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Least-Squares Linear Estimation for Multirate Uncertain Systems subject to DoS Attacks

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Abstract: This paper investigates the least-squares linear estimation problem for multirate systems with stochastic parameter matrices, under the influence of random denial-of-service (DoS) attacks. These attacks can severely impair the performance of estimation algorithms by causing intermittent loss of measurement data. To counteract the adverse effect of DoS attacks, two compensation strategies –hold-input and prediction compensation– are used. For each of these strategies, specific recursive filtering and smoothing algorithms are designed. A key advantage of the proposed methodology is its ability to operate without requiring a detailed signal evolution model, relying only on the mean and covariance functions of the involved processes. The effectiveness of the proposed approaches is validated through numerical simulations, which highlight how common network-induced phenomena, such as missing observations, can be incorporated into the framework of systems with random parameter matrices and, additionally, they provide insights into estimation performance under different attack probabilities.

Keywords: multirate systems; least-squares estimation; random parameter matrices; DoS attacks; compensation strategies; hold-input; prediction compensation

1. Introduction

Least-squares (LS) estimation is one of the fundamental techniques in stochastic signal processing, widely used for the optimal reconstruction of stochastic signals from noisy observations. By minimizing the mean-squared error, LS estimators provide a computationally efficient framework suitable for a broad range of applications. Over the years, significant advancements have been made in adapting LS and other estimation methods to account for challenges such as complex dynamics, noise correlation, randomly missing or fading observations, stochastic disturbances, communication constraints or adversarial attacks, among others (see e.g. [1–5] and references therein).

In systems where parameter matrices are randomly varying, the estimation problem becomes significantly more challenging due to the added uncertainty and variability in system dynamics. Such scenarios often arise in practical applications, such as digital control of chemical processes, radar control, navigation systems and economic systems, due e.g. to physical constraints, environmental complexities, changes in subsystem interconnections and random component failures or repairs. Recent advancements have extended estimation frameworks to address systems with random parameter matrices, incorporating techniques such as adaptive filtering, stochastic modeling, and robust optimization. These developments complement the broader efforts to tackle issues like noise correlation, random delays, packet dropouts and adversarial attacks (as highlighted in [2, 4] and [6–8]), further enhancing the applicability of signal estimation algorithms in real-world systems.

Multirate (MR) systems are frequently found in practical engineering applications, where different sampling rates are utilized based on the distinct physical characteristics of different components, with the goal of optimizing both system performance and resource utilization [9]. For example, in biomedical signal processing, different biological signals require different sampling rates for efficient processing and analysis. Estimation problems related to MR systems have been explored in the literature under various scenarios (see [10] and references therein). For instance, in [11] and [12], the H_∞ filtering problem under denial-of-service (DoS) attacks and the distributed recursive filtering problem with packet losses have been studied for MR linear systems, respectively. However, existing studies typically rely on the state-space model of the system, which may not always be feasible in practical applications. This limitation motivates us to propose an alternative approach based on covariance information.

The increasing occurrence of adversarial attacks in practical systems highlights the critical need for algorithms capable of delivering robust performance even in the presence of malicious interference [13]. Among such threats, DoS attacks are particularly destructive, as they aim to deplete network bandwidth and system resources, effectively blocking communication between system components. This disruption diminishes data availability and undermines the effectiveness of estimation algorithms. To mitigate the risks posed by DoS attacks, various solutions have been proposed in the literature. For instance, [14] introduced a distributed dimensionality reduction fusion filter tailored for cyber-physical systems. In [15], a secure consensus control strategy was proposed for leader-following multiagent systems, leveraging observer estimates to minimize the impact of DoS attacks. The study in [16] addressed periodic DoS attacks, offering sufficient stability conditions for the system under consideration. A stochastic model based on Bernoulli random variables was used in [17] to characterize the occurrence of DoS attacks and a fusion estimation algorithm was designed. Furthermore, [18] employed a predictive compensation approach to counteract packet losses caused by DoS attacks, thereby preserving fusion estimation accuracy. A more recent work in [19] proposed a protocol-based distributed fusion filter for networked systems with stochastic uncertainties and DoS attacks.

Although substantial advancements have been achieved in the study of LS linear estimation problems for different types of systems, the interaction of MR sampling, system uncertainty, and communication disruptions presents compelling challenges that motivate the current investigation. The key contributions of this work are summarized as follows. 1) LS linear estimation algorithms are formulated for time-varying MR systems that incorporate stochastic parameter matrices and random DoS attacks. 2) Two compensation strategies –hold-input (HI) and prediction compensation (PC)– are introduced to counteract the adverse effects of random DoS attacks, thereby improving data integrity and estimation reliability. 3) By replacing the observation process by the equivalent sequence of uncorrelated innovation terms, recursive filtering and smoothing algorithms are developed, along with explicit formulas for the estimation error covariance matrices, that enable a quantitative assessment of the estimators' accuracy. This innovation approach simplifies mathematical derivations, while preserving optimality in the LS sense.

The rest of the paper is organized as follows. Section 2 introduces the problem statement and describes the mathematical model and necessary assumptions. Section 3 details the core contributions, including the development of recursive filtering and smoothing algorithms for the two examined compensation strategies. Section 4 provides a simulation experiment to show the effectiveness of the proposed estimation methods and assess how attack probabilities influence their performance. Finally, Section 5 concludes by summarizing the principal insights and contributions of the study.

2. Problem Statement and Mathematical Model

Our aim is to address the LS linear estimation problem in discrete-time MR sampling systems subject to random variations in the measurement model. In the MR sampling scenario, we assume that the signal to be estimated is updated uniformly at a certain rate, while the measured outputs are sampled uniformly at a slower rate. Additionally, it is considered that the measurement transmission is subject to DoS attacks, which randomly disable the system normal operation, leading to packet losses. Covariance-based recursive algorithms for LS linear filtering and fixed-point smoothing estimators will be developed using an innovation approach. To mitigate the negative impact of lost information caused by DoS attacks, two compensation strategies will be introduced: the HI strategy, which compensates for lost measurements by utilizing the most recent received measurement, and the PC strategy, which replaces lost measurements with their prediction estimates.

Consider an n_x-dimensional stochastic signal process, $\{x_n\}_{n\geqslant 1}$, and assume that we aim to estimate this signal based on noisy observations. The evolution dynamics of the signal process are not required and, therefore, can remain unknown; however, it is assumed that its first and second-order moments exist and adhere to the following assumption.

Assumption 1: For all $n\geqslant 1$, $\mathbb{E}[x_n]=0$ and $\mathbb{E}\left[x_n x_m^T\right]=\Phi_n\Psi_m^T$, $1\leqslant m\leqslant n$, where Φ_n and Ψ_m are known $n_x\times N$ matrices.

Remark 1. *Although Assumption 1 on the signal covariance function might seem restrictive, it actually covers a broad range of real-world scenarios. For instance, non-stationary signals can satisfy this assumption when the state-space model is given as* $x_n=A_{n-1}x_{n-1}+w_{n-1}$, $n\geqslant 1$, *assuming non-singular transition matrices and white noise independent of the initial condition. In this case, the signal covariance function can be expressed as* $\mathbb{E}\left[x_n x_m^T\right]=A_{n,m}\mathbb{E}\left[x_m x_m^T\right]$, $m\leqslant n$, *where* $A_{n,m}=A_{n-1}\cdots A_m$. *So, by taking, for example,* $\Phi_n=A_{n,0}$ *and* $\Psi_m^T=\mathbb{E}\left[x_m x_m^T\right](A_{m,0}^{-1})^T$, *Assumption 1 holds. Similarly, for state-space models with stationary signals,* $x_n = Ax_{n-1} + w_{n-1}$, $n \geqslant 1$, *assuming non-singularity and independence, the covariance function can be written as* $\mathbb{E}\left[x_n x_m^T\right] = A^{n-m}\mathbb{E}\left[x_m x_m^T\right]$, $m \leqslant n$. *Hence, by taking* $\Phi_n = A^n$ *and* $\Psi_m^T = \mathbb{E}\left[x_m x_m^T\right](A^{-m})^T$, *Assumption 1 is also satisfied. Moreover, processes with finite-dimensional, possibly time-variant state-space models often have semi-separable covariance functions,* $\mathbb{E}\left[x_n x_m^T\right] = \sum_{h=1}^{r} \phi_n^h \psi_m^{hT}$, $m \leqslant n$, *and this structure is a particular case of Assumption 1, just taking* $\Phi_n = \left(\phi_n^1, \phi_n^2, \cdots, \phi_n^r\right)$ *and* $\Psi_m = \left(\psi_m^1, \psi_m^2, \cdots, \psi_m^r\right)$. *Additionally, uncertain systems with state-dependent multiplicative noise, such as those considered in [8], also satisfy Assumption 1. Therefore, the proposed estimation scheme based on the signal covariance factorization established in Assumption 1 is applicable to a wide variety of signal models, without the need to design a specific algorithm for each case. Finally, it is worth noting that, although a state-space model can be generated from covariance information, when only covariances are available, it is preferable to directly address the estimation problem using them, as this approach eliminates the need for prior identification of the state-space model.*

2.1. Measurement Model

The measurement model is described by the following equation:

$$z_{kn} = C_{kn} x_{kn} + v_{kn}, \quad n \geqslant 1, \tag{1}$$

where z_{kn} is the n_z-dimensional measurement at the sampling time kn and k is a fixed positive integer that defines the MR sampling. If we denote ρ_x as the signal update rate and ρ_z as the measurement sampling rate, then $\rho_x = k\rho_z$, meaning that the signal update frequency is k times faster than the measurement sampling frequency. The following assumptions are required.

