement noise. Initially, by introducing an equivalent auxiliary system and a new indefinite quadratic performance function, they transformed the design of the fault estimator into a quadratic minimization problem. Subsequently, by utilizing the projection theorem and 2D Riccati-like difference equation, they have provided explicit conditions for the existence and determination of the fault estimate.

It is noteworthy that most of the existing results [5, 25–27, 28] in H_∞ estimation problem are presented using Linear Matrix Inequality (LMI). However, LMI techniques can only provide implicit results with a high degree of conservatism. The Krein space approach [29, 30] can provide explicit solutions in a recursive form. Unlike the conventional Hilbert space, Krein space is a special type of linear indefinite metric space, where the inner product can be positive definite, negative definite, or indefinite. This allows the design of H_∞ filter to be transformed into minimization problem of certain indefinite quadratic form, which can then be solved by deriving extend Kalman filter in Krein space using the projection theorem. Ref. [24, 31] have been the first to extend the application of the Krein space approach to 2D H_∞ estimation.

As control systems increasingly depend on complex network environments, various network-induced phenomena, such as network delay, packet loss, communication uncertainty, and information quantization, have become an unavoidable challenge. This paper considers the scenario where both random perturbation and asynchronous multi-channel delays result in incomplete information. For the first time in 2D systems, by appropriately modifying the quadratic performance function, the design of the robust H_∞ filter is transformed into a minimization problem of a certain indefinite quadratic form. This approach circumvents the high complexity associated with state augmentation approach and the strong conservatism of LMI technique.

Motivated by the discussions above, this paper aims to explore the H_∞ estimation problem for a class of 2D uncertain systems, in the presence of random perturbations and asynchronous multi-channel delays. The three pressing challenges to be addressed are as follows:

1) How can random perturbations be reflected in the dual stochastic system in Krein space through a given quadratic performance function?

2) How to exploit incomplete multi-channel delay information?

3) How to provide the necessary and sufficient conditions for the existence of a robust filter and present an explicit recursive estimation framework? In response to these challenges, the main contributions of this paper are summarized as follows:

1) A modified quadratic performance function considering random perturbation matrix terms is given.

2) An equivalent multi-channel delay-free observation systems are established by using the partition reconstruction approach.

3) A further improved quadratic performance function based on reconstructed observation and noise sequence is given.

4) A novel recursive estimation framework is developed for the concerned dual stochastic system in Krein space.

A comparative analysis between our work and existing works is provided in Table 1.

Table 1 Comparisons between our work and existing works

Work	2D system	H_∞ filtering	Uncertain	Delay
[8]	√	√	×	×
[7]	√	×	√	×
[17]	√	√	×	√(unidirectional)
[26]	√	√	×	×
This paper	√	√	√	√(directional,asynchronous)

The remainder of this paper is organized as follows. Section 2 presents the system description and the H_∞ estimation problem to be solved. In Section 3, gives the details of multi-channel delay partition reconstruction and initial system duality. Section 4 designs a robust H_∞ filter and the necessary and sufficient conditions for its existence. Section 5 provides a numerical example to validate the efficacy of the proposed recursive estimation algorithm. Finally, Section 6 concludes the paper.

Notations: The notation used here is normative. $\mathcal{R}^n$ is the n-dimensional real Euclidean space and $l_2([0,V],[0,V];\mathcal{R}^n)$ is the space of square summable vector sequences on $[0,V]\times[0,V]$ with values on $\mathcal{R}^n$. I indicates the identity matrix and $\mathbf{0}$ denotes the zero matrix with suitable dimensions. A^{-1} and A^{T} stand, respectively, the inverse and transpose of A. For real matrices A and B, $A>B$ ($A<B$) infers that $A-B>0$ ($A-B<0$) means $A-B$ is positive (negative) definite. $\mathcal{E}\{\cdot\}$ represent mathematical expectation operator. $[a,b]$ represents the set $[a,a+1,\cdots,b]$ for $a\leqslant b$. $\mathrm{col}_{1\leqslant c\leqslant s}\{a_c\}$ represents a column vector with $a_1,a_2,\cdots,a_s$ as an element. $\langle a,b\rangle$ is the inner product of vector a and b. $\|a\|_A^2$ is the l_2-norm with weight matrix A (i.e. $\|a\|_A^2=a^{\mathsf{T}}Aa$). Elements in the Euclidean space and Krein space are written by regular letters and boldface letters, respectively.

2. Problem Formulation

2.1. System Description

The plant under consideration is a linear shift-varying 2D FMLSS system of the form

$$\begin{cases} x_{i+1,j+1} = (A^1_{i+1,j}+\check{A}^1_{i+1,j})x_{i+1,j} \\ \qquad\qquad +(A^2_{i,j+1}+\check{A}^2_{i,j+1})x_{i,j+1} \\ \qquad\qquad +B^1_{i+1,j}w_{i+1,j}+B^2_{i,j+1}w_{i,j+1} \\ z_{i,j}=T_{i,j}x_{i,j} \end{cases} \tag{1}$$

for a finite horizon $\mathcal{N}=\{(i,j)|0\leqslant i\leqslant V, 0\leqslant j\leqslant V\}$ with V is given positive integer, $x_{i,j}\in\mathcal{R}^{n_x}$ and $z_{i,j}\in\mathcal{R}^{n_z}$ represent the system state and the signal to be estimated. $w_{i,j}\in l_2([0,V],[0,V];\mathcal{R}^{n_w})$ represents the system noise.

Remark 1: In control systems, finite-horizon filtering offers distinct practical advantages over infinite-horizon filtering. Specifically, when dealing with transient disturbances (such as intermittent sensor failures), the infinite-horizon approach tends to distribute error accumulation across the entire historical dataset, leading to persistent deviations in system state estimation. In contrast, the finite-horizon approach optimizes the performance function within a finite temporal window, thereby minimizing the maximum filtering error within that window. This mechanism ensures reliable filtering even in the presence of instantaneous high-intensity disturbances (such as electromagnetic pulse interference), significantly enhancing system robustness in dynamically perturbed environments.

In a network environment, multi-source observation systems are considered an effective means to improve system reliability. Due to the limitations of network resources, capacity, and service capabilities, the actual observation often experiences delays during channel transmission. This multi-channel asynchronous delay is typically described as

$$\begin{aligned} y^{(s)}_{i,j} &= (C^{(s)}_{i-\varsigma_s,j-\tau_s}+\check{C}^{(s)}_{i-\varsigma_s,j-\tau_s})x_{i-\varsigma_s,j-\tau_s} \\ &\quad + v^{(s)}_{i-\varsigma_s,j-\tau_s} \end{aligned} \tag{2}$$

where $y^{(s)}_{i,j}\in\mathcal{R}^{n_y}(s\in[1,N])$ represent the observations received from the s-th sensor with N is the number of sensor. $v^{(s)}_{i,j}\in l_2([0,V],[0,V];\mathcal{R}^{n_v})$ represents the observation noise. ς_s and τ_s are the communication delays in horizontal and vertical coordinates, respectively. $A^{\jmath}_{i,j}, B^{\jmath}_{i,j}, C^{(s)}_{i,j}$, and $T_{i,j}$ are known shift-varying matrices of suitable dimensions. $\check{A}^{\jmath}_{i,j}(\jmath\in[1,2])$ and $\check{C}^{(s)}_{i,j}$ represent random perturbation matrices.

Assumption 1: The communication delays ς_s and τ_s being a strictly increasing order: $0=\varsigma_1=\tau_1<\varsigma_2\leqslant\tau_2<\cdots<\varsigma_N\leqslant\tau_N$.

Assumption 2: The initial condition of (1) for the considered finite horizon $\mathcal{N}$ represent $x_0=[x^{\mathsf{T}}_{0,0},x^{\mathsf{T}}_{1,0},\cdots,x^{\mathsf{T}}_{V,0},$

$x_{0,1}^{\mathsf{T}}, \cdots, x_{0,V}^{\mathsf{T}}]^{\mathsf{T}}$ and $\|x_0\| < \infty$.

Remark 2: In complex network environments, the routes that information takes may vary due to network configurations and decisions made by dynamic routing algorithm. The differences in the length and transmission rates of various routes, as well as the time differences in processing and forwarding data packets by intermediate nodes (such as routers and switches), can cause communication delays in multi-channel transmission. In particular, when the multi-channel delay in horizontal and vertical coordinates satisfies $0 = \varsigma_1 = \tau_1 \leqslant \varsigma_2 = \tau_2 \leqslant \cdots \leqslant \varsigma_N = \tau_N$, then the asynchronous multi-channel delay considered in this paper regresses to the synchronous multi-channel delay caused by same-time memory buffer queuing. Obviously, the model in (2) is general and aligns well with engineering practice.

Assumption 3: The random perturbation matrices satisfying

$$\begin{bmatrix} \check{A}_{i,j}^{\jmath} & \check{C}_{i,j}^{(s)} \end{bmatrix} = \begin{bmatrix} M_{i,j}^{\jmath} & N_{i,j}^{(s)} \end{bmatrix} L_{i,j} E_{i,j}$$

where $M_{i,j}^{\jmath}, N_{i,j}^{(s)}$, and $E_{i,j}$ are known shift-varying matrices of suitable dimensions. $L_{i,j}$ is the unknown perturbation item satisfying $L_{i,j}^{\mathsf{T}} L_{i,j} \leqslant I$.

Remark 3: In actual control systems, various random uncertainties that are difficult to predict or quantify (environmental noise, component aging, parameter drift, etc.) are prevalent. These uncertainties are typically incorporated into the system's mathematical model in the form of random perturbation. Specifically, the dynamic behavior of a control system (encompassing multiple state variables such as position, velocity, and temperature) is influenced by the combined effect of dual input signals: one being the actively adjustable control input signal by the designer, and the other being uncontrollable external random disturbances. To more accurately describe this coupled interaction mechanism, modern control theory often adopts a modeling approach where the disturbance matrix is coupled with the system state. This modeling method not only precisely characterizes the dynamic dependence of the system state on external disturbances but also faithfully reflects the intrinsic nature of the real-time response, where control inputs and random disturbances interact with each other.

2.2. Original Observation with Asynchronous Delays Description

A set of observations received on different channels at the same horizon are provided as follows. Let $\mathcal{Y}_{l,k}^{s}$ represent the original observation with asynchronous delay at the (l,k) horizon. For clarity, define the following set

$$\begin{aligned}
\mathcal{M}_1^{+} &= \{(l,k)|\varsigma_1 \leqslant l < \varsigma_2, \tau_2 \leqslant k \leqslant j\},\\
\mathcal{M}_1^{\circ} &= \{(l,k)|\varsigma_1 \leqslant l < \varsigma_2, \tau_1 \leqslant k < \tau_2\},\\
\mathcal{M}_1^{-} &= \{(l,k)|\varsigma_2 \leqslant l \leqslant i, \tau_1 \leqslant k < \tau_2\},\\
&\vdots\\
\mathcal{M}_{N-1}^{+} &= \{(l,k)|\varsigma_{N-1} \leqslant l < \varsigma_N, \tau_N \leqslant k \leqslant j\},\\
\mathcal{M}_{N-1}^{\circ} &= \{(l,k)|\varsigma_{N-1} \leqslant l < \varsigma_N, \tau_{N-1} \leqslant k < \tau_N\},\\
\mathcal{M}_{N-1}^{-} &= \{(l,k)|\varsigma_N \leqslant l \leqslant i, \tau_{N-1} \leqslant k < \tau_N\},\\
\mathcal{M}_N^{\circ} &= \{(l,k)|\varsigma_N \leqslant l \leqslant i, \tau_N \leqslant k \leqslant j\},\\
\mathcal{M}_s &= \mathcal{M}_s^{+} \cup \mathcal{M}_s^{\circ} \cup \mathcal{M}_s^{-} (s \in [1, N-1]), \mathcal{M}_N = \mathcal{M}_N^{\circ}.
\end{aligned}$$

The original observation with asynchronous delays and the corresponding original observation noise $\mathcal{V}_{l,k}^{s}$ can be describe as

$$\mathcal{Y}_{l,k}^{s} = \begin{bmatrix} (y_{l,k}^{(1)})^{\mathsf{T}} & \cdots & (y_{l,k}^{(s)})^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \tag{3}$$

$$\mathcal{V}_{l,k}^{s} = \begin{bmatrix} (v_{l-\varsigma_1,k-\tau_1}^{(1)})^{\mathsf{T}} & \cdots & (v_{l-\varsigma_s,k-\tau_s}^{(s)})^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \tag{4}$$

where $s \in \{1, \cdots, N\}$ for $(l,k) \in \{\mathcal{M}_1, \cdots, \mathcal{M}_N\}$. It can be seen from the obtained original observation sequence, due to the influence of multi-channel delays, the observation information received by each channel within the same horizon is incomplete and reflects different state information.

2.3. H_∞ Performance Index

In accordance with the original observation sequence with asynchronous delays $\{\mathcal{Y}_{l,k}^{s} | (l,k) \in \mathcal{N}\}$ and the original observation noise sequence $\{\mathcal{V}_{l,k}^{s} | (l,k) \in \mathcal{N}\}$, the H_∞ estimation problem under a given scalar $\gamma > 0$ is stated as:

$$\sup_{(x_0, w_{l,k}, \mathcal{V}_{l,k}^{s}) \neq 0} \frac{\mathcal{X}}{\mathcal{Z}} \leqslant \gamma^2 \tag{5}$$

where $\mathcal{X} = \sum_{(l,k)\in\mathcal{N}} \|\hat{z}_{l,k} - z_{l,k}\|^2$ and $\mathcal{Z} = \|x_0\|^2_{P_0^{-1}} + \sum_{(l,k)\in\mathcal{N}} \left\{\|w_{l,k}\|^2_{R_{l,k}^{-1}} + \|\mathcal{V}^s_{l,k}\|^2_{(\mathcal{Q}^s_{l,k})^{-1}}\right\}$. $\hat{z}_{l,k}$ represent the estimation of $z_{l,k}$. $P_0, R_{l,k}$, and $\mathcal{Q}^s_{l,k}$ are given positive definite symmetric matrix which reflect the relative uncertainty of the initial state, input noise, and observation noise.

Denote quadratic performance function,

$$J_{z,\mathcal{N}}(x_0, \mathcal{V}^s_{l,k}, \mathcal{Y}^s_{l,k}) = \mathcal{Z} - \frac{1}{\gamma^2}\mathcal{X}.$$

For the sake of simplicity, it is abbreviated as $J_{\mathcal{N}}$. We can easy to know that the inequality (5) holds if and only if $J_{\mathcal{N}} > 0$. Taking full account of system (1) uncertainty, $J_{\mathcal{N}}$ is modified to $\bar{J}_{\mathcal{N}}$

$$\bar{J}_{\mathcal{N}} = J_{\mathcal{N}} + \Delta J_{\mathcal{N}} \tag{6}$$

where $\Delta J_{\mathcal{N}} = \sum_{(l,k)\in\mathcal{N}} \left\{\|L_{l,k}E_{l,k}x_{l,k}\|^2 - \|E_{l,k}x_{l,k}\|^2\right\}$. In light of the unknown perturbation matrix $L(l,k)$ property, it is obvious that $J_{\mathcal{N}} \geqslant \bar{J}_{\mathcal{N}}$ and $J_{\mathcal{N}} = \bar{J}_{\mathcal{N}}|_{E_{l,k}=0}$, so the verification condition of (5) can be modified to verify $\bar{J}_{\mathcal{N}} > 0$. Define the minima of $\bar{J}_{\mathcal{N}}$ and $J_{\mathcal{N}}$ as $\bar{J}^*_{\mathcal{N}}$ and $J^*_{\mathcal{N}}$, we can easily find that $J^*_{\mathcal{N}} \geqslant \bar{J}^*_{\mathcal{N}}$ and $J^*_{\mathcal{N}} = \bar{J}^*_{\mathcal{N}}|_{E_{l,k}=0}$. Thereby, taking full consideration of asynchronous observation delays and system uncertainty, the key to the H_∞ estimation problem considered in this paper is to obtain $\bar{J}^*_{\mathcal{N}}$.

Remark 4: As well known that systems often undergo the influence of uncertainty [7], such as random perturbation, and improper handling of these can degrade the estimation performance of the system. Concerning the random perturbation model proposed in Assumption 3, references [5, 20] have employed the state augmentation method. However, to reduce system complexity and ensure robustness, this paper takes a different approach by modifying the quadratic performance function in light of a thorough consideration of random perturbation, to achieve an equivalent dualization of the original system according to the modified quadratic performance function.

