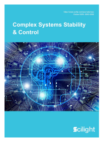
Scilight

Complex Systems Stability & Control
https://www.sciltp.com/journals/cssc



Article

Mean-Square Convergence and Stability of the Backward Euler Method for SDDEs with Highly Nonlinear Growing Coefficients

Zhuoqi Liu [1] and Shuaibin Gao [2,*]

[1] Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
[2] School of Mathematics and Statistics, South-Central Minzu University, Wuhan 430074, China
* Correspondence: shuaibingao@163.com

How To Cite: Liu, Z.; Gao, S. Mean-Square Convergence and Stability of the Backward Euler Method for SDDEs with Highly Nonlinear Growing Coefficients. *Complex Systems Stability & Control* **2025**, *1*(1), 2.

Received: 9 August 2025
Revised: 29 August 2025
Accepted: 1 September 2025
Published: 3 September 2025

Abstract: By virtue of the novel technique, this paper focuses on the mean-square convergence of the backward Euler method (BEM) for stochastic differential delay equations (SDDEs) without using the moment boundedness of numerical solutions. The convergence rate for SDDE whose drift and diffusion coefficients can both grow polynomially is shown. Furthermore, under fairly general conditions, the novel technique is applied to prove that the BEM can inherit the exponential mean-square stability with a simple proof. At last, some numerical experiments are implemented to illustrate the reliability of the theories.

Keywords: the backward Euler method; stochastic differential delay equations; mean-square convergence; mean-square stability

1. Introduction

Stochastic differential equations (SDEs) have been investigated by many scholars due to their extensive applications in many fields, including control problems, finance, biology, population model, communication, etc. (see [1–3]). However, it is difficult to obtain the exact solutions to SDEs. So using numerical methods to aquire approximations is a meaningful way to analyze the properties of solutions. It is well known that Euler-Maruyama (EM) method is one of the most popular numerical methods for SDEs [2,3]. Unfortunately, the divergence of EM method for SDEs with super-linear coefficients was proved in [4]. Whereafter, different kinds of modified EM methods have been established to approximate nonlinear SDEs, such as truncated EM method [5,6], tamed EM method [7,8], stopped EM method [9], multilevel EM method [10], projected EM method [11] and others. Furthermore, the implicit methods have also been studied and developed on account of their better convergence rates in the last decades [12–14].

The scholars use stochastic differential delay equations (SDDEs) to describe a class of more applicative systems which not only depend on the present states but also depend on the past states [3,15]. Like SDEs, the numerical methods of SDDEs have also been widely discussed. The modified EM methods for SDDEs were analyzed in [16–20], while the implicit EM methods were investigated in [21–25]. It is worth noting that the diffusion coefficients of the equations in [21–25] can not grow super-linearly, which has a adverse effect on the development of the implicit methods. In order to eliminate this adverse effect, the papers [26–30] focused on studying the backward Euler method (BEM) and split-step method for SDDEs respectively, whose drift and diffusion coefficients can grow super-linearly. However, the convergence rate was not shown in [27]; there is no constant in the diffusion coefficient in [26,28], which is a strong constraint. Therefore, by exploiting the novel technique, the first goal of our paper is to investigate the strong convergence rate of the BEM for SDDEs with highly nonlinear drift and diffusion coefficients under the weaker conditions. Therefore, the first motivation of this paper is to investigate the strong convergence rate of the BEM for SDDEs under the weaker conditions.

As is known to all, in addition to the convergence, the long-time stability of the numerical solution is also worth studying. The stabilities of implicit EM methods for SDEs were given in [31–37]. In the rest of this paragraph, we only discuss the stabilities of the implicit methods for SDDEs. When the diffusion coefficients satisfy $|g(x(t), x(t-\tau))|^2 \le c_1|x(t)|^2 + c_2|x(t-\tau)|^2$ (where c_1, c_2 are positive constants and $x(t-\tau)$ is the delay term), the stabilities of the BEM solutions to SDDEs were studied in [38–47]. The theories in [26–28] can not cover the equations with diffusion coefficient $g(x(t), x(t-\tau)) = x(t-\tau)$. Similarly, to a degree, Assumption 2.2 in [48] is also a bit strong. Moreover, the stabilities of split-step methods were analyzed in [24,25,49–51]. Especially, it should be noted that the constraint of coefficients was relaxed in [52], but the locally Lipschitz conditions of coefficients were used in the proof process. Hence, the second motivation of this paper is to prove that the numerical solutions of BEM for SDDEs are exponentially mean-square stable without using the locally Lipschitz conditions.

Let's summarize the main contributions of this paper compared to the existing literature. Under the fairly general conditions, by borrowing the ideas presented in [12,14,53], this paper investigates the strong convergence rate and exponential mean-square stability of the BEM for SDDEs with highly nonlinear drift and diffusion coefficients.

This paper is organized as follows. In Section 2, we introduce some necessary notations and prove that the global error in mean-square sence is controlled by the local error. Section 3 gives the convergence rates of BEM for SDDEs driven by multiplicative noise without using the moment boundedness of numerical solutions. In Section 4, under a stronger condition, the convergence rate is given by a much simpler proof. In Section 5, we present the exponential mean-square stability of BEM. In Section 6, three numerical examples are considered to illustrate the reliability of the theories.

2. Error Bounds for BEM

Let $|\cdot|$ and $<\cdot,\cdot>$ denote the Euclidean norm and the inner product of vectors in $\mathbb{R}^d$. We use $\mathbb{N}$ to denote the set of all positive integers. If $A \in \mathbb{R}^{d\times m}$ is a matrix, its trace norm is denoted by $\|A\| = \sqrt{trace(A^TA)}$, where A^T is the transpose of matrix A, $d, m \in \mathbb{N}$. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ stand for a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions. Let $\mathbb{E}$ be the probability expectation with respect to $\mathbb{P}$. Let $\mathcal{L}^r = \mathcal{L}^r(\Omega, \mathcal{F}, \mathbb{P})$ be the family of random variables ξ satisfying $\mathbb{E}|\xi|^r < \infty$. Let $\mathcal{C}([-\tau, 0]; \mathbb{R}^d)$ stand for the family of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^d$ with the norm $|\varphi|_\infty = \sup_{-\tau\le\theta\le 0}|\varphi(\theta)|$. And denote by C a generic positive constant which is independent of time stepsize.

Now, consider the nonlinear SDDE of the form:

$$dx(t) = \alpha\left(x(t), x(t-\tau)\right)dt + \beta\left(x(t), x(t-\tau)\right)dW(t), \tag{1}$$

on $t \ge 0$ with the initial data

$$x_0 = \varphi = \{\varphi(\theta) : -\tau \le \theta \le 0\} \in \mathcal{C}([-\tau, 0]; \mathbb{R}^d), \tag{2}$$

where $\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\beta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$. Here, $W(t)$ is an m-dimensional Brownian motion.

Now we construct the BEM for SDDEs. Suppose that there exist two positive integers N, M such that $\Delta = \frac{T}{N} = \frac{\tau}{M}$, where Δ is the step size. Define

$$\begin{cases} Z_n = \varphi(n\Delta), \quad n = -M, -M+1, ..., 0, \\ Z_n = Z_{n-1} + \alpha(Z_n, Z_{n-M})\Delta + \beta(Z_{n-1}, Z_{n-1-M})\Delta W_{n-1}, \quad n = 1, 2, ..., N, \end{cases} \tag{3}$$

where $\Delta W_{n-1} := W(t_n) - W(t_{n-1})$. To use the novel technique, we introduce a crucial equality

$$2\langle x-y, x\rangle = |x|^2 - |y|^2 + |x-y|^2, \qquad \forall x, y \in \mathbb{R}^d, \tag{4}$$

which will play an important role in our paper. Moreover, the discussions of the comparison between explicit and implicit numerical methods can be found in [14,53–55] and references therein. Before analyzing the errors between exact solutions and numerical solutions, some necessary assumptions should be imposed.

Assumption 1. *There exist constants $K > 0$ and $q > 2$ such that*

$$\langle x-\bar{x}, \alpha(x,y) - \alpha(\bar{x},\bar{y})\rangle + \frac{q-1}{2}\|\beta(x,y) - \beta(\bar{x},\bar{y})\|^2 \le K(|x-\bar{x}|^2 + |y-\bar{y}|^2),$$

for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$.