Assumption 2: $\left\{C_{kn}\right\}_{n \geqslant 1}$ is a sequence of independent random parameter matrices, whose entries are scalar stochastic processes with known first and second-order moments. We will denote $\overline{C}_{kn} = \mathbb{E}[C_{kn}]$.

Assumption 3: The measurement noise $\{v_{kn}\}_{n \geqslant 1}$ is a white second-order process with zero mean and known second-order moments. We will denote $R_{kn} = \mathbb{E}[v_{kn} v_{kn}^T]$.

Remark 2. *The existence of second-order moments stated in Assumption 2 ensures that the expected value of* $C_{kn} A_{kn} C_{kn}^T$ *is well-defined for any random matrix* A_{kn} *with mean* $\overline{A}_{kn}$. *Furthermore, when* A_{kn} *is independent of* C_{kn}, *the* (r,s)*-th entry of this expectation is calculated as* $\sum_{a=1}^{n_x} \sum_{b=1}^{n_x} E\left[C_{kn}^{(r,a)} C_{kn}^{(s,b)}\right] \overline{A}_{kn}^{(a,b)}$, *where* $C_{kn}^{(i,j)}$ *and* $\overline{A}_{kn}^{(i,j)}$ *denote the* (i,j)*-th entries of* C_{kn} *and* $\overline{A}_{kn}$, *respectively.*

Remark 3. *The inclusion of random parameter matrices in the proposed measurement model provides a comprehensive and unified framework for representing real-world uncertainties that cannot be effectively captured using deterministic system parameters. A key example is the phenomenon commonly referred to as missing measurements or uncertain observations, where sensor measurements may either contain information about the signal to be estimated or consist purely of noise. This situation commonly arises due to sensor saturation, limited sensing capability, or temporary sensor failures, among other factors. By incorporating random parameter matrices, the proposed observation model not only accounts for missing measurements but also addresses other common challenges in stochastic systems, including fading measurements, multiplicative noise, and their combined effects. Furthermore, systems affected by random delays or packet dropouts in transmission can be reformulated as equivalent stochastic systems with random measurement matrices. As a result, this framework is highly applicable to a broad range of real-world systems, including digital control of chemical processes, human-operated systems, economic models, and stochastically sampled digital control systems.*

2.2. Randomly Occurring DoS Attacks

Physical systems usually operate under adversarial threats aimed at disrupting or manipulating the accuracy of measurements. In such scenarios, the observations utilized for estimation can differ substantially from the true data, demanding an appropriate mathematical framework to account for the impact of potential attacks on the measurement outputs. This paper considers that the system is exposed to DoS attacks, which can randomly succeed or fail, leading to random packet losses. This fact is described by a sequence of Bernoulli random variables, $\{\gamma_{kn}\}_{n \geqslant 1}$, where $\gamma_{kn} = 1$ represents a successful attack (packet loss), and $\gamma_{kn} = 0$ indicates normal operation (no packet loss). Under this framework, the observations affected by DoS attacks are modeled as:

$$y_{kn} = (1 - \gamma_{kn}) z_{kn}, \quad n \geqslant 1.$$

It will be assumed that $\gamma_k = 0$, specifying that $y_k = z_k$; in other words, the initial observation corresponds to the first received measurement, which is fixed as the reference time point. The following assumptions are considered.

Assumption 4: $\{\gamma_{kn}\}_{n\geqslant 1}$ is a sequence of independent Bernoulli random variables with known probabilities. We will denote $\overline{\gamma}_{kn} = \mathbb{P}(\gamma_{kn} = 1)$, $n\geqslant 1$.

Assumption 5: The processes $\{x_n\}_{n\geqslant 1}$, $\{C_{kn}\}_{n\geqslant 1}$, $\{\gamma_{kn}\}_{n\geqslant 1}$ and $\{v_{kn}\}_{n\geqslant 1}$ are mutually independent.

2.3. Compensation Strategies for DoS-induced Measurement Losses

Measurement losses caused by DoS attacks disrupt the flow of reliable data, compromising the accuracy and performance of estimation processes. To mitigate these disruptions, compensation strategies are essential for ensuring continuity and reliability in system operation, even under adverse conditions. Two distinct strategies for addressing these losses are the HI strategy and the PC strategy. Both strategies offer valuable solutions, with their applicability depending on the specific characteristics of the system and the required level of performance.

- *HI strategy*. This approach compensates for lost measurements by reusing the most recent successfully received data. For this reason, it is also referred to as *zero-order hold strategy* in the literature. Its simplicity and ease of implementation make it particularly suitable for systems with limited computational resources or where measurement variations are relatively slow. After the compensation, the observations available are

$$y^H_{kn} = (1-\gamma_{kn})z_{kn} + \gamma_{kn} y^H_{k(n-1)}, \quad n\geqslant 2; \qquad y^H_k = z_k. \tag{2}$$

- *PC strategy*. This method estimates lost data based on historical information. While more computationally intensive, it is well-suited for applications requiring higher accuracy and where system characteristics can be reliably modeled. The compensated observations are formulated as

$$y^P_{kn} = (1-\gamma_{kn})z_{kn} + \gamma_{kn} \widehat{z}^P_{kn/k(n-1)}, \quad n\geqslant 2; \qquad y^P_k = z_k, \tag{3}$$

where $\widehat{z}^P_{kn/k(n-1)}$ denotes the prediction estimate of z_{kn} based on the previous available observations $\left\{y^P_{ki} : i = 1,2,\cdots,n-1\right\}$. This prediction estimate will be described later (see (13)); specifically, it is calculated from $\hat{x}^P_{kn/k(n-1)}$ –the prediction estimate of the signal– as $\hat{z}^P_{kn/k(n-1)} = \overline{C}_{kn}\hat{x}^P_{kn/k(n-1)}$.

In the next section, these compensation strategies will be integrated into the estimation framework to address the challenges posed by DoS-induced measurement losses and ensure reliable signal estimation even in the presence of adversarial disruptions.

3. Main Results

In this section, we develop recursive LS filtering and fixed-point smoothing algorithms tailored to the two compensation strategies introduced in Section 2. For notational convenience, we will write y^*_{kn}, with $* = H$ and P, to comprehensively denote the observation at time kn under the HI and the PC strategies, respectively.

3.1. LS Estimation

For each $n\geqslant 1$, our aim is to obtain the LS linear estimator of the signal x_n based on the observations available till time n, $\widehat{x}^*_{n/n}$, that is the filtering estimator. If n is less than the first sampling time ($n < k$), no observations are available and, consequently, $\widehat{x}^*_{n/n} = \mathbb{E}[x_n] = 0$. Otherwise, the time n lies between two consecutive multiples of k. Let $t\geqslant 1$ be a positive integer such that $kt\leqslant n < k(t+1)$; that is, kt is the largest multiple of k less than or equal to n and $k(t+1)$ is the smallest multiple of k greater than n. This setup partitions the timeline into intervals of length k, corresponding to the sampling periods of the observations. The observations are only available at multiples of k, so t represents the total number of observations available at time n. Clearly, the LS linear filter, $\widehat{x}^*_{n/n}$, will depend on the observations up to time kt, which are $\left\{y^*_{ki} : i = 1,2,\cdots,t\right\}$. So, our aim is to derive a recursive algorithm for the estimators $\widehat{x}^*_{n/kt}$, with $k\leqslant kt\leqslant n$.

To simplify the complexity of mathematical derivations, the LS estimation problem will be approached by the innovation method. The innovation sequence comprises the measurement residuals that capture the new information gained from each observation relative to its prediction estimate. In other words, at each sampling step kn, the innovation is defined as the difference between the actual observation and its predicted value based on prior information. Specifically, for $n\geqslant 2$, the innovation is defined as $\mu^*_{kn} = y^*_{kn} - \widehat{y}^*_{kn/k(n-1)}$, where $\widehat{y}^*_{kn/k(n-1)}$ is the LS linear estimator of y^*_{kn} based on the set of previous observations $\left\{y^*_{ki} : i = 1,2,\cdots,n-1\right\}$, while for the initial sampling time ($n = 1$) the innovation is $\mu^*_k = y^*_k$.

