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Article

Sampled-Data Based Containment Control for a Class of Nonlinear Multiagent Systems With Dynamic Leaders and Control Saturation

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Abstract: This article focuses on examining the sampled-data based containment control (CC) issue for nonlinear multiagent systems (MASs) with dynamic leaders and input saturation. The proposed control protocol requires that the information is exchanged and calculated only at the sampling instants with the aim of conserving communication resources, and the protocol incorporates the control saturation as well. The CC is analyzed by means of the algebraic graph theory, M-matrix theory and Halanay-type inequality, etc. Some criteria are derived to ensure the MAS can realize the CC under the control protocol, and in the meantime, a CC region is also given ensuring that all the followers with their initial stacked states in it will converge ultimately to the convex hull formed by the leaders. Furthermore, the design of the control gain can be carried out by searching for feasible solutions to a group of matrix inequalities. Finally, a numerical illustration is provided to substantiate the efficacy of the theoretical findings.

Keywords: nonlinear multiagent system; containment control; sampled-data; input saturation

1. Introduction

In recent decades, the cooperative control problem for MASs has garnered substantial research interest due to its extensive prospective applications in unmanned aerial vehicles, satellite formation flying, distributed sensor networks, and so on [1−5]. The primary target of cooperative control for an MAS is to devise an appropriate control protocol or algorithm such that each agent adjusts its state by communicating or exchanging information only with its neighbours to accomplish a coordinated task (achieve a collective goal).

In the context of cooperative control, consensus is regarded as one of the key and fundamental issues. Usually, according to whether there exist leaders, one or multiple, in MASs, consensus problems are classified into three categories: *leaderless consensus*, *leader-follower consensus*, and *containment control (CC)*. In a MAS, a so-called leader is defined as an agent having no neighbors, that is, a leader doesn't obtain any information from other agents; as leaders, their primary role is to guide other agents to achieve a coordinated objective. In general, leaderless consensus problem does not involve any leader, and its aim is to design a protocol allowing agents to achieve an interesting goal through local interactions between agents; leader-following consensus deals with the collective behavior which is guided by one single leader, that is, the aim of leader-follower consensus is to devise an appropriate control algorithm enabling that followers can track the leader; CC involves multiple leaders, with its primary aim being driven to the followers within the convex hull formed by the leaders. Early research on consensus primarily focused on leaderless consensus and leader-follower consensus. These two aspects have been comprehensively explored, as can be seen in, for example, [6−15] and the references cited therein. Lately, the CC problem has emerged as a prominent topic in the cooperative control of MASs [16−20]. The investigation of CC is driven by the observation of numerous natural phenomena as well as practical applications. For instance, consider a scenario where a set of autonomous vehicles (or agents) need to traverse a hazardous zone. In this situation, only a subset of the agents, acting as leaders, are outfitted with the essential sensors. These leaders are responsible for detecting the suspicious targets and guiding those agents without sensors (specified as followers) into the safety area—the convex hull formed by the final positions of the leaders. The concept of CC was first put forward in [21]. In this work, the issue of guiding a group of mobile robots to a pre-determined target location was explored within the framework of partial difference equations, and a hybrid Stop-Go control policy was introduced. This strategy was designed to guarantee that the followers would be contained within the convex leader polytope, provided that the leaders are stationary and the interaction graph is connected. In [22], the issue of CC was explored for a second-order MAS guided by multiple leaders under random switching topologies.

It should be noted that the aforementioned works, by implication, assume that the signals are continuously measured and exchanged over MASs. In reality, due to limited communication resources and cost considerations, it is not always feasible or necessary to require exchanging information continuously and updating its control signal in real-time. In fact, since the bandwidth of a typical communication network is always limited, constant communication would cause the network jam or network breakdown, and might also incur high communication costs even if the bandwidth is adequate. To treat this issue, sampled-data control may be a viable alternative to continuous-time control. The sampled-data-based distributed control of MASs is currently made easier and more practical due to the recent technological improvements in computing and communication resources. Nowadays, most control systems exploit digital computers as their crucial components and computer controlled systems are getting more and more popular in engineering practice. Particularly, autonomous agents, for instance, mobile robots, frequently come equipped with digital microprocessors. These microprocessors serve to orchestrate data acquisition, facilitate communication with other agents, and regulate control actuation in accordance with specific rules. In such scenarios, information is sampled, and exchanged at some discrete instants, which leads to the sampled-data systems. A sampled-data control system can be categorized as a hybrid system in which, although the system itself operates as a continuous process, the control law utilizes data sampled only at discrete time instants. In the recent years, sampled-data-based control systems have received widespread attention. Quite a few results concerning sampled-data-based control of MASs have appeared in the literature [4, 5, 23–27].

On the other hand, due to physical constraints, the saturation phenomenon is ubiquitous and usually inevitable in system control and design [28–30]. The presence of saturation definitely brings about the impact on dynamics of the overall system to a certain extent. In a controlled system, input saturations, if not properly addressed, could result in severe performance deterioration or even instability of the system [30]. Consequently, a wide range of control issues for systems with saturation nonlinearities have received significant attention in the past few decades, and several results regarding the cooperative control issues of MASs with input saturation have been reported in the recent literature. For instance, in [31], the global consensus problem was studied for a class of discrete-time MASs with input saturation under fixed undirected topologies. In [32], the consensus control issue was investigated for a general class of MASs with actuator imperfections consisting of both actuator saturations and actuator faults. In [3], CC problem was analyzed for both first-order and second-order MASs subject to input saturation.