Assumption 2. *Suppose that the SDDE (1) admits a unique $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted solution with continuous sample paths, and $\sup_{s\in[0,T]}\mathbb{E}|x(s)|^2 < \infty$, $\sup_{s\in[0,T]}\mathbb{E}|\alpha(x(s), x(s-\tau))|^2 < \infty$. Furthermore, assume that the BEM* *admits a unique* $\{\mathcal{F}_{t_n}\}_{n\in\{1,2,\cdots,N\}}$*-adapted solution*$\{Z_n\}_{n=0}^N$ *as well.*

Lemma 1. *Let Assumption 2 hold. For* $n \in \{1, 2, ..., N\}$*, we have*

$$\begin{aligned}
&\big|[x(t_n) - Z_n] - \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})]\big|^2 \\
=&\big|[x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})]\big|^2 \\
&- \Delta^2\big|\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big|^2 \\
&+ \big|[\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\big|^2 + |\mathcal{R}_n|^2 \\
&+ 2\Delta\langle x(t_{n-1}) - Z_{n-1}, \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\rangle \\
&+ 2\langle [x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})], \\
&\quad [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\rangle \\
&+ 2\langle [x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})], \mathcal{R}_n\rangle \\
&+ 2\Delta\langle \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M}), \\
&\quad [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\rangle \\
&+ 2\Delta\langle \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M}), \mathcal{R}_n\rangle \\
&+ 2\langle [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}, \mathcal{R}_n\rangle,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\mathcal{R}_n =& \int_{t_{n-1}}^{t_n} [\alpha(x(s), x(s-\tau)) - \alpha(x(t_n), x(t_{n-M}))]ds \\
&+ \int_{t_{n-1}}^{t_n} [\beta(x(s), x(s-\tau)) - \beta(x(t_{n-1}), x(t_{n-1-M}))]dW(s).
\end{aligned} \tag{6}$$

Proof. From (1) and (3), one can see that

$$\begin{aligned}
&[x(t_n) - Z_n] - \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})] \\
=&[x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})] \\
&+ \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})] \\
&+ [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1} + \mathcal{R}_n,
\end{aligned}$$

where $\mathcal{R}_n$ is defined by (6). Then we have

$$\begin{aligned}
&\big|[x(t_n) - Z_n] - \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})]\big|^2 \\
=&\big|[x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})]\big|^2 \\
&+ \Delta^2\big|\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big|^2 \\
&+ \big|[\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\big|^2 + |\mathcal{R}_n|^2 \\
&+ 2\Delta\langle x(t_{n-1}) - Z_{n-1}, \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\rangle \\
&- 2\Delta^2\big|\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big|^2 \\
&+ 2\langle [x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})], \\
&\quad [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\rangle \\
&+ 2\langle [x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})], \mathcal{R}_n\rangle \\
&+ 2\Delta\langle \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M}), \\
&\quad [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\rangle \\
&+ 2\Delta\langle \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M}), \mathcal{R}_n\rangle \\
&+ 2\langle [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}, \mathcal{R}_n\rangle.
\end{aligned}$$

Rearranging the above equation gives the result. □

Lemma 2. *Let Assumptions 1 and 2 hold and* $0 < 2K\Delta < 1$*. Then for all* $n \in \{1, 2, ..., N\}$*, we derive that*

$$\mathbb{E}\big|\mathcal{R}_n\big|^2 < \infty, \qquad \mathbb{E}\big|x(t_n) - Z_n\big|^2 < \infty,$$
$$\mathbb{E}\big|[x(t_n) - Z_n] - \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})]\big|^2 < \infty,$$
$$\mathbb{E}\big|\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})\big|^2 < \infty,$$
$$\mathbb{E}\big|\beta(x(t_n), x(t_{n-M})) - \beta(Z_n, Z_{n-M})]\Delta W_{n-1}\big|^2 < \infty.$$

Proof. For all $n \in \{1, 2, ..., N\}$, proving $\mathbb{E}|\mathcal{R}_n|^2 < \infty$ is relatively simple. We have already known that $\sup_{s\in[0,T]} \mathbb{E}|x(s)|^2 < \infty$ and $\sup_{s\in[0,T]} \mathbb{E}|\alpha(x(s), x(s-\tau))|^2 < \infty$. By Assumption 1, for $s \in [t_{n-1}, t_n]$, we get

$$\begin{aligned}
&\mathbb{E}\big\|\beta(x(s), x(s-\tau)) - \beta(x(t_{n-1}), x(t_{n-1-M}))\big\|^2 \\
\leq&\frac{2K}{q-1}\mathbb{E}\big|x(s) - x(t_{n-1})\big|^2 + \frac{2K}{q-1}\mathbb{E}\big|x(s-\tau) - x(t_{n-1-M})\big|^2 \\
&- \frac{2}{q-1}\mathbb{E}\big\langle x(s) - x(t_{n-1}), \alpha(x(s), x(s-\tau)) - \alpha(x(t_{n-1}), x(t_{n-1-M}))\big\rangle \\
\leq&\frac{2K+1}{q-1}\mathbb{E}\big|x(s) - x(t_{n-1})\big|^2 + \frac{2K}{q-1}\mathbb{E}\big|x(s-\tau) - x(t_{n-1-M})\big|^2 \\
&+ \frac{1}{q-1}\mathbb{E}\big|\alpha(x(s), x(s-\tau)) - \alpha(x(t_{n-1}), x(t_{n-1-M}))\big|^2 \\
<&\infty.
\end{aligned}$$

Relying on the obtained inequality above, for $n \in \{1, 2, ..., N\}$, $\mathbb{E}|\mathcal{R}_n|^2 < \infty$ can be proved by using the Itô isometry. Taking expectations on both sides of (5) and using Hölder's inequality yield that

$$\begin{aligned}
&\mathbb{E}\big|[x(t_n) - Z_n] - \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})]\big|^2 \\
\leq&2\mathbb{E}\big|[x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})]\big|^2 \\
&+ 2\mathbb{E}\big|[\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\big|^2 + 4\mathbb{E}|\mathcal{R}_n|^2 \\
&+ 2\Delta\mathbb{E}\big\langle x(t_{n-1}) - Z_{n-1}, \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big\rangle \\
&+ 2\mathbb{E}\big\langle [x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})], \\
&\quad [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\big\rangle \\
&+ 2\Delta\mathbb{E}\big\langle \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M}), \\
&\quad [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\big\rangle.
\end{aligned} \tag{7}$$

Next, we use the inductive reasoning to prove $\mathbb{E}\big|[x(t_n) - Z_n] - \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})]\big|^2 < \infty$. According to (7), we know that $\mathbb{E}\big|[x(t_1) - Z_1] - \Delta[\alpha(x(t_1), x(t_{1-M})) - \alpha(Z_1, Z_{1-M})]\big|^2 \leq 4\mathbb{E}|\mathcal{R}_1|^2 < \infty$. Suppose that $\mathbb{E}\big|[x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})]\big|^2 < \infty$, then we get from Assumption 1 that

$$\begin{aligned}
&\mathbb{E}\big|[x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})]\big|^2 \\
=&\mathbb{E}\big|x(t_{n-1}) - Z_{n-1}\big|^2 + \Delta^2\mathbb{E}\big|\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big|^2 \\
&- 2\Delta\mathbb{E}\big\langle x(t_{n-1}) - Z_{n-1}, \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big\rangle \\
\geq&\mathbb{E}\big|x(t_{n-1}) - Z_{n-1}\big|^2 - 2K\Delta\mathbb{E}\big|x(t_{n-1}) - Z_{n-1}\big|^2 - 2K\Delta\mathbb{E}\big|x(t_{n-1-M}) - Z_{n-1-M}\big|^2 \\
&+ \Delta^2\mathbb{E}\big|\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\big|[x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})]\big|^2 \\
&+ 2K\Delta\mathbb{E}\big|x(t_{n-1-M}) - Z_{n-1-M}\big|^2 \\
\geq&(1 - 2K\Delta)\mathbb{E}\big|x(t_{n-1}) - Z_{n-1}\big|^2 \\
&+ \Delta^2\mathbb{E}\big|\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big|^2.
\end{aligned}$$

Based on the conditions, we have

$$\begin{aligned}\infty >&(1-2K\Delta)\mathbb{E}\big|x(t_{n-1})-Z_{n-1}\big|^2\\&+\Delta^2\mathbb{E}\big|\alpha(x(t_{n-1}),x(t_{n-1-M}))-\alpha(Z_{n-1},Z_{n-1-M})\big|^2.\end{aligned}$$

Therefore, for $0\le 2K\mathit{\Delta}<1$, one can derive that

$$\begin{aligned}&\mathbb{E}\big|x(t_{n-1})-Z_{n-1}\big|^2<\infty,\\&\mathbb{E}\big|\alpha(x(t_{n-1}),x(t_{n-1-M}))-\alpha(Z_{n-1},Z_{n-1-M})\big|^2<\infty,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\big\|\beta(x(t_{n-1}),x(t_{n-1-M}))-\beta(Z_{n-1},Z_{n-1-M})\big\|^2\\\le&\frac{2K+1}{q-1}\mathbb{E}\big|x(t_{n-1})-Z_{n-1}\big|^2+\frac{2K}{q-1}\mathbb{E}\big|x(t_{n-1-M})-Z_{n-1-M}\big|^2\\&+\frac{1}{q-1}\mathbb{E}\big|\alpha(x(t_{n-1}),x(t_{n-1-M}))-\alpha(Z_{n-1},Z_{n-1-M})\big|^2\\<&\infty.\end{aligned}$$

Moreover, using the properties of Brownian motion and Itô's isometry gives that

$$\begin{aligned}&\mathbb{E}\big|[\beta(x(t_{n-1}),x(t_{n-1-M}))-\beta(Z_{n-1},Z_{n-1-M})]\Delta W_{n-1}\big|^2\\=&\Delta\mathbb{E}\big\|\beta(x(t_{n-1}),x(t_{n-1-M}))-\beta(Z_{n-1},Z_{n-1-M})\big\|^2,\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\big\langle[x(t_{n-1})-Z_{n-1}]-\Delta[\alpha(x(t_{n-1}),x(t_{n-1-M}))-\alpha(Z_{n-1},Z_{n-1-M})],\\&[\beta(x(t_{n-1}),x(t_{n-1-M}))-\beta(Z_{n-1},Z_{n-1-M})]\Delta W_{n-1}\big\rangle=0,\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\big\langle\alpha(x(t_{n-1}),x(t_{n-1-M}))-\alpha(Z_{n-1},Z_{n-1-M}),\\&\quad[\beta(x(t_{n-1}),x(t_{n-1-M}))-\beta(Z_{n-1},Z_{n-1-M})]\Delta W_{n-1}\big\rangle=0.\end{aligned}$$