The innovation sequence is a white process that spans the same linear space as the observation sequence. This property ensures that the innovation sequence fully encapsulates the information provided by the observations, allowing the estimation problem to be formulated recursively and solved efficiently with reduced computational complexity using the innovations. Actually, the LS linear estimator $\widehat{x}^*_{n/kt}$ is expressed as the following linear combination of the innovations:

$$\widehat{x}^*_{n/kt} = \sum_{j=1}^{t} \mathcal{X}^*_{n,kj}\left(\Pi^*_{kj}\right)^{-1}\mu^*_{kj}, \quad k\leqslant kt\leqslant n, \tag{4}$$

with $\mathcal{X}^*_{n,kj} = \mathbb{E}\left[x_n(\mu^*_{kj})^T\right]$ and $\Pi^*_{kj} = \mathbb{E}\left[\mu^*_{kj}(\mu^*_{kj})^T\right]$. The product $\mathcal{X}^*_{n,kj}\left(\Pi^*_{kj}\right)^{-1}$ represents the gain coefficient that weights the contribution of the innovation μ^*_{kj} at each sampling step kj. From this expression, the derivation of the estimators involves computing the innovations μ^*_{kj} and their covariance matrices Π^*_{kj}, which are clearly affected by the choice of compensation strategy. The following propositions provide the formulas for the innovations and their covariance matrices under the HI (Proposition 1) and PC (Proposition 2) strategies.

Proposition 1. *Under the HI strategy, consider the model equations (1) and (2) and assumptions 1-5. The innovation μ^H_{kj} is given by*

$$\mu^H_{kj} = y^H_{kj} - (1-\overline{\gamma}_{kj})\overline{C}_{kj}\widehat{x}^H_{kj/k(j-1)} - \overline{\gamma}_{kj}y^H_{k(j-1)}, \quad j\geqslant 2; \qquad \mu^H_k = y^H_k, \tag{5}$$

where $\widehat{x}^H_{kj/k(j-1)}$ is the prediction estimator of the signal x_{kj} based on the prior observations $\{y^H_{ki} : i = 1,2,\cdots, j-1\}$.

The innovation covariance matrix satisfies

$$\begin{aligned}\Pi^H_{kj} = {} & \Sigma^{y^H}_{kj} - (1-\overline{\gamma}_{kj})^2\overline{C}_{kj}\Sigma^{\widehat{x}^H}_{kj/k(j-1)}\overline{C}^T_{kj} - \overline{\gamma}^2_{kj}\Sigma^{y^H}_{k(j-1)} \\ & - \overline{\gamma}_{kj}(1-\overline{\gamma}_{kj})\left(\overline{C}_{kj}\Sigma^{\widehat{x}^H y^H}_{kj/k(j-1)} + (\Sigma^{\widehat{x}^H y^H}_{kj/k(j-1)})^T\overline{C}^T_{kj}\right), \quad j\geqslant 2; \qquad \Pi^H_k = \Sigma^{y^H}_k,\end{aligned} \tag{6}$$

where $\Sigma^{\widehat{x}^H}_{kj/k(j-1)} = \mathbb{E}\left[\widehat{x}^H_{kj/k(j-1)}(\widehat{x}^H_{kj/k(j-1)})^T\right]$ is the predictor covariance matrix, and $\Sigma^{\widehat{x}^H y^H}_{kj/k(j-1)} = \mathbb{E}\left[\widehat{x}^H_{kj/k(j-1)}(y^H_{k(j-1)})^T\right]$ is the cross-covariance between the signal predictor and the last observation. The observation covariance, $\Sigma^{y^H}_{kj} = \mathbb{E}\left[y^H_{kj}(y^H_{kj})^T\right]$, is recursively calculated by

$$\Sigma^{y^H}_{kj} = (1-\overline{\gamma}_{kj})\Sigma^z_{kj} + \overline{\gamma}_{kj}\Sigma^{y^H}_{k(j-1)}, \quad j\geqslant 2; \qquad \Sigma^{y^H}_k = \Sigma^z_k, \tag{7}$$

with

$$\Sigma^z_{kj} = \mathbb{E}[C_{kj}\Phi_{kj}\Psi^T_{kj}C^T_{kj}] + R_{kj}, \quad j\geqslant 1. \tag{8}$$

Proof. To obtain the innovation $\mu^H_{kj} = y^H_{kj} - \widehat{y}^H_{kj/k(j-1)}$, we need to calculate the observation predictor $\widehat{y}^H_{kj/k(j-1)}$. From (2), we have

$$\widehat{y}^H_{kj/k(j-1)} = (1-\overline{\gamma}_{kj})\widehat{z}^H_{kj/k(j-1)} + \overline{\gamma}_{kj}y^H_{k(j-1)}, \quad j\geqslant 2. \tag{9}$$

According to (1) and Assumption 5, we can write $\widehat{z}^H_{kj/k(j-1)} = \overline{C}_{kj}\widehat{x}^H_{kj/k(j-1)}$. Combining this with (9) yields expression (5) for the innovation. From it, the innovation covariance is derived taking into account that, from the Orthogonal Projection Lemma (OPL), $\Pi^H_{kj} = \mathbb{E}[y^H_{kj}(y^H_{kj})^T] - \mathbb{E}[\widehat{y}^H_{kj/k(j-1)}(\widehat{y}^H_{kj/k(j-1)})^T]$. Using (9) and the independence hypotheses specified in Assumption 5, we have

$$\begin{aligned}\mathbb{E}[\widehat{y}^H_{kj/k(j-1)}(\widehat{y}^H_{kj/k(j-1)})^T] = {} & (1-\overline{\gamma}_{kj})^2\overline{C}_{kj}\mathbb{E}\left[\widehat{x}^H_{kj/k(j-1)}(\widehat{x}^H_{kj/k(j-1)})^T\right]\overline{C}^T_{kj} \\ & + \overline{\gamma}^2_{kj}\mathbb{E}[y^H_{k(j-1)}(y^H_{k(j-1)})^T] \\ & + \overline{\gamma}_{kj}(1-\overline{\gamma}_{kj})\overline{C}_{kj}\mathbb{E}\left[\widehat{x}^H_{kj/k(j-1)}(y^H_{k(j-1)})^T\right] \\ & + \overline{\gamma}_{kj}(1-\overline{\gamma}_{kj})\mathbb{E}\left[y^H_{k(j-1)}(\widehat{x}^H_{kj/k(j-1)})^T\right]\overline{C}^T_{kj},\end{aligned}$$

from which equation (6) is obtained.

Expression (7) for the observation covariance is easily obtained from (2), according to the independence assumptions and using that $\mathbb{E}[\gamma^2_{kj}] = \overline{\gamma}_{kj}$, $\mathbb{E}[(1-\gamma_{kj})^2] = 1-\overline{\gamma}_{kj}$ and $\mathbb{E}[\gamma_{kj}(1-\gamma_{kj})] = 0$. Finally, (8) is straightforward from equation (1) and the model assumptions. This completes the proof.

Proposition 2. *Under the PC strategy, consider the model equations (1) and (3) and assumptions 1-5. The innovation μ^P_{kj} is computed as*

$$\mu^P_{kj} = y^P_{kj} - \overline{C}_{kj}\widehat{x}^P_{kj/k(j-1)}, \quad j\geqslant 2; \qquad \mu^P_k = y^P_k, \tag{10}$$

where $\widehat{x}^P_{kj/k(j-1)}$ is the prediction estimator of the signal x_{kj} based on the prior observations $\{y^P_{ki} : i = 1,2,\cdots, j-1\}$.

The innovation covariance matrix is given by

$$\Pi^P_{kj} = \Sigma^{y^P}_{kj} - \overline{C}_{kj}\Sigma^{\widehat{x}^P}_{kj/k(j-1)}\overline{C}^T_{kj}, \quad j\geqslant 2; \qquad \Pi^P_k = \Sigma^{y^P}_k, \tag{11}$$

where $\Sigma^{\widehat{x}^P}_{kj/k(j-1)} = \mathbb{E}\left[\widehat{x}^P_{kj/k(j-1)}(\widehat{x}^P_{kj/k(j-1)})^T\right]$ *is the predictor covariance matrix and* $\Sigma^{y^P}_{kj} = \mathbb{E}\left[y^P_{kj}(y^P_{kj})^T\right]$ *is the observation covariance, which satisfies*

$$\Sigma^{y^P}_{kj} = (1-\overline{\gamma}_{kj})\Sigma^z_{kj} + \overline{\gamma}_{kj}\overline{C}_{kj}\Sigma^{\widehat{x}^P}_{kj/k(j-1)}\overline{C}^T_{kj}, \quad j\geqslant 2; \qquad \Sigma^{y^P}_k = \Sigma^z_k, \tag{12}$$

where Σ^z_{kj}, $j\geqslant 1$, *is given in (8).*

Proof. To derive the innovation $\mu^P_{kj} = y^P_{kj} - \widehat{y}^P_{kj/k(j-1)}$, it is necessary to compute the observation predictor $\widehat{y}^P_{kj/k(j-1)}$. From (3), we obtain:

$$\widehat{y}^P_{kj/k(j-1)} = (1-\overline{\gamma}_{kj})\widehat{z}^P_{kj/k(j-1)} + \overline{\gamma}_{kj}\widehat{z}^P_{kj/k(j-1)}, \quad j\geqslant 2.$$

Consequently, $\widehat{y}^P_{kj/k(j-1)} = \widehat{z}^P_{kj/k(j-1)}$, which using (1) and Assumption 5, can be expressed as

$$\widehat{z}^P_{kj/k(j-1)} = \overline{C}_{kj}\widehat{x}^P_{kj/k(j-1)}, \tag{13}$$

and expression (10) for the innovation is immediately obtained. From (10), the innovation covariance in expression (11) is directly obtained by using the OPL and applying the independence hypotheses outlined in Assumption 5.