3. Preliminaries

3.1. Reconstruction of Observation with Asynchronous Delays

To estimate $z_{i,j}$ by taking full advantage of all observations received on different channels, the observations will be partition reconstructed as follows. For the convenience of discussion, we consider the case of $\{(i,j)|0\leqslant i\leqslant\varsigma_N, 0\leqslant j\leqslant\tau_N\} \subseteq \mathcal{N}$. Denote $i_s = i - \varsigma_{N-s}, j_s = j - \tau_{N-s} (s \in [1,N])$ and define the following sets

$$\begin{aligned}
&\mathcal{N}_1^\circ = \{(l,k)|0\leqslant l\leqslant i_1, 0\leqslant k\leqslant j_1\},\\
&\mathcal{N}_2^- = \{(l,k)|0\leqslant l\leqslant i_1, j_1 < k\leqslant j_2\},\\
&\mathcal{N}_2^\circ = \{(l,k)|i_1 < l\leqslant i_2, j_1 < k\leqslant j_2\},\\
&\mathcal{N}_2^+ = \{(l,k)|i_1 < l\leqslant i_2, 0\leqslant k\leqslant j_1\},\\
&\qquad\vdots\\
&\mathcal{N}_N^- = \{(l,k)|0\leqslant l\leqslant i_{N-1}, j_{N-1} < k\leqslant j_N\},\\
&\mathcal{N}_N^\circ = \{(l,k)|i_{N-1} < l\leqslant i_N, j_{N-1} < k\leqslant j_N\},\\
&\mathcal{N}_N^+ = \{(l,k)|i_{N-1} < l\leqslant i_N, 0\leqslant k\leqslant j_{N-1}\},\\
&\mathcal{N}_1 = \mathcal{N}_1^\circ, \mathcal{N}_s = \mathcal{N}_s^- \cup \mathcal{N}_s^\circ \cup \mathcal{N}_s^+ (s \in [2,N]).
\end{aligned}$$

The original observation with asynchronous delays can be reconstructed as follows:

$$y^s_{l,k} = \begin{bmatrix} y^{(1)}_{l+\varsigma_1,k+\tau_1} \\ y^{(2)}_{l+\varsigma_2,k+\tau_2} \\ \vdots \\ y^{(s)}_{l+\varsigma_s,k+\tau_s} \end{bmatrix} \tag{7}$$

where $s \in \{1,\cdots,N\}$ for $(l,k) \in \{\mathcal{N}_N,\cdots,\mathcal{N}_1\}$.

It is easy to see from (1) that $y_s(l,k)$ satisfy

$$y^s_{l,k} = \left(C^s_{l,k} + \check{C}^s_{l,k}\right)x_{l,k} + v^s_{l,k} \tag{8}$$

where $C^s_{l,k} = \mathrm{col}_{1\leqslant\mathcal{J}\leqslant s}(C^{(\mathcal{J})}_{l,k}), \check{C}^s_{l,k} = \mathrm{col}_{1\leqslant\mathcal{J}\leqslant s}(\check{C}^{(\mathcal{J})}_{l,k}) = N^s_{l,k}L_{l,k}E_{l,k}, N^s_{l,k} = \mathrm{col}_{1\leqslant\mathcal{J}\leqslant s}(N^{(\mathcal{J})}_{l,k})$, and $v^s_{l,k} = \mathrm{col}_{1\leqslant\mathcal{J}\leqslant s}(v^{(\mathcal{J})}_{l,k})$.

In line with the reconstructed observation noise sequence, $\bar{J}_U$ can be further modified as

$$\bar{\mathcal{J}}_U = \mathcal{J} + \Delta J_{\mathcal{N}} \tag{9}$$

where $\mathcal{J} = \mathcal{Z} - \frac{1}{\gamma^2}\bar{\mathcal{X}}$ and $\bar{\mathcal{X}} = \|x_0\|^2_{P_0^{-1}} + \sum_{(l,k)\in\mathcal{N}} \{\|w(l,k)\|^2_{R_{l,k}^{-1}} + \|v^s_{l,k}\|^2_{(Q^s_{l,k})^{-1}}\}$. Define the minima of $\bar{\mathcal{J}}_U$ as $\bar{\mathcal{J}}_U^*$. In fact, (6) and (9) are equivalent, as we will prove later.

The following lemma will show that the reconstructed observation sequence have the same information as the original observation sequence with asynchronous delays.

Lemma 1: The linear space spanned by the reconstructed observation sequences and the linear space spanned by the original observation sequences with asynchronous delays are equivalent, *i.e.*

$$\begin{aligned}&\mathcal{L}\{\{y^N_{l,k}|(l,k)\in\mathcal{N}_1\};\{y^{N-1}_{l,k}|(l,k)\in\mathcal{N}_2\};\cdots;\{y^1_{l,k}|(l,k)\in\mathcal{N}_N\}\}\\ &= \mathcal{L}\{\{\mathcal{Y}^1_{l,k}|(l,k)\in\mathcal{M}_1;\{\mathcal{Y}^2_{l,k}|(l,k)\in\mathcal{M}_2\};\cdots;\{\mathcal{Y}^N_{l,k}|(l,k)\in\mathcal{M}_N\}.\end{aligned}$$

Proof: See Appendix A. □

Remark 5: The new observation sequence $\{\{y^N_{l,k}|(l,k)\in\mathcal{N}_1\};\cdots;\{y^1_{l,k}|(l,k)\in\mathcal{N}_N\}\}$ is named as the reconstructed observation sequence of $\{\mathcal{Y}^1_{l,k}|(l,k)\in\mathcal{M}_1;\cdots;\{\mathcal{Y}^N_{l,k}|(l,k)\in\mathcal{M}_N\}\}$. It can be proven in a similar way that the original observation noise sequence $\{\mathcal{V}^1_{l,k}|(l,k)\in\mathcal{M}_1;\cdots;\{\mathcal{V}^N_{l,k}|(l,k)\in\mathcal{M}_N\}\}$ and the reconstructed observation noise sequence $\{\{v^N_{l,k}|(l,k)\in\mathcal{N}_1\};\cdots;\{v^1_{l,k}|(l,k)\in\mathcal{N}_N\}\}$ contain the same information, due to space limitations, detail description will not be given here. Hence, $\bar{\mathcal{J}}_U$ and $\bar{J}_U$ are equivalent (*i.e.* $\bar{\mathcal{J}}_U^* = \bar{J}_U^*$).

The robust H_∞ estimation problem of the 2D FMLSS systems (1) can be reformulated as follows.

Problem: Given the reconstructed observation sequence $\{\{y^N_{l,k}|(l,k)\in\mathcal{N}_1\};\cdots;\{y^1_{l,k}|(l,k)\in\mathcal{N}_N\}\}$ and scalar $\gamma>0$, design the appropriate filter of $z_{i,j}$ such that:

1). the modified uncertain quadratic performance function $\bar{\mathcal{J}}_{\mathcal{N}}$ has a minimum $\bar{\mathcal{J}}^*_{\mathcal{N}}$ with respect to $x_0, \{w_{l,k}|(l,k)\in\mathcal{N}\}$, and $\{\{v^N_{l,k}|(l,k)\in\mathcal{N}_1\};\cdots;\{v^1_{l,k}|(l,k)\in\mathcal{N}_N\}\}$;

2). $\bar{\mathcal{J}}_{\mathcal{N}} *|_{E_{l,k}=0}$ is positive.

3.2. Equivalent Transformation in Krein Space

In order to convert the H_∞ estimation problem into the equivalent problem in Kerin space, we introduce a 'fictitious' observation system $\hat{z}_{l,k} = T_{l,k}x_{l,k} + r_{l,k}$, where $r_{l,k} = \hat{z}_{l,k} - z_{l,k}$. Then, fully consider the impact of system uncertainty and asynchronous multi-channel delays, and combine the fictitious observation system to rewrite system (1) and (2) into a compact form related to $\bar{\mathcal{J}}_{\mathcal{N}}$,

$$\begin{cases} x_{l+1,k+1} = A^1_{l+1,k}x_{l+1,k} + A^2_{l,k+1}x_{l,k+1} \\ \qquad\qquad + \bar{B}^1_{l+1,k}\bar{w}_{l+1,k} + \bar{B}^2_{l,k+1}\bar{w}_{l,k+1} \\ \bar{y}^s_{l,k} = \bar{C}^s_{l,k}x_{l,k} + \bar{D}^s_{l,k}\bar{v}^s_{l,k} \end{cases} \tag{10}$$

where $s\in\{1,\cdots,N\}$ for $(l,k)\in\{\mathcal{N}_N,\cdots,\mathcal{N}_1\}$ and

$$\begin{aligned}&\bar{B}^1_{l,k} = [B^1_{l,k}\ M^1_{l,k}],\ \bar{B}^2_{l,k} = [B^2_{l,k}\ M^2_{l,k}]\\ &\bar{C}^s_{l,k} = [(C^s_{l,k})^{\mathsf{T}}\ E^{\mathsf{T}}_{l,k}\ T^{\mathsf{T}}_{l,k}]^{\mathsf{T}}\\ &\bar{D}^s_{l,k} = \begin{bmatrix} I & N^s_{l,k} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix}\\ &\bar{w}_{l,k} = [w^{\mathsf{T}}_{l,k}\quad (L_{l,k}E_{l,k}x_{l,k})^{\mathsf{T}}]^{\mathsf{T}},\ \bar{y}^s_{l,k} = [(y^s_{l,k})^{\mathsf{T}}\quad \mathbf{0}\quad \hat{z}^{\mathsf{T}}_{l,k}]^{\mathsf{T}}\\ &\bar{v}^s_{l,k} = [(v^s_{l,k})^{\mathsf{T}}\quad (L_{l,k}E_{l,k}x_{l,k})^{\mathsf{T}}\quad (-E_{l,k}x_{l,k})^{\mathsf{T}}\quad r^{\mathsf{T}}_{l,k}]^{\mathsf{T}}.\end{aligned}$$

In view of (10), $\bar{\mathcal{J}}_{\mathcal{N}}$ can be appropriately reformulated as a compact quadratic form,

$$\begin{aligned}\bar{\mathcal{J}}_{\mathcal{N}} =&\|x\|^2_{P_0^{-1}} - \sum_{(l,k)\in\mathcal{N}}\|E_{l,k}x_{l,k}\|^2 - \frac{1}{\gamma^2}\mathcal{X}\\ &+\sum_{(l,k)\in\mathcal{N}}\Bigg\{\|w_{l,k}\|^2_{R^{-1}_{l,k}}\\ &+\left\|\begin{bmatrix} L_{l,k}E_{l,k}x_{l,k} \\ v^s_{l,k} + N^s_{l,k}L_{l,k}E_{l,k}x_{l,k}\end{bmatrix}\right\|^2_{(\Lambda^s_{l,k})^{-1}}\Bigg\}\\ =&\|x\|^2_{P_0^{-1}} + \sum_{(l,k)\in\mathcal{N}}\left\|\begin{bmatrix}\bar{w}_{l,k} \\ \bar{D}^s_{l,k}\bar{v}^s_{l,k}\end{bmatrix}\right\|^2_{(\Theta^s_{l,k})^{-1}}\end{aligned} \tag{11}$$

where

$$\Lambda_{l,k}^{s}=\begin{bmatrix} I & (N_{l,k}^{s})^{\mathsf{T}} \\ N_{l,k}^{s} & Q_{l,k}^{s}+N_{l,k}^{s}(N_{l,k}^{s})^{\mathsf{T}} \end{bmatrix}$$
$$\Theta_{l,k}^{s}=\begin{bmatrix} \bar{R}_{l,k} & \bar{S}_{l,k}^{s}(\bar{D}_{l,k}^{s})^{\mathsf{T}} \\ \bar{D}_{l,k}^{s}(\bar{S}_{l,k}^{s})^{\mathsf{T}} & \bar{D}_{l,k}^{s}\bar{Q}_{l,k}^{s}(\bar{D}_{l,k}^{s})^{\mathsf{T}} \end{bmatrix}$$
$$\bar{R}_{l,k}=\begin{bmatrix} R_{l,k} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix},\quad \bar{S}_{l,k}^{s}=\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \end{bmatrix}$$
$$\bar{Q}_{l,k}^{s}=\begin{bmatrix} Q_{l,k}^{s} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^{2}I \end{bmatrix}.$$

Below, the $\bar{w}_{l,k}, \bar{v}_{l,k}^{s}$, and $\bar{y}_{l,k}^{s}$ in each area are collected in the same order, denote

$$\bar{w}_{\mathcal{N}}=\mathrm{col}\{\underbrace{\bar{w}_{0,0},\cdots,\bar{w}_{i_1,j_1}}_{\mathcal{N}_1},\cdots,\underbrace{\bar{w}_{0,j_{N-1}+1},\cdots,\bar{w}_{i_N,j_N}}_{\mathcal{N}_N}\}$$
$$\bar{v}_{\mathcal{N}}=\mathrm{col}\{\underbrace{\bar{v}_{0,0}^{N},\cdots,\bar{v}_{i_1,j_1}^{N}}_{\mathcal{N}_1},\cdots,\underbrace{\bar{v}_{0,j_{N-1}+1}^{1},\cdots,\bar{v}_{i_N,j_N}^{1}}_{\mathcal{N}_N}\}$$
$$\bar{y}_{\mathcal{N}}=\mathrm{col}\{\underbrace{\bar{y}_{0,0}^{N},\cdots,\bar{y}_{i_1,j_1}^{N}}_{\mathcal{N}_1},\cdots,\underbrace{\bar{y}_{0,j_{N-1}+1}^{1},\cdots,\bar{y}_{i_N,j_N}^{1}}_{\mathcal{N}_N}\}$$
$$\bar{R}_{\mathcal{N}}=\mathrm{diag}\{\underbrace{\bar{R}_{0,0},\cdots,\bar{R}_{i_1,j_1}}_{\mathcal{N}_1},\cdots,\underbrace{\bar{R}_{0,j_{N-1}+1},\cdots,\bar{R}_{i_N,j_N})}_{\mathcal{N}_N}\}$$
$$\bar{D}_{\mathcal{N}}=\mathrm{diag}\{\underbrace{\bar{D}_{0,0}^{N},\cdots,\bar{D}_{i_1,j_1}^{N}}_{\mathcal{N}_1},\cdots,\underbrace{\bar{D}_{0,j_{N-1}+1}^{1},\cdots,\bar{D}_{i_N,j_N}^{1}}_{\mathcal{N}_N}\}$$
$$\bar{Q}_{\mathcal{N}}=\mathrm{diag}\{\underbrace{\bar{Q}_{0,0}^{N},\cdots,\bar{Q}_{i_1,j_1}^{N}}_{\mathcal{N}_1},\cdots,\underbrace{\bar{Q}_{0,j_{N-1}+1}^{1},\cdots,\bar{Q}_{i_N,j_N}^{1}}_{\mathcal{N}_N}\}$$
$$\bar{S}_{\mathcal{N}}=\mathrm{diag}\{\underbrace{\bar{S}_{0,0}^{N},\cdots,\bar{S}_{i_1,j_1}^{N}}_{\mathcal{N}_1},\cdots,\underbrace{\bar{S}_{0,j_{N-1}+1}^{1},\cdots,\bar{S}_{i_N,j_N}^{1}}_{\mathcal{N}_N}\}$$

where $i = j = V$. $\bar{\mathcal{J}}_{\mathcal{N}}$ can be reformulated as

$$\bar{\mathcal{J}}_{\mathcal{N}}=\left\|\begin{bmatrix} x_0 \\ \bar{w}_{\mathcal{N}} \\ \bar{D}_{\mathcal{N}}\bar{v}_{\mathcal{N}} \end{bmatrix}\right\|_{\bar{\Theta}^{-1}}^{2} \tag{12}$$

and

$$\bar{\Theta}=\begin{bmatrix} P_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{R}_{\mathcal{N}} & \bar{S}_{\mathcal{N}}\bar{D}_{\mathcal{N}}^{\mathsf{T}} \\ \mathbf{0} & \bar{D}_{\mathcal{N}}\bar{S}_{\mathcal{N}}^{\mathsf{T}} & \bar{D}_{\mathcal{N}}\bar{Q}_{\mathcal{N}}\bar{D}_{\mathcal{N}}^{\mathsf{T}} \end{bmatrix}.$$

As far, the purpose of the above derivation is to explicitly appear the reconstructed observation in the modified quadratic performance function $\bar{\mathcal{J}}_{\mathcal{N}}$. The following lemma is given to further advance the realization of this purpose.

Lemma 2: Consider the systems (10), for $x_0, \bar{w}_{\mathcal{N}}, \bar{v}_{\mathcal{N}}$, and $\bar{y}_{\mathcal{N}}$, there exist operators $k_0, k_{\bar{w}}$, and $k_{\bar{v}} = \bar{D}_{\mathcal{N}}$, s.t.

$$\bar{y}_{\mathcal{N}} = k_0 x_0 + k_{\bar{w}}\bar{w}_{\mathcal{N}} + k_{\bar{v}}\bar{v}_{\mathcal{N}}. \tag{13}$$

Proof: Within each partition, the reconstructed observation $\bar{y}_s$ can be expressed as a linear combination of $x_0, \bar{w}$, and $\bar{v}_s$. The proof resembles that in [24], due to space limitations, it is unmentioned. By stacking all the reconstructed observations together, we can prove this lemma. □

On account of (13), it is easy to obtain a nondegenerate transformation

$$\begin{bmatrix} x_0 \\ \bar{w}_{\mathcal{N}} \\ \bar{y}_{\mathcal{N}} \end{bmatrix} = \Pi \begin{bmatrix} x_0 \\ \bar{w}_{\mathcal{N}} \\ \bar{D}_{\mathcal{N}}\bar{v}_{\mathcal{N}} \end{bmatrix} \tag{14}$$

where

$$\Pi = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ k_0 & k_{\bar{w}} & I \end{bmatrix}$$

Π is a lower triangular matrix and the determinant is not equal to 0. The modified quadratic performance function $\bar{\mathcal{J}}_{\mathcal{N}}$ can obtain an equivalent transformation

$$\bar{\mathcal{J}}_{\mathcal{N}} = \left\| \begin{bmatrix} x_0 \\ \bar{w}_{\mathcal{N}} \\ \bar{y}_{\mathcal{N}} \end{bmatrix} \right\|^2_{\bar{\Pi}^{-1}} \tag{15}$$

where $\bar{\Pi}$ and $\bar{\Theta}$ are congruence matrix regarding Π ($i.e.$ $\bar{\Pi} = \Pi\bar{\Theta}\Pi^{\mathsf{T}}$). As far, the optimization problem of H_∞ performance index can be equivalently transformed into the H_2 estimation problem of stochastic systems in krein space.