By Assumption 1, we get from (7) that

$$\begin{aligned}&\mathbb{E}\big|[x(t_n)-Z_n]-\Delta[\alpha(x(t_n),x(t_{n-M}))-\alpha(Z_n,Z_{n-M})]\big|^2]\\\le&2\mathbb{E}\big|[x(t_{n-1})-Z_{n-1}]-\Delta[\alpha(x(t_{n-1}),x(t_{n-1-M}))-\alpha(Z_{n-1},Z_{n-1-M})]\big|^2\\&+2\Delta\mathbb{E}\big\|\beta(x(t_{n-1}),x(t_{n-1-M}))-\beta(Z_{n-1},Z_{n-1-M})\big\|^2\\&+4\mathbb{E}|\mathcal{R}_n|^2+2\Delta\mathbb{E}\big\langle x(t_{n-1})-Z_{n-1},\alpha(x(t_{n-1}),x(t_{n-1-M}))-\alpha(Z_{n-1},Z_{n-1-M})\big\rangle\\\le&2\mathbb{E}\big|[x(t_{n-1})-Z_{n-1}]-\Delta[\alpha(x(t_{n-1}),x(t_{n-1-M}))-\alpha(Z_{n-1},Z_{n-1-M})]\big|^2\\&+2\Delta\mathbb{E}\big\|\beta(x(t_{n-1}),x(t_{n-1-M}))-\beta(Z_{n-1},Z_{n-1-M})\big\|^2\\&+4\mathbb{E}|\mathcal{R}_n|^2+2K\Delta\mathbb{E}\big|x(t_{n-1})-Z_{n-1}\big|^2+2K\Delta\mathbb{E}\big|x(t_{n-1-M})-Z_{n-1-M}\big|^2\\<&\infty.\end{aligned}$$

By the induction reasoning, we know that

$$\mathbb{E}\big|[x(t_n)-Z_n]-\Delta[\alpha(x(t_n),x(t_{n-M}))-\alpha(Z_n,Z_{n-M})]\big|^2<\infty.$$

Then, for all $n\in\{1,2,...,N\}$, the following results can be acquired

$$\mathbb{E}\big|x(t_n)-Z_n\big|^2<\infty,$$

$$\mathbb{E}\big|\alpha(x(t_n),x(t_{n-M}))-\alpha(Z_n,Z_{n-M})\big|^2<\infty,$$

$$\mathbb{E}\big\|\beta(x(t_n),x(t_{n-M}))-\beta(Z_n,Z_{n-M})\big\|^2<\infty.$$

□

We will prove the following theorem by Lemmata 1 and 2.

Theorem 1. *Let Assumptions 1 and 2 hold and $0 < 2K\Delta < 1$. Then for all $n \in \{1, 2, ..., N\}$, there is a constant C independent of n, such that*

$$\mathbb{E}|x(t_n) - Z_n|^2 \le C\left(\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_i|^2 + \Delta^{-1}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i \mid \mathcal{F}_{t_{i-1}})|^2\big]\right), \tag{8}$$

where $\mathcal{R}_i$ is defined by (6).

Proof. By Lemma 1 and the Hölder inequality, we derive that

$$\begin{aligned}
&\mathbb{E}\big|[x(t_n) - Z_n] - \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})]\big|^2\\
\le&\mathbb{E}\big|[x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})]\big|^2\\
&+ \Delta\mathbb{E}\big\|\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})\big\|^2\\
&+ \mathbb{E}|\mathcal{R}_n|^2 + 2\Delta\mathbb{E}\big\langle x(t_{n-1}) - Z_{n-1}, \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big\rangle\\
&+ 2\mathbb{E}\big\langle [x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})], \mathcal{R}_n\big\rangle\\
&+ 2\Delta\mathbb{E}\big\langle \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M}), \mathcal{R}_n\big\rangle\\
&+ (q-2)\mathbb{E}\big|[\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}\big|^2 + \frac{1}{q-2}\mathbb{E}|\mathcal{R}_n|^2\\
=&\mathbb{E}\big|[x(t_{n-1}) - Z_{n-1}] - [\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})]\big|^2\\
&+ 2\Delta\mathbb{E}\big\langle x(t_{n-1}) - Z_{n-1}, \alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})\big\rangle\\
&+ (q-1)\Delta\mathbb{E}\big|\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})\big|^2 + \frac{q-1}{q-2}\mathbb{E}|\mathcal{R}_n|^2\\
&+ 2\mathbb{E}\big\langle x(t_{n-1}) - Z_{n-1}, \mathcal{R}_n\big\rangle.
\end{aligned}$$

Using Assumption 1 and the property of conditional expectation leads to

$$\begin{aligned}
&\mathbb{E}\big|[x(t_n) - Z_n] - \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})]\big|^2\\
\le&\mathbb{E}\big|[x(t_{n-1}) - Z_{n-1}] - \Delta[\alpha(x(t_{n-1}), x(t_{n-1-M})) - \alpha(Z_{n-1}, Z_{n-1-M})]\big|^2\\
&+ 2K\Delta\mathbb{E}\big|x(t_{n-1}) - Z_{n-1}\big|^2 + 2K\Delta\mathbb{E}\big|x(t_{n-1-M}) - Z_{n-1-M}\big|^2\\
&+ \Delta\mathbb{E}\big|x(t_{n-1}) - Z_{n-1}\big|^2 + \frac{1}{\Delta}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_n \mid \mathcal{F}_{t_{n-1}})|^2\big] + \frac{q-1}{q-2}\mathbb{E}|\mathcal{R}_n|^2.
\end{aligned}$$

Since $\mathbb{E}\big|[x(t_0) - Z_0] - \Delta[\alpha(x(t_0), x(-\tau)) - \alpha(Z_0, Z_{-M})]\big| = 0$, by iterating, we have

$$\begin{aligned}
&\mathbb{E}\big|[x(t_n) - Z_n] - \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})]\big|^2\\
\le&\mathbb{E}\big|[x(t_0) - Z_0] - \Delta[\alpha(x(t_0), x(-\tau)) - \alpha(Z_0, Z_{-M})]\big|^2 + (2K+1)\Delta\sum_{i=0}^{n-1}\mathbb{E}\big|x(t_i) - Z_i\big|^2\\
&+ 2K\Delta\sum_{i=0}^{n-1}\mathbb{E}\big|x(t_{i-M}) - Z_{i-M}\big|^2 + \frac{1}{\Delta}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i \mid \mathcal{F}_{t_{i-1}})|^2\big] + \frac{q-1}{q-2}\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_i|^2\\
=&(2K+1)\Delta\sum_{i=0}^{n-1}\mathbb{E}\big|x(t_i) - Z_i\big|^2 + 2K\Delta\sum_{i=-M}^{n-1-M}\mathbb{E}\big|x(t_i) - Z_i\big|^2\\
&+ \frac{1}{\Delta}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i \mid \mathcal{F}_{t_{i-1}})|^2\big] + \frac{q-1}{q-2}\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_i|^2,
\end{aligned}$$

where we used the fact that

$$\sum_{i=0}^{n-1}\mathbb{E}\big|x(t_{i-M}) - Z_{i-M}\big|^2 = \sum_{i=-M}^{n-1-M}\mathbb{E}\big|x(t_i) - Z_i\big|^2.$$

Using Assumption 1 leads to

$$\begin{aligned}
&\mathbb{E}\big|[x(t_n)-Z_n]-\Delta[\alpha(x(t_n),x(t_{n-M}))-\alpha(Z_n,Z_{n-M})]\big|^2\\
=&\mathbb{E}\big|x(t_n)-Z_n\big|^2-2\Delta\mathbb{E}\big\langle x(t_n)-Z_n,\alpha(x(t_n),x(t_{n-M}))-\alpha(Z_n,Z_{n-M})\big\rangle\\
&+\Delta^2\mathbb{E}\big|\alpha(x(t_n),x(t_{n-M}))-\alpha(Z_n,Z_{n-M})\big|^2\\
\geq&(1-2K\Delta)\mathbb{E}\big|x(t_n)-Z_n\big|^2-2K\Delta\mathbb{E}\big|x(t_{n-M})-Z_{n-M}\big|^2.
\end{aligned}$$

Combining these inequalities yields that

$$\begin{aligned}
&(1-2K\Delta)\mathbb{E}\big|x(t_n)-Z_n\big|^2\\
\leq&(2K+1)\Delta\sum_{i=0}^{n-1}\mathbb{E}\big|x(t_i)-Z_i\big|^2+2K\Delta\sum_{i=-M}^{-1}\mathbb{E}\big|x(t_i)-Z_i\big|^2\\
&+2K\Delta\sum_{i=0}^{n-1-M}\mathbb{E}\big|x(t_i)-Z_i\big|^2+2K\Delta\mathbb{E}\big|x(t_{n-M})-Z_{n-M}\big|^2\\
&+\frac{1}{\Delta}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i\mid\mathcal{F}_{t_{i-1}})|^2\big]+\frac{q-1}{q-2}\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_i|^2\\
\leq&(2K+1)\Delta\sum_{i=0}^{n-1}\mathbb{E}\big|x(t_i)-Z_i\big|^2+2K\Delta\sum_{i=0}^{n-M}\mathbb{E}\big|x(t_i)-Z_i\big|^2\\
&+\frac{1}{\Delta}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i\mid\mathcal{F}_{t_{i-1}})|^2\big]+\frac{q-1}{q-2}\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_i|^2\\
\leq&(4K+1)\Delta\sum_{i=0}^{n-1}\mathbb{E}\big|x(t_i)-Z_i\big|^2+\frac{1}{\Delta}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i\mid\mathcal{F}_{t_{i-1}})|^2\big]+\frac{q-1}{q-2}\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_i|^2.
\end{aligned}$$

By the discrete-type Gronwall inequality, we get the desired result. □

3. Strong Convergence Rates for SDDEs

The strong convergence rate of BEM for SDDE with super-linear coefficients is dicussed in this section. In order to analyze the convergence rate, we need to make additional assumptions.