Although a substantial amount of research findings have been reported in the literature, the issue of CC has not been thoroughly explored and still poses significant challenges. This paper examines the CC issue for MASs with sampled-data and control saturation. The main contribution points can be recapitulated as follows.

1) A novel CC protocol is proposed for a class of MASs with multiple active leaders, where agents' states are sampled before their local information exchanges, and the control signals suffer from the saturation constraint. Compared with [4, 5, 24–26], this article explores the effect of control saturation on performance of CC. In contrast to [3], it analyzes in depth the impact of sampled-data-based control.

2) An analysis framework is developed to deal with the CC for nonlinear MASs with sampled data and input saturation. Different from the commonly used *input delay approach* [2], a Halanay-type inequality over a finite interval and its property are derived and applied to handle the sampled-data MASs. The advantage of this Halanay-type inequality based approach is that the resulting CC criteria have less decision variables and are hence easier to check.

3) The CC region under the given protocol is specified such that all the followers with their stacked states starting from inside of it will converge ultimately to the convex hull spanned by the leaders. In addition, an upper bound of the sampling period is acquired to guarantee the realization of the CC for the concerned MAS.

Notations: $\mathbf{R}^p$ and $\mathbf{R}^{p\times q}$ denote, respectively, the p-dimensional Euclidean space and the set of all $p\times q$ real matrices. For matrices R and S with $R=R^T$ and $S=S^T$, the notation $R>S$ (respectively, $R\geqslant S$) means that $R-S$ is positive definite (respectively, semi-definite). For a matrix S with $S=S^T$, $\lambda_{\max}(S)$ (respectively, $\lambda_{\min}(P)$) is its maximum (respectively, minimum) eigenvalue. For $\forall\zeta,\ \eta\in\mathbf{R}^p$, the notation $\zeta\geq\eta$ (respectively, $\zeta\leq\eta$) means $\zeta_i\geqslant\eta_i$ (respectively, $\zeta_i\leqslant\eta_i$) for $i=1,2,\cdots,p$.

2. Problem Formulation

Consider the following nonlinear MAS with N followers and M leaders in the form:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + f(t, x_i(t)) + Bu_i(t), & i \in \mathcal{V}_{\mathcal{F}}, \\ \dot{x}_i(t) = Ax_i(t) + f(t, x_i(t)), & i \in \mathcal{V}_{\mathcal{L}}. \end{cases} \tag{1}$$

where $x_i(t) \in \mathbf{R}^n$ is the state of the i-th agent; $\mathcal{V}_F = \{1, 2, \cdots, N\}$ and $\mathcal{V}_{\mathcal{F}} = \{N+1, N+2, \cdots, N+M\}$ denote, respectively, the leader set and the follower set; $u_i(t) \in \mathbf{R}^m$ represents the control protocol subject to input saturation; $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is a nonlinear vector-valued function; $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ are known constant matrices.

Remark 1. In MAS (1), for simplicity, the first N nodes are assumed to be followers and the last M nodes leaders. For more general cases, all of the following analysis is valid except that we need to introduce a suitable permutation transformation.

In this paper, we focus on the CC with multiple dynamic leaders and fixed directed network topology. The protocol is given by

$$u_i(t) = \mathrm{sat}\Big(K \sum_{j=1}^{N+M} a_{ij}(x_j(t_q) - x_i(t_q))\Big), \quad t \in [t_q, t_{q+1}), \quad i \in \mathcal{V}_F, \tag{2}$$

where $t_q = q\hbar, q = 0, 1, 2, \cdots$, denote the sampling instants with sampling period $\hbar$; K is the control gain; a_{ij} is the (i, j)-th entry within the adjacency matrix $\mathbf{A}$, which corresponds to the graph $\mathcal{G}$; $\mathrm{sat}(\cdot) : \mathbf{R}^m \to \mathbf{R}^m$ is a standard saturation function, defined by $\mathrm{sat(u)} = [\mathrm{sat(u_1), sat(u_2)}, \cdots, \mathrm{sat(u_m)}]^{\mathrm{T}}$, and $\mathrm{sat}(u_s) = \mathrm{sign}(u_s)\min\{1, |u_s|\}$. Note that the control protocol relies on periodically sampled input and suffers from control saturation.

Definition 1 ([33]). Let Y be a linear space. A subset Ψ of Y is said to be convex if for each pair of points of Ψ, the segment joining them also belongs to Ψ, namely, for any $\eta, \zeta \in \Psi$ and $0 < \mu < 1$, the point $\mu\eta + (1-\mu)\zeta \in \Psi$.

Definition 2 ([33]). If S is any set in X, the convex hull of S or the convex hull spanned by S, denoted by $\mathrm{Co}(S)$, is defined to be the intersection of all convex sets that contain S.