Introduce the following stochastic systems in krein space,

$$\begin{cases} \mathbf{x}_{l+1,k+1} = A^1_{l+1,k}\mathbf{x}_{l+1,k} + A^2_{l,k+1}\mathbf{x}_{l,k+1} \\ \qquad\qquad + \bar{B}^1_{l+1,k}\bar{\mathbf{w}}_{l+1,k} + \bar{B}^2_{l,k+1}\bar{\mathbf{w}}_{l,k+1} \\ \bar{\mathbf{y}}^s_{l,k} = \bar{C}^s_{l,k}\mathbf{x}_{l,k} + \bar{D}^s_{l,k}\bar{\mathbf{v}}^s_{l,k} \end{cases} \tag{16}$$

where $s \in \{1, \cdots, N\}$ for $(l,k) \in \{\mathcal{N}_N, \cdots, \mathcal{N}_1\}$. The system matrix parameters here are the same as (10), $\mathbf{x}_0, \bar{\mathbf{w}}_{l,k}$, and $\bar{\mathbf{v}}^s_{l,k}$ are zero-mean white noises sequence in Krein space, set $\zeta_{l,k} = \mathrm{col}\{x_0, \bar{\mathbf{w}}_{l,k}, \bar{\mathbf{v}}^s_{l,k}\}$, for any $(l,k),(r,t) \in \mathcal{N}$,

$$\langle \zeta_{l,k}, \zeta_{r,t} \rangle = \begin{bmatrix} P_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{R}_{l,k} & \bar{S}^s_{l,k} \\ \mathbf{0} & (\bar{S}^s_{l,k})^{\mathsf{T}} & \bar{Q}^s_{l,k} \end{bmatrix} \delta_{l,r}\delta_{k,t}.$$

Assume that $\bar{\mathbf{w}}_{\mathcal{N}}, \bar{\mathbf{v}}_{\mathcal{N}}, \bar{\mathbf{y}}_{\mathcal{N}}$ is arranged in the form of $\bar{w}_{\mathcal{N}}, \bar{v}_{\mathcal{N}}, \bar{y}_{\mathcal{N}}$ to collect $\bar{\mathbf{w}}_{l,k}, \bar{\mathbf{v}}^s_{l,k}, \bar{\mathbf{y}}^s_{l,k}$, one has

$$\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \bar{\mathbf{w}}_{\mathcal{N}} \\ \bar{\mathbf{y}}_{\mathcal{N}} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \bar{\mathbf{w}}_{\mathcal{N}} \\ \bar{\mathbf{y}}_{\mathcal{N}} \end{bmatrix} \right\rangle = \bar{\Pi} \tag{17}$$

$$\bar{\mathcal{J}}_{\mathcal{N}} = \begin{bmatrix} \mathbf{x}_0 \\ \bar{\mathbf{w}}_{\mathcal{N}} \\ \bar{\mathbf{y}}_{\mathcal{N}} \end{bmatrix}^{\mathsf{T}} \left\langle \begin{bmatrix} \mathbf{x}_0 \\ \bar{\mathbf{w}}_{\mathcal{N}} \\ \bar{\mathbf{y}}_{\mathcal{N}} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \bar{\mathbf{w}}_{\mathcal{N}} \\ \bar{\mathbf{y}}_{\mathcal{N}} \end{bmatrix} \right\rangle^{-1} \begin{bmatrix} \mathbf{x}_0 \\ \bar{\mathbf{w}}_{\mathcal{N}} \\ \bar{\mathbf{y}}_{\mathcal{N}} \end{bmatrix}. \tag{18}$$

Remark 6: It is imperative to underscore that a meticulous analysis of (7), (16), and (17) can significantly elucidate the complex challenges that asynchronous multi-channel delays present to the problem of H_∞ estimation. This paper aims to comprehensively utilize all information received through different channels to meet the H_∞ quadratic performance index and achieve efficient estimation of $z_{l,k}$. However, due to the observations in the channel is affected by asynchronous delay. $\bar{\mathcal{J}}_{\mathcal{N}}$ cannot be reduced to a compact form, and all received observations cannot be effectively utilized to estimate $z_{l,k}$. By partitioning and reconstructing the observations, the study has fundamentally overcome the presented challenges. With the incorporation of a stochastic systems (16), the objective of this paper is to design appropriate filter within each partition and to derive sufficient conditions that meet criteria *Problem 1)* and *Problem 2)*.

4. Main Results

4.1. The Estimate of $\mathbf{x}_{l,k}$

Before proceeding *Problem 1)* and *Problem 2)* further, let us first give an estimate of $\mathbf{x}_{l,k}$, which is necessary to derive the following main results.

According to the reconstruction partition of the observations, a N-step estimator of $\mathbf{x}_{l,k}$ is proposed as follows,

<u>Step 1 : For $(l,k) \in \mathcal{N}_1$,</u>

$$\hat{\mathbf{x}}^{1-}_{l,k} = A^1_{l,k-1}\hat{\mathbf{x}}^1_{l,k-1} + A^2_{l-1,k}\hat{\mathbf{x}}^1_{l-1,k} \tag{19a}$$

$$\hat{\mathbf{x}}^1_{l,k} = \hat{\mathbf{x}}^{1-}_{l,k} + K^1_{l,k}\left(\bar{\mathbf{y}}^N_{l,k} - \bar{C}^N_{l,k}\hat{\mathbf{x}}^{1-}_{l,k}\right) \tag{19b}$$

where $\hat{\mathbf{x}}^{1-}_{l,k}$ is the one-step prediction of $\mathbf{x}_{l,k}$, $\hat{\mathbf{x}}^1_{l,k}$ is the corresponding updated estimate with the initial conditions

$\hat{\mathbf{x}}^1_{l,0}=\mathbf{x}_{l,0}$ and $\hat{\mathbf{x}}^1_{0,k}=\mathbf{x}_{0,k}$.

Step 2 : For $(l,k)\in\mathcal{N}_2$,

$$\hat{\mathbf{x}}^{2-}_{l,k}=A^1_{l,k-1}\hat{\mathbf{x}}^2_{l,k-1}+A^2_{l-1,k}\hat{\mathbf{x}}^2_{l-1,k} \tag{20a}$$

$$\hat{\mathbf{x}}^2_{l,k}=\hat{\mathbf{x}}^{2-}_{l,k}+K^2_{l,k}\left(\bar{\mathbf{y}}^{N-1}_{l,k}-\bar{C}^{N-1}_{l,k}\hat{\mathbf{x}}^{2-}_{l,k}\right) \tag{20b}$$

where $\hat{\mathbf{x}}^{2-}_{l,k}$ is the one-step prediction of $\mathbf{x}_{l,k}$, $\hat{\mathbf{x}}^2_{l,k}$ is the corresponding updated estimate with the initial conditions $\hat{\mathbf{x}}^2_{l,0}=\mathbf{x}_{l,0}(i_1+1\leqslant l\leqslant i_2),\hat{\mathbf{x}}^2_{l,j_1}=\hat{\mathbf{x}}^1_{l,j_1}(0\leqslant l\leqslant i_1)$ and $\hat{\mathbf{x}}^2_{0,k}=\mathbf{x}_{0,k}(j_1+1\leqslant k\leqslant j_2),\hat{\mathbf{x}}^2_{i_1,k}=\hat{\mathbf{x}}^1_{i_1,k}(0\leqslant k\leqslant j_1)$.

Step N : For $(l,k)\in\mathcal{N}_N$,

$$\hat{\mathbf{x}}^{N-}_{l,k}=A^1_{l,k-1}\hat{\mathbf{x}}^N_{l,k-1}+A^2_{l-1,k}\hat{\mathbf{x}}^N_{l-1,k} \tag{21a}$$

$$\hat{\mathbf{x}}^N_{l,k}=\hat{\mathbf{x}}^{N-}_{l,k}+K^N_{l,k}\left(\bar{\mathbf{y}}^1_{l,k}-\bar{C}^1_{l,k}\hat{\mathbf{x}}^{N-}_{l,k}\right) \tag{21b}$$

where $\hat{\mathbf{x}}^{N-}_{l,k}$ is the one-step prediction of $\mathbf{x}_{l,k}$, $\hat{\mathbf{x}}^N_{l,k}$ is the corresponding updated estimate with the initial conditions $\hat{\mathbf{x}}^N_{l,0}=\mathbf{x}_{l,0}(i_{N-1}+1\leqslant l\leqslant i_N),\hat{\mathbf{x}}^N_{l,j_{N-1}}=\hat{\mathbf{x}}^{N-1}_{l,j_{N-1}}(0\leqslant l\leqslant i_{N-1})$ and $\hat{\mathbf{x}}^N_{0,k}=\mathbf{x}_{0,k}(j_{N-1}+1\leqslant k\leqslant j_N),\hat{\mathbf{x}}^N_{i_{N-1},k}=\hat{\mathbf{x}}^{N-1}_{i_{N-1},k}(0\leqslant k\leqslant j_{N-1})$.

In (19)–(21), the $K^1_{l,k},K^2_{l,k},\cdots,K^N_{l,k}$ are the estimation gains to be determined.

To facilitate subsequent analysis, define and the one-step prediction error and the estimation error as,

$$\tilde{\mathbf{x}}^{\iota-}_{l,k}=x_{l,k}-\hat{\mathbf{x}}^{\iota-}_{l,k},\ \tilde{\mathbf{x}}^{\iota}_{l,k}=x_{l,k}-\hat{\mathbf{x}}^{\iota}_{l,k}$$

and the innovation,

$$\begin{aligned}\phi^\iota_{l,k}&=\bar{\mathbf{y}}^\iota_{l,k}-\bar{C}^\iota_{l,k}\hat{\mathbf{x}}^{(N+1-\iota)-}_{l,k}\\&=\bar{C}^\iota_{l,k}\tilde{\mathbf{x}}^{(N+1-\iota)-}_{l,k}+\bar{D}^\iota_{l,k}\bar{\mathbf{v}}^\iota_{l,k}.\end{aligned} \tag{22}$$

Then, by combining (16) and (19)–(21), one has

Step 1 : For $(l,k)\in\mathcal{N}_1$,

$$\begin{aligned}\tilde{\mathbf{x}}^{1-}(l,k)=&A^1_{l,k-1}\tilde{\mathbf{x}}^1_{l,k-1}+A^2_{l-1,k}\tilde{\mathbf{x}}^1_{l-1,k}\\&+\bar{B}^1_{l,k-1}\bar{\mathbf{w}}_{l,k-1}+\bar{B}^2_{l-1,k}\bar{\mathbf{w}}_{l-1,k}\end{aligned} \tag{23a}$$

$$\tilde{\mathbf{x}}^1_{l,k}=(I-K^1_{l,k}\bar{C}^N_{l,k})\tilde{\mathbf{x}}^{1-}_{l,k}-K^1_{l,k}\bar{D}^N_{l,k}\bar{\mathbf{v}}^N_{l,k} \tag{23b}$$

Step 2 : For $(l,k)\in\mathcal{N}_2$,

$$\begin{aligned}\tilde{\mathbf{x}}^{2-}_{l,k}=&A^1_{l,k-1}\tilde{\mathbf{x}}^2_{l,k-1}+A^2_{l-1,k}\tilde{\mathbf{x}}^2_{l-1,k}\\&+\bar{B}^1_{l,k-1}\bar{\mathbf{w}}_{l,k-1}+\bar{B}^2_{l-1,k}\bar{\mathbf{w}}_{l-1,k}\end{aligned} \tag{24a}$$

$$\tilde{\mathbf{x}}^2_{l,k}=(I-K^2_{l,k}\bar{C}^{N-1}_{l,k})\tilde{\mathbf{x}}^{2-}_{l,k}-K^2_{l,k}\bar{D}^{N-1}_{l,k}\bar{\mathbf{v}}^{N-1}_{l,k} \tag{24b}$$

Step N : For $(l,k)\in\mathcal{N}_N$,

$$\begin{aligned}\tilde{\mathbf{x}}^{N-}_{l,k}=&A^1_{l,k-1}\tilde{\mathbf{x}}^N_{l,k-1}+A^2_{l-1,k}\tilde{\mathbf{x}}^N_{l-1,k}\\&+\bar{B}^1_{l,k-1}\bar{\mathbf{w}}_{l,k-1}+\bar{B}^2_{l-1,k}\bar{\mathbf{w}}_{l-1,k}\end{aligned} \tag{25a}$$

$$\tilde{\mathbf{x}}^N_{l,k}=(I-K^N_{l,k}\bar{C}^1_{l,k})\tilde{\mathbf{x}}^{N-}_{l,k}-K^N_{l,k}\bar{D}^1_{l,k}\bar{\mathbf{v}}^1_{l,k} \tag{25a}$$

The following lemma details that the proposed N-step estimator (19)–(21) can achieve an unbiased estimate of $\mathbf{x}_{l,k}$.

Lemma 3: For the stochastic systems (16), the proposed N-step estimator of the form (19)–(21) are unbiased.

Proof: To prove the unbiasedness of the proposed estimator, we need to prove that (23)–(25) are zero-mean variable. Prove it separately in each partition, the proof resembles that in [32], it is unmentioned. □

To facilitate subsequent discussions, define

$$\mathbf{P}^{\iota-}_{l,k}\triangleq\mathcal{E}\{\tilde{\mathbf{x}}^{\iota-}_{l,k}(\tilde{\mathbf{x}}^{\iota-}_{l,k})^{\mathsf{T}}\} \tag{26}$$

$$\mathbf{P}^{\iota}_{l,k}\triangleq\mathcal{E}\{\tilde{\mathbf{x}}^{\iota}_{l,k}(\tilde{\mathbf{x}}^{\iota}_{l,k})^{\mathsf{T}}\} \tag{27}$$

The following lemma gives the recursive evolution of one-step prediction error covariance and estimation error covariance.

Lemma 4: For the stochastic systems (16), the one-step prediction error covariance $\mathbf{P}^{\iota-}_{l,k}$ and estimation error covariance $\mathbf{P}^{\iota}_{l,k}$ are the solution of the following Riccati-like equations,

Step 1 : For $(l,k)\in\mathcal{N}_1$,

$$\begin{aligned}\mathbf{P}^{1-}_{l,k} =& A^1_{l,k-1}\mathbf{P}^1_{l,k-1}(A^1_{l,k-1})^{\top} + A^2_{l-1,k}\mathbf{P}^1_{l-1,k}(A^2_{l-1,k})^{\top}\\ &+\bar{B}^1_{l,k-1}\bar{R}_{l,k-1}(\bar{B}^1_{l,k-1})^{\top}+\bar{B}^2_{l-1,k}\bar{R}_{l-1,k}(\bar{B}^2_{l-1,k})^{\top}\\ &+A^1_{l,k-1}\mathcal{E}\{\tilde{\mathbf{x}}^1_{l,k-1}(\tilde{\mathbf{x}}^1_{l-1,k})^{\top}\}(A^2_{l-1,k})^{\top}\\ &+A^2_{l-1,k}\mathcal{E}\{\tilde{\mathbf{x}}^1_{l-1,k}(\tilde{\mathbf{x}}^1_{l,k-1})^{\top}\}(A^1_{l,k-1})^{\top}\end{aligned} \tag{28a}$$

$$\begin{aligned}\mathbf{P}^1_{l,k} =& \left(I-K^1_{l,k}\bar{C}^N_{l,k}\right)\mathbf{P}^{1-}_{l,k}\left(I-K^1_{l,k}\bar{C}^N_{l,k}\right)^{\top}\\ &+K^1_{l,k}\bar{D}^N_{l,k}\bar{Q}^N_{l,k}(\bar{D}^N_{l,k})^{\top}(K^1_{l,k})^{\top}\end{aligned} \tag{28b}$$

with the initial conditions $\mathbf{P}^1_{l,0}=\mathbf{P}_{l,0}$ and $\mathbf{P}^1_{0,k}=\mathbf{P}_{0,k}$.