Assumption 3. *Let Assumption 1 hold, and there exist three constants $\rho\in[1,\infty)$, $\bar{p}\in[4\rho-2,\infty)$, $K_1\in(0,\infty)$ such that*

$$\big|\alpha(x,y)-\alpha(\bar{x},\bar{y})\big|\leq K_1\big(1+|x|^{\rho-1}+|y|^{\rho-1}+|\bar{x}|^{\rho-1}+|\bar{y}|^{\rho-1}\big)\big(|x-\bar{x}|+|y-\bar{y}|\big),$$

$$\big\langle x,\alpha(x,y)\big\rangle+\frac{(\bar{p}-1)}{2}\big\|\beta(x,y)\big\|^2\leq K_1\big(1+|x|^2+|y|^2\big),$$

for any $x,y,\bar{x},\bar{y}\in\mathbb{R}^d$.

Utilizing Assumptions 1 and 3 leads to

$$\|\beta(x,y)-\beta(\bar{x},\bar{y})\|^2\leq C(1+|x|^{\rho-1}+|y|^{\rho-1}+|\bar{x}|^{\rho-1}+|\bar{y}|^{\rho-1})(|x-\bar{x}|^2+|y-\bar{y}|^2),\tag{9}$$

$$|\alpha(x,y)|\leq C\left(1+|x|^{\rho}+|y|^{\rho}\right),\tag{10}$$

$$\|\beta(x,y)\|^2\leq C\left(1+|x|^{\rho+1}+|y|^{\rho+1}\right),\tag{11}$$

and

$$\langle x-\bar{x},\alpha(x,y)-\alpha(\bar{x},y)\rangle\leq K|x-\bar{x}|^2,\tag{12}$$

for any $x,y,\bar{x},\bar{y}\in\mathbb{R}^d$.

It is worth noting that Assumption 3 suffices to imply Assumption 2, which means that SDDE (1) admits a unique solution satisfying $\sup_{s\in[0,T]}\mathbb{E}|x(s)|^2<\infty$, $\sup_{s\in[0,T]}\mathbb{E}\big|\alpha(x(s),x(s-\tau))\big|^2<\infty$, see [18]. When

$K\Delta < 1$, (3) has a unique solution due to the fixed point theorem. Then the BEM is well defined [33]. Moreover, under Assumption 3, for any $p \in [2, \bar{p}]$, we know from [17] that the exact solution of SDDE (1) with the initial data (2) satisfies

$$\sup_{t\in[0,T]} \mathbb{E}|x(t)|^p < \infty. \tag{13}$$

Assumption 4. *There exists a constant $K_0 > 0$ such that the initial value φ satisfies*

$$|\varphi(\theta_2) - \varphi(\theta_1)| \le K_0|\theta_2 - \theta_1|^{\frac{1}{2}}, \qquad -\tau \le \theta_1, \theta_2 \le 0.$$

Lemma 3. *Let Assumption 3 hold. For any $0 \le t_1 < t_2 \le T$, there is a constant C dependent of K, K_1, γ, ρ but independent of n such that*

$$(\mathbb{E}\,|x(t_2) - x(t_1)|^\gamma)^{\frac{1}{\gamma}} \le C(t_2 - t_1)^{\frac{1}{2}}, \qquad \forall \gamma \in [2, \bar{p}/\rho].$$

Proof. Using an elementary inequality $|a + b|^\gamma \le 2^{\gamma-1}(|a|^\gamma + |b|^\gamma)$ for any $a, b \in \mathbb{R}^d$, it is easy to get that

$$\mathbb{E}\,|x(t_2) - x(t_1)|^\gamma \le 2^{\gamma-1}\mathbb{E}\left|\int_{t_1}^{t_2} \alpha\left(x(s), x(s-\tau)\right) ds\right|^\gamma + 2^{\gamma-1}\mathbb{E}\left|\int_{t_1}^{t_2} \beta\left(x(s), x(s-\tau)\right) dW(s)\right|^\gamma.$$

Then by Assumption 3, Hölder's inequality and Theorem 1.7.1 in [3], one can derive that

$$\begin{aligned}
&\mathbb{E}\,|x(t_2) - x(t_1)|^\gamma \\
\le&[2(t_2 - t_1)]^{\gamma-1}\mathbb{E}\int_{t_1}^{t_2} |\alpha\left(x(s), x(s-\tau)\right)|^\gamma ds \\
&+ \frac{1}{2}[2\gamma(\gamma-1)]^{\frac{\gamma}{2}}(t_2 - t_1)^{\frac{\gamma-2}{2}}\mathbb{E}\int_{t_1}^{t_2} \|\beta\left(x(s), x(s-\tau)\right)\|^\gamma ds \\
\le&C(t_2 - t_1)^{\gamma-1}\int_{t_1}^{t_2} \mathbb{E}\left(1 + |x(s)|^{\gamma\rho} + |x(s-\tau)|^{\gamma\rho}\right) ds \\
&+ C(t_2 - t_1)^{\frac{\gamma-2}{2}}\int_{t_1}^{t_2} \mathbb{E}\left(1 + |x(s)|^{\frac{\gamma(\rho+1)}{2}} + |x(s-\tau)|^{\frac{\gamma(\rho+1)}{2}}\right) ds \\
\le&C(t_2 - t_1)^{\frac{\gamma}{2}}.
\end{aligned}$$

□

Theorem 2. *Let Assumptions 3 and 4 hold and $\Delta \in (0, \frac{1}{2K})$. Then for the exact solution x(t) to SDDE (1) and the numerical solution Z_n to BEM (3), there is a constant C independent of n, Δ such that*

$$\sup_{0\le n\le N} \mathbb{E}|x(t_n) - Z_n|^2 \le C\Delta.$$

Proof. According to Theorem 1, to obtain the convergence rate, what we need is estimating two terms $\mathbb{E}[|\mathbb{E}(\mathcal{R}_i \mid \mathcal{F}_{t_{i-1}})|^2]$ and $\mathbb{E}|\mathcal{R}_i|^2$, $i \in \{1, 2, ..., N\}$. By (6), we have

$$\begin{aligned}
\mathbb{E}|\mathcal{R}_i|^2 =& \mathbb{E}\Big|\int_{t_{i-1}}^{t_i} [\alpha(x(s), x(s-\tau)) - \alpha(x(t_i), x(t_{i-M}))]ds \\
&+ \int_{t_{i-1}}^{t_i} [\beta(x(s), x(s-\tau)) - \beta(x(t_{i-1}), x(t_{i-1-M}))]dW(s)\Big|^2 \\
\le&2\mathbb{E}\Big|\int_{t_{i-1}}^{t_i} [\alpha(x(s), x(s-\tau)) - \alpha(x(t_i), x(t_{i-M}))]ds\Big|^2 \\
&+ 2\mathbb{E}\Big|\int_{t_{i-1}}^{t_i} [\beta(x(s), x(s-\tau)) - \beta(x(t_{i-1}), x(t_{i-1-M}))]dW(s)\Big|^2.
\end{aligned}$$

Now we estimate the first term on the right side of the inequality. By the Hölder inequality and Assumptions 3 and 4, we derive that

$$
\begin{aligned}
&\mathbb{E}\left|\int_{t_{i-1}}^{t_i}[\alpha(x(s),x(s-\tau))-\alpha(x(t_i),x(t_{i-M}))]ds\right|^2\\
\leq& C\Delta\mathbb{E}\int_{t_{i-1}}^{t_i}\big|\alpha(x(s),x(s-\tau))-\alpha(x(t_i),x(t_{i-M}))\big|^2ds\\
\leq& C\Delta\int_{t_{i-1}}^{t_i}\mathbb{E}\big[\big(1+|x(s)|^{2\rho-2}+|x(s-\tau)|^{2\rho-2}+|x(t_i)|^{2\rho-2}+|x(t_{i-M})|^{2\rho-2}\big)\\
&\qquad\big(|x(s)-x(t_i)|^2+|x(s-\tau)-x(t_{i-M})|^2\big)\big]ds\\
\leq& C\Delta\int_{t_{i-1}}^{t_i}\big(1+\mathbb{E}|x(s)|^{\frac{\bar{p}(2\rho-2)}{\bar{p}-2\rho}}+\mathbb{E}|x(s-\tau)|^{\frac{\bar{p}(2\rho-2)}{\bar{p}-2\rho}}+\mathbb{E}|x(t_i)|^{\frac{\bar{p}(2\rho-2)}{\bar{p}-2\rho}}+\mathbb{E}|x(t_{i-M})|^{\frac{\bar{p}(2\rho-2)}{\bar{p}-2\rho}}\big)^{\frac{\bar{p}-2\rho}{\bar{p}}}\\
&\qquad\cdot\big[\big(\mathbb{E}|x(s)-x(t_i)|^{\frac{\bar{p}}{\rho}}\big)^{\frac{2\rho}{\bar{p}}}+\big(\mathbb{E}|x(s-\tau)-x(t_{i-M})|^{\frac{\bar{p}}{\rho}}\big)^{\frac{2\rho}{\bar{p}}}\big]ds\\
\leq& C\Delta^3.
\end{aligned}
\tag{14}
$$

Employing the Itô isometry gives that

$$
\begin{aligned}
&\mathbb{E}\left|\int_{t_{i-1}}^{t_i}[\beta(x(s),x(s-\tau))-\beta(x(t_{i-1}),x(t_{i-1-M}))]dW(s)\right|^2\\
=&\mathbb{E}\int_{t_{i-1}}^{t_i}\big\|\beta(x(s),x(s-\tau))-\beta(x(t_{i-1}),x(t_{i-1-M}))\big\|^2ds\\
\leq& C\int_{t_{i-1}}^{t_i}\mathbb{E}\Big[\big(1+|x(s)|^{\rho-1}+|x(s-\tau)|^{\rho-1}+|x(t_i)|^{\rho-1}+|x(t_{i-M})|^{\rho-1}\big)\\
&\qquad\big(|x(s)-x(t_i)|^2+|x(s-\tau)-x(t_{i-M})|^2\big)\Big]ds\\
\leq& C\Delta^2.
\end{aligned}
$$