Lemma 1 ([33]). *The convex hull* $\mathrm{Co}(S)$ *of* S *consists of all points that are expressible in the form* $\lambda_1 x_l + \lambda_2 x_2 + \cdots + \lambda_k x_k$, *where* $x_1, x_2, \cdots, x_k$ *are any points of* S, $\lambda_i > 0$ *for each* i *and* $\sum_i \lambda_i = 1$. *The index* k *is not fixed.*

Let the stacked state vectors of all the followers and the leaders be denoted, respectively, by

$$x_F(t) = \left[x_1^T(t), x_2^T(t), \cdots, x_N^T(t)\right]^T,$$

and

$$x_L(t) = \left[x_{N+1}^T(t), x_{N+2}^T(t), \cdots, x_{N+M}^T(t)\right]^T.$$

Definition 3 ([34]). Let $x_L(0)$ be an initial stacked vector, A a subset of $\mathbf{R}^{Nn}$. The MAS (1) with control strategy (2) is said to achieve CC relative to A, if all followers with their initial stacked vector $x_F(0) \in A$ will converge ultimately to the convex hull spanned by the leaders as $t \to \infty$, namely, $\forall s \in \mathcal{V}_F$

$$\lim_{t\to\infty} \mathrm{dist}\big(x_s(t), \mathrm{Co}\big(x_r(t),\ r \in \mathcal{V}_L\big)\big) = 0.$$

The set A is called a *CC region*.

In this paper, we aim to establish the conditions that ensure all followers will ultimately enter the convex hull formed by the leaders.

3. Main Results

Before going further, we need the following assumptions and lemmas.

Assumption 1 ([16]). Assume that for each follower, there is at least one leader that has a directed path to the follower.

Assumption 2. For nonlinear function $f(\cdot)$ in (1), there is a matrix $\Sigma > 0$ satisfying

$$\begin{aligned} &\Big(f(t,\zeta) - \sum_{s=N+1}^{N+M} \sigma_{s-N} f(t, \varpi_s)\Big)^T \Big(f(t,\zeta) - \sum_{s=N+1}^{N+M} \sigma_{s-N} f(t, \varpi_s)\Big) \\ &\leqslant \Big(\zeta - \sum_{s=N+1}^{N+M} \sigma_{s-N}\varpi_s\Big)^T \Sigma \Big(\zeta - \sum_{s=N+1}^{N+M} \sigma_{s-N}\varpi_s\Big), \end{aligned} \tag{3}$$

for all $\zeta, \varpi_s \in \mathbf{R}^n$, and all $\sigma_s \geqslant 0$ $(s = 1, 2, \cdots, M)$ with $\sum_{s=1}^{M} \sigma_s = 1$.

Remark 2. It is straightforward to verify that Assumption 2 implies that nonlinear function f satisfies the Lipschitz condition, but the converse is not necessarily true. Assumption 2 will be used in the analysis of CC. When Σ in (3) is replaced with a positive scalar constant, say $\sqrt{\delta}$, Assumption 2 reduces to the following special case:

$$\left\|f(t,\zeta)-\sum_{s=N+1}^{N+M}\sigma_{s-N}f(t,\varpi_s)\right\|\leqslant\delta\left\|\zeta-\sum_{s=N+1}^{N+M}\sigma_{s-N}\varpi_s\right\|, \tag{4}$$

for all $\zeta,\varpi_s\in\mathbf{R}^n$, and all $\sigma_s\geqslant0$ $(s=1,2,\cdots,M)$ with $\sum_{s=1}^{M}\sigma_s=1$. The condition (4) was first proposed in [35], and has later been applied in quite a few papers such as [36–38].

Since the leaders have no neighbor, the Laplacian matrix $\mathcal{L}=[l_{ij}]$ of the multiagent system (1) can be broken down into

$$\mathcal{L}=\begin{bmatrix}\mathcal{L}_1 & \mathcal{L}_2\\ 0_{M\times N} & 0_{M\times M}\end{bmatrix}, \tag{5}$$

where $\mathcal{L}_1\in R^{N\times N},\mathcal{L}_2\in R^{N\times M}$.

Lemma 2 ([39]). *If Assumption 1 holds, the matrix $\mathcal{L}_1$ is a nonsingular M-matrix and the matrix $-\mathcal{L}_1^{-1}\mathcal{L}_2$ is non-negative. Moreover, each row of $-\mathcal{L}_1^{-1}\mathcal{L}_2$ has a sum equal to one.*

Lemma 3 ([40]). *Let $\psi(x)\triangleq\mathrm{sat}(x)-x$, for $x\in\mathbf{R}^p$. Then, for any $v,w\in\mathbf{R}^p$ with $-1_p\le v-w\le 1_p$, the following relationship holds:*

$$\psi^T(v)\Phi(\psi(v)+w)\le 0, \tag{6}$$

for any matrix $\Phi=\mathrm{diag}\{\phi_1,\phi_2,\cdots,\phi_p\}>0$.

Lemma 4 ([41]). *For any symmetric matrices $\Omega,\Pi\in\mathbf{R}^{m\times m}$ with $\Omega>0$, the following inequality holds:*

$$\lambda_{\min}\left(\Omega^{-1}\Pi\right)z^T\Omega z\leqslant z^T\Pi z\leqslant\lambda_{\max}\left(\Omega^{-1}\Pi\right)z^T\Omega z, \tag{7}$$

where $z\in\mathbf{R}^m$.

Lemma 5 ([42, 43]). *Suppose M is a positive semi-definite matrix, $\varrho(\cdot):(-\infty,\tau]\to[0,+\infty)$ is a scalar function, $F(\cdot):(-\infty,\tau]\to\mathbf{R}^n$ is a vector function and the integrations concerned are well defined. Then the following inequality is true:*

$$\left(\int_{-\infty}^{\tau}\varrho(\rho)F(\rho)d\rho\right)^T M\left(\int_{-\infty}^{\tau}\varrho(\rho)F(\rho)d\rho\right)\leqslant\int_{-\infty}^{\tau}\varrho(\rho)d\rho\left(\int_{-\infty}^{\tau}\varrho(\rho)F^T(\rho)MF(\rho)d\rho\right). \tag{8}$$

The primary result of this paper is presented below.