The observation covariance in (12) is easily derived from (3), using (13) and the model assumptions. This concludes the proof.

Once we have calculated the innovations and their covariance matrices, the following lemma provides the cross-correlation terms in (4), $\mathcal{X}^*_{n,kj} = \mathbb{E}\left[x_n(\mu^*_{kj})^T\right]$, $k\leqslant kj\leqslant n$, under the HI ($*=H$) and the PC ($*=P$) strategies. These coefficients will play a crucial role in the derivation of the estimators.

Lemma 1. *Under the HI* ($*=H$) *and the PC* ($*=P$) *strategies, the coefficients* $\mathcal{X}^*_{n,kj} = \mathbb{E}\left[x_n(\mu^*_{kj})^T\right]$, $k\leqslant kj\leqslant n$, *satisfy*

$$\mathcal{X}^*_{n,kj} = (1-\overline{\gamma}_{kj})\left(\Phi_n\Psi^T_{kj} - \sum_{i=1}^{j-1}\mathcal{X}^*_{n,ki}(\Pi^*_{ki})^{-1}(\mathcal{X}^*_{kj,ki})^T\right)\overline{C}^T_{kj}, \quad j\geqslant 2; \qquad \mathcal{X}^*_{n,k} = \Phi_n\Psi^T_k\overline{C}^T_k. \tag{14}$$

Proof. Since the innovations are influenced by the selected compensation strategy, the coefficients $\mathcal{X}^*_{n,kj}$ must be derived separately, for $*=H,P$.

(a) Under the HI strategy ($*=H$), using (5), we can write

$$\mathcal{X}^H_{n,kj} = \mathbb{E}\left[x_n(y^H_{kj})^T\right] - (1-\overline{\gamma}_{kj})\mathbb{E}\left[x_n(\widehat{x}^H_{kj/k(j-1)})^T\right]\overline{C}^T_{kj} - \overline{\gamma}_{kj}\mathbb{E}\left[x_n(y^H_{k(j-1)})^T\right], \quad k\leqslant kj\leqslant n. \tag{15}$$

From (2) and Assumption 1, it follows that

$$\mathbb{E}\left[x_n(y^H_{kj})^T\right] = (1-\overline{\gamma}_{kj})\Phi_n\Psi^T_{kj}\overline{C}^T_{kj} + \overline{\gamma}_{kj}\mathbb{E}\left[x_n(y^H_{k(j-1)})^T\right], \quad j\geqslant 1.$$

Additionally, using (4), we obtain

$$\mathbb{E}\left[x_n(\widehat{x}^H_{kj/k(j-1)})^T\right] = \sum_{i=1}^{j-1}\mathcal{X}^H_{n,ki}\left(\Pi^H_{ki}\right)^{-1}\left(\mathcal{X}^H_{kj,ki}\right)^T, \quad j\geqslant 2.$$

Substituting the above expectations into (15), we derive expression (14) for $*=H$.

(b) Under the PC strategy ($*=P$), using (10), we can express

$$\mathcal{X}^P_{n,kj} = \mathbb{E}\left[x_n(y^P_{kj})^T\right] - \mathbb{E}\left[x_n(\widehat{x}^P_{kj/k(j-1)})^T\right]\overline{C}^T_{kj}, \quad k\leqslant kj\leqslant n. \tag{16}$$

According to (1), (3) and the model assumptions, it follows that

$$\mathbb{E}\left[x_n(y^P_{kj})^T\right] = (1-\overline{\gamma}_{kj})\Phi_n\Psi^T_{kj}\overline{C}^T_{kj} + \overline{\gamma}_{kj}\mathbb{E}\left[x_n(\widehat{x}^P_{kj/k(j-1)})^T\right]\overline{C}^T_{kj}, \quad j\geqslant 1.$$

Furthermore, using (4), we also find

$$\mathbb{E}\left[x_n(\widehat{x}^P_{kj/k(j-1)})^T\right] = \sum_{i=1}^{j-1}\mathcal{X}^P_{n,ki}\left(\Pi^P_{ki}\right)^{-1}\left(\mathcal{X}^P_{kj,ki}\right)^T, \quad j\geqslant 2.$$

Substituting these expectations into (16), we obtain expression (14) for $* = P$.

The following theorem provides a comprehensive formula for the estimators $\widehat{x}^*_{n/kt}$, $k \leqslant kt \leqslant n$, and the associated error covariance matrices, $\Sigma^{\widetilde{x}^*}_{n/kt} = \mathbb{E}\left[\widetilde{x}^*_{n/kt}(\widetilde{x}^*_{n/kt})^T\right]$, with $\widetilde{x}^*_{n/kt} = x_n - \widehat{x}^*_{n/kt}$, under both the HI and the PC strategies.

Theorem 1. *Under the HI ($* = H$) and the PC ($* = P$) strategies, the LS estimators $\widehat{x}^*_{n/kt}$, $k \leqslant kt \leqslant n$, are calculated by*

$$\widehat{x}^*_{n/kt} = \Phi_n f^*_{kt}, \quad k \leqslant kt \leqslant n, \tag{17}$$

and the estimation error covariance matrices $\Sigma^{\widetilde{x}^*}_{n/kt}$ satisfy

$$\Sigma^{\widetilde{x}^*}_{n/kt} = \Phi_n\left(\Psi_n - \Phi_n \Sigma^{f^*}_{kt}\right)^T, \quad k \leqslant kt \leqslant n. \tag{18}$$

The vectors f^*_{kt} and the matrices $\Sigma^{f^*}_{kt}$ are recursively obtained by

$$f^*_{kt} = f^*_{k(t-1)} + F^*_{kt}(\Pi^*_{kt})^{-1}\mu^*_{kt}, \quad t \geqslant 1; \qquad f^*_0 = 0, \tag{19}$$

$$\Sigma^{f^*}_{kt} = \Sigma^{f^*}_{k(t-1)} + F^*_{kt}(\Pi^*_{kt})^{-1}(F^*_{kt})^T, \quad t \geqslant 1; \qquad \Sigma^{f^*}_0 = 0, \tag{20}$$

with

$$F^*_{kt} = (1 - \overline{\gamma}_{kt})\left(\Psi_{kt} - \Phi_{kt}\Sigma^{f^*}_{k(t-1)}\right)^T \overline{C}^T_{kt}, \quad t \geqslant 1. \tag{21}$$

Proof. From (4), the determination of the estimators requires the calculation of the innovations and their covariance matrices, given in Propositions 1 and 2, together with the coefficients $\mathcal{X}^*_{n,kj}$. Using Lemma 1 and defining

$$F^*_{kj} = (1 - \overline{\gamma}_{kj})\left(\Psi^T_{kj} - \sum_{i=1}^{j-1} F^*_{ki}(\Pi^*_{ki})^{-1}(\mathcal{X}^*_{kj,ki})^T\right)\overline{C}^T_{kj}, \quad j \geqslant 2; \qquad F^*_k = \Psi^T_k \overline{C}^T_k, \tag{22}$$

the following factorization is obtained:

$$\mathcal{X}^*_{n,kj} = \Phi_n F^*_{kj}, \quad k \leqslant kj \leqslant n. \tag{23}$$

Hence, by defining

$$f^*_{kt} = \sum_{j=1}^{t} F^*_{kj}(\Pi^*_{kj})^{-1}(\mu^*_{kj})^T, \quad t \geqslant 1; \qquad f^*_0 = 0, \tag{24}$$

the recursion (19) is immediately obtained and expression (17) is deduced using (4), (23) and (24).

Using the OPL, the estimation error covariance matrices satisfy $\Sigma^{\widetilde{x}^*}_{n/kt} = \mathbb{E}[x_n x_n^T] - \mathbb{E}[\widehat{x}^*_{n/kt}(\widehat{x}^*_{n/kt})^T]$, which, according to Assumption 1 and (17), leads to (18), with $\Sigma^{f^*}_{kt} = \mathbb{E}\left[f^*_{kt}(f^*_{kt})^T\right]$.

Employing (24) and noting that the innovation sequence is a white process, the matrices $\Sigma^{f^*}_{kt}$ can be computed as

$$\Sigma^{f^*}_{kt} = \sum_{j=1}^{t} F^*_{kj}(\Pi^*_{kj})^{-1}(F^*_{kj})^T, \quad t \geqslant 1; \qquad \Sigma^{f^*}_0 = 0, \tag{25}$$

from which (20) is straightforward. Finally, by combining (25) with (22) and (23), expression (21) is derived and the proof is complete.

The following corollary provides the prediction covariances $\Sigma^{\widehat{x}^*}_{kj/k(j-1)}$ and the cross-covariances $\Sigma^{\widehat{x}^H y^H}_{kj/k(j-1)}$, that are necessary to calculate the innovation covariance matrices in Propositions 1 and 2.