Thus, by Theorem 1, we draw a conclusion that

$$
\mathbb{E}|\mathcal{R}_i|^2\leq C\Delta^2. \tag{15}
$$

Moreover,

$$
\begin{aligned}
\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i\mid\mathcal{F}_{t_{i-1}})|^2\big]&=\mathbb{E}\Big[\Big|\mathbb{E}\Big(\int_{t_{i-1}}^{t_i}[\alpha(x(s),x(s-\tau))-\alpha(x(t_i),x(t_{i-M}))]ds\mid\mathcal{F}_{t_{i-1}}\Big)\Big|^2\Big]\\
&\leq\mathbb{E}\Big|\int_{t_{i-1}}^{t_i}[\alpha(x(s),x(s-\tau))-\alpha(x(t_i),x(t_{i-M}))]ds\Big|^2\\
&\leq C\Delta^3,
\end{aligned}
\tag{16}
$$

where we used the fact that

$$
\mathbb{E}\left(\int_{t_{i-1}}^{t_i}[\beta(x(s),x(s-\tau))-\beta(x(t_{i-1}),x(t_{i-1-M}))]dW(s)\mid\mathcal{F}_{t_{i-1}}\right)=0.
$$

Combining (15) and (16) gives the result. □

4. Convergence Rate under the Stronger Condition

In this section, a simpler proof process is exploited to show the convergence rate of BEM (3) for SDDE with multiplicative noise (1) under stronger condition .

Assumption 5. *There exist constants $K_3, K_4 > 0$ such that*

$$
\langle x-\bar{x},\alpha(x,y)-\alpha(\bar{x},\bar{y})\rangle+\|\beta(x,y)-\beta(\bar{x},\bar{y})\|^2\leq -K_3|x-\bar{x}|^2+K_4|y-\bar{y}|^2,
$$

for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$.

Theorem 3. *Let Assumptions 3–5 hold and $\Delta \in (0, \frac{1}{K})$. Then for the exact solution x(t) to SDDE (1) and the numerical solution Z_n to BEM (3), there is a constant C independent of n, Δ satisfying*

$$\sup_{0\le n\le N} \mathbb{E}|x(t_n) - Z_n|^2 \le C\Delta.$$

Proof. According to (1) and (3), we have

$$\begin{aligned}
&x(t_n) - Z_n \\
=&[x(t_{n-1}) - Z_{n-1}] + \Delta[\alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M})] \\
&+ [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1} + \mathcal{R}_n,
\end{aligned}$$

where $\mathcal{R}_n$ is defined by (6). Then

$$\begin{aligned}
&2\mathbb{E}\big\langle [x(t_n) - Z_n] - [x(t_{n-1}) - Z_{n-1}], x(t_n) - Z_n\big\rangle \\
=&2\Delta\mathbb{E}\big\langle \alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M}), x(t_n) - Z_n\big\rangle \\
&+ 2\mathbb{E}\big\langle [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}, x(t_n) - Z_n\big\rangle \\
&+ 2\mathbb{E}\big\langle \mathcal{R}_n, x(t_n) - Z_n\big\rangle \\
=&2\Delta\mathbb{E}\big\langle \alpha(x(t_n), x(t_{n-M})) - \alpha(Z_n, Z_{n-M}), x(t_n) - Z_n\big\rangle \\
&+ 2\mathbb{E}\big\langle [\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})]\Delta W_{n-1}, \\
&\quad [x(t_n) - Z_n] - [x(t_{n-1}) - Z_{n-1}]\big\rangle \\
&+ 2\mathbb{E}\big\langle \mathbb{E}\left(\mathcal{R}_n \mid \mathcal{F}_{t_{i-1}}\right), x(t_{n-1}) - Z_{n-1}\big\rangle \\
&+ 2\mathbb{E}\big\langle \mathcal{R}_n, [x(t_n) - Z_n] - [x(t_{n-1}) - Z_{n-1}]\big\rangle.
\end{aligned}$$

By Young's inequality and Assumption 5, there exists a positive constant ε_* such that

$$\begin{aligned}
&\mathbb{E}\big|x(t_n) - Z_n\big|^2 - \mathbb{E}\big|x(t_{n-1}) - Z_{n-1}\big|^2 + \mathbb{E}\big|[x(t_n) - Z_n] - [x(t_{n-1}) - Z_{n-1}]\big|^2 \\
\le& - 2K_3\Delta\mathbb{E}\big|x(t_n) - Z_n\big|^2 + 2K_4\Delta\mathbb{E}\big|x(t_{n-M}) - Z_{n-M}\big|^2 \\
&- 2\Delta\mathbb{E}\big\|\beta(x(t_n), x(t_{n-M})) - \beta(Z_n, Z_{n-M})\big\|^2 \\
&+ 2\Delta\mathbb{E}\big\|\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})\big\|^2 \\
&+ 2\mathbb{E}|\mathcal{R}_n|^2 + \mathbb{E}\big|[x(t_n) - Z_n] - [x(t_{n-1}) - Z_{n-1}]\big|^2 \\
&+ \frac{1}{\varepsilon_*^2}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i \mid \mathcal{F}_{t_{i-1}})|^2\big] + \varepsilon_*^2\mathbb{E}\big|x(t_{n-1}) - Z_{n-1}\big|^2,
\end{aligned}$$

where we used the equliaty (4). Then the inequality can be rearranged as

$$\begin{aligned}
&\big(1 + 2K_3\Delta\big)\mathbb{E}\big|x(t_n) - Z_n\big|^2 + 2\Delta\mathbb{E}\big\|\beta(x(t_n), x(t_{n-M})) - \beta(Z_n, Z_{n-M})\big\|^2 \\
\le&\big(1 + \varepsilon_*^2\big)\mathbb{E}\big|x(t_{n-1}) - Z_{n-1}\big|^2 \\
&+ 2\Delta\mathbb{E}\big\|\beta(x(t_{n-1}), x(t_{n-1-M})) - \beta(Z_{n-1}, Z_{n-1-M})\big\|^2 \\
&+ 2K_4\Delta\mathbb{E}\big|x(t_{n-M}) - Z_{n-M}\big|^2 + 2\mathbb{E}|\mathcal{R}_n|^2 + \frac{1}{\varepsilon_*^2}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i \mid \mathcal{F}_{t_{i-1}})|^2\big].
\end{aligned} \tag{17}$$

By choosing $\varepsilon_* = \sqrt{2K_3\Delta}$ and denoting $G_n = \big(1 + 2K_3\Delta\big)\mathbb{E}\big|x(t_n) - Z_n\big|^2 + 2\Delta\mathbb{E}\big\|\beta(x(t_n), x(t_{n-M})) - \beta(Z_n, Z_{n-M})\big\|^2$, we get that

$$G_n - G_{n-1} \le 2K_4\Delta\mathbb{E}\big|x(t_{n-M}) - Z_{n-M}\big|^2 + 2\mathbb{E}|\mathcal{R}_n|^2 + \frac{1}{2K_3\Delta}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i \mid \mathcal{F}_{t_{i-1}})|^2\big].$$

It is easy to see that

$$\begin{aligned}
G_n \leq & 2K_4\Delta\sum_{i=1}^{n}\mathbb{E}\big|x(t_{i-M})-Z_{i-M}\big|^2+2\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_n|^2 \\
& +\frac{1}{2K_3\Delta}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i\mid\mathcal{F}_{t_{i-1}})|^2\big] \\
\leq & 2K_4\Delta\sum_{i=1-M}^{0}\mathbb{E}\big|x(t_i)-Z_i\big|^2+2K_4\Delta\sum_{i=1}^{n-1}\mathbb{E}\big|x(t_i)-Z_i\big|^2 \\
& +2\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_n|^2+\frac{1}{2K_3\Delta}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i\mid\mathcal{F}_{t_{i-1}})|^2\big] \\
\leq & 2K_4\Delta\sum_{i=1}^{n-1}\mathbb{E}\big|x(t_i)-Z_i\big|^2+2\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_n|^2+\frac{1}{2K_3\Delta}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i\mid\mathcal{F}_{t_{i-1}})|^2\big],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathbb{E}\big|x(t_n)-Z_n\big|^2 \\
\leq & 2K_4\Delta\sum_{i=1}^{n-1}\mathbb{E}\big|x(t_i)-Z_i\big|^2+2\sum_{i=1}^{n}\mathbb{E}|\mathcal{R}_n|^2+\frac{1}{2K_3\Delta}\sum_{i=1}^{n}\mathbb{E}\big[|\mathbb{E}(\mathcal{R}_i\mid\mathcal{F}_{t_{i-1}})|^2\big].
\end{aligned}$$

Using the discrete-type Gronwall inequality leads to the convergence rate. □

Remark 1. *The reason why the technique in Theorem 3 can simplify the proof process under the stronger condition is that: in (17), when* $1+2K_3\Delta>0$ *holds, the subsequent proof process can be given. If Assumption 1 holds but Assumption 5 does not,* $1+2K_3\Delta$ *will change into* $1-2K_3\Delta$*, then we can not use this technique to get the desired result.*

5. Mean-Square Stability of BEM

This section will show that the BEM can inherit the exponential mean-square stability under the fairly general conditions.

Assumption 6. *There exist some constants* $l>1$, $\Gamma>2$, $c_1>c_2>0, c_3>c_4>0$ *such that*

$$\big\langle x,\alpha(x,y)\big\rangle+\frac{l}{2}\big\|\beta(x,y)\big\|^2\leq -c_1|x|^2+c_2|y|^2-c_3|x|^\Gamma+c_4|y|^\Gamma,$$

for all $x,y\in\mathbb{R}^d$.