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Let K be a given gain matrix, and $\alpha>\beta>0$ be two given scalar constants. Then the MAS (1) with the protocol (2) achieves the CC with a CC region $A=\left\{x\in\mathbf{R}^{Nn}\,\middle|\,(x+\mathcal{L}_1{}^{-1}\mathcal{L}_2x_L(0))^T(I_N\otimes P)(x+\mathcal{L}_1{}^{-1}\mathcal{L}_2x_L(0))\leqslant1\right\}$, if there exist a matrix $P>0$, a diagonal matrix $\Phi>0$, a matrix G and a number $\vartheta>0$ satisfying the following LMIs:*

$$\begin{bmatrix}-I_N\otimes P & \left(L_{1(i)}\otimes(G+K)_j\right)^T\\ * & -1\end{bmatrix}<0, \tag{9}$$

$$\begin{bmatrix}\Xi & 0 & I_N\otimes(PB)\\ * & -\beta(I_N\otimes P) & -\mathcal{L}_1^T\otimes(G^T\Phi)\\ * & * & -2(I_N\otimes\Phi)\end{bmatrix}<0, \tag{10}$$

and the sampling period $\hbar$ satisfies

$$\hbar<\frac{\alpha-\beta}{3\rho_1+\rho_2}, \tag{11}$$

where

$$\begin{aligned}\Xi=&I_N\otimes\left(\alpha P+PA+A^TP+\vartheta PP+\vartheta^{-1}\Sigma\right)-\mathcal{L}_1\otimes(PBK)-\mathcal{L}_1^T\otimes(K^TB^TP),\\ \rho_1=&\lambda_{\max}\left(\mathcal{L}_1\mathcal{L}_1^T\right),\\ \rho_2=&\lambda_{\max}\left(P^{-1}(BKA)^TPBKA\right)+\lambda_{\max}\left((BK)^TPBK\right)\lambda_{\max}\left(P^{-1}\Sigma\right)\\ &+\rho_1\lambda_{\max}\left(P^{-1}(BKB)^TPBKB\right)\lambda_{\max}\left(P^{-1}K^TK\right).\end{aligned}$$

Proof. To begin with, denote the CC error by

$$e(t) = x_F(t) + ((\mathcal{L}_1^{-1}\mathcal{L}_2) \otimes I_n)x_L(t).$$

Then by Lemma 2, to prove the system (1) with the protocol (2) achieves the CC with a CC region A, it suffices to prove $e(t) \to 0$ as $t \to \infty$ for any $x_F(0) \in A$. So, our next objective is to explore the evolution of $e(t)$. To simplify the notation, we denote

$$\begin{aligned}
z_i(t) &= \sum_{j=1}^{M+N} a_{ij}(x_j(t) - x_i(t)),\\
z(t) &= \left[z_1^T(t), z_2^T(t), \cdots, z_N^T(t)\right]^T,\\
\widehat{F}(t, x_F) &= \left[f^T(t, x_1), f^T(t, x_2), \cdots, f^T(t, x_N)\right]^T,\\
\widehat{F}(t, x_L) &= \left[f^T(t, x_{N+1}), f^T(t, x_{N+2}), \cdots, f^T(t, x_{N+M})\right]^T,\\
F(t, e(t)) &= \widehat{F}(t, x_F(t)) + ((\mathcal{L}_1^{-1}\mathcal{L}_2) \otimes I_n)\widehat{F}(t, x_L(t)).
\end{aligned}$$

Then, the MAS (1) under the protocol (2) can be expressed as the following compact form:

$$\begin{cases} \dot{x}_F(t) = (I_N \otimes A)x_F(t) + \widehat{F}(t, x_F(t)) + (I_N \otimes B)\mathrm{sat}((I_N \otimes K)z(t_q)),\ t \in [t_q,\ t_{q+1}),\\ \dot{x}_L(t) = (I_M \otimes A)x_L(t) + \widehat{F}(t, x_L(t)). \end{cases} \tag{12}$$

Note that $(I_N \otimes A) + ((\mathcal{L}_1^{-1}\mathcal{L}_2) \otimes I_n)(I_M \otimes A) = (I_N \otimes A) + (\mathcal{L}_1^{-1}\mathcal{L}_2) \otimes A = (I_N \otimes A)(I_{Nn} + (\mathcal{L}_1^{-1}\mathcal{L}_2) \otimes I_n)$. Then, from (12), it follows that

$$\frac{de(t)}{dt} = (I_N \otimes A)e(t) + F(t, e(t)) + (I_N \otimes B)\,\mathrm{sat}\left((I_N \otimes K)z\left(t_q\right)\right),\ t \in [t_q,\ t_{q+1}). \tag{13}$$