Corollary 1. *The prediction covariance matrices $\Sigma^{\widehat{x}^*}_{kj/k(j-1)}$ in Proposition 1 ($* = H$) and Proposition 2 ($* = P$) are given by*

$$\Sigma^{\widehat{x}^*}_{kj/k(j-1)} = \Phi_{kj}\Sigma^{f^*}_{k(j-1)}\Phi^T_{kj}, \quad j \geqslant 2. \tag{26}$$

The cross-covariance matrices $\Sigma^{\widehat{x}^H y^H}_{kj/k(j-1)}$ in Proposition 1 are calculated as

$$\Sigma^{\widehat{x}^H y^H}_{kj/k(j-1)} = \Phi_{kj}\Sigma^{f^H y^H}_{k(j-1)}, \quad j\geqslant 2, \tag{27}$$

where $\Sigma^{f^H y^H}_{kj}$ *satisfies the following recursion:*

$$\Sigma^{f^H y^H}_{kj} = F^H_{kj} + (1-\overline{\gamma}_{kj})\Sigma^{f^H}_{k(j-1)}\Phi^T_{kj}\overline{C}^T_{kj} + \overline{\gamma}_{kj}\Sigma^{f^H y^H}_{k(j-1)}, \quad j\geqslant 2; \qquad \Sigma^{f^H y^H}_{k} = F^H_k. \tag{28}$$

Proof. Expression (26) is directly derived from (17) by setting $n = kj$ and $t = j-1$. Using (17) again, the cross-covariance matrices $\Sigma^{\widehat{x}^H y^H}_{kj/k(j-1)} = \mathbb{E}\big[\widehat{x}^H_{kj/k(j-1)}(y^H_{k(j-1)})^T\big]$ clearly satisfy (27), with $\Sigma^{f^H y^H}_{kj} = \mathbb{E}\big[f^H_{kj}(y^H_{kj})^T\big]$. To derive (28), we decompose $\Sigma^{f^H y^H}_{kj}$ as

$$\Sigma^{f^H y^H}_{kj} = \mathbb{E}\big[f^H_{kj}(\mu^H_{kj})^T\big] + \mathbb{E}\big[f^H_{kj}(\widehat{y}^H_{kj/k(j-1)})^T\big].$$

Using (24) and noting that the innovation is a white process, it follows that $\mathbb{E}\big[f^H_{kj}(\mu^H_{kj})^T\big] = F^H_{kj}$ and $\mathbb{E}\big[f^H_{kj}(\widehat{y}^H_{kj/k(j-1)})^T\big] = \mathbb{E}\big[f^H_{k(j-1)}(\widehat{y}^H_{kj/k(j-1)})^T\big]$. Finally, applying (9), taking into account that $\widehat{z}^H_{kj/k(j-1)} = \overline{C}_{kj}\widehat{x}^H_{kj/k(j-1)}$, and using (17), we compute

$$\mathbb{E}\big[f^H_{k(j-1)}(\widehat{y}^H_{kj/k(j-1)})^T\big] = (1-\overline{\gamma}_{kj})\Sigma^{f^H}_{k(j-1)}\Phi^T_{kj}\overline{C}^T_{kj} + \overline{\gamma}_{kj}\Sigma^{f^H y^H}_{k(j-1)}.$$

Combining these results we obtain (28), thus concluding the proof.

Remark 4. *The computational procedure to obtain the estimators* $\widehat{x}^*_{n/kt}$ *and the associated error covariance matrices* $\Sigma^{\widetilde{x}^*}_{n/kt}$*, for* $k\leqslant kt\leqslant n < k(t+1)$*, is summarized as follows:*

Step 1. *Initialize the algorithm with* $F^*_k = \Psi^T_k\overline{C}^T_k$, $\Sigma^{y^*}_k = \mathbb{E}[C_k\Phi_k\Psi^T_k C^T_k] + R_k$, $\Pi^*_k = \Sigma^{y^*}_k$, $\mu^*_k = y^*_k$, $f^*_k = F^*_k(\Pi^*_k)^{-1}\mu^*_k$ *and* $\Sigma^{f^*}_k = F^*_k(\Pi^*_k)^{-1}(F^*_k)^T$ *(under the HI strategy, also include* $\Sigma^{f^H y^H}_k = F^H_k$*). Compute the estimator value,* $\widehat{x}^*_{n/k} = \Phi_n f^*_k$*, and the error covariance matrix,* $\Sigma^{\widetilde{x}^*}_{n/k} = \Phi_n\big(\Psi_n - \Phi_n\Sigma^{f^*}_k\big)^T$*. Set* $t = 2$*.*

Step 2. *Use (17) to obtain the predictor value* $\widehat{x}^*_{kt/k(t-1)} = \Phi_{kt}f^*_{k(t-1)}$ *and, from it, compute the innovation* μ^*_{kt} *using (5) or (10), depending on the selected compensation strategy (HI or PC, respectively).*

Step 3. *Compute the matrix* F^*_{kt} *using equation (21).*

Step 4. *Evaluate the covariance matrices* $\Sigma^{\widehat{x}^*}_{kt/k(t-1)}$ *using (26),* Σ^z_{kt} *using (8) and* $\Sigma^{y^*}_{kt}$ *using (7) or (12), depending on the chosen compensation strategy (HI or PC, respectively). Under the HI strategy, also compute* $\Sigma^{\widehat{x}^H y^H}_{kt/k(t-1)}$ *from (27), with* $\Sigma^{f^H y^H}_{kt}$ *given in (28).*

Step 5. *Determine the innovation covariance matrix* Π^*_{kt} *by (6) or (11), depending on the selected compensation strategy (HI or PC, respectively).*

Step 6. *Compute* f^*_{kt} *from (19) and* $\Sigma^{f^*}_{kt}$ *from (20).*

Step 7. *For* $kt\leqslant n < k(t+1)$*, obtain the estimator value,* $\widehat{x}^*_{n/kt}$*, using (17), and the error covariance matrix,* $\Sigma^{\widetilde{x}^*}_{n/kt}$*, using equation (18).*

Step 8. *Set* $t = t+1$ *and return to* Step 2.

Remark 5. *The proposed estimation scheme –like the Kalman filter– provides the LS linear estimator. However, the key advantage is that our approach does not require full knowledge of the state-space model, relying only on covariance information. This distinction results in a recursive structure that differs from conventional Kalman-like estimation algorithms based on state-space models, while still yielding optimal linear estimators.*

3.2. LS Smoothing

Given a fixed time $n\geqslant k$, let $t\geqslant 1$ be a nonnegative integer such that $kt\leqslant n < k(t+1)$. Under the two proposed compensation strategies ($* = H, P$), our purpose is to design the smoothing estimators of x_n by updating the estimators $\widehat{x}^*_{n/kt}$ with the incoming observations $y^*_{k(t+1)}, y^*_{k(t+2)}, \cdots$.

Theorem 2. *Under the HI ($* = H$) and the PC ($* = P$) strategies, the LS fixed-point smoothers* $\widehat{x}^*_{n/k(t+L)}$*,* $L\geqslant 1$*, and the associated error covariance matrices* $\Sigma^{\widetilde{x}^*}_{n/k(t+L)} = \mathbb{E}\big[(x_n - \widehat{x}^*_{n/k(t+L)})(x_n - \widehat{x}^*_{n/k(t+L)})^T\big]$ *satisfy the following recursions*

$$\widehat{x}^*_{n/k(t+L)} = \widehat{x}^*_{n/k(t+L-1)} + \mathcal{X}^*_{n,k(t+L)}(\Pi^*_{k(t+L)})^{-1}\mu^*_{k(t+L)}, \quad L\geqslant 1, \tag{29}$$

$$\Sigma^{\widetilde{x}^*}_{n/k(t+L)} = \Sigma^{\widetilde{x}^*}_{n/k(t+L-1)} - \mathcal{X}^*_{n,k(t+L)}(\Pi^*_{k(t+L)})^{-1}(\mathcal{X}^*_{n,k(t+L)})^T, \quad L\geqslant 1, \tag{30}$$

whose initial conditions are $\widehat{x}^*_{n/kt}$ *and* $\Sigma^{\widetilde{x}^*}_{n/kt}$, *respectively, both given in Theorem 1.*

The coefficients $\mathcal{X}^*_{n,k(t+L)}$ *are computed by*

$$\mathcal{X}^*_{n,k(t+L)} = (1-\overline{\gamma}_{k(t+L)})\big(\Psi_n - \mathfrak{M}^*_{n,k(t+L-1)}\big)\Phi^T_{k(t+L)}\overline{C}^T_{k(t+L)}, \quad L \geqslant 1, \tag{31}$$

where the matrices $\mathfrak{M}^*_{n,k(t+L)}$ *are recursively calculated by*

$$\mathfrak{M}^*_{n,k(t+L)} = \mathfrak{M}^*_{n,k(t+L-1)} + \mathcal{X}^*_{n,k(t+L)}(\Pi^*_{k(t+L)})^{-1}(F^*_{k(t+L)})^T, \quad L \geqslant 1, \qquad \mathfrak{M}^*_{n,kt} = \Phi_n \Sigma^{f^*}_{kt}. \tag{32}$$

Proof. Similar to (4), the LS fixed-point smoother is expressed as a linear combination of the innovations:

$$\widehat{x}^*_{n/k(t+L)} = \sum_{j=1}^{t+L} \mathcal{X}^*_{n,kj}(\Pi^*_{kj})^{-1}\mu^*_{kj}, \quad L \geqslant 1. \tag{33}$$

From this, expression (29) is immediately derived. Using (29), the smoothing error, $\widetilde{x}^*_{n/k(t+L)} = x_n - \widehat{x}^*_{n/k(t+L)}$, can be expressed as $\widetilde{x}^*_{n/k(t+L)} = \widetilde{x}^*_{n/k(t+L-1)} - \mathcal{X}^*_{n,k(t+L)}(\Pi^*_{k(t+L)})^{-1}\mu^*_{k(t+L)}$, from which the smoothing error covariance (30) is straightforwardly obtained using the OPL.