Definition 1. *The exact solution of (1) is said to be exponentially mean-square stable if there exists a constant* $\varepsilon>0$ *such that*

$$\limsup_{t\to\infty}\frac{\log\mathbb{E}|x(t)|^2}{t}\leq-\varepsilon.$$

Definition 2. *The numerical solution defined by (3) is said to be exponentially mean-square stable if there exists a constant* $\varepsilon>0$ *such that, for any* $\Delta<\frac{1}{K}$,

$$\limsup_{n\to\infty}\frac{\log\mathbb{E}|Z_n|^2}{n\Delta}\leq-\varepsilon.$$

Theorem 4. *Under Assumption 6, the exact solution of (1) is exponentially mean-square stable.*

The proof of above theorem is the same as Theorem 3.1 in [56], so we omit it. Then we simply prove that the numerical solution to BEM is exponentially mean-square stable by using the novel technique.

Theorem 5. *Let Assumption 6 hold. Then there exists a sufficiently small* $\Delta<\min\{\frac{l-1}{2c_1},\Delta^*,\frac{1}{K}\}$ *such that*

$$\limsup_{n\to\infty}\frac{\log\mathbb{E}|Z_n|^2}{n\Delta}\leq-\varepsilon,$$

where ε satisfies $\varepsilon + 2c_2 e^{\varepsilon\tau} < 2c_1$, $\varepsilon \le \frac{1}{\tau}\log(\frac{c_3}{c_4})$, and Δ^* is the root of $1 + 2c_1\Delta - e^{\varepsilon\Delta} - 2c_2\Delta e^{\varepsilon\tau} = 0$.

Proof. Obviously, we get from (3) and (4) that

$$
\begin{aligned}
&2\mathbb{E}\big\langle Z_n - Z_{n-1}, Z_n\big\rangle \\
=&2\Delta\mathbb{E}\big\langle \alpha(Z_n, Z_{n-M}), Z_n\big\rangle + 2\mathbb{E}\big\langle \beta(Z_{n-1}, Z_{n-1-M})\Delta W_{n-1}, Z_n\big\rangle \\
=&2\Delta\mathbb{E}\big\langle \alpha(Z_n, Z_{n-M}), Z_n\big\rangle + 2\mathbb{E}\big\langle \beta(Z_{n-1}, Z_{n-1-M})\Delta W_{n-1}, Z_n - Z_{n-1}\big\rangle \\
\le& - 2c_1\Delta\mathbb{E}|Z_n|^2 + 2c_2\Delta\mathbb{E}|Z_{n-M}|^2 - 2c_3\Delta\mathbb{E}|Z_n|^\Gamma + 2c_4\Delta\mathbb{E}|Z_{n-M}|^\Gamma \\
&- l\Delta\mathbb{E}\big\|\beta(Z_n, Z_{n-M})\big\|^2 + \Delta\mathbb{E}\big\|\beta(Z_{n-1}, Z_{n-1-M})\big\|^2 + \mathbb{E}\big|Z_n - Z_{n-1}\big|^2.
\end{aligned}
$$

Rearranging this inequality leads to

$$
\begin{aligned}
&\big(1 + 2c_1\Delta\big)\mathbb{E}|Z_n|^2 + l\Delta\mathbb{E}\big\|\beta(Z_n, Z_{n-M})\big\|^2 \\
\le&\mathbb{E}|Z_{n-1}|^2 + \Delta\mathbb{E}\big\|\beta(Z_{n-1}, Z_{n-1-M})\big\|^2 + 2c_2\Delta\mathbb{E}|Z_{n-M}|^2 - 2c_3\Delta\mathbb{E}|Z_n|^\Gamma + 2c_4\Delta\mathbb{E}|Z_{n-M}|^\Gamma.
\end{aligned}
$$

Due to $\Delta < \frac{l-1}{2c_1}$, it is easy to get

$$
\big(1 + 2c_1\Delta\big)F_n \le F_{n-1} + 2c_2\Delta\mathbb{E}|Z_{n-M}|^2 - 2c_3\Delta\mathbb{E}|Z_n|^\Gamma + 2c_4\Delta\mathbb{E}|Z_{n-M}|^\Gamma,
$$

where $F_n = \mathbb{E}|Z_n|^2 + \Delta\mathbb{E}\big\|\beta(Z_n, Z_{n-M})\big\|^2$. By multiplying both sides by $e^{\varepsilon n\Delta}$ and subtracting $\big(1 + 2c_1\Delta\big)e^{\varepsilon(n-1)\Delta}F_{n-1}$ from two sides, we obtain that

$$
\begin{aligned}
&\big(1 + 2c_1\Delta\big)e^{\varepsilon n\Delta}F_n - \big(1 + 2c_1\Delta\big)e^{\varepsilon(n-1)\Delta}F_{n-1} \\
\le&\big(e^{\varepsilon\Delta} - (1 + 2c_1\Delta)\big)e^{\varepsilon(n-1)\Delta}F_{n-1} + 2c_2\Delta e^{\varepsilon n\Delta}\mathbb{E}|Z_{n-M}|^2 \\
&- 2c_3\Delta e^{\varepsilon n\Delta}\mathbb{E}|Z_n|^\Gamma + 2c_4\Delta e^{\varepsilon n\Delta}\mathbb{E}|Z_{n-M}|^\Gamma.
\end{aligned}
$$

Then it is not difficult to get that

$$
\begin{aligned}
&e^{\varepsilon n\Delta}\mathbb{E}|Z_n|^2 \le e^{\varepsilon n\Delta}F_n \\
\le&\big(1 + 2c_1\Delta\big)F_0 + \big(e^{\varepsilon\Delta} - (1 + 2c_1\Delta)\big)\sum_{i=0}^{n-1} e^{\varepsilon i\Delta}F_i + 2c_2\Delta\sum_{i=1}^{n} e^{\varepsilon i\Delta}\mathbb{E}|Z_{i-M}|^2 \\
&- 2c_3\Delta\sum_{i=1}^{n} e^{\varepsilon i\Delta}\mathbb{E}|Z_i|^\Gamma + 2c_4\Delta\sum_{i=1}^{n} e^{\varepsilon i\Delta}\mathbb{E}|Z_{i-M}|^\Gamma \\
\le&\big(1 + 2c_1\Delta\big)\Big(\mathbb{E}|Z_0|^2 + \Delta\mathbb{E}\big\|\beta(Z_0, Z_{-M})\big\|^2\Big) \\
&+ \big(e^{\varepsilon\Delta} - (1 + 2c_1\Delta)\big)\sum_{i=0}^{n-1} e^{\varepsilon i\Delta}\Big(\mathbb{E}|Z_i|^2 + \Delta\mathbb{E}\big\|\beta(Z_i, Z_{i-M})\big\|^2\Big) \\
&+ 2c_2\Delta\sum_{i=1-M}^{n-M} e^{\varepsilon(i+M)\Delta}\mathbb{E}|Z_i|^2 - 2c_3\Delta\sum_{i=1}^{n} e^{\varepsilon i\Delta}\mathbb{E}|Z_i|^\Gamma + 2c_4\Delta\sum_{i=1-M}^{n-M} e^{\varepsilon(i+M)\Delta}\mathbb{E}|Z_i|^\Gamma \\
\le&e^{\varepsilon\Delta}\Big(\mathbb{E}|Z_0|^2 + \Delta\mathbb{E}\big\|\beta(Z_0, Z_{-M})\big\|^2\Big) + 2c_2\Delta e^{\varepsilon\tau}\mathbb{E}|Z_0|^2 + 2c_4\Delta e^{\varepsilon\tau}\mathbb{E}|Z_0|^\Gamma \\
&+ 2c_2\Delta e^{\varepsilon\tau}\sum_{i=1-M}^{-1} e^{\varepsilon i\Delta}\mathbb{E}|Z_i|^2 + 2c_4\Delta e^{\varepsilon\tau}\sum_{i=1-M}^{-1} e^{\varepsilon i\Delta}\mathbb{E}|Z_i|^\Gamma \\
&- \big(1 + 2c_1\Delta - e^{\varepsilon\Delta}\big)\sum_{i=1}^{n-1} \Delta e^{\varepsilon i\Delta}\mathbb{E}\big\|\beta(Z_i, Z_{i-M})\big\|^2 \\
&- \big(1 + 2c_1\Delta - e^{\varepsilon\Delta} - 2c_2\Delta e^{\varepsilon\tau}\big)\sum_{i=1}^{n-1} \mathbb{E}e^{\varepsilon i\Delta}|Z_i|^2 - \big(2c_3\Delta - 2c_4\Delta e^{\varepsilon\tau}\big)\sum_{i=1}^{n-1} e^{\varepsilon i\Delta}\mathbb{E}|Z_i|^\Gamma.
\end{aligned}
$$

Define

$$
f(\Delta) = 1 + 2c_1\Delta - e^{\varepsilon\Delta} - 2c_2\Delta e^{\varepsilon\tau} \quad \text{and} \quad g(\Delta) = 2c_3\Delta - 2c_4\Delta e^{\varepsilon\tau}.
$$

Then one can see that

$$f'(\Delta) = 2c_1 - \varepsilon e^{\varepsilon\Delta} - 2c_2 e^{\varepsilon\tau}, \quad f''(\Delta) = -\varepsilon^2 e^{\varepsilon\Delta},$$

and

$$g'(\Delta) = 2c_3 - 2c_4 e^{\varepsilon\tau}.$$

We observe that $f(0) = 0$ and $f'(0) > 0$ when $\varepsilon + 2c_2 e^{\varepsilon\tau} < 2c_1$. It means that $f(\Delta)$ is an increasing function in a sufficiently small interval. In addition, $f''(\Delta) < 0$ implies $f(\Delta)$ is concave function, so there exists a Δ' such that $f'(\Delta') = 0$. Then $f(\Delta)$ increases strictly when $\Delta < \Delta'$ and $f(\Delta)$ decreases strictly when $\Delta > \Delta'$. Therefore, there exists a $\Delta^* > \Delta'$ satisfying $f(\Delta^*) = 0$, and for all $\Delta < \Delta^*$,

$$1 + 2c_1\Delta - e^{\varepsilon\Delta} - 2c_2\Delta e^{\varepsilon\tau} > 0.$$

Next, we analyze $g(\Delta)$ in the same way. We see that $g(0) = 0$ and $g'(\Delta) > 0$ when $c_3 > c_4$, $\varepsilon \le \frac{1}{\tau}\log(\frac{c_3}{c_4})$, so $g(\Delta) > 0$.