To analyze the CC of the MAS (1) with the protocol (2), we construct the following Lyapunov function:

$$V(t) = e^T(t)(I_N \otimes P)e(t). \tag{14}$$

Taking the time derivative of $V(t)$ along a given trajectory of (13) leads to

$$\begin{aligned}
\dot{V}(t) &= 2e^T(t)(I_N \otimes P)\dot{e}(t)\\
&= 2e^T(t)(I_N \otimes P)\left[(I_N \otimes A)e(t) + F(t, e(t)) + (I_N \otimes B)\mathrm{sat}\left((I_N \otimes K)z\left(t_q\right)\right)\right]\\
&= 2e^T(t)(I_N \otimes PA)e(t) + 2e^T(t)(I_N \otimes P)F(t, e(t)) + 2e^T(t)(I_N \otimes PB)\psi((I_N \otimes K)z(t_q))\\
&\quad + 2e^T(t)(I_N \otimes (PBK))z(t) - 2e^T(t)(I_N \otimes (PBK))\Delta z(t),\quad t \in [t_q,\ t_{q+1}),
\end{aligned} \tag{15}$$

where $\Delta z(t) = z(t) - z(t_q)$.

By using Young's inequality, it is easy to see that

$$\begin{aligned}
&2e^T(t)(I_N \otimes P)F(t, e(t))\\
\leqslant\ &\vartheta e^T(t)(I_N \otimes PP)e(t) + \vartheta^{-1}F^T(t, e(t))F(t, e(t)).
\end{aligned} \tag{16}$$

According to Assumption 2, one has

$$F^T(t, e(t))F(t, e(t)) \le e^T(t)(I_N \otimes \Sigma)e(t), \tag{17}$$

which, together with (16), leads to

$$\begin{aligned}
&2e^T(t)(I_N \otimes P)F(t, e(t))\\
\leqslant\ &\vartheta e^T(t)(I_N \otimes PP)e(t) + \vartheta^{-1}e^T(t)(I_N \otimes \Sigma)e(t).
\end{aligned} \tag{18}$$

Note that

$$\begin{aligned}
z(t) = \begin{bmatrix} z_1(t)\\ z_2(t)\\ \vdots\\ z_N(t) \end{bmatrix} &= -\begin{bmatrix} \mathcal{L}_1 \otimes I_n & \mathcal{L}_2 \otimes I_n \end{bmatrix}\begin{bmatrix} x_F(t)\\ x_L(t) \end{bmatrix}\\
&= -(\mathcal{L}_1 \otimes I_n)\begin{bmatrix} I_{Nn} & (\mathcal{L}_1^{-1}\mathcal{L}_2) \otimes I_n \end{bmatrix}\begin{bmatrix} x_F(t)\\ x_L(t) \end{bmatrix}\\
&= -(\mathcal{L}_1 \otimes I_n)e(t).
\end{aligned} \tag{19}$$

From (19), it follows that

$$\begin{aligned} & 2e^T(t)(I_N \otimes (PBK))z(t) \\ =& 2e^T(t)(I_N \otimes (PBK))(-(\mathcal{L}_1 \otimes I_n)e(t)) \\ =& -2e^T(t)(\mathcal{L}_1 \otimes (PBK))e(t). \end{aligned} \tag{20}$$

Notice that

$$\begin{aligned} \Delta z(t) =& z(t) - z(t_q) \\ =& \int_{t_q}^{t} \dot{z}(\theta) d\theta \\ =& -\int_{t_q}^{t} (\mathcal{L}_1 \otimes I_n)\dot{e}(\theta) d\theta \quad (\text{ by } (19)\ ) \\ =& -2e^T(t)(\mathcal{L}_1 \otimes (PBK)) \int_{t_q}^{t} \big[(I_N \otimes A)e(\theta) + F(\theta, e(\theta)) \\ & + (I_N \otimes B)\,\mathrm{sat}\left((I_N \otimes K)z(t_q)\right)\big] d\theta. \end{aligned} \tag{21}$$

From (21), we arrive at

$$\begin{aligned} =& -2e^T(t)(I_N \otimes (PBK))\Delta z(t) \\ =& 2e^T(t)(\mathcal{L}_1 \otimes (PBK)) \int_{t_q}^{t} \big[(I_N \otimes A)e(\theta) + F(\theta, e(\theta)) \\ & + (I_N \otimes B)\,\mathrm{sat}\left((I_N \otimes K)z(t_q)\right)\big] d\theta. \end{aligned} \tag{22}$$

By employing Young's inequality and Lemmas 4 and 5, we deduce that

$$\begin{aligned} & 2e^T(t)(\mathcal{L}_1 \otimes (PBK)) \int_{t_q}^{t} (I_N \otimes A)e(\theta) d\theta \\ \leqslant & (t - t_q)\, e^T(t)\left((\mathcal{L}_1 \mathcal{L}_1^T) \otimes P\right)e(t) \\ & + \frac{1}{t - t_q}\left(\int_{t_q}^{t} e(\theta) d\theta\right)^T \left(I_N \otimes \left((BKA)^T P(BKA)\right)\right) \int_{t_q}^{t} e(\theta) d\theta \\ \leqslant & (t - t_q)\, e^T(t)\left((\mathcal{L}_1 \mathcal{L}_1^T) \otimes P\right)e(t) \\ & + \int_{t_q}^{t} e^T(\theta)\left(I_N \otimes (BKA)^T P(BKA)\right)e(\theta) d\theta \\ \leqslant & (t - t_q)\left[\lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^T\right)V(t) + \lambda_{\max}\left(P^{-1}(BKA)^T PBKA\right) \max_{\theta \in [t_q, t]} V(\theta)\right], \end{aligned} \tag{23}$$