Step 2 : For $(l,k)\in\mathcal{N}_2$,

$$\begin{aligned}\mathbf{P}^{2-}_{l,k} =& A^1_{l,k-1}\mathbf{P}^2_{l,k-1}(A^1_{l,k-1})^{\top} + A^2_{l-1,k}\mathbf{P}^2_{l-1,k}(A^2_{l-1,k})^{\top}\\ &+\bar{B}^1_{l,k-1}\bar{R}_{l,k-1}(\bar{B}^1_{l,k-1})^{\top}+\bar{B}^2_{l-1,k}\bar{R}_{l-1,k}(\bar{B}^2_{l-1,k})^{\top}\\ &+A^1_{l,k-1}\mathcal{E}\{\tilde{\mathbf{x}}^2_{l,k-1}(\tilde{\mathbf{x}}^2_{l-1,k})^{\top}\}(A^2_{l-1,k})^{\top}\\ &+A^2_{l-1,k}\mathcal{E}\{\tilde{\mathbf{x}}^2_{l-1,k}(\tilde{\mathbf{x}}^2_{l,k-1})^{\top}\}(A^1_{l,k-1})^{\top}\end{aligned} \tag{29a}$$

$$\begin{aligned}\mathbf{P}^2_{l,k} =& \left(I-K^2_{l,k}\bar{C}^{N-1}_{l,k}\right)\mathbf{P}^{2-}_{l,k}\left(I-K^2_{l,k}\bar{C}^{N-1}_{l,k}\right)^{\top}\\ &+K^2_{l,k}\bar{D}^{N-1}_{l,k}\bar{Q}^{N-1}(l,k)(\bar{D}^{N-1}_{l,k})^{\top}K^{\top}_2(l,k)\end{aligned} \tag{29b}$$

with the initial conditions $\mathbf{P}^2_{l,0}=\mathbf{P}_{l,0}(i_1+1\leqslant l\leqslant i_2),\mathbf{P}^2_{l,j_1}=\mathbf{P}^1_{l,j_1}(0\leqslant l\leqslant i_1)$ and $\mathbf{P}^2_{0,k}=\mathbf{P}_{0,k}(j_1+1\leqslant k\leqslant j_2),\mathbf{P}^2_{i_1,k}=\mathbf{P}^1_{i_1,k}(0\leqslant k\leqslant j_1)$.

Step N : For $(l,k)\in\mathcal{N}_N$,

$$\begin{aligned}\mathbf{P}^{N-}_{l,k} =& A^1_{l,k-1}\mathbf{P}^N_{l,k-1}(A^1_{l,k-1})^{\top} + A^2_{l-1,k}\mathbf{P}^N_{l-1,k}(A^2_{l-1,k})^{\top}\\ &+\bar{B}^1_{l,k-1}\bar{R}_{l,k-1}(\bar{B}^1_{l,k-1})^{\top}+\bar{B}^2_{l-1,k}\bar{R}_{l-1,k}(\bar{B}^2_{l-1,k})^{\top}\\ &+A^1_{l,k-1}\mathcal{E}\{\tilde{\mathbf{x}}^N_{l,k-1}(\tilde{\mathbf{x}}^N_{l-1,k})^{\top}\}(A^2_{l-1,k})^{\top}\\ &+A^2_{l-1,k}\mathcal{E}\{\tilde{\mathbf{x}}^N_{l-1,k}(\tilde{\mathbf{x}}^N_{l,k-1})^{\top}\}(A^1_{l,k-1})^{\top}\end{aligned} \tag{30a}$$

$$\begin{aligned}\mathbf{P}^N_{l,k} =& \left(I-K^2_{l,k}\bar{C}^{N-1}_{l,k}\right)\mathbf{P}^{N-}_{l,k}\left(I-K^N_{l,k}\bar{C}^1_{l,k}\right)^{\top}\\ &+K^N_{l,k}\bar{D}^1_{l,k}\bar{Q}^1_{l,k}(\bar{D}^1_{l,k})^{\top}(K^N_{l,k})^{\top}\end{aligned} \tag{30b}$$

with the initial conditions $\mathbf{P}^N_{l,0}=\mathbf{P}_{l,0}(i_{N-1}+1\leqslant l\leqslant i_N),\mathbf{P}^N_{l,j_{N-1}}=\mathbf{P}^{N-1}_{l,j_{N-1}}(0\leqslant l\leqslant i_{N-1})$ and $\mathbf{P}^N_{0,k}=\mathbf{P}_{0,k}(j_{N-1}+1\leqslant k\leqslant j_N)$, $\mathbf{P}^N_{i_{N-1},k}=\mathbf{P}^{N-1}_{i_{N-1},k}(0\leqslant k\leqslant j_{N-1})$.

Proof: For simplicity, only {Step 1} is verified. In terms of the statistical property of $\bar{\mathbf{w}}_{l,k}$ and $\bar{\mathbf{v}}^{\iota}_{l,k}$, $\tilde{\mathbf{x}}^{(N+1-\iota)-}_{l,k}$ is uncorrelated with $\bar{\mathbf{w}}_{l,k}$ and $\tilde{\mathbf{x}}^{(N+1-\iota)-}_{l,k}$ is uncorrelated with $\bar{\mathbf{v}}^{\iota}_{l,k}$ for every $(l,k)\in\{(e,s)|e>l \text{ or } s>k\}\cup(l,k)\subseteq\mathcal{N}_1$, substituting (23b) and (23c) into (25) and (26), it is easy to know that (28a) and (28b) are hold. □

In the following theorem, through the completing squares approach the estimation gain parameters are given in the minimum mean square sense.

Theorem 1: For the stochastic systems (16), the recursive evolution of the estimation gain parameters and minimum estimation error covariance are given as follows:

Step 1 : For $(l,k)\in\mathcal{N}_1$,

$$\begin{aligned}K^1_{l,k} =&\ \mathbf{P}^{1-}_{l,k}(\bar{C}^N_{l,k})^{\top}\left(\bar{\mathcal{Q}}^N\left(\mathbf{P}^{1-}_{l,k}\right)\right)^{-1}\\ \mathbf{P}^1_{l,k} =&\ \mathbf{P}^{1-}_{l,k}-K^1_{l,k}\bar{C}^N_{l,k}\mathbf{P}^{1-}_{l,k}\end{aligned} \tag{31a}$$

Step 2 : For $(l,k)\in\mathcal{N}_2$,

$$K^2_{l,k}=\mathbf{P}^{2-}_{l,k}(\bar{C}^{N-1}_{l,k})^{\top}\left(\bar{\mathcal{Q}}^{N-1}\left(\mathbf{P}^{2-}_{l,k}\right)\right)^{-1} \tag{32a}$$

$$\mathbf{P}^2_{l,k}=\mathbf{P}^{2-}_{l,k}-K^2_{l,k}\bar{C}^{N-1}_{l,k}\mathbf{P}^{2-}_{l,k} \tag{32b}$$

Step N : For $(l,k)\in\mathcal{N}_N$,

$$K^N_{l,k}=\mathbf{P}^{N-}_{l,k}(\bar{C}^1_{l,k})^{\mathsf{T}}\left(\bar{\mathcal{Q}}^1\left(\mathbf{P}^{N-}_{l,k}\right)\right)^{-1} \tag{33a}$$

$$\mathbf{P}^N_{l,k}=\mathbf{P}^{N-}_{l,k}-K^N_{l,k}\bar{C}^1_{l,k}\mathbf{P}^{N-}_{l,k} \tag{33b}$$

where

$$\bar{\mathcal{Q}}^\iota\left(\mathbf{P}^{(N+1-\iota)-}_{l,k}\right)=\bar{C}^\iota_{l,k}\mathbf{P}^{(N+1-\iota)-}_{l,k}(\bar{C}^\iota_{l,k})^{\mathsf{T}}+\bar{D}^\iota_{l,k}\bar{Q}^\iota_{l,k}(\bar{D}^\iota_{l,k})^{\mathsf{T}}.$$

Proof: In view of (23), it is easy to conclude that

$$\begin{aligned}\mathbf{P}^1_{l,k}=&\mathbf{P}^{1-}_{l,k}-K^1_{l,k}\bar{C}^N_{l,k}\mathbf{P}^{1-}_{l,k}-\mathbf{P}^{1-}_{l,k}\\&\times(\bar{C}^N_{l,k})^{\mathsf{T}}(K^1_{l,k})^{\mathsf{T}}+K^1_{l,k}\bar{\mathcal{Q}}^1\left(\mathbf{P}^{1-}_{l,k}\right)(K^1_{l,k})^{\mathsf{T}},\end{aligned}$$

by completing squares approach on the above equation, one has

$$\begin{aligned}\mathbf{P}^1_{l,k}=&\mathbf{P}^{1-}_{l,k}+\left[K^1_{l,k}-\mathcal{K}^1_{l,k}\right]\bar{\mathcal{Q}}^1\left(\mathbf{P}^{1-}_{l,k}\right)\left[K^1_{l,k}-\mathcal{K}^1_{l,k}\right]^{\mathsf{T}}\\&-\mathcal{K}^1_{l,k}\bar{\mathcal{Q}}^1\left(\mathbf{P}^{1-}_{l,k}\right)(\mathcal{K}^1_{l,k})^{\mathsf{T}}.\end{aligned}$$

It is straightforward to see that $\mathbf{P}^1_{l,k}$ is minimized if and only if $K^1_{l,k}=\mathcal{K}^1_{l,k}$, (31a) is valid. The above formula can be further expressed as,

$$\begin{aligned}\mathbf{P}^1_{l,k}&=\mathbf{P}^{1-}_{l,k}-\mathcal{K}^1_{l,k}\bar{\mathcal{Q}}^1\left(\mathbf{P}^{1-}_{l,k}\right)(\mathcal{K}^1_{l,k})^{\mathsf{T}}\\&=\mathbf{P}^{1-}_{l,k}-K^1_{l,k}\bar{C}^N_{l,k}\mathbf{P}^{1-}_{l,k}.\end{aligned}$$

Thereby, the assert (31b) holds. Other partitions can be proved similarly. That completes the proof of this theorem. □

Remark 7: This paper uses compact technology to convert the modified quadratic performance function into a certain equivalent minimal problem of indefinite quadratic form. Due to the observation of the Krein space stochastic system being divided into N regions for reconstruction, an N-step recursive estimator of $\mathbf{x}_{l,k}$ is proposed. Meanwhile, Lemma 4 reveals that the estimation error covariance for each region is recursively computed, and the estimation error covariance of adjacent regions provides the recursive initial value for the next region. For the H_∞ estimation problem considered in this paper, the estimation scheme for $\mathbf{x}_{l,k}$ that we have developed offers the advantages of being less conservative in the estimation algorithm and recursively verifiable.

Subsequently, some sufficient conditions are established to guarantee the boundedness of the minimized filtering error covariance (i.e., $\mathbf{P}^\iota_{l,k},\iota=1,2,\cdots,N$). For simplicity, we only discuss $(l,k)\in\mathcal{N}^\circ_0$. To begin with, let us make the following assumption.

Assumption 4: For every $l,k\in\mathcal{N}^o_0,\iota\in[1,2]$, and given positive scalars $\bar{a}^{(\iota)},\bar{b}^\iota,\bar{r}$ and o, the following inequalities hold:

$$\|\bar{A}^{(\iota)}_{l,k}\|\leqslant\bar{a}^{(\iota)},\ \|\bar{B}^{(\iota)}_{l,k}\|\leqslant\bar{b}^{(\iota)},\ \|\bar{R}_{l,k}\|\leqslant\bar{r}.$$

Lemma 5: Under Assumption 1, the minimal filtering error covariance obeys the following inequality:

For $(l,k)\in\mathcal{N}^\circ_0$,

$$\begin{aligned}\|\mathbf{P}^1_{l,k}\|\leqslant&\sum_{s=1}^{l}\psi_1\chi_{l-s,k-1}\|\mathbf{P}^1_{s,0}\|+\sum_{t=1}^{l}\psi_2\chi_{l-1,k-t}\|\mathbf{P}^1_{0,t}\|\\&+\sum_{s=0}^{l-1}\sum_{t=0}^{k-1}\chi_{l-s-1,k_2-t-1}\varpi_0.\end{aligned} \tag{34}$$

where $\psi_1=(1+o)(\bar{a}^{(1)})^2,\psi_2=(1+o^{-1})(\bar{a}^{(2)})^2$, and $\varpi_0=((\bar{b}^{(1)})^2+\bar{b}^{(2)})^2\bar{r}$, $\chi(\cdot,\cdot)$ with $\chi(0,0)=1$ can be recursively calculated by

$$\begin{aligned}&\chi_{0,k}=\psi_1\chi_{0,k-1},\ \chi_{l,0}=\psi_2\chi_{l-1,0},\\&\chi_{l,k}=\psi_1\chi_{l,k-1}+\psi_2\chi_{l-1,k}.\end{aligned}$$

Proof: Reviewing the the one-step prediction error covariance, it is easy to know that

$$\|\mathbf{P}^{1-}_{l,k}\| \leqslant (1+o)(\bar{a}^{(1)})^2\|\mathbf{P}^{1}_{l,k-1}\| + (1+o^{-1})(\bar{a}^{(2)})^2\|\mathbf{P}^{1}_{l-1,k}\| + ((\bar{b}^{(1)})^2 + +\bar{b}^{(2)})^2)\bar{r}.$$

Then, according to the minimal filtering error covariance,

$$\|\mathbf{P}^{1}_{l,k}\| \leqslant \|\mathbf{P}^{1-}_{l,k}\| \leqslant \psi_1\|\mathbf{P}^{1}_{l,k-1}\| + \psi_2\|\mathbf{P}^{1}_{l-1,k}\| + \varpi_0. \tag{35}$$

In the following, we would provide a rigorous mathematical induction to prove (34).

Initial Step: From (35), it is clear that (34) is true for $(l,k)=(1,1)$.

Inductive Step: Suppose that (4) holds for $(l,k)\in\{(l_0,k_0)|1\leqslant l_0\leqslant i_1, 1\leqslant k_0\leqslant j_1; l_0+k_0=\mathfrak{b}\}$ with $\mathfrak{b}\in[2, i_1+j_1-1]$. Then, we need to check if (34) holds for $(l,k)\in\{(l_0,k_0)|1\leqslant l_0\leqslant i_1, 1\leqslant k_0\leqslant j_1; l_0+k_0=\mathfrak{b}+1\}$. Clearly, one has

$$\begin{aligned}
\mathbf{P}^{1}_{l,k} \leqslant\ & \psi_1\|\mathbf{P}^{1}_{l,k-1}\| + \psi_2\|\mathbf{P}^{1}_{l-1,k}\| \\
\leqslant\ & \psi_1\left\{\sum_{s=1}^{l}\psi_1\chi_{l-s,k-2}\|\mathbf{P}^{1}_{s,0}\| + \sum_{t=1}^{k-1}\psi_2\chi_{l-1,k-t-1}\right. \\
& \left.\times\|\mathbf{P}^{1}_{0,t}\| + \sum_{s=0}^{l-1}\sum_{t=0}^{k-2}\chi_{l-s-1,k-t-2}\varpi_0\right\} \\
& + \psi_2\left\{\sum_{s=1}^{l-1}\psi_1\chi_{l-s-1,k-1}\|\mathbf{P}^{1}_{s,0}\| + \sum_{t=1}^{k}\psi_2\chi_{l-2,k-t}\right. \\
& \left.\times\|\mathbf{P}^{1}_{0,t}\| + \sum_{s=0}^{l-2}\sum_{t=0}^{k-1}\chi_{l-s-2,k-t-1}\varpi_0\right\} + \varpi_0 \\
=\ & \sum_{s=1}^{l}\psi_1\chi_{l-s,k-1}\|\mathbf{P}^{1}_{s,0} + \sum_{t=1}^{k}\psi_2\chi_{l-1,k-t}\|\mathbf{P}^{1}_{0,t}\| \\
& + \sum_{s=0}^{l-1}\sum_{t=0}^{k-1}\chi_{l-s-1,k-t-1}\varpi_0.
\end{aligned}$$

The proof is now complete. □

4.2. The Sufficient and Necessary Condition for Problem 1)

On the basis of the above discussion, we will give the detailed derivation of the necessary and sufficient conditions for *Problem 1)* as follows.