To sum up, we obtain that for all sufficiently small $\Delta < \min\{\frac{l-1}{2c_1}, \Delta^*, \frac{1}{K}\}$, there exists a positive constant Υ independent of n such that

$$e^{\varepsilon n\Delta}\mathbb{E}|Z_n|^2 \le \Upsilon,$$

which means that

$$\limsup_{n\to\infty} \frac{\log \mathbb{E}|Z_n|^2}{n\Delta} \le -\varepsilon.$$

□

Remark 2. *Note that if the corresponding assumptions in [27,28,56] are replaced by Assumption 6, the stability results can not be obtained. But, on the contrary, we can also get Theorem 5 under the corresponding assumptions in [27,28,56].*

Remark 3. *The locally Lipschitz conditions of drift and diffusion coefficients were used in the proof process in [52]. The advantage of our proof is that, by borrowing the technique in [53], we can still get the exponential mean-square stability of BEM without using the locally Lipschitz conditions.*

6. Numerical Experiments

Example 1. *Consider the following scalar nonlinear SDDE*

$$dx(t) = \big(x(t) - 4x(t)^3 + x(t-\tau)\big)\,dt + \big(x(t)^2 - x(t-\tau) + 2\big)\,dW(t). \tag{18}$$

Here, $t \in [0,1]$, $\tau = 0.25$, and the initial data $\varphi(\theta) = |\theta|^{\frac{1}{2}} + 1$ for $\theta \in [-0.25, 0]$. Now we verify that the drift and diffusion coeficients fulfill Assumption 1. Let q = 3, then

$$\begin{aligned}
&\langle x - \bar{x}, \alpha(x,y) - \alpha(\bar{x},\bar{y})\rangle + \frac{(q-1)}{2}|\beta(x,y) - \beta(\bar{x},\bar{y})|^2 \\
\le& |x-\bar{x}|^2 + \frac{1}{2}|x-\bar{x}|^2 + \frac{1}{2}|y-\bar{y}|^2 + 2|y-\bar{y}|^2 - 4|x-\bar{x}|^2(x^2 + x\bar{x} + \bar{x}^2) \\
&+ 2|x-\bar{x}|^2(x^2 + 2x\bar{x} + \bar{x}^2) \\
\le& |x-\bar{x}|^2 + \frac{1}{2}|x-\bar{x}|^2 + \frac{1}{2}|y-\bar{y}|^2 + 2|y-\bar{y}|^2 - 2|x-\bar{x}|^2(x^2 + \bar{x}^2) \\
\le& \frac{3}{2}|x-\bar{x}|^2 + \frac{5}{2}|y-\bar{y}|^2,
\end{aligned}$$

and Assumption 3 is simple to be tested as well.

In order to check the theory in Theorem 2, we perform a numerical experiment with four different stepsizes $\Delta = 2^{-13}$, 2^{-12}, 2^{-11}, 2^{-10} at $T = 1$. The numerical solution with stepsize $\Delta = 2^{-15}$ is regarded as the exact solution of this experiment since it is difficult to be expressed explicitly. Then mean-square error can be estimated by computing the average of 500 sample paths' errors between exact solutions and numerical solutions. Figure 1 *illustrates the mean-square error which is defined by*

$$(\mathbb{E}|x(T)-Z_N|^2)^{\frac{1}{2}} \approx \left(\frac{1}{500}\sum_{i=1}^{500}|x^i(T)-Z_N^i|^2\right)^{\frac{1}{2}}.$$

From Figure 1, we observe that the root mean-square error line and the reference line are visually parallel, indicating a mean-square convergence order of 0.5 for the BEM, which means that the numerical experiment is consistent with the theoretical result in Theorem 2.

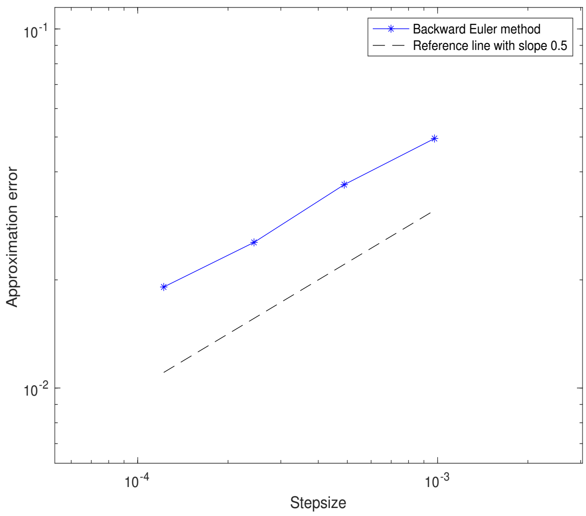


Figure 1. Convergence rate of BEM for (18).

Example 2. *Consider the following scalar nonlinear SDDE*

$$dx(t) = \left(-2x(t) - 4x(t)^3 + x(t-\tau)\right)dt + \left(x(t)^2 + \frac{1}{2}x(t-\tau)\right)dW(t), \tag{19}$$

on $t \geq 0$. Here, the initial data $\varphi(\theta) = |\theta|^{\frac{1}{2}} + 3$, $t \in [-\tau, 0]$, $\tau = 0.2$. Let $l = 2$, then

$$\langle x, \alpha(x,y)\rangle = x^T(-2x - 4x^3 + y) \leq -\frac{3}{2}|x|^2 - 4|x|^4 + \frac{1}{2}|y|^2,$$

$$|\beta(x,y)|^2 = |x^2 + \frac{1}{2}y|^2 \leq 2|x|^4 + \frac{1}{2}|y|^2,$$

which means that the coefficients satisfy Assumption 6.

We compute the average of the numerical solutions simulated by 1000 sample paths with stepsize $\Delta = 0.1$ and plot it in Figure 2. From Figure 2, we observe that the simulated trajectory of the BEM applied to the scalar nonlinear SDDE (19) consistently decays toward zero over time. And this numerical behavior confirms that the BEM discretization preserves mean-square stability for SDDE (19). Furthermore, we compare the tamed EM method (TEM) in [16] with the BEM in this paper. In Table 1, we provide the values of the TEM and the BEM over time. It can be seen that when stepsize $\Delta = 0.1$, the stability performance of the BEM is better than that of TEM, which means that the BEM has more relaxed constraints on the stepsize Δ.

Table 1. Numerical solutions simulated by 1000 sample paths with stepsize $\Delta = 0.1$.

t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
TEM	37.48	7860.54	6.22×10^9	4.21×10^{22}	1.58×10^{49}	3.81×10^{102}	INF	INF
BEM	2.74	1.19	0.71	0.44	0.29	0.21	0.15	0.11
t	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
TEM	INF	INF	INF	INF	INF	INF	INF	INF
BEM	0.09	0.07	0.06	0.05	0.04	0.03	0.03	0.02

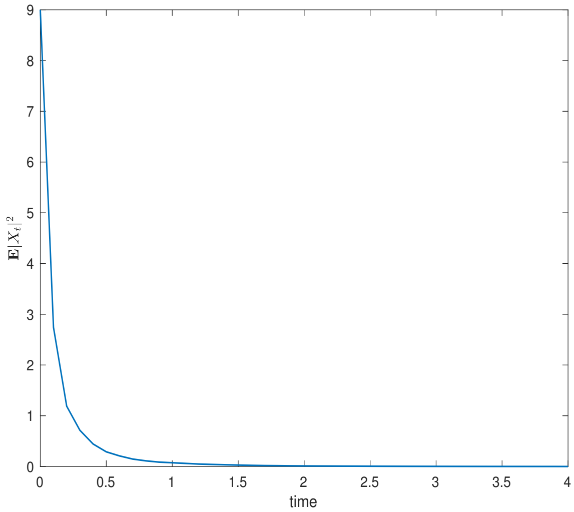


Figure 2. Numerical simulation of BEM for (19).

Author Contributions

Z.L. and S.G. contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding

This work is supported by the Fundamental Research Funds for the Central Universities, South-Central MinZu University (CZQ25020).

Data Availability Statement

No data was used for the research described in the paper.

Conflicts of Interest

The authors declare no competing interests.