Also, by utilizing Young's inequality and Lemmas 4 and 5, together with Assumption 2, we have

$$\begin{aligned} & 2e^T(t)(\mathcal{L}_1 \otimes (PBK)) \int_{t_q}^{t} F(\theta, e(\theta)) d\theta \\ \leqslant & (t - t_q)e^T(t)((\mathcal{L}_1 \mathcal{L}_1^T) \otimes P)e(t) \\ & + \int_{t_q}^{t} F^T(\theta, e(\theta))\left(I_N \otimes \left((BK)^T P(BK)\right)\right)F(\theta, e(\theta)) d\theta \\ \leqslant & (t - t_q)\,\lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^T\right)V(t) \\ & + \lambda_{\max}\left((BK)^T P(BK)\right) \int_{t_q}^{t} F^T(\theta, e(\theta))F(\theta, e(\theta)) d\theta \\ \leqslant & (t - t_q)\,\lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^T\right)V(t) \\ & + \lambda_{\max}\left((BK)^T P(BK)\right) \int_{t_q}^{t} e^T(\theta)\left(I_N \otimes \Sigma\right)e(\theta) d\theta \\ \leqslant & (t - t_q)\,\lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^T\right)V(t) \\ & + (t - t_q)\lambda_{\max}\left((BK)^T P(BK)\right)\lambda_{\max}\left(P^{-1}\Sigma\right) \max_{\theta \in [t_q, t]} V(\theta). \end{aligned} \tag{24}$$

Similarly, we learn that

$$
\begin{aligned}
&2e^{T}(t)(\mathcal{L}_1 \otimes (PBK)) \int_{t_q}^{t} (I_N \otimes B)\,\mathrm{sat}\left((I_N \otimes K) z\left(t_q\right)\right) d\theta \\
&\leqslant \left(t-t_q\right) e^{T}(t) \left((\mathcal{L}_1 \mathcal{L}_1^{T}) \otimes P\right) e(t) \\
&\quad + \frac{1}{t-t_q} \left(\int_{t_q}^{t} \mathrm{sat}\left((I_N \otimes K) z\left(t_q\right)\right) d\theta \right)^{T} \left(I_N \otimes \left((BKB)^{T} P (BKB)\right)\right) \int_{t_q}^{t} \mathrm{sat}\left((I_N \otimes K) z\left(t_q\right)\right) d\theta \\
&\leqslant \left(t-t_q\right) e^{T}(t) \left((\mathcal{L}_1 \mathcal{L}_1^{T}) \otimes P\right) e(t) \\
&\quad + \int_{t_q}^{t} \left(\mathrm{sat}((I_N \otimes K) z(t_q))\right)^{T} \left(I_N \otimes \left((BKB)^{T} P (BKB)\right)\right) \mathrm{sat}\left((I_N \otimes K)\right) z(t_q))\right) d\theta \\
&\leqslant \left(t-t_q\right) \lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^{T}\right) V(t) \\
&\quad + \left(t-t_q\right) \lambda_{\max}\left((BKB)^{T} P (BKB)\right) \lambda_{\max}\left(P^{-1} K^{T} K\right) \max_{\theta \in [t_q, t]} V(\theta).
\end{aligned}
\tag{25}
$$

Inserting (23), (24), and (25) into (22) gives that

$$
\begin{aligned}
&-2e^{T}(t)(I_N \otimes (PBK)) \Delta z(t) \\
&\leqslant (t-t_q)\left[\lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^{T}\right) V(t) + \lambda_{\max}\left(P^{-1}(BKA)^{T} PBKA\right) \max_{\theta \in [t_q, t]} V(\theta)\right] \\
&\quad + \left(t-t_q\right) \lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^{T}\right) V(t) + (t-t_q) \lambda_{\max}\left((BK)^{T} P (BK)\right) \lambda_{\max}\left(P^{-1} \Sigma\right) \max_{\theta \in [t_q, t]} V(\theta) \\
&\quad + \left(t-t_q\right) \lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^{T}\right) V(t) + \left(t-t_q\right) \lambda_{\max}\left((BKB)^{T} P (BKB)\right) \lambda_{\max}\left(P^{-1} K^{T} K\right) \max_{\theta \in [t_q, t]} V(\theta).
\end{aligned}
\tag{26}
$$

Substituting (18), (20), and (26) into (15) yields that

$$
\begin{aligned}
\dot{V}(t) \leqslant\, & e^{T}(t) \left(I_N \otimes \left(PA + A^{T} P + \vartheta PP + \vartheta^{-1} \Sigma\right)\right) e(t) \\
& - e^{T}(t) \left(\mathcal{L}_1 \otimes (PBK) + \mathcal{L}_1^{T} \otimes (K^{T} B^{T} P)\right) e(t) \\
& + 2e^{T}(t) (I_N \otimes (PB)) \psi\left((I_N \otimes K) z(t_q)\right) \\
& + (t-t_q)\left[\lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^{T}\right) V(t) + \lambda_{\max}\left(P^{-1}(BKA)^{T} PBKA\right) \max_{\theta \in [t_q, t]} V(\theta)\right] \\
& + \left(t-t_q\right) \left[\lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^{T}\right) V(t) + \lambda_{\max}\left((BK)^{T} P (BK)\right) \lambda_{\max}\left(P^{-1} \Sigma\right) \max_{\theta \in [t_q, t]} V(\theta)\right] \\
& + \left(t-t_q\right) \left[\lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^{T}\right) V(t) + \lambda_{\max}\left((BKB)^{T} P (BKB)\right) \lambda_{\max}\left(P^{-1} K^{T} K\right) \max_{\theta \in [t_q, t]} V(\theta)\right].
\end{aligned}
\tag{27}
$$