Theorem 2: For a given scalar $\gamma>0$, the final existence condition holds if and only if the following steps are satisfied:

Step 1 : For $(l,k)\in\mathcal{N}_1$,

$$\text{①}\, C^N_{l,k}\mathbf{P}^{1-}_{l,k}(C^N_{l,k})^{\mathsf{T}} + Q^N_{l,k} + N^N_{l,k}(N^N_{l,k})^{\mathsf{T}} > 0$$

$$\begin{aligned}
\text{②}\, & \begin{bmatrix} E_{l,k} \\ T_{l,k} \end{bmatrix}\mathbf{P}^{1-}_{l,k}\begin{bmatrix} E_{l,k} \\ T_{l,k} \end{bmatrix}^{\mathsf{T}} + \begin{bmatrix} -I & \mathbf{0} \\ \mathbf{0} & -\gamma^2 I \end{bmatrix} \\
& - \begin{bmatrix} E_{l,k} \\ T_{l,k} \end{bmatrix}\mathbf{P}^{1-}_{l,k}(C^N_{l,k})^{\mathsf{T}}\left[C^N_{l,k}\mathbf{P}^{1-}_{l,k}(C^N_{l,k})^{\mathsf{T}}\right. \\
& \left. + Q^N_{l,k} + N^N_{l,k}(N^N_{l,k})^{\mathsf{T}}\right]^{-1} C^N_{l,k}\mathbf{P}^{1-}_{l,k}\begin{bmatrix} E_{l,k} \\ T_{l,k} \end{bmatrix}^{\mathsf{T}} < 0
\end{aligned}$$

Step 2 : For $(l,k)\in\mathcal{N}_2$,

$$\text{①}\, C^{N-1}_{l,k}\mathbf{P}^{2-}_{l,k}(C^{N-1}_{l,k})^{\mathsf{T}} + Q^{N-1}_{l,k} + N^{N-1}_{l,k}(N^{N-1}_{l,k})^{\mathsf{T}} > 0$$

$$\begin{aligned}
\text{②}\, & \begin{bmatrix} E_{l,k} \\ T_{l,k} \end{bmatrix}\mathbf{P}^{2-}_{l,k}\begin{bmatrix} E_{l,k} \\ T_{l,k} \end{bmatrix}^{\mathsf{T}} + \begin{bmatrix} -I & \mathbf{0} \\ \mathbf{0} & -\gamma^2 I \end{bmatrix} \\
& - \begin{bmatrix} E_{l,k} \\ T_{l,k} \end{bmatrix}\mathbf{P}^{2-}_{l,k}(C^{N-1}_{l,k})^{\mathsf{T}}\left[C^{N-1}_{l,k}\mathbf{P}^{2-}_{l,k}(C^{N-1}_{l,k})^{\mathsf{T}}\right. \\
& \left. + Q^{N-1}_{l,k} + N^{N-1}_{l,k}(N^{N-1}_{l,k})^{\mathsf{T}}\right]^{-1} C^{N-1}_{l,k}\mathbf{P}^{2-}_{l,k}\begin{bmatrix} E_{l,k} \\ T_{l,k} \end{bmatrix}^{\mathsf{T}} < 0
\end{aligned}$$

Step N : For $(l,k)\in\mathcal{N}_N$,

$$
\begin{aligned}
&①C^1_{l,k}\mathbf{P}^{N-}_{l,k}(C^1_{l,k})^{\mathsf{T}}+Q^1_{l,k}+N^1_{l,k}(N^1_{l,k})^{\mathsf{T}}>0\\
&② \begin{bmatrix}E_{l,k}\\T_{l,k}\end{bmatrix}\mathbf{P}^{N-}_{l,k}\begin{bmatrix}E_{l,k}\\T_{l,k}\end{bmatrix}^{\mathsf{T}}+\begin{bmatrix}-I&\mathbf{0}\\\mathbf{0}&-\gamma^2I\end{bmatrix}\\
&\quad-\begin{bmatrix}E_{l,k}\\T_{l,k}\end{bmatrix}\mathbf{P}^{N-}_{l,k}(C^1_{l,k})^{\mathsf{T}}\left[C^1_{l,k}\mathbf{P}^{N-}_{l,k}(C^1_{l,k})^{\mathsf{T}}\right.\\
&\quad\left.+Q^1_{l,k}+N^1_{l,k}(N^1_{l,k})^{\mathsf{T}}\right]^{-1}C^1_{l,k}\mathbf{P}^{N-}_{l,k}\begin{bmatrix}E_{l,k}\\T_{l,k}\end{bmatrix}^{\mathsf{T}}<0
\end{aligned}
$$

$\bar{\mathcal{J}}_{\mathcal{N}}$ has a minimum on $x_0,\{w_{l,k}|(l,k)\in\mathcal{N}\}$, and $\{\{v^N_{l,k}|(l,k)\in\mathcal{N}_1\};\cdots;\{v^1_{l,k}|(l,k)\in\mathcal{N}_N\}\}$ as

$$
\begin{aligned}
\bar{\mathcal{J}}^*_{\mathcal{N}}=&\sum\nolimits_{\iota=1}^{N}\sum\nolimits_{(l,k)\in\mathcal{N}_\iota}\{(\bar{y}^{N+1-\iota}_{l,k}-\hat{\bar{y}}^{N+1-\iota}_{l,k})^{\mathsf{T}}\\
&\times\langle\phi^{N+1-\iota}_{l,k},\phi^{N+1-\iota}_{l,k}\rangle^{-1}(\bar{y}^{N+1-\iota}_{l,k}-\hat{\bar{y}}^{N+1-\iota}_{l,k})\}
\end{aligned}
\tag{36}
$$

where $\hat{\bar{y}}^{\iota}_{l,k}=\bar{C}^{\iota}_{l,k}\hat{x}^{(N+1-\iota)-}_{l,k}$.

Proof: See Appendix B. □

Remark 8: In fact, Theorem 2 is essential for the estimation problems considered in this paper. Only if it can be guaranteed that the modified uncertain quadratic performance function $\mathcal{J}_{\mathcal{N}}$ has a minimum value with respect to $x_0,\{w_{l,k}|(l,k)\in\mathcal{N}\}$, and $\{\{v^N_{l,k}|(l,k)\in\mathcal{N}_1\};\cdots;\{v^1_{l,k}|(l,k)\in\mathcal{N}_N\}\}$, can a robust filter be further designed to realize that the minimum value is positive when $E_{l,k}=0$. Therefore, the verification of Theorem 2 should be given priority in the simulation results.

4.3. The robust H_∞ filter

Below, the design detail of the robust filter is given under the condition that *problem 2)* is established.

Theorem 3: For the systems (10), the 2D robust H_∞ filter is

Step 1 : For $(l,k)\in\mathcal{N}_1$,

$$
\hat{z}^1_{l,k}=T_{l,k}\hat{x}^{1-}_{l,k}+H^N_{l,k}\left(y^N_{l,k}-C^N_{l,k}\hat{x}^{1-}_{l,k}\right)\tag{37a}
$$

$$
\begin{aligned}
H^N_{l,k}=&T_{l,k}\mathbf{P}^{1-}_{l,k}(C^N_{l,k})^{\mathsf{T}}\left(C^N_{l,k}\mathbf{P}^{1-}_{l,k}(C^N_{l,k})^{\mathsf{T}}\right.\\
&\left.+Q^N_{l,k}+N^N_{l,k}(N^N_{l,k})^{\mathsf{T}}\right)^{-1}
\end{aligned}
\tag{37b}
$$

with the initial conditions $\hat{z}^1_{l,0}=z_{l,0}$ and $\hat{z}^1_{0,k}=z_{0,k}$.

Step 2 : For $(l,k)\in\mathcal{N}_2$,

$$
\hat{z}^2_{l,k}=T_{l,k}\hat{x}^{2-}_{l,k}+H^{N-1}_{l,k}\left(y^{N-1}_{l,k}-C^{N-1}_{l,k}\hat{x}^{2-}_{l,k}\right)\tag{38a}
$$

$$
\begin{aligned}
H^{N-1}_{l,k}=&T_{l,k}\mathbf{P}^{2-}_{l,k}(C^{N-1}_{l,k})^{\mathsf{T}}\left(C^{N-1}_{l,k}\mathbf{P}^{2-}_{l,k}(C^{N-1}_{l,k})^{\mathsf{T}}\right.\\
&\left.+Q^{N-1}_{l,k}+N^{N-1}_{l,k}(N^{N-1}_{l,k})^{\mathsf{T}}\right)^{-1}
\end{aligned}
\tag{38b}
$$

with the initial conditions $\hat{z}^2_{l,0}=z_{l,0}(i_1+1\leqslant l\leqslant i_2),\hat{z}^2_{l,j_1}=\hat{z}^1_{l,j_1}(0\leqslant l\leqslant i_1)$ and $\hat{z}^2_{0,k}=z_{0,k}(j_1+1\leqslant k\leqslant j_2),\hat{z}^2_{i_1,k}=\hat{z}^1_{i_1,k}$ $(0\leqslant k\leqslant j_1)$.

Step N : For $(l,k)\in\mathcal{N}_N$,

$$
\hat{z}^N_{l,k}=T_{l,k}\hat{x}^{N-}_{l,k}+H^1_{l,k}\left(y^1_{l,k}-C^1_{l,k}\hat{x}^{N-}_{l,k}\right)\tag{39a}
$$

$$
\begin{aligned}
H^1_{l,k}=&T_{l,k}\mathbf{P}^{N-}_{l,k}(C^1_{l,k})^{\mathsf{T}}\left(C^1_{l,k}\mathbf{P}^{N-}_{l,k}(C^1_{l,k})^{\mathsf{T}}\right.\\
&\left.+Q^1_{l,k}+N^1_{l,k}(N^1_{l,k})^{\mathsf{T}}\right)^{-1}
\end{aligned}
\tag{39b}
$$

with the initial conditions $\hat{z}^N_{l,0}=z_{l,0}(i_{N-1}+1\leqslant l\leqslant i_N),\hat{z}^N_{l,j_{N-1}}=\hat{z}^{N-1}_{l,j_{N-1}}(0\leqslant l\leqslant i_{N-1})$ and $\hat{z}^N_{0,k}=z_{0,k}(j_{N-1}+1\leqslant k\leqslant j_N),\hat{z}^N_{i_{N-1},k}=\hat{z}^{N-1}_{i_{N-1},k}(0\leqslant k\leqslant j_{N-1})$.

Proof: See Appendix C. □

Remark 9: It is worth noting that the robust filter (37)-(39) established is not unique as can be seen from (46), but this paper provides a fixed form of partition recursive calculation. $\hat{x}^{\iota-}_{l,k}$ can be obtained from the estimate of $\mathbf{x}_{l,k}$ in the stochastic systems (16) (i.e. $\hat{x}_{l,k}^{\iota-} = \hat{\mathbf{x}}_{l,k}^{\iota-}$). Theorem 1 and Theorem 3 give the H_∞ recursive estimation algorithm for 2D uncertain systems with multi-channel observation delays. Theorem 2 gives the necessary and sufficient conditions to ensure that the estimation algorithm is established.

Remark 10: The core idea of this paper is to appropriately design the quadratic performance function $J_{\mathcal{N}}$, and dualize the original system to an auxiliary stochastic system in Krein space, achieving the correspondence between the weight matrix of the l_2-norm in the quadratic performance function and the covariance matrix of the stochastic system variables in Krein space. However, due to the influence of external random disturbance and incomplete information, the quadratic performance function $J_{\mathcal{N}}$ established solely for the H_∞ index does not fully reflect the uncertainty factors of the system. This paper makes two modifications to the quadratic performance function $J_{\mathcal{N}}$, respectively, for external random disturbance and the incomplete information caused by multi-channel delays. The first modification adds a random disturbance term to the quadratic performance function, ensuring that disturbances are considered synchronously when the original system is equivalently dualized according to the quadratic performance function, thus avoiding variable scaling or signal enhancement. The second modification replaces the original observation and observation noise sequences within the quadratic performance function with reconstructed observation and observation noise sequences. This ensures that information is complete in the real-time recursive computation of the estimation algorithm.

Remark 11: The specific algorithm operating in the $\mathcal{N}_1$ region is given in Algorithm 1.

Algorithm 1: The $\mathcal{H}_\infty$ estimation algorithm in $\mathcal{N}_1$ region

Input: Initial conditions, compact partial parameters, and $\{y_{0,0}^N, \cdots, y_{i_1,j_1}^N\}, \{v_{0,0}^N, \cdots, v_{i_1,j_1}^N\}$

Output: The signal $\hat{z}_{i,j}^1$ and H_∞ performance index $\mathcal{J}_{\mathcal{N}\infty}^*$

Set $i = 1$ and $j = 1$;

while $i \in [1, i_1]$ and $j \in [1, j_1]$ **do**

  Calculate one-step prediction $\hat{\mathbf{x}}_{i,j}^{1-}$;

  Calculate one-step prediction error $\tilde{\mathbf{x}}_{i,j}^{1-}$;

  Calculate one-step prediction error covariance $\mathbf{P}_{i,j}^{1-}$;

  Calculate estimation gains $K_{i,j}^1$ and $H_{i,j}^1$;

  Update estimation signal $\hat{z}_{i,j}^1$;

  Compact parameters $\bar{y}_{i,j}^N$ and $\bar{v}_{i,j}^N$;

  Update estimation signal $\hat{\mathbf{x}}_{i,j}^1$;

  Update estimation error $\tilde{\mathbf{x}}_{i,j}^1$;

  Update estimation error covariance $\mathbf{P}_{i,j}^1$;

  Calculate H_∞ performance index $\mathcal{J}_{\mathcal{N}_1}^*$;

  if $i \in [1, i_1]$ and $j = j_1$ **then**

    Set $i = i + 1$ and $j = 1$;

  else

    Set $j = j + 1$;

  end

end

Return $\hat{z}_{i,j}^1$ and $\mathcal{J}_{\mathcal{N}_1}^*$.

4.4. Computational Complexity Analysis

This section compares the computational cost (floating-point operations of the algorithm) of the traditional augmentation approach and the partition reconstruction approach proposed in this paper.

To facilitate this, suppose that $N = 2$.

Note the *Problem* in this paper can be solved by introducing the augmentation state as ($x_{i,j}^a \in \mathbf{R}^{n_x(2\varsigma_1\tau_1+\varsigma_1+\tau_1+1)}$)

$$
\begin{aligned}
x_{i,j}^a = \big[& x_{i,j}^{\mathsf{T}}\ x_{i-1,j}^{\mathsf{T}} \cdots x_{i-\varsigma_1,j}^{\mathsf{T}}\ x_{i,j-1}^{\mathsf{T}} \cdots x_{i,j-\tau_1}^{\mathsf{T}} \\
& x_{i-1,j-1}^{\mathsf{T}} \cdots x_{i-\varsigma_1,j-1}^{\mathsf{T}}\ x_{i-1,j-1}^{\mathsf{T}} \cdots x_{i-1,j-\tau_1}^{\mathsf{T}} \\
& x_{i-1,j-2}^{\mathsf{T}} \cdots x_{i-\varsigma_1,j-2}^{\mathsf{T}}\ x_{i-2,j-1}^{\mathsf{T}} \cdots x_{i-2,j-\tau_1}^{\mathsf{T}} \\
& \quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\
& x_{i-1,j-\tau_1}^{\mathsf{T}} \cdots x_{i-\varsigma_1,j-\tau_1}^{\mathsf{T}}\ x_{i-\varsigma_1,j-1}^{\mathsf{T}} \cdots x_{i-\varsigma_1,j-\tau_1}^{\mathsf{T}} \big]^{\mathsf{T}}
\end{aligned}
$$

Then, the augmented shift-varying 2D FMLSS systems is

$$\begin{aligned} x^a_{i,j} &= (A^{a1}_{i,j-1} + \check{A}^{a1}_{i,j-1})x^a_{i,j-1} \\ &\quad + (A^{a2}_{i-1,j} + \check{A}^{a2}_{i-1,j})x^a_{i-1,j} \\ &\quad + B^{a1}_{i,j-1}w^a_{i,j-1} + B^{a2}_{i-1,j}w^a_{i-1,j} \\ y^a_{i,j} &= (C^a_{i,j} + \check{C}^a_{i,j})x^a_{i,j} + v^a_{i,j} \\ z^a_{i,j} &= T^a_{i,j}x^a_{i,j} \end{aligned}$$

where

$$A^{a1}_{i,j} = \left[\begin{array}{cccc|ccc} A^1_{i,j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{array}\right], \check{A}^{a1}_{i,j} = \left[\begin{array}{c|c} \check{A}^1_{i,j} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right].$$

$$A^{a2}_{i,j} = \left[\begin{array}{cccc|ccc} A^2_{i,j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right], \check{A}^{a2}_{i,j} = \left[\begin{array}{c|c} \check{A}^2_{i,j} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right].$$

$$B^{a1}_{i,j} = \left[\begin{array}{c|c} B^1_{i,j} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right], B^{a2}_{i,j} = \left[\begin{array}{c|c} B^2_{i,j} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right], w_{i,j} = \left[\begin{array}{c} w_{i,j} \\ \hline \mathbf{0} \end{array}\right]$$

$$y^a_{i,j} = \left[\begin{array}{c} y^{(1)}_{i,j} \\ \hline y^{(2)}_{i,j} \end{array}\right], C^a_{i,j} = \left[\begin{array}{cc|c} C^{(1)}_{i,j} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & C^{(2)}_{i-\varsigma_1,j-\tau_1} \end{array}\right]$$

$$\check{C}^a_{i,j} = \left[\begin{array}{cc|c} \check{C}^{(1)}_{i,j} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \check{C}^{(2)}_{i-\varsigma_1,j-\tau_1} \end{array}\right], v^a_{i,j} = \left[\begin{array}{c} v^{(1)}_{i,j} \\ \hline v^{(2)}_{i-\varsigma_1,j-\tau_1} \end{array}\right]$$

$$z^a_{i,j} = z_{i,j}, T^a_{i,j} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \end{array}\right]$$

To further verify the advantages of the proposed partition reorganization approach over the state augmentation approach, we calculate their total floating-point operations.

$$\begin{aligned} S^t_r &= 8n_x^2 + 8n_xn_y + 4n_xn_z - 2n_x \\ S^t_a &= 8n_x^2(2\varsigma_1\tau_1 + \varsigma_1 + \tau_1 + 1)^2 + 8n_x(2\varsigma_1\tau_1 + \varsigma_1 \\ &\quad + \tau_1 + 1)n_y + 4n_x(2\varsigma_1\tau_1 + \varsigma_1 + \tau_1 + 1)n_z \\ &\quad - 2n_x(2\varsigma_1\tau_1 + \varsigma_1 + \tau_1 + 1) \end{aligned}$$

where S^t_a and S^t_r represent the total computational complexity of the augmentation system and the reconstruction system at step 3 and step 9 in Algorithm 1, respectively. The details of the computational complexity of each step in the proposed algorithm are provided in Table 2. When $n_x = n_y = n_z = n = 4, \varsigma_1 = 2, \tau_1 = 3$, $S^t_r = 312, S^t_a = 44784$. Thus when the multi-channel asynchronous delay ς_1 and τ_2 are large, we have $S^t_a \gg S^t_r$, which implies that the computation cost of the presented algorithm is more effective.