References

1. Allen, E. *Modeling with Itô Stochastic Differential Equations*; Springer: Dordrecht, The Netherlands, 2007.
2. Kloeden, P.E.; Platen, E. *Numerical Solution of Stochastic Differential Equations*; Springer: Berlin, Germany, 1992.
3. Mao, X. *Stochastic Differential Equations and Applications*, 2nd ed.; Horwood: Chichester, UK, 2007.
4. Hutzenthaler, M.; Jentzen, A.; Kloeden, P.E. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2011**, *467*, 1563–1576.
5. Li, X.; Mao, X.; Yin, G. Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in pth moment and stability. *IMA J. Numer. Anal.* **2019**, *39*, 847–892.
6. Mao, X. The truncated Euler?Maruyama method for stochastic differential equations. *J. Comput. Appl. Math.* **2015**, *290*, 370–384.
7. Hutzenthaler, M.; Jentzen, A.; Kloeden, P.E. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *Ann. Appl. Probab.* **2012**, *22*, 1611–1641.
8. Sabanis, S. Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients. *Ann. Appl. Probab.* **2016**, *26*, 2083–2105.
9. Liu, W.; Mao, X. Strong convergence of the stopped Euler?Maruyama method for nonlinear stochastic differential equations. *Appl. Math. Comput.* **2013**, *223*, 389–400.
10. Anderson, D.F.; Higham, D.J.; Sun, Y. Multilevel Monte Carlo for stochastic differential equations with small noise. *SIAM J. Numer. Anal.* **2016**, *54*, 505–529.
11. Beyn, W.; Isaak, E.; Kruse, R. Stochastic C-stability and B-consistency of explicit and implicit Euler-type schemes. *J. Sci. Comput.* **2016**, *67*, 955–987.

12. Andersson, A.; Kruse, R. Mean-square convergence of the BDF2?Maruyama and backward Euler schemes for SDE satisfying a global monotonicity condition. *BIT* **2017**, *57*, 21–53.
13. Szpruch, L.; Mao, X.; Higham, D.J.; et al. Numerical simulation of a strongly nonlinear Ait-Sahalia-type interest rate model. *BIT* **2011**, *51*, 405–425.
14. Wang, X.; Wu, J.; Dong, B. Mean-square convergence rates of stochastic theta methods for SDEs under a coupled monotonicity condition. *BIT* **2020**, *60*, 759–790.
15. Buckwar, E.; Pikovsky, A.; Scheutzow, M. Stochastic dynamics with delay and memory-Preface. *Stoch. Dyn.* **2005**, *5*, III–IV.
16. Deng, S.; Fei, C.; Fei, W.; et al. Tamed EM schemes for neutral stochastic differential delay equations with superlinear diffusion coefficients. *J. Comput. Appl. Math.* **2021**, *388*, 113269.
17. Fei, W.; Hu, L.; Mao, X.; et al. Advances in the truncated Euler-Maruyama method for stochastic differential delay equations. *Commun. Pure Appl. Anal.* **2020**, *19*, 2081–2100.
18. Guo, Q.; Mao, X.; Yue, R. The truncated Euler-Maruyama method for stochastic differential delay equations. *Numer. Algorithms* **2018**, *78*, 599–624.
19. Li, M.; Huang, C. Projected Euler-Maruyama method for stochastic delay differential equations under a global monotonicity condition. *Appl. Math. Comput.* **2020**, *366*, 124733.
20. Song, G.; Hu, J.; Gao, S.; et al. The strong convergence and stability of explicit approximations for nonlinear stochastic delay differential equations. *Numer. Algorithms* **2022**, *89*, 855–883.
21. Gan, S.; Schurz, H.; Zhang, H. Mean square convergence of stochastic θ-methods for nonlinear neutral stochastic differential delay equations. *Int. J. Numer. Anal. Model.* **2011**, *8*, 201–213.
22. Liu, M.; Cao, W.; Fan, Z. Convergence and stability of the semi-implicit Euler method for a linear stochastic differential delay equation. *J. Comput. Appl. Math.* **2004**, *170*, 255–268.
23. Wang, L.; Mei, C.; Xue, H. The semi-implicit Euler method for stochastic differential delay equation with jumps. *Appl. Math. Comput.* **2007**, *192*, 567–578.
24. Wang, X.; Gan, S. The improved split-step backward Euler method for stochastic differential delay equations. *Int. J. Comput. Math.* **2011**, *88*, 2359–2378.
25. Zhang, H.; Gan, S.; Hu, L. The split-step backward Euler method for linear stochastic delay differential equations. *J. Comput. Appl. Math.* **2009**, *225*, 558–568.
26. Zhang, C.; Xie, Y. Backward Euler-Maruyama method applied to nonlinear hybrid stochastic differential equations with time-variable delay. *Sci. China Math.* **2019**, *62*, 597–616.
27. Zhou, S. Strong convergence and stability of backward Euler-Maruyama scheme for highly nonlinear hybrid stochastic differential delay equation. *Calcolo* **2015**, *52*, 445–473.
28. Zhou, S.; Jin, H. Numerical solution to highly nonlinear neutral-type stochastic differential equation. *Appl. Numer. Math.* **2019**, *140*, 48–75.
29. Yan, Z.; Xiao, A.; Tang, X. Strong convergence of the split-step theta method for neutral stochastic delay differential equations. *Appl. Numer. Math.* **2017**, *120*, 215–232.
30. Yue, C.; Zhao, L. Strong convergence of the split-step backward Euler method for stochastic delay differential equations with a nonlinear diffusion coefficient. *J. Comput. Appl. Math.* **2021**, *382*, 113087.
31. Higham, D.J. Mean-square and asymptotic stability of stochastic theta method. *SIAM J. Numer. Anal.* **2000**, *38*, 753–769.
32. Higham, D.J.; Mao, X.; Yuan, C. Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations. *SIAM J. Numer. Anal.* **2007**, *45*, 592–609.
33. Mao, X.; Shen, Y.; Gray, A. Almost sure exponential stability of backward Euler-Maruyama discretizations for hybrid stochastic differential equations. *J. Comput. Appl. Math.* **2011**, *235*, 1213–1226.
34. Mao, X.; Szpruch, L. Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients. *J. Comput. Appl. Math.* **2013**, *238*, 14–28.
35. Zhu, Q.; Song, S.; Tang, T. Mean square exponential stability of stochastic nonlinear delay systems. *Internat. J. Control* **2017**, *90*, 2384–2393.
36. Liu, L.; Zhu, Q. Almost sure exponential stability of numerical solutions to stochastic delay Hopfield neural networks. *Appl. Math. Comput.* **2015**, *266*, 698–712.
37. Liu, L.; Zhu, Q. Mean square stability of two classes of theta method for neutral stochastic differential delay equations. *J. Comput. Appl. Math.* **2016**, *305*, 55–67.
38. Farkhondeh Rouz, O. Preserving asymptotic mean-square stability of stochastic theta scheme for systems of stochastic delay differential equations. *Comput. Methods Differ. Equ.* **2020**, *8*, 468–479.
39. Huang, C. Mean square stability and dissipativity of two classes of theta methods for systems of stochastic delay differential equations. *J. Comput. Appl. Math.* **2014**, *259*, 77–86.
40. Li, Q.; Gan, S. Almost sure exponential stability of numerical solutions for stochastic delay differential equations with jumps. *J. Appl. Math. Comput.* **2011**, *37*, 541–557.
41. Milošević, M. Convergence and almost sure exponential stability of implicit numerical methods for a class of highly nonlinear neutral stochastic differential equations with constant delay. *J. Comput. Appl. Math.* **2015**, *280*, 248–264.

42. Milošević, M. Implicit numerical methods for highly nonlinear neutral stochastic differential equations with time-dependent delay. *Appl. Math. Comput.* **2014**, *244*, 741–760.
43. Qu, X.; Huang, C. Delay-dependent exponential stability of the backward Euler method for nonlinear stochastic delay differential equations. *Int. J. Comput. Math.* **2012**, *89*, 1039–1050.
44. Wang, W.; Chen, Y. Mean-square stability of semi-implicit Euler method for nonlinear neutral stochastic delay differential equations. *Appl. Numer. Math.* **2011**, *61*, 696–701.
45. Wu, F.; Mao, X.; Szpruch, L. Almost sure exponential stability of numerical solutions for stochastic delay differential equations. *Numer. Math.* **2010**, *115*, 681–697.
46. Yu, Z. The improved stability analysis of the backward Euler method for neutral stochastic delay differential equations. *Int. J. Comput. Math.* **2013**, *90*, 1489–1494.
47. Zong, X.; Wu, F. Exponential stability of the exact and numerical solutions for neutral stochastic delay differential equations. *Appl. Math. Model.* **2016**, *40*, 19–30.
48. Chen, L.; Wu, F. Almost sure exponential stability of the backward Euler-Maruyama scheme for stochastic delay differential equations with monotone-type condition. *J. Comput. Appl. Math.* **2015**, *282*, 44–53.
49. Cao, W.; Hao, P.; Zhang, Z. Split-step θ-method for stochastic delay differential equations. *Appl. Numer. Math.* **2014**, *76*, 19–33.
50. Mo, H.; Deng, F.; Zhang, C. Exponential stability of the split-step θ-method for neutral stochastic delay differential equations with jumps. *Appl. Math. Comput.* **2017**, *315*, 85–95.
51. Zhao, G.; Liu, M. Numerical methods for nonlinear stochastic delay differential equations with jumps. *Appl. Math. Comput.* **2014**, *233*, 222–231.
52. Zong, X.; Wu, F.; Huang, C. Theta schemes for SDDEs with non-globally Lipschitz continuous coefficients. *J. Comput. Appl. Math.* **2015**, *278*, 258–277.
53. Liu, W.; Mao, X.; Wu, Y. The backward Euler-Maruyama method for invariant measures of stochastic differential equations with super-linear coefficients. *Appl. Numer. Math.* **2023**, *184*, 137–150.
54. Higham, D.J. Stochastic ordinary differential equations in applied and computational mathematics. *IMA J. Appl. Math.* **2011**, *76*, 449–474.
55. Milstein, G.N.; Tretyakov, M.V. *Stochastic Numerics for Mathematical Physics*; Springer: Berlin, Germany, 2004.
56. Mo, H.; Liu, L.; Xing, M.; et al. Exponential stability of implicit numerical solution for nonlinear neutral stochastic differential equations with time-varying delay and Poisson jumps. *Math. Methods Appl. Sci.* **2021**, *44*, 5574–5592.