By the *Schur Complement Lemma*, the condition (9) implies that

$$
\left(\mathcal{L}_{1(i)} \otimes (G+K)_{(j)}\right)^{T} \left(\mathcal{L}_{1(i)} \otimes (G+K)_{(j)}\right) \leqslant I_N \otimes P. \tag{28}
$$

When $x_F(t_0) \in A$, it is clear that

$$
e^{T}(t_0)(I_N \otimes P) e(t_0) \leqslant 0. \tag{29}
$$

From (28) and (29), it follows readily

$$
e^{T}(t_0) \left(\mathcal{L}_{1(i)} \otimes (G+K)_{(j)}\right)^{T} \left(\mathcal{L}_{1(i)} \otimes (G+K)_{(j)}\right) e(t_0) \leqslant 1, \tag{30}
$$

Thus,

$$
-1 \leqslant \left(\mathcal{L}_{1(i)} \otimes (G+K)_{(j)}\right) e(t_0) \leqslant 1, \tag{31}
$$

which implies

$$
-1_{nN} \preceq (\mathcal{L}_1 \otimes (G+K)) e(t_0) \preceq 1_{nN}. \tag{32}
$$

Note that

$$
\begin{aligned}
&(I_N \otimes K) z(t_0) - (\mathcal{L}_1 \otimes G) e(t_0)) \\
&= (I_N \otimes K)\left(-(\mathcal{L}_1 \otimes I_n) e(t_0)\right) - (\mathcal{L}_1 \otimes G) e(t_0)) \quad \text{(by (19))} \\
&= -(\mathcal{L}_1 \otimes (G+K)) e(t_0).
\end{aligned}
\tag{33}
$$

Thus, from (32) and (33), one has

$$-1_{nN} \preceq (I_N \otimes K)z(t_0) - (\mathcal{L}_1 \otimes G)e(t_0)) \preceq 1_{nN}. \tag{34}$$

By Lemma 3 and (34), we get

$$2\psi^T((I_N \otimes K)z(t_0))(I_N \otimes \Phi)(\psi((I_N \otimes K)z(t_0)) + (\mathcal{L}_1 \otimes G)e(t_0)) \leqslant 0. \tag{35}$$

From (27) and (35), we can deduce that, for $t \in [t_0,\ t_1)$,

$$\begin{aligned}
\dot{V}(t) \leqslant & e^T(t)\left(I_N \otimes \left(PA + A^T P + \vartheta PP + \vartheta^{-1}\Sigma\right)\right)e(t) \\
& - e^T(t)\left(\mathcal{L}_1 \otimes (PBK) + \mathcal{L}_1^T \otimes (K^T B^T P)\right)e(t) \\
& + 2e^T(t)(I_N \otimes (PB))\psi((I_N \otimes K)z(t_0)) \\
& + (t - t_0)\left[\lambda_{\max}\left(\mathcal{L}_1\mathcal{L}_1^T\right)V(t) + \lambda_{\max}\left(P^{-1}(BKA)^T PBKA\right)\max_{\theta\in[t_0,t]} V(\theta)\right] \\
& + (t - t_0)\left[\lambda_{\max}\left(\mathcal{L}_1\mathcal{L}_1^T\right)V(t) + \lambda_{\max}\left((BK)^T P(BK)\right)\lambda_{\max}\left(P^{-1}\Sigma\right)\max_{\theta\in[t_0,t]} V(\theta)\right] \\
& + (t - t_0)\left[\lambda_{\max}\left(\mathcal{L}_1\mathcal{L}_1^T\right)V(t) + \lambda_{\max}\left((BKB)^T P(BKB)\right)\lambda_{\max}\left(P^{-1}K^T K\right)\max_{\theta\in[t_0,t]} V(\theta)\right] \\
& - 2\psi^T((I_N \otimes K)z(t_0))(I_N \otimes \Phi)(\psi((I_N \otimes K)z(t_0)) + (\mathcal{L}_1 \otimes G)e(t_0)) \\
= & \eta^T(t)\Xi_0\eta(t) + 3\rho_1(t - t_0)V(t) + \beta V(t_0) + \rho_2(t - t_0)\max_{\theta\in[t_0,t]} V(\theta),
\end{aligned} \tag{36}$$

where

$$\eta(t) = \left[e^T(t),\ e^T(t_0),\ \psi^T\left((I_N \otimes K)z(t_0)\right)\right]^T,$$

$$\Xi_0 = \begin{bmatrix} \Pi_0 & 0 & I_N \otimes (PB) \\ * & -\beta(I_N \otimes P) & -\mathcal{L}_1^T \otimes (G^T\Phi) \\ * & * & -2(I_N \otimes \Phi) \end{bmatrix}$$

with $\Pi_0 = I_N \otimes \left(PA + A^T P + \vartheta PP + \vartheta^{-1}\Sigma\right) - \mathcal{L}_1 \otimes (PBK) - \mathcal{L}_1^T \otimes (K^T B^T P)$.