Table 2 The total computational complexity

Approach	Step	Floating Point Operations
S_r	3	$2n_x(2n_x-1)$
S_r	9	$4n_x(2n_y+n_x+n_z)$
S_a	3	$2n_x(2\varsigma_1\tau_1+\varsigma_1+\tau_1+1)[2n_x(2\varsigma_1\tau_1+\varsigma_1+\tau_1+1)-1]$
S_a	9	$4n_x(2\varsigma_1\tau_1+\varsigma_1+\tau_1+1)[2n_y+n_x(2\varsigma_1\tau_1+\varsigma_1+\tau_1+1)+n_z]$

5. Numerical Example

In this section, the effectiveness of the proposed H_∞ estimation approach is demonstrated through a numerical example.

Example 1: Consider a shift-invariant 2D uncertain systems with the following parameters:

$$A^1=\begin{bmatrix}0.5&0.1&0\\0.2&0.3&0.1\\0.3&0.1&0.1\end{bmatrix},\ A^2=\begin{bmatrix}0.1&0&0.1\\0.2&0.2&0.2\\0.4&0.4&0\end{bmatrix}$$
$$B^1=\begin{bmatrix}0.4\\0.6\\0.1\end{bmatrix},\ B^2=\begin{bmatrix}0.6\\0.2\\0.2\end{bmatrix},\ C^{(1)}=\begin{bmatrix}7&2&3\end{bmatrix}$$
$$C^{(2)}=\begin{bmatrix}8&1&4\end{bmatrix},\ C^{(3)}=\begin{bmatrix}9&2&3\end{bmatrix},\ T=\begin{bmatrix}2&1&1\end{bmatrix}.$$

The relevant parameters of the random disturbance are set to

$$M^1=\begin{bmatrix}0.01\\0.06\\0.08\end{bmatrix},\ M^2=\begin{bmatrix}0.06\\0.07\\0.05\end{bmatrix},\ L=0.2,\ N^{(1)}=0.05$$
$$N^{(2)}=0.04,\ N^{(3)}=0.07,\ E=\begin{bmatrix}0.2&0.8&0.6\end{bmatrix}.$$

In performance index, $P_0=I_3$, $R_{i,j}=I$, $\mathcal{Q}_{i,j}^{(1)}=\mathcal{Q}_{i,j}^{(2)}=\mathcal{Q}_{i,j}^{(3)}=I$ and the specified attenuation level $\gamma=2$, the input noise and observation noise satisfy $w_{i,j}=0.5o(i,j)$, where $o_{i,j}$ is a random white noise and its energy is less than 1, $v_{i,j}^{(1)}=0.5\cos(0.05\pi i), v_{i,j}^{(2)}=0.4\cos(0.04\pi j), v_{i,j}^{(3)}=0.6\sin(0.06\pi i)$. The asynchronous multi-channel delays $\varsigma_1=\tau=0, \varsigma_2=5, \tau_2=7, \varsigma_3=19, \tau_3=23$. Let the initial conditions be given as follows for any $(i,j)\in\mathcal{N}=[0,50]\times[0,50]$:

$$\begin{cases}\mathbf{x}_{i,0}=\mathbf{x}_{0,j}=[1\quad 1\quad 1]^{\mathrm{T}}\\ z_{i,0}=z_{0,j}=[1\quad 1\quad 1]^{\mathrm{T}}\\ \mathbf{P}_{i,0}=\mathbf{P}_{0,j}=0.01\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\end{cases}.$$

In view of the given multi-channel delay parameters, the partition of $\mathcal{N}$ is divided as follows, $\mathcal{N}_1^{\circ}=[0,31]\times[0,27]$, $\mathcal{N}_2^{+}=(31,45]\times[0,27]$, $\mathcal{N}_2^{\circ}=(31,45]\times(27,43]$, $\mathcal{N}_2^{-}=[0,31]\times(27,43]$, $\mathcal{N}_3=(45,50]\times[0,43]$, $\mathcal{N}_{\circ}=(45,50]\times(43,50]$, $\mathcal{N}_3^{-}=[0,45]\times(43,50]$. According to Algorithm 1, the estimate of $z_{i,j}$ can be recursive calculated. First, verify whether *Problem (1)* is hold according to Theorem 2. To save space, some of the matrix parameters are shown in Table 3 and Table 4, where $\mathrm{aux1}_{i,j}$ and $\mathrm{aux2}_{i,j}$ are the auxiliary matrices. Table 5 shows the partial estimation gain parameter $H_{i,j}$.

Table 3 *Problem (1)* of necessary and sufficient conditions ① matrix

$\mathrm{aux1}_{1,1}=\begin{bmatrix}81.9&87.2&96.5\\87.3&95.5&104.1\\96.5&104.1&116.3\end{bmatrix}$	$\mathrm{aux1}_{1,2}=\begin{bmatrix}56.6&59.3&66.2\\59.3&64.5&70.6\\66.2&70.6&80.0\end{bmatrix}$	$\mathrm{aux1}_{1,3}=\begin{bmatrix}56.6&59.3&66.1\\59.2&64.5&70.5\\66.1&70.5&79.9\end{bmatrix}$	$\mathrm{aux1}_{1,4}=\begin{bmatrix}56.5&59.1&66.0\\59.1&64.4&70.4\\66.0&70.4&79.8\end{bmatrix}$
$\mathrm{aux1}_{2,1}=\begin{bmatrix}73.3&77.7&87.1\\77.7&84.9&93.8\\87.1&93.8&106.2\end{bmatrix}$	$\mathrm{aux1}_{2,2}=\begin{bmatrix}47.6&49.3&56.5\\49.4&53.5&59.9\\56.5&59.9&69.4\end{bmatrix}$	$\mathrm{aux1}_{2,3}=\begin{bmatrix}47.6&49.3&56.3\\49.3&53.5&59.7\\56.3&59.8&69.2\end{bmatrix}$	$\mathrm{aux1}_{2,4}=\begin{bmatrix}47.7&49.3&56.4\\49.3&53.4&59.8\\56.4&59.8&69.3\end{bmatrix}$

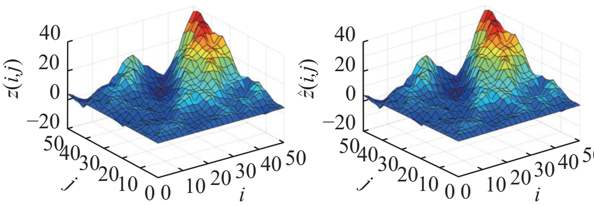
Table 4 *Problem (1)* of necessary and sufficient conditions ② matrix

$\mathrm{aux}\,2_{1,1} = \begin{bmatrix} -0.82 & 0.12 \\ 0.12 & -3.89 \end{bmatrix}$	$\mathrm{aux}\,2_{1,2} = \begin{bmatrix} -0.85 & 0.10 \\ 0.10 & -3.91 \end{bmatrix}$	$\mathrm{aux}\,2_{1,3} = \begin{bmatrix} -0.85 & 0.10 \\ 0.10 & -3.90 \end{bmatrix}$	$\mathrm{aux}\,2_{1,4} = \begin{bmatrix} -0.84 & 0.10 \\ 0.10 & -3.91 \end{bmatrix}$
$\mathrm{aux}\,2_{2,1} = \begin{bmatrix} -0.90 & 0.08 \\ 0.08 & -3.91 \end{bmatrix}$	$\mathrm{aux}\,2_{2,2} = \begin{bmatrix} -0.93 & 0.06 \\ 0.06 & -3.93 \end{bmatrix}$	$\mathrm{aux}\,2_{2,3} = \begin{bmatrix} -0.93 & 0.06 \\ 0.06 & -3.93 \end{bmatrix}$	$\mathrm{aux}\,2_{2,4} = \begin{bmatrix} -0.92 & 0.07 \\ 0.07 & -3.93 \end{bmatrix}$

Table 5 The estimation gain matrix

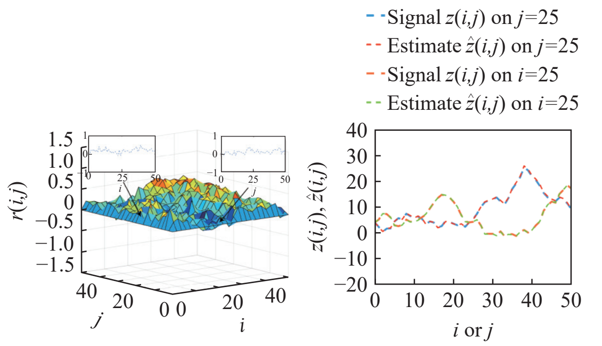
$H_{1,1}$ = [0.159 0.055 0.0900]	$H_{1,2}$ = [0.146 0.070 0.089]	$H_{1,3}$ = [0.146 0.071 0.089]	$H_{1,4}$ = [0.147 0.070 0.089]
$H_{2,1}$ = [0.142 0.034 0.118]	$H_{2,2}$ = [0.128 0.048 0.119]	$H_{2,3}$ = [0.128 0.049 0.119]	$H_{2,4}$ = [0.132 0.049 0.115]

The simulation results are shown in Figures 1–4. Figure 1 displays $z_{i,j}$ and its estimate $\hat{z}_{i,j}$. Intuitively, $z_{i,j}$ and $\hat{z}_{i,j}$ are approximately the same. Further, we verified this from the perspective of estimate error, the estimate error and cross-section $j = 25$ or $i = 25$ are depicted in Figure 2, it can be seen from Figure 2a that $r_{i,j}$ is between $[-0.5, 0.5]$, and from Figure 2b, it can be concluded that the trajectories of the $z_{i,j}$ and $\hat{z}_{i,j}$ cross-section overlap, indicating that the estimation algorithm in this paper is effective. The Problem (1) verification of necessary and sufficient conditions ① and ② for Theorem 2 are plotted in Figure 3, where $\mathrm{Pos}_{i,j}$ and $\mathrm{Nos}_{i,j}$ are indicator matrices of $\mathrm{aux}1_{i,j}$ and $\mathrm{aux}2_{i,j}$, the value is 1 when it is a positive definite matrix, and the value is -1 when it is a negative definite matrix. Figure 4 displays the H_∞ performance index and the trace of estimation gain, it is obvious from Figure 4a that $\sup_{(x_0, w_{i,j}, v^s_{i,j}) \neq 0} \frac{\mathcal{X}}{\mathcal{Z}} \leqslant 0$, due to the existence of multi-channel delays, the quadratic performance index gradually increases in a step-like manner. And the trace of the estimation gain converges in a step-like manner is depicted in Figure 4b. The simulation results Figures 1–4 show that the algorithm has good estimation performance.



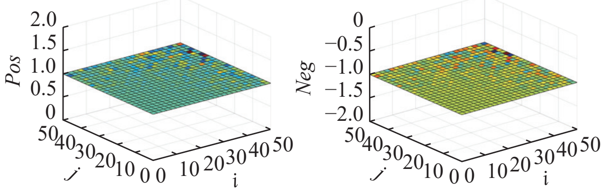
(a) Signal $z_{i,j}$ (b) Estimate $\hat{z}_{i,j}$

Figure 1. Signal and its estimate.



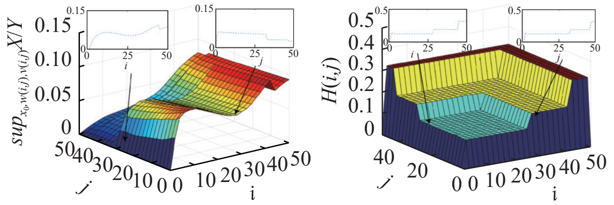
(a) Estimateerror $r_{i,j}$ (b) Cross-section of $\hat{z}_{i,j}$ and $z_{i,j}$

Figure 2. Estimate error and cross-section $j = 25$ or $i = 25$.



(a) Indicator matrices $Pos_{i,j}$ (b) Indicator matrices $Neg_{i,j}$

Figure 3. Indicator matrices.



(a) The H_∞ quadratic performance indexsup $sup_{x_0,w(i,j),v(i,j)}\neq X/Y$ (b) The trace of estimation gain pa-rameter $\mathrm{tr}\{H_{i,j}\}$

Figure 4. The H_∞ performance index and the trace of estimation gain.

Example 2: Consider the metal rolling process as shown in Figure 5:

$$\ddot{R}_{i,j} + \frac{z}{M}R_{i,j} = \frac{z}{H}\ddot{R}_{i-1,j} + \frac{z}{M}R_{i-1,j} + \frac{z}{Mn}F_M$$

where $R_{i,j}$ represents the $i-$th roll-gap thickness at time j, and the other variables in the metal rolling process are shown in Table 6. Through the backward difference technique, the differential equation can be transformed into a 2D FMLSS system.

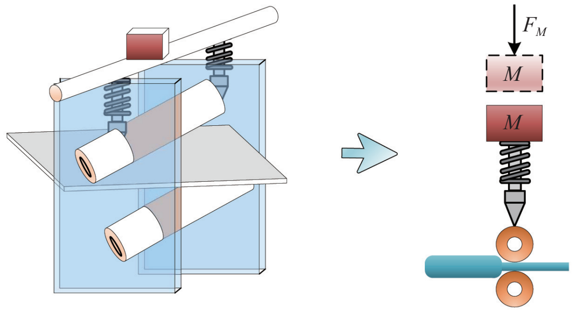


Figure 5. Metal rolling process.

Table 6 Variables in the metal rolling process

M	mass of the roll-gap adjusting mechanism	50 kg
H	stiffness of the adjusting mechanism spring	3000 N/mm
n	hardness of the metal strip	500 N/mm
z	composite stiffness $\left(\frac{Hn}{H+n}\right)$	$\frac{3000}{7}$ N/mm
F_M	force of the motor	/

Considering the actual metal rolling process, the system may be affected by disturbances caused by non-ideal rolling conditions. Moreover, due to variations in roll gap thickness, the heating model under consideration may experience fluctuations in system parameters. When using sensors to measure the actual rolling thickness, the measurement signals are often subjected to asynchronous delays caused by motor temperature increases. Therefore, the relevant system parameters are

$$A^1_{i,j} = \begin{bmatrix} 1 & -0.02999 & 0.2090 & -0.1045 & 0.0149 \\ 0 & e^{(-i)} & 1\sin(j) & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{(-0.1j)} \\ 0 & 0 & 0 & 0 & 0 \\ 0.01\cos(i+j) & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2_{i,j} = \begin{bmatrix} 0 & 0 & 0 & -0.02\sin(j) & 0 \\ 0 & 0 & \cos(i) & 0 & 0 \\ 1 & -0.0299 & 0.2090 & -0.1045 & 0.0149 \\ 0 & 0 & 1 & 0 & 0.1e^{-(i+j)} \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B^1_{i,j} = \begin{bmatrix} -0.02 \\ 0.05e^{-(i)} \\ 0.03 \\ 0 \\ 0.1 \end{bmatrix}, B^2_{i,j} = \begin{bmatrix} 0 \\ 0.1 \\ 0.05 \\ 0 \\ 0.1 \end{bmatrix}, C^{(1)}_{i,j} = \begin{bmatrix} 0.01 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}}$$

In performance index, $P_0 = I_5, R_{i,j} = 0.4I, Q^{(1)}_{i,j} = Q^{(2)}_{i,j} = Q^{(3)}_{i,j} = 0.9I$ and the specified attenuation level $\gamma = 1$, the input noise and observation noise satisfy $w_{i,j} = 0.49o(i,j)$, where $o_{i,j}$ is a random white noise and its energy is less than 1, $v^{(1)}_{i,j} = 0.3\cos(0.06\pi i), v^{(2)}_{i,j} = 0.7\cos(0.07\pi j), v^{(3)}_{i,j} = 0.5\sin(0.05\pi i)$. The asynchronous multi-channel delays $\varsigma_1 = \tau = 0, \varsigma_2 = 7, \tau_2 = 9, \varsigma_3 = 24, \tau_3 = 28$. Let the initial conditions be given as follows for any $(i,j) \in \mathcal{N} = [0,50]\times[0,50]$: $\mathbf{x}_{i,0} = \mathbf{x}_{0,j} = z_{i,0} = z_{0,j} = [1\ 1\ 1\ 1\ 1]^{\mathsf{T}}, \mathbf{P}_{i,0} = \mathbf{P}_{0,j} = 0.01I_5$.

$$C^{(2)}_{i,j} = \begin{bmatrix} 0.06 \\ 0 \\ 0.01\text{sgn}(j-i) \\ 0 \\ 0 \end{bmatrix}^{\mathsf{T}}, \ C^{(3)}_{i,j} = \begin{bmatrix} 0 \\ 0.01+0.01\cos(j) \\ 0.04 \\ 0 \\ 0 \end{bmatrix}^{\mathsf{T}}$$

$$T_{i,j} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}^{\mathsf{T}}, \ M^1_{i,j} = \begin{bmatrix} 0.01 \\ 0 \\ 0.02 \\ 0 \\ 0 \end{bmatrix}, \ M^2_{i,j} = \begin{bmatrix} 0 \\ 0.05 \\ 0 \\ 0 \\ 0.01 \end{bmatrix}, \ L_{i,j} = 0.1$$

$$N^{(1)}_{i,j} = 0.03,\ N^{(2)}_{i,j} = 0.06,\ N^{(3)}_{i,j} = 0.01$$

$$E_{i,j} = \begin{bmatrix} 0.12 & 0.21 & 0 & 0 & 0.43 \end{bmatrix}$$

In view of the given multi-channel delay parameters, the partition of $\mathcal{N}$ is divided as follows, $\mathcal{N}^{\circ}_1 = [0,26]\times[0,22], \mathcal{N}^{+}_2 = (26,43]\times[0,22], \mathcal{N}^{\circ}_2 = (26,43]\times(22,41], \mathcal{N}^{-}_2 = [0,26]\times(22,41], \mathcal{N}_3 = (43,50]\times[0,41], \mathcal{N}_{\circ} = (43,50]\times(41,50], \mathcal{N}^{-}_3 = [0,43]\times(41,50]$.