To compute the coefficients $\mathcal{X}^*_{n,k(t+L)} = \mathbb{E}\left[x_n(\mu^*_{k(t+L)})^T\right]$, we observe that

$$\begin{aligned}\mathcal{X}^*_{n,k(t+L)} &= \mathbb{E}\left[x_n(y^*_{k(t+L)})^T\right] - \mathbb{E}\left[x_n(\widehat{y}^*_{k(t+L)/k(t+L-1)})^T\right] \\ &= (1-\overline{\gamma}_{k(t+L)})\Psi_n\Phi^T_{k(t+L)}\overline{C}^T_{k(t+L)} - \mathbb{E}\left[x_n(f^*_{k(t+L-1)})^T\right]\Phi^T_{k(t+L)}\overline{C}^T_{k(t+L)},\end{aligned}$$

and (31) follows, just defining $\mathfrak{M}^*_{n,k(t+L)} = \mathbb{E}\left[x_n(f^*_{k(t+L)})^T\right]$.

Finally, the recursion (32) for the matrices $\mathfrak{M}^*_{n,k(t+L)}$ is derived using (19) and its initial condition is easily obtained from the OPL and expression (17). This completes the proof.

Remark 6. *Theorem 2 provides the formulas to compute the smoothing estimators* $\widehat{x}^*_{n/k(t+L)}$ *for* $n \geqslant k$. *For* $n < k$, *a similar reasoning leads to the following recursions for the smoothers* $\widehat{x}^*_{n/kL}$ *and their covariance matrices* $\Sigma^{\widetilde{x}^*}_{n/kL}$:

$$\widehat{x}^*_{n/kL} = \widehat{x}^*_{n/k(L-1)} + \mathcal{X}^*_{n,kL}(\Pi^*_{kL})^{-1}\mu^*_{kL}, \quad L \geqslant 2,$$

$$\Sigma^{\widetilde{x}^*}_{n/kL} = \Sigma^{\widetilde{x}^*}_{n/k(L-1)} - \mathcal{X}^*_{n,kL}(\Pi^*_{kL})^{-1}(\mathcal{X}^*_{n,kL})^T, \quad L \geqslant 2,$$

whose initial conditions are $\widehat{x}^*_{n/k} = \mathcal{X}^*_{n,k}(\Pi^*_k)^{-1}\mu^*_k$ *and* $\Sigma^{\widetilde{x}^*}_{n/k} = \Phi_n\Psi_n^T - \mathcal{X}^*_{n,k}(\Pi^*_k)^{-1}(\mathcal{X}^*_{n,k})^T$, *respectively.*

The coefficients $\mathcal{X}^*_{n,kL}$ *are computed by*

$$\mathcal{X}^*_{n,kL} = (1-\overline{\gamma}_{kL})\big(\Psi_n - \mathfrak{M}^*_{n,k(L-1)}\big)\Phi^T_{kL}\overline{C}^T_{kL}, \quad L \geqslant 1,$$

with

$$\mathfrak{M}^*_{n,kL} = \mathfrak{M}^*_{n,k(L-1)} + \mathcal{X}^*_{n,kL}(\Pi^*_{kL})^{-1}(F^*_{kL})^T, \quad L \geqslant 1, \qquad \mathfrak{M}^*_{n,0} = 0.$$

4. Numerical Simulation Example

In this section, a simulation example is considered to assess the effectiveness of the proposed estimation methods and analyze the impact of attack probabilities on their performance. Specifically, consider a two-dimensional stochastic signal process $\{x_n\}_{n\geqslant 1}$. For simulation purposes, the following signal model is adopted [20]:

$$x_{n+1} = Fx_n + Gw_n, \quad n \geqslant 1,$$

where

$$F = \begin{pmatrix} 0.95 & 0.1 \\ 0 & 0.95 \end{pmatrix}, \qquad G = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}.$$

The initial signal x_0 is a zero-mean Gaussian vector with covariance matrix $\mathbb{E}\left[x_0x_0^T\right] = 0.1I_2$, where I_2 denotes de 2×2 identity matrix. The sequence $\{w_n\}_{n\geqslant 1}$ is a zero-mean white Gaussian noise with variance $\mathbb{E}\left[w_n^2\right] = 0.1$ and both x_0 and $\{w_n\}_{n\geqslant 1}$ are assumed to be mutually independent. From this, the signal covariance function $\mathbb{E}\left[x_nx_m^T\right]$ can be factorized as follows:

$$\mathbb{E}\left[x_n x_m^T\right] = F^{n-m}\mathbb{E}\left[x_m x_m^T\right] = F^n F^{-m}\Sigma_m^x, \quad 1 \leqslant m \leqslant n,$$

verifying *Assumption 1* with $\Phi_n = F^n$ and $\Psi_m^T = F^{-m}\Sigma_m^x$, where the matrix $\Sigma_m^x = \mathbb{E}\left[x_m x_m^T\right]$ can be derived recursively by

$$\Sigma_m^x = F\Sigma_{m-1}^x F^T + 0.1GG^T, \quad m \geqslant 1, \qquad \Sigma_0^x = 0.1I_2.$$

Consider that the measurement model is given by the following equation

$$z_{kn} = \theta_{kn} C x_{kn} + v_{kn}, \ \ n \geqslant 1,$$

where $C = (0.5, 1)$ and $\{\theta_{kn}\}_{n\geqslant 1}$ is a sequence of independent identically distributed (i.i.d.) Bernoulli random variables, such that $\mathbb{P}(\theta_{kn} = 1) = \overline{\theta}$. The measurement noise $\{v_{kn}\}_{n\geqslant 1}$ is a zero-mean white Gaussian sequence with variance $R_{kn} = 0.1$. Note that this is a specific case of the measurement model (1), with $C_{kn} = \theta_{kn} C$, representing situations where signal information may be randomly missed in certain observations, resulting in measurements that consist only of noise (*missing measurements*).

According to the theoretical model, these observations are assumed to be affected by DoS attacks, whose random success or failure is modeled by Bernoulli random variables $\{\gamma_{kn}\}_{n\geqslant 1}$, and we assume that these variables are i.i.d. with probability $\mathbb{P}(\gamma_{kn} = 1) = \overline{\gamma}$.

The current simulation study aims to: analyze the impact of the measurement sampling rate on estimation accuracy; demonstrate the feasibility and effectiveness of the derived estimators through error covariance comparisons; evaluate their performance in relation to the probability that the measurements consist only of noise, $1 - \overline{\theta}$; and assess the influence of successful attack probability, $\overline{\gamma}$, on estimation accuracy.

First, to analyze the impact of the measurement sampling rate on estimation accuracy, Figure 1 displays the filtering error variances under the two compensation strategies, HI and PC, for different values of k ($k = 1, 2, 3, 5, 9$), while keeping both probabilities $\overline{\theta}$ and $\overline{\gamma}$ fixed at 0.5. As expected, this figure shows that the error variances of both signal components increase with k, indicating that estimation accuracy deteriorates as the measurement sampling frequency decreases. Additionally, this figure reveals that the error variances exhibit a sawtooth pattern, with increasing phases between consecutive multiples of k (where the set of observations is not updated, and the filter acts as a predictor based on increasingly outdated observations) and reaching local minimum values at multiples of k (where the set of observations is updated, allowing the estimator to incorporate the new information). Notably, although this sawtooth behavior prevents the error variances from stabilizing at a single value, it allows us to define "error variance bands", determined by their lower and upper bounds, whose behavior can stabilize –as it actually does in this example– when the sampling time n is sufficiently large.

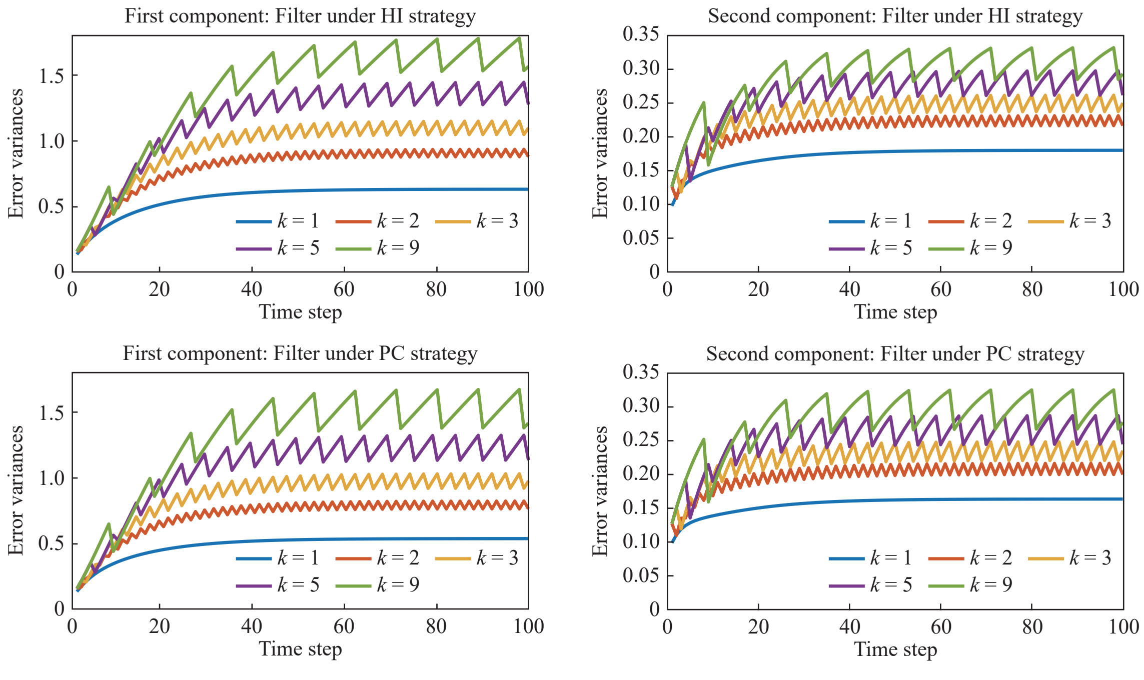

Figure 1. Filtering error variances, under HI and PC compensation strategies, for different measurement sampling rates.

Table 1 displays the stabilized upper bound values of the filtering error variances for different values of k and fixed probabilities $\overline{\gamma} = \overline{\theta} = 0.5$. These results support the comments made about Figure 1 and show a reduction in error variance when using the PC strategy compared to HI, demonstrating its superior estimation accuracy.

Table 1 Stabilized upper bound values of the filtering error variances

	First component		Second component	
	HI	**PC**	**HI**	**PC**
$k = 1$	0.6297	0.5354	0.1801	0.1625
$k = 2$	0.9343	0.8213	0.2314	0.2153
$k = 3$	1.1491	1.0285	0.2618	0.2472
$k = 5$	1.4448	1.3207	0.2979	0.2857
$k = 9$	1.7808	1.6682	0.3321	0.3237

In order to compare the accuracy of the filtering and smoothing estimators, the associated error variance bands are computed for both components of the signal process, keeping $\overline{\theta}$ and $\overline{\gamma}$ fixed at the same value 0.5. Furthermore, the sampling period is selected as $k = 3$, meaning that the signal update frequency is three times faster than the measurement sampling frequency. The results, presented in Figure 2, are analyzed under the two compensation strategies, HI and PC. Three key aspects are worth highlighting. First, the fixed-point smoothers consistently outperform the filters, exhibiting lower error variances. Second, the values of fixed-point smoothing error variances decrease as the number of available observations increases, further improving estimation accuracy. Lastly, the PC strategy achieves a greater reduction in error variance compared to the HI strategy. The comparative advantages of the PC strategy become even more apparent in Figure 3, which presents the filtering and smoothing ($L = 3$) error variance bands and demonstrates its superiority compared to the HI strategy.

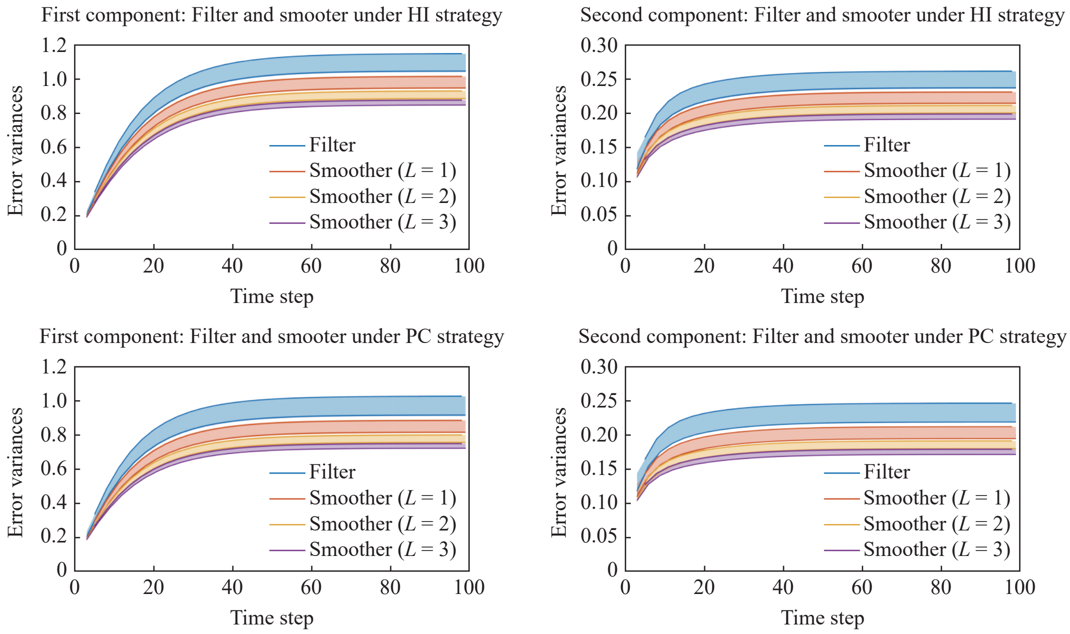

Figure 2. Filtering and smoothing ($L = 1,2,3$) error variance bands, when $\overline{\theta} = \overline{\gamma} = 0.5$ and $k = 3$.

Next, we compare the performance of the estimators versus the probability that the observations contain the signal, $\overline{\theta}$. Figure 4 shows the stabilized lower and upper bound values for both filtering and smoothing error variances, considering different values of $\overline{\theta}$, ranging from 0.1 to 0.9, while assuming that the attack probability is $\overline{\gamma} = 0.5$ and the sampling period is $k = 3$. As expected, better estimation performance is observed as $\overline{\theta}$ increases. In other words, the filtering and smoothing estimators become more accurate when the probability that the measurements consist only of noise, $1 - \overline{\theta}$, decreases.

Finally, the influence of successful attack probability, $\overline{\gamma}$, on the accuracy of the filtering and smoothing estimation schemes is analyzed. For this purpose, considering $\overline{\theta} = 0.5$ and $k = 3$, Figure 5 presents the stabilized lower and upper bound values of the error variances for different probabilities $\overline{\gamma}$, ranging from 0.1 to 0.9. The results demonstrate a clear relationship between increasing attack probability and reduced estimation performance, as evidenced by the increasing trend of the error variances for both filtering and smoothing. In line with previous findings, both Figure 4 and Figure 5 confirm that smoothing estimators outperform filtering estimators and the superiority of the PC strategy over the HI strategy is also reaffirmed.

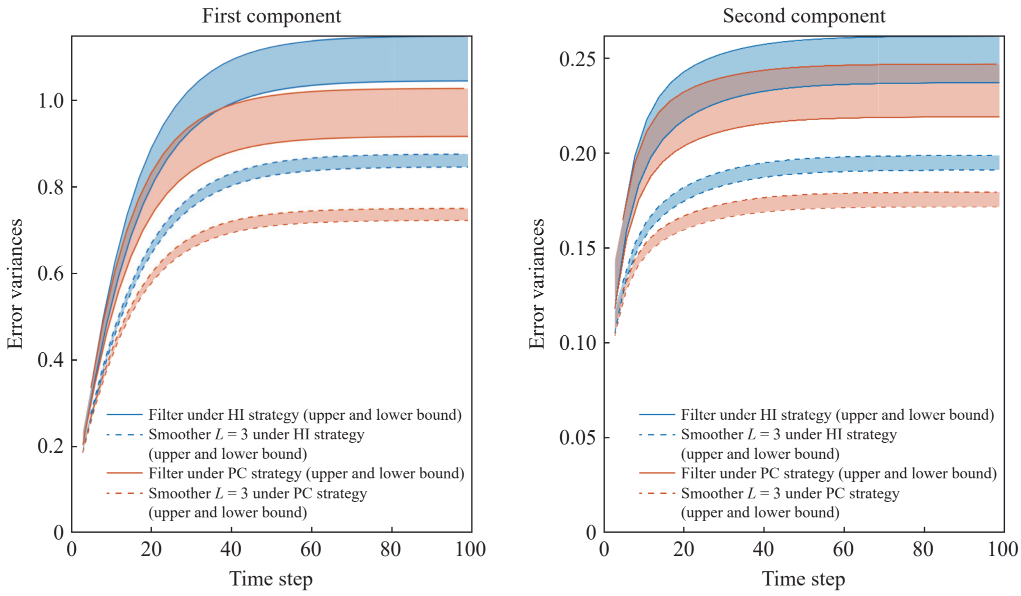

Figure 3. Comparison of filtering and smoothing ($L = 3$) error variance bands under the HI and PC compensation strategies, when $\bar{\theta} = \bar{\gamma} = 0.5$ and $k = 3$.