The simulation results are shown in Figures 6–8. Figure 6 displays estimate $\hat{z}_{i,j}$ and cross-section on $j = 25$ or $i = 25$. Further, the Problem (1) verification of necessary and sufficient conditions ① and ② for Theorem 2 are plotted in Figure 7. Figure 8 displays the estimate error and H_∞ performance index, it is obvious from Figure 8b that $\sup_{(x_0, w_{i,j}, v^s_{i,j})\neq 0} \frac{\mathcal{X}}{\mathcal{Z}} \leqslant 0$. The simulation results Figures 6–8 show that the algorithm has good estimation performance.

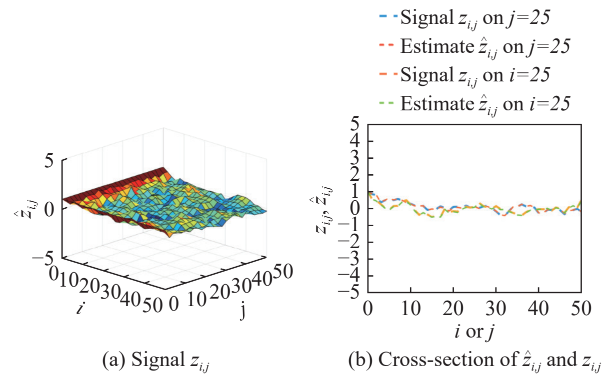


Figure 6. Signal and cross-section on $j = 20$ and $i = 25$.

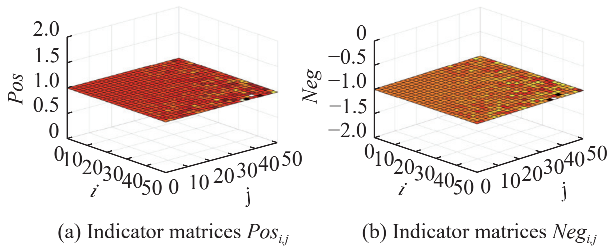


Figure 7. Indicator matrices.

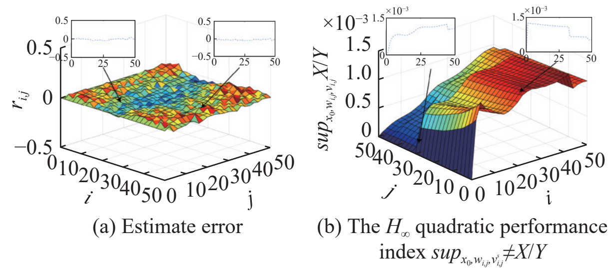
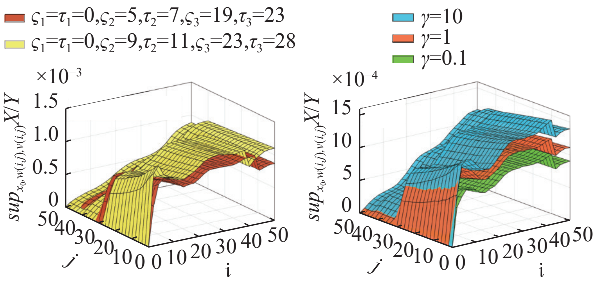


Figure 8. Estimate error and The H_∞ performance index.

Using the system parameters of Example 2, we consider the effects of different disturbance attenuation level γ and delay coefficients on H_∞ performance.

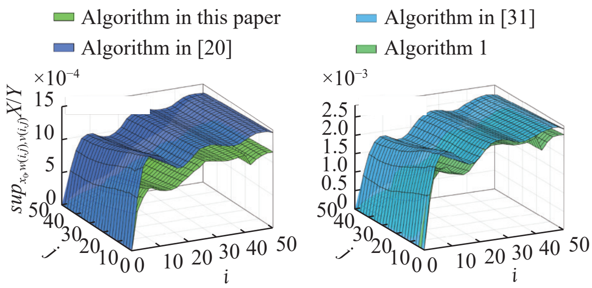
Remark 12: Different delay coefficients affect the observations received in the current horizon. As shown in Figure 9a, a larger delay coefficient results in greater information lag in the current channel, reducing the available information for subsequent state estimation. Consequently, this leads to a slower arrival the H_∞ performance under a given γ. By setting the disturbance attenuation level γ, the induced gain of the transfer function from uncertainty signals to estimation errors can be strictly bounded below a given positive scalar γ. The value of γ directly reflects the system's suppression strength against uncertain disturbances: a smaller γ enforces stricter robustness constraints, whereas a larger γ permits looser disturbance attenuation boundaries as shown in Figure 9b.



(a) H_∞ performance for different delay coefficients (b) H_∞ performance for different disturbance attenuation level

Figure 9. performance index.

The H_∞ performance function is shown in Figure 10, which is obtained by the algorithm proposed in Algorithm 1, [20], and [5]. The effectiveness of the algorithm can be evaluated by determining whether the H_∞ performance converges. In the simulation, the parameters, delay coefficients, and initial conditions are kept consistent. It is clearly shown that Algorithm 1 not only achieves excellent convergence, but also shows excellent performance in effectively solving the H_∞ filtering problem for delayed system.



(a) Case 1. (b) Case 2.

Figure 10. The H_∞ performance for different Algorithm.

6. Conclusions

This paper has explored the robust H_∞ filtering problem for a class of 2D uncertain systems under asynchronous multi-channel delays. A partition reconstruction approach has been employed to transform the original delayed system into an equivalent multi-channel observation delay-free systems. By making two modifications to the quadratic performance function, an equivalence relation between the design of the H_∞ filter and the minimization problem of an indefinite quadratic form has been established. Subsequently, a robust H_∞ filter has been designed within a 2D Krein space stochastic system by minimizing the quadratic performance metric, and the sufficient and necessary conditions for its existence have been presented. The limitation of this paper is that the proposed filtering framework cannot be extended to 2D systems with both state and observation delays simultaneously, as the reconstruction approach is unable to account for the effects of state delays. Future research directions will include extending the proposed H_∞ estimation approach to more general systems with more complex network-induced phenomena, such as non-logarithmic sensor resolution, dynamic bias, communication scheduling [34–37].

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proof of Lemma 1

Proof: In accordance with (4) and (7), we can know that $y^s_{l,k}$ is a linear combination of $\mathcal{Y}^N_{l,k}$, i.e.

$$
\begin{aligned}
y^1_{l,k} &= \begin{bmatrix} I_{n_y} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \mathcal{Y}^N_{l+\varsigma_1,k+\tau_1} \\
y^2_{l,k} &= \begin{bmatrix} I_{n_y} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \mathcal{Y}^N_{l+\varsigma_1,k+\tau_1} \\
&\quad + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & I_{n_y} & \cdots & \mathbf{0} \end{bmatrix} \mathcal{Y}^N_{l+\varsigma_2,k+\tau_2} \\
&\cdots\cdots \\
y^N_{l,k} &= \begin{bmatrix} I_{n_y} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \mathcal{Y}^N_{l+\varsigma_1,k+\tau_1} \\
&\quad + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & I_{n_y} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \mathcal{Y}^N_{l+\varsigma_2,k+\tau_2} \\
&\quad + \cdots + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & I_{n_y} \end{bmatrix} \mathcal{Y}^N_{l+\varsigma_N,k+\tau_N}.
\end{aligned}
$$

Moreover, it is obvious that $\mathcal{Y}^N_{l,k}$ is also a linear combination of $y^s_{l,k}$, i.e.

$$
\begin{aligned}
\mathcal{Y}^N_{l,k} &= \begin{bmatrix} I_{n_y} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} y^1_{l-\varsigma_1,k-\tau_1} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{n_y} \\ \vdots & \vdots \\ \mathbf{0} & \mathbf{0} \end{bmatrix} y^2_{l-\varsigma_2,k-\tau_2} \\
&\quad + \cdots + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & I_{n_y} \end{bmatrix} y^N_{l-\varsigma_N,k-\tau_N}.
\end{aligned}
$$

Following the above proof approach, it can be seen that $y^s_{l,k}$ and $\mathcal{Y}^{\jmath}_{l,k}(\jmath \in [1, N-1])$ can also be expressed linearly with each other. □

Appendix B. Proof of Lemma 2

Proof: We first give the calculation of the two covariance matrices. In light of (13) and (17), it is easy to know that

$$
\bar{\Pi} = \left\langle \begin{bmatrix} \mathbf{x}_0 \\ \bar{\mathbf{w}}_{\mathcal{N}} \\ \hline \bar{\mathbf{y}}_{\mathcal{N}} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \bar{\mathbf{w}}_{\mathcal{N}} \\ \hline \bar{\mathbf{y}}_{\mathcal{N}} \end{bmatrix} \right\rangle = \begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} \\ \bar{\Pi}_{21} & \bar{\Pi}_{22} \end{bmatrix}
$$

where

$$\bar{\Pi}_{11}=\begin{bmatrix}P_0 & \mathbf{0}\\ \mathbf{0} & \bar{R}_{\mathcal{N}}\end{bmatrix}\quad \bar{\Pi}_{12}=\begin{bmatrix}P_0k_0^{\mathsf{T}}\\ \bar{R}_{\mathcal{N}}k_{\bar{w}}^{\mathsf{T}}+\bar{S}_{\mathcal{N}}k_{\bar{v}}^{\mathsf{T}}\end{bmatrix},$$
$$\bar{\Pi}_{22}=k_0P_0k_0^{\mathsf{T}}+k_{\bar{w}}\bar{R}_{\mathcal{N}}k_{\bar{w}}^{\mathsf{T}}+k_{\bar{v}}\bar{Q}_{\mathcal{N}}k_{\bar{v}}^{\mathsf{T}}+k_{\bar{w}}\bar{S}_{\mathcal{N}}k_{\bar{v}}^{\mathsf{T}}+k_{\bar{v}}\bar{S}_{\mathcal{N}}^{\mathsf{T}}k_{\bar{w}}^{\mathsf{T}}.$$

Then, in line with (22), one has

$$\phi_{l,k}^{\iota}=\begin{bmatrix}C_{l,k}^{\iota}\\ E_{l,k}\\ T_{l,k}\end{bmatrix}\tilde{\mathbf{x}}^{(N+1-\iota)-}(l,k)+\begin{bmatrix}\mathbf{v}_{l,k}^{\iota}+N_{l,k}^{\iota}L_{l,k}E_{l,k}\mathbf{x}_{l,k}\\ \hline -E_{l,k}\mathbf{x}_{l,k}\\ \mathbf{r}_{l,k}\end{bmatrix}\tag{40}$$

$$\Phi_{l,k}^{\iota}\triangleq\langle\phi_{l,k}^{\iota},\phi_{l,k}^{\iota}\rangle=\begin{bmatrix}\Phi_{l,k}^{\iota,11} & \Phi_{l,k}^{\iota,12}\\ \Phi_{l,k}^{\iota,21} & \Phi_{l,k}^{\iota,22}\end{bmatrix}\tag{41}$$

where

$$\Phi_{l,k}^{\iota,11}=C_{l,k}^{\iota}\mathbf{P}_{l,k}^{(N+1-\iota)-}(C_{l,k}^{\iota})^{\mathsf{T}}+Q_{l,k}^{\iota}+N_{l,k}^{\iota}(N_{l,k}^{\iota})^{\mathsf{T}}$$
$$\Phi_{l,k}^{\iota,12}=C_{l,k}^{\iota}\mathbf{P}_{l,k}^{(N+1-\iota)-}\begin{bmatrix}E_{l,k}\\ T_{l,k}\end{bmatrix}^{\mathsf{T}}$$
$$\Phi_{l,k}^{\iota,22}=\begin{bmatrix}E_{l,k}\\ T_{l,k}\end{bmatrix}\mathbf{P}_{l,k}^{(N+1-\iota)-}\begin{bmatrix}E_{l,k}\\ T_{l,k}\end{bmatrix}^{\mathsf{T}}+\begin{bmatrix}-I & \mathbf{0}\\ \mathbf{0} & -\gamma^2 I\end{bmatrix}$$

It can be verified that the sequential principal formulas of each order of $\Phi_{l,k}^{\iota}$ are not zero, and the unique trigonometric decomposition of $\Phi_{l,k}^{\iota}$ can be obtained,

$$\Phi_{l,k}^{\iota}=\begin{bmatrix}I & \mathbf{0}\\ \Phi_{l,k}^{\iota,21}\left(\Phi_{l,k}^{\iota,11}\right)^{-1} & I\end{bmatrix}\begin{bmatrix}\Phi_{l,k}^{\iota,11} & \mathbf{0}\\ \mathbf{0} & \Phi_{l,k}^{\iota,2}\end{bmatrix}\times\begin{bmatrix}I & \mathbf{0}\\ \Phi_{l,k}^{\iota,21}\left(\Phi_{l,k}^{\iota,11}\right)^{-1} & I\end{bmatrix}^{\mathsf{T}}$$

where $\Phi_{l,k}^{\iota,2}=\Phi_{l,k}^{\iota,22}-\Phi_{l,k}^{\iota,21}\left(\Phi_{l,k}^{\iota,11}\right)^{-1}\Phi_{l,k}^{\iota,12}$. The necessary and sufficient condition for the existence of the minimum of (18) is: for $\bar{\Pi}$, when $\bar{\Pi}_{11}>0$, the inertia index of $\bar{\Pi}_{22}$ and $\bar{\Pi}_{22}-\bar{\Pi}_{21}\bar{\Pi}_{11}^{-1}\bar{\Pi}_{12}$ are equal, i.e. $\mathbf{I}(\bar{\Pi}_{22})=\mathbf{I}(k_{\bar{v}}\bar{Q}_{\mathcal{N}}k_{\bar{v}}^{\mathsf{T}}+k_{\bar{v}}\bar{S}_{\mathcal{N}}^{\mathsf{T}}\bar{R}_{\mathcal{N}}^{-1}\bar{S}_{\mathcal{N}}k_{\bar{v}}^{\mathsf{T}})$. In accordance with [33], $\mathbf{I}(\bar{\Pi}_{22})=\mathbf{I}(\langle\bar{\mathbf{y}}_{\mathcal{N}},\bar{\mathbf{y}}_{\mathcal{N}}\rangle)$ is equal to $\mathbf{I}(\Phi_{\mathcal{N}})$, where $\Phi_{\mathcal{N}}\triangleq\mathrm{diag}\{\Phi_{0,0}^{N},\cdots,\Phi_{i_1,j_1}^{N},\Phi_{0,j_1+1}^{N-1},\cdots,\Phi_{i_2,j_2}^{N-1},\cdots,\Phi_{0,j_{N-1}}^{1},\cdots,\Phi_{i_N,j_N}^{1}\}$. And $\bar{\mathcal{J}}_{\mathcal{N}}$ has a minimum on $x_0,\{w_{l,k}|(l,k)\in\mathcal{N}\}$, and $\{\{v_{l,k}^{N}|(l,k)\in\mathcal{N}_1\};\cdots;\{v_{l,k}^{1}|(l,k)\in\mathcal{N}_N\}\}$ as

$$\bar{\mathcal{J}}_{\mathcal{N}}^{*}=\sum_{\iota=1}^{N}\sum_{(l,k)\in\mathcal{N}_{\iota}}\{(\bar{y}_{l,k}^{N+1-\iota}-\hat{\bar{y}}_{l,k}^{N+1-\iota})^{\mathsf{T}}\times\langle\phi_{l,k}^{N+1-\iota},\phi_{l,k}^{N+1-\iota}\rangle^{-1}(\bar{y}_{l,k}^{N+1-\iota}-\hat{\bar{y}}_{l,k}^{N+1-\iota})\}$$