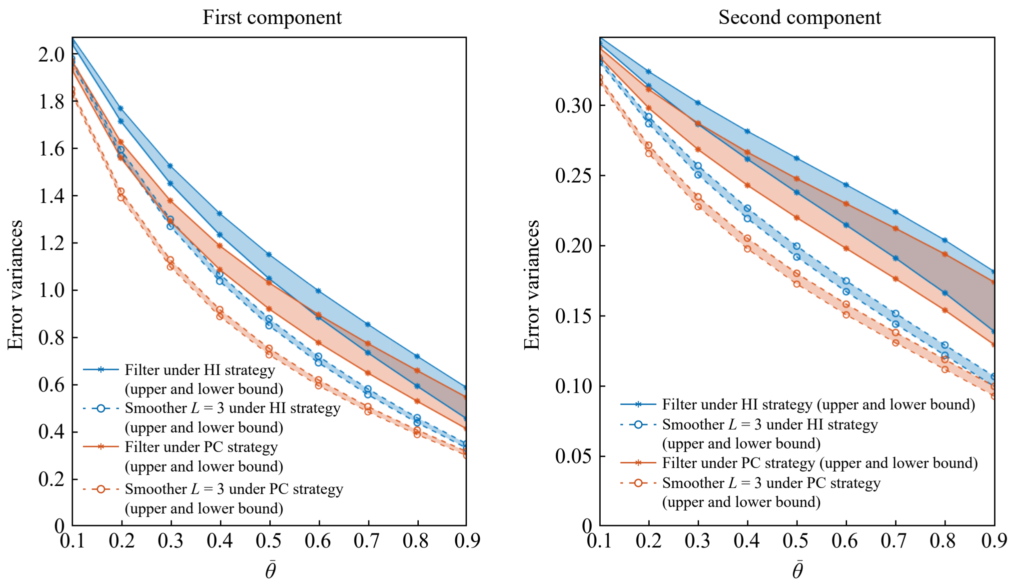

Figure 4. Stabilized lower and upper bound values of the filtering and smoothing error variances versus $\bar{\theta}$, when $\bar{\gamma} = 0.5$ and $k = 3$.

Our final aim in this section is to show the superior performance of the proposed estimation method in scenarios involving random parameter variations (e.g. missing measurements) and DoS attacks. To this end, we compare the estimators derived in this paper (Theorem 1) with those presented in [21] for multirate systems with multiple random measurement time delays, where missing measurements can be considered a particular case.

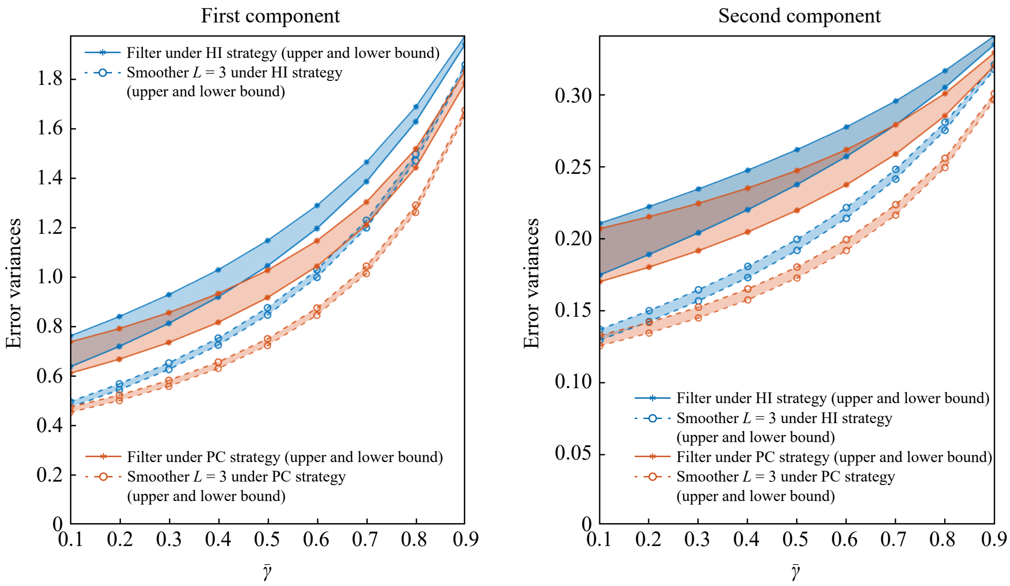

Figure 5. Stabilized lower and upper bound values of the filtering and smoothing error variances versus $\bar{\gamma}$, when $\bar{\theta} = 0.5$ and $k = 3$.

For both signal components, the comparison is based on the empirical values of the mean-squared error (MSE) at each time instant. These values are calculated over five thousand independent simulations as follows:

$$\mathrm{MSE}_{a,n} = \frac{1}{5000}\sum_{s=1}^{5000}\left(x_{a,n}^{(s)} - \widehat{x}_{a,n/kt}^{(s)}\right)^2, \quad 1 \leqslant n \leqslant 100, \ a = 1, 2,$$

where, for each sampling time n and the sth simulation run, $x_{a,n}^{(s)}$ denotes the ath component of the simulated signal, and $\widehat{x}_{a,n/kt}^{(s)}$ represents its corresponding estimate.

Since the estimator in [21] does not account for DoS attacks, the proposed method is expected to yield better performance. Indeed, assuming $\bar{\theta} = \bar{\gamma} = 0.5$ and $k = 3$, this is confirmed by the results presented in Figure 6, which show that the empirical MSE values for both signal components are consistently lower for the proposed estimates (under HI and PC strategies) compared to those in [21].

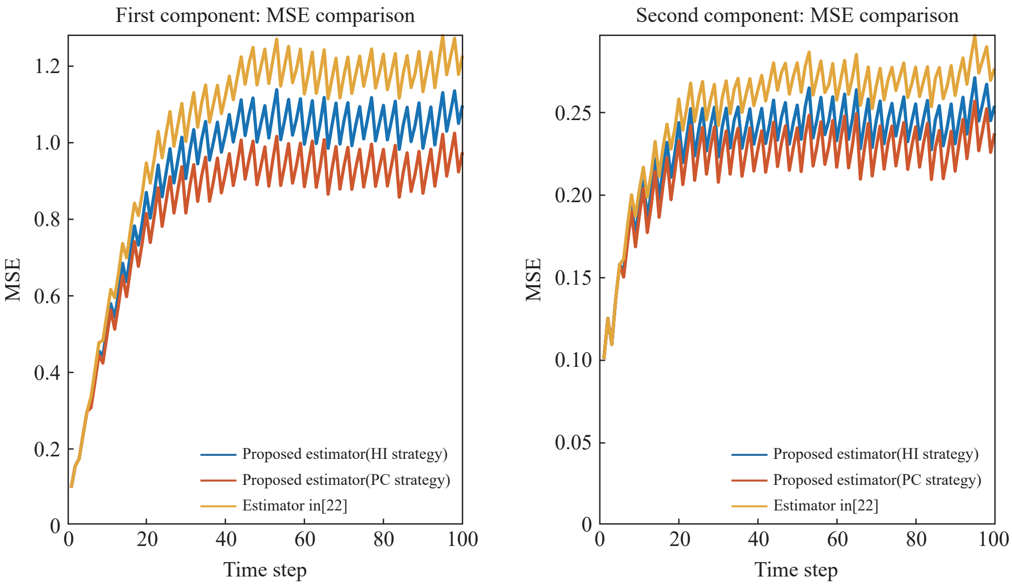

Figure 6. MSE comparison of proposed estimators and estimator in [21].

5. Conclusions

This paper has addressed the problem of LS linear estimation for time-varying MR systems with stochastic parameter matrices under the influence of randomly occurring DoS attacks. Through an innovation approach, recursive filtering and smoothing algorithms have been developed under both the HI and the PC strategies. Moreover, explicit expressions for the estimation error covariance matrices have been derived, providing a rigorous mechanism to assess the accuracy of the estimators. Simulation results have demonstrated that the theoretical model accommodates missing measurements as a specific case and have validated the effectiveness of the proposed methods, highlighting the influence of attack probabilities on estimation performance.

The findings of this study open interesting directions for future research. One potential direction is the extension of the proposed framework to multisensor systems, where fusion estimation techniques can be employed to improve accuracy under DoS attacks and other communication constraints. Another valuable research direction is the development of quadratic estimators, especially for situations where higher estimation accuracy is crucial. Additionally, the design of distributed estimation algorithms for large-scale networked systems represents an important area of exploration.

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