In order to facilitate the analysis, the following will no longer superimpose the various variables, and only analyze partition $\mathcal{N}_1$. Then, we give the necessary and sufficient conditions for $\mathbf{I}(\bar{\Theta}_{l,k}^{1}\triangleq\langle\bar{\mathbf{y}}_{l,k}^{N}-\hat{\bar{\mathbf{y}}}_{l,k}^{N},\bar{\mathbf{y}}_{l,k}^{N}-\hat{\bar{\mathbf{y}}}_{l,k}^{N}\rangle)$ to be equal to $\mathbf{I}(\Upsilon_{l,k}^{N}\triangleq\bar{D}_{l,k}^{N}\bar{Q}_{l,k}^{N}(\bar{D}_{l,k}^{N})^{\mathsf{T}}-\bar{D}_{l,k}^{N}(\bar{S}_{l,k}^{N})^{\mathsf{T}}(\bar{R}_{l,k})^{-1}\bar{S}_{l,k}^{N}(\bar{D}_{l,k}^{N})^{\mathsf{T}})$. Refer to the matrix definition above, it is easy to see that

$$\Upsilon_{l,k}^{N}=\begin{bmatrix}Q_{l,k}^{N} & & \\ & -I & \\ & & -\gamma^2 I\end{bmatrix}\tag{42}$$

$$\bar{\Theta}_{l,k}^{1}=\begin{bmatrix}\bar{\theta}_{l,k}^{1,11} & \bar{\theta}_{l,k}^{1,12}\\ \bar{\theta}_{l,k}^{1,21} & \bar{\theta}_{l,k}^{1,22}\end{bmatrix}=\begin{bmatrix}I & \mathbf{0}\\ \bar{\theta}_{l,k}^{1,21}(\bar{\pi}_{l,k}^{1,11})^{-1} & I\end{bmatrix}\begin{bmatrix}\bar{\theta}_{l,k}^{1,11} & \mathbf{0}\\ \mathbf{0} & \bar{\theta}_{l,k}^{1,3}\end{bmatrix}\begin{bmatrix}I & \mathbf{0}\\ \bar{\theta}_{l,k}^{1,21}(\bar{\theta}_{l,k}^{1,11})^{-1} & I\end{bmatrix}^{\mathsf{T}}.\tag{43}$$

In view of (40) and (41), $\bar{\theta}_{l,k}^{1,11}=\Phi_{l,k}^{N,11},\bar{\theta}_{l,k}^{1,12}=\Phi_{l,k}^{N,12},\bar{\pi}_{l,k}^{1,22}=\Phi_{l,k}^{N,22}$, and $\bar{\theta}_{l,k}^{1,3}=\bar{\theta}_{l,k}^{1,22}-\bar{\theta}_{l,k}^{1,21}(\bar{\pi}_{l,k}^{1,11})^{-1}\bar{\pi}_{l,k}^{1,12}$. It can be seen from (42) and (43) that $\mathbf{I}(\bar{\Theta}_{l,k}^{1})=\mathbf{I}(\Upsilon_{l,k}^{N})$ if and only if $\bar{\theta}_{l,k}^{1,11}>0$ and $\bar{\theta}_{l,k}^{1,3}<0$. □

Appendix C. Proof of Theorem 3

Proof: For simplicity, the following proof is only carried out in *Step 1*. According to (36), one derives

$$\bar{\mathcal{J}}^*_{\mathcal{N}_1} = \sum_{(l,k)\in\mathcal{N}_1} \{(\bar{y}^N_{l,k} - \hat{\bar{y}}^N_{l,k})^{\mathrm{T}} (\Phi^N_{l,k})^{-1} (\bar{y}^N_{l,k} - \hat{\bar{y}}^N_{l,k})\} \tag{44}$$

and

$$\begin{aligned}\Phi^N_{l,k} = &\begin{bmatrix} I & \mathbf{0} \\ \Phi^{N,21}_{l,k}(\Phi^{N,11}_{l,k})^{-1} & I \end{bmatrix} \begin{bmatrix} \Phi^{N,11}_{l,k} & \mathbf{0} \\ \mathbf{0} & \Phi^{N,2}_{l,k} \end{bmatrix} \\ &\times \begin{bmatrix} I & \mathbf{0} \\ \Phi^{N,21}_{l,k}(\Phi^{N,11}_{l,k})^{-1} & I \end{bmatrix}^{\mathrm{T}}\end{aligned} \tag{45}$$

Reviewing the definition of correlation matrix yields

$$\begin{aligned}&\begin{bmatrix} I & \mathbf{0} \\ \Phi^{N,21}_{l,k}(\Phi^{N,11}_{l,k})^{-1} & I \end{bmatrix}^{-1} (\bar{y}^N_{l,k} - \hat{\bar{y}}^N_{l,k}) \\ &= \begin{bmatrix} y^N_{l,k} - C^N_{l,k}\hat{x}^{1-}_{l,k} \\ \hat{z}^a_{l,k} - z^p_{l,k} \end{bmatrix}\end{aligned}$$

where $\hat{z}^a_{l,k} = [\mathbf{0}\ \hat{z}^{\mathrm{T}}_{l,k}]^{\mathrm{T}}$ and $z^p_{l,k} = [E^{\mathrm{T}}_{l,k}\ T^{\mathrm{T}}_{l,k}]^{\mathrm{T}}\hat{x}^{1-}_{l,k} + \Phi^{N,21}_{l,k}(\Phi^{N,11}_{l,k})^{-1}(y^N_{l,k} - C^N_{l,k}\hat{x}^{1-}_{l,k})$. Therefore, substituting (45) into (44) we get

$$\begin{aligned}\bar{\mathcal{J}}^*_{\mathcal{N}_1} = &\sum_{(l,k)\in\mathcal{N}_1} \{\|y^N_{l,k} - C^N_{l,k}\hat{x}^{1-}_{l,k}\|^2_{(\Phi^{N,11}_{l,k})^{-1}} \\ &+ \|\hat{z}^a_{l,k} - z^p_{l,k}\|^2_{(\Phi^{N,2}_{l,k})^{-1}}\}.\end{aligned} \tag{46}$$

If Theorem 2 hold then $\Phi^{N,11}_{l,k} > 0$ and $\Phi^{N,2}_{l,k} < 0$. It is obvious from (46) that $\bar{\mathcal{J}}^*_{\mathcal{N}_1}$ takes the minimum if and only if $\hat{z}^a_{l,k} = z^p_{l,k}$. When $E_{l,k} = 0$ holds, exactly $\hat{z}^a_{l,k} - z^p_{l,k} = 0$ and $\bar{\mathcal{J}}^*_{\mathcal{N}_1}|_{E_{l,k}=0} > 0$. Finally, it comes $\bar{\mathcal{J}}^*_{\mathcal{N}}|_{E_{l,k}=0} = \sum_{\iota=1}^{N} \bar{\mathcal{J}}^*_{\mathcal{N}_\iota}|_{E_{l,k}=0} > 0$. □

References

1. Ahn, C.K.; Wu, L.G.; Shi, P. Stochastic stability analysis for 2-D Roesser systems with multiplicative noise. Automatica, **2016**, *69*: 356-363. doi: 10.1016/j.automatica.2016.03.006
2. Chinimilli, P.T.; Redkar, S.; Sugar, T. A two-dimensional feature space-based approach for human locomotion recognition. IEEE Sens. J., **2019**, *19*: 4271-4282. doi: 10.1109/JSEN.2019.2895289
3. Li, H.; Wang, S.Q.; Shi, H.Y.; *et al*. Two-dimensional iterative learning robust asynchronous switching predictive control for multiphase batch processes with time-varying delays. IEEE Trans. Syst. Man Cybern. Syst., **2023**, *53*: 6488-6502. doi: 10.1109/TSMC.2023.3284078
4. Dabkowski, P.; Galkowskiy, K.; Rogers, E.; *et al*. Iterative learning control based on relaxed 2-D systems stability criteria. IEEE Trans. Control Syst. Technol., **2013**, *21*: 1016-1023. doi: 10.1109/TCST.2012.2198477
5. Luo, Y.Q.; Wang, Z.D.; Wei, G.L.; *et al*. Robust H_∞ filtering for a class of two-dimensional uncertain fuzzy systems with randomly occurring mixed delays. IEEE Trans. Fuzzy Syst., **2017**, *25*: 70-83. doi: 10.1109/TFUZZ.2016.2556001
6. Liu, Q.Y.; Wang, Z.D.; Dong, H.L.; *et al*. Remote estimation for energy harvesting systems under multiplicative noises: A binary encoding scheme with probabilistic bit flips. IEEE Trans. Autom. Control, **2023**, *68*: 343-354. doi: 10.1109/TAC.2022.3170540
7. Zhao, D.; Ding, S.X.; Karimi, H.R.; *et al*. Robust H_∞ filtering for two-dimensional uncertain linear discrete time-varying systems: A Krein space-based method. IEEE Trans. Autom. Control, **2019**, *64*: 5124-5131. doi: 10.1109/TAC.2019.2908699
8. Xing, J.M.; Chi, R.H.; Lin, N. Adaptive iterative learning control for 2D nonlinear systems with nonrepetitive uncertainties. Int. J. Robust Nonlinear Control, **2021**, *31*: 1168-1180. doi: 10.1002/rnc.5347
9. Ye, L.W.; Zhao, Z.G.; Liu, F. Optimal control of two-dimensional Roesser model: Solution based on reinforcement learning. IEEE Trans. Autom. Control, **2024**, *69*: 5424-5430. doi: 10.1109/TAC.2024.3357743
10. Zhao, D.; Ahn, C.K.; Paszke, W.; *et al*. Fault diagnosability analysis of two-dimensional linear discrete systems. IEEE Trans. Autom. Control, **2021**, *66*: 826-832. doi: 10.1109/TAC.2020.2986054
11. Roesser, R. A discrete state-space model for linear image processing. IEEE Trans. Autom. Control, **1975**, *20*: 1-10. doi: 10.1109/TAC.1975.1100844
12. Heath, W.P. Orthogonal functions for cross-directional control of Web forming processes. Automatica, **1996**, *32*: 183-198. doi: 10.1016/0005-1098(96)85548-8
13. Lo, W.C.; Wang, L.; Li, B.W. Thermal transistor: Heat flux switching and modulating. J. Phys. Soc. Japan, **2008**, *77*: 054402. doi: 10.1143/jpsj.77.054402
14. Bose, N.K. *Multidimensional Systems Theory and Applications*, 2nd ed.; Kluwer Academic Publishers: Dordrecht, 2003.
15. Chen, B.; Hu, G.Q.; Ho, D.W.C.; *et al*. Distributed Kalman filtering for time-varying discrete sequential systems. Automatica, **2019**, *99*: 228-236. doi: 10.1016/j.automatica.2018.10.025
16. Choi, H.D.; Ahn, C.K.; Karimi, H.R.; *et al*. Filtering of discrete-time switched neural networks ensuring exponential dissipative and l_2-l_∞ performances. IEEE Trans. Cybern., **2017**, *47*: 3195-3207. doi: 10.1109/TCYB.2017.2655725
17. Wang, F.; Liang, J.L.; Liu, X.H. H_∞ state estimation for time-varying networks with probabilistic delay in measurements. In *Proceedings of the 22nd International Conference on Automation and Computing (ICAC), Colchester, UK, 7–8 September 2016*; IEEE: New York, 2016; pp. 1–6. doi:10.1109/IConAC.2016.7604885
18. Wang, Y.A.; Shen, B.; Zou, L.; *et al*. A survey on recent advances in distributed filtering over sensor networks subject to communication constraints. Int. J. Network Dyn. Intell., **2023**, *2*: 100007. doi: 10.53941/ijndi0201007

19. Wang, Y.; Liu, H.J.; Tan, H.L. An overview of filtering for sampled-data systems under communication constraints. Int. J. Network Dyn. Intell., **2023**, *2*: 100011. doi: 10.53941/ijndi.2023.100011
20. Yi, X.J.; Yu, H.Y.; Fang, Z.Y.; *et al*. Probability-guaranteed state estimation for nonlinear delayed systems under mixed attacks. Int. J. Syst. Sci., **2023**, *54*: 2059-2071. doi: 10.1080/00207721.2023.2216274
21. Zhu, K.Q.; Wang, Z.D.; Dong, H.L.; *et al*. Set-membership filtering for two-dimensional systems with dynamic event-triggered mechanism. Automatica, **2022**, *143*: 110416. doi: 10.1016/j.automatica.2022.110416
22. Cheng, P.; Chen, H.T.; He, S.P.; *et al*. Asynchronous deconvolution filtering for 2-D Markov jump systems with packet loss compensation. IEEE Trans. Autom. Sci. Eng., **2024**, *21*: 4165-4176. doi: 10.1109/TASE.2023.3292891
23. Song, J.; Wang, Z.D.; Niu, Y.G.; *et al*. Co-design of dissipative deconvolution filter and round-robin protocol for networked 2-D digital systems: Optimization and application. IEEE Trans. Syst. Man Cybern. Syst., **2023**, *53*: 6316-6328. doi: 10.1109/TSMC.2023.3284223
24. Zhao, D.; Wang, Y.Q.; Li, Y. Y.; *et al*. H_∞ fault estimation for 2-D linear discrete time-varying systems based on Krein space method. IEEE Trans. Syst. Man Cybern. Syst., **2018**, *48*: 2070-2079. doi: 10.1109/TSMC.2017.2723623
25. Cheng, P.; Wang, H.; Stojanovic, V.; *et al*. Asynchronous fault detection observer for 2-D Markov jump systems. IEEE Trans. Cybern., **2022**, *52*: 13623-13634. doi: 10.1109/TCYB.2021.3112699
26. Li, D.H.; Liang, J.L.; Wang, F. Robust H_∞ filtering for 2D systems with RON under the stochastic communication protocol. IET Control Theory Appl., **2020**, *14*: 2795-2804. doi: 10.1049/iet-cta.2019.0932
27. Li, M.Q.; Chen, W.B.; Shao, Y. Mixed H_∞-based finite-time passive filtering for a class of uncertain nonlinear singular systems. Opt. Control Appl. Methods, **2024**, *45*: 1122-1139. doi: 10.1002/oca.3092
28. Yang, R.N.; Zheng, W. X. H_∞ filtering for discrete-time 2-D switched systems: An extended average dwell time approach. Automatica, **2018**, *98*: 302-313. doi: 10.1016/j.automatica.2018.09.013
29. Hassibi, B.; Sayed, A.H.; Kailath, T. Linear estimation in Krein spaces. I. Theory. IEEE Trans. Autom. Control, **1996**, *41*: 18-33. doi: 10.1109/9.481605
30. Hassibi, B.; Sayed, A.H.; Kailath, T. Linear estimation in Krein spaces. II. Applications. IEEE Trans. Autom. Control, **1996**, *41*: 34-49. doi: 10.1109/9.481606
31. Zhao, D.; Ding, S.X.; Wang, Y.Q.; *et al*. Krein-space based robust H_∞ fault estimation for two-dimensional uncertain linear discrete time-varying systems. Syst. Control Lett., **2018**, *115*: 41-47. doi: 10.1016/j.sysconle.2018.03.005
32. Wang, W.; Chen, Y.; Xu, J.J.; *et al*. Finite-horizon estimation for 2-D systems with time-correlated multiplicative noises: A recursive iterative coupled estimator design. Int. J. Robust Nonlinear Control, **2024**, *34*: 8013-8032. doi: 10.1002/rnc.7375
33. Chen, H.W.; Wang, Z.D.; Shen, B.; *et al*. Distributed recursive filtering over sensor networks with nonlogarithmic sensor resolution. IEEE Trans. Autom. Control, **2022**, *67*: 5408-5415. doi: 10.1109/TAC.2021.3115473
34. Wang, F.; Wang, Z.D.; Liang, J.L.; *et al*. Recursive filtering for two-dimensional systems with amplify-and-forward relays: Handling degraded measurements and dynamic biases. Inf. Fusion, **2024**, *108*: 102368. doi: 10.1016/j.inffus.2024.102368
35. Wang, F.; Liang, J.L.; Lam, J.; *et al*. Robust filtering for 2-D systems with uncertain-variance noises and Weighted try-once-discard protocols. IEEE Trans. Syst. Man Cybern. Syst., **2023**, *53*: 2914-2924. doi: 10.1109/TSMC.2022.3219919
36. Shen, B.; Wang, X.L.; Zou, L. Maximum correntropy Kalman filtering for non-Gaussian systems with state saturations and stochastic nonlinearities. IEEE/CAA J. Autom. Sin., **2023**, *10*: 1223-1233. doi: 10.1109/JAS.2023.123195
37. Yang, R.; Ntogramatzidis, L.; Cantoni, M. On Kalman filtering for 2-D Fornasini-Marchesini models. In *Proceeding International Workshop on Multidimensional (nD) Systems, Thessaloniki, Greece, 29 June–1 July 2009*; IEEE: New York, 2009; pp. 1–8. doi:10.1109/NDS.2009.5195928

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