From (9) and (36), we obtain, for $t \in [t_0, t_1)$,

$$\begin{aligned}
\dot{V}(t) &\leqslant -\alpha V(t) + 3\rho_1(t - t_0)V(t) + \beta V(t_0) + \rho_2(t - t_0)\max_{\theta\in[t_0,t]} V(\theta) \\
&\leqslant -\alpha V(t) + 3\rho_1\hbar V(t) + \beta V(t_0) + \rho_2\hbar\max_{\theta\in[t_0,t]} V(\theta).
\end{aligned} \tag{37}$$

Next, we prove by contradiction that

$$\max_{s\in[t_0,t_1)} V(s) = V(t_0). \tag{38}$$

We only need to analyze the case when $V(t_0) \neq 0$, otherwise the conclusion is obvious. To this end, suppose (38) is not true, then there is a $t^* \in [t_0,\ t_1)$ such that

$$V(t^*) > V(t_0). \tag{39}$$

From (11) and (37), we attain

$$\dot{V}(t_0) \leqslant \left[-\alpha + \beta + 3\rho_1\hbar + \rho_2\hbar\right]V(t_0) < 0. \tag{40}$$

Then, there is $\epsilon \in (0, t^* - t_0)$ such that

$$V(t) < V(t_0),\ \text{for } t_0 < t < t_0 + \epsilon. \tag{41}$$

Denote $\bar{t} = \inf\{t > t_0 \mid V(t) = V(t_0)\}$. Then $t_0 < \bar{t} < t^*$, and clearly

$$\max_{t\in[t_0,\ \bar{t}]} V(t) = V(t_0) = V(\bar{t}), \tag{42}$$

which implies

$$\dot{V}(\bar{t}) \geqslant 0. \tag{43}$$

In addition, from (37) and (42), it can be inferred that

$$\dot{V}(\bar{t}) < 0, \tag{44}$$

contradicting (43). Therefore, (38) holds.

From (37) and (38), one has

$$\dot{V}(t) \leqslant -\gamma_1 V(t) + \gamma_2 V(t_0), \text{ for } t \in [t_0,\ t_1), \tag{45}$$

where $\gamma_1 = \alpha - 3\rho_1 \hbar$, and $\gamma_2 = \rho_2 \hbar + \beta$.

From (45) and by using the *comparison principle*, we get

$$\begin{aligned} V(t) \leqslant & e^{-\gamma_1(t-t_0)} V(t_0) + \int_{t_0}^{t} e^{-\gamma_1(t-s)} \gamma_2 V(t_0) ds \\ = & e^{-\gamma_1(t-t_0)} V(t_0) + \frac{\gamma_2}{\gamma_1}\left(1 - e^{-\gamma_1(t-t_0)}\right) V(t_0) \\ = & \left[\frac{\gamma_2}{\gamma_1} + \left(1 - \frac{\gamma_2}{\gamma_1}\right) e^{-\gamma_1(t-t_0)}\right] V(t_0),\ t \in [t_0,\ t_1). \end{aligned} \tag{46}$$

We can repeat this procedure recursively to get, for any nonnegative integer q,

$$V(t) \leqslant \left[\frac{\gamma_2}{\gamma_1} + \left(1 - \frac{\gamma_2}{\gamma_1}\right) e^{-\gamma_1(t-t_q)}\right] V(t_q), \quad \text{for } t \in [t_q,\ t_{q+1}). \tag{47}$$

Denote $\zeta = \frac{\gamma_2}{\gamma_1} + \left(1 - \frac{\gamma_2}{\gamma_1}\right) e^{-\gamma_1 \hbar}$, then $0 < \zeta < 1$.

Now, for any $t > 0$, there is a unique nonnegative integer κ and $0 \le \tilde{t} < \hbar$ such that $t = \kappa\hbar + \tilde{t}$. Then, we arrive at

$$\begin{aligned} V(t) \leqslant & \left[\frac{\gamma_2}{\gamma_1} + \left(1 - \frac{\gamma_2}{\gamma_1}\right) e^{-\gamma_1(t-t_\kappa)}\right] V(t_\kappa) \\ \leqslant & V(t_\kappa) \le \zeta V(t_{\kappa-1}) \le \zeta^2 V(t_{\kappa-2}) \le \cdots \le \zeta^\kappa V(t_0). \end{aligned} \tag{48}$$

Note that $\kappa \ge \frac{t-\hbar}{\hbar}$, then from (48), we get

$$V(t) \leqslant \zeta^{\frac{t-\hbar}{\hbar}} V(t_0) \le V(t_0) e^{\frac{\ln\zeta}{\hbar}(t-\hbar)}. \tag{49}$$

This implies that $V(t) \to 0$, as $t \to 0$; accordingly one has $e(t) \to 0$, as $t \to 0$. Therefore, the MAS (1) with the protocol (2) reaches the CC with the CC region A.

Remark 3. (37) can be regarded as a Halanay-type inequality over a finite interval. This inequality, together with its derivative (38), plays a crucial role for handling the sampled-data systems.

Up till now, we have carried out analysis of the conditions for the concerned MAS to achieve CC. Now we turn to solve the following design problem of the CC.

Theorem 2. *Suppose that Assumptions 1 and 2 hold. Let $\alpha > \beta > 0$ be two given scalar constants. Then the MAS* (1) *with the protocol* (2) *achieves the CC with a CC region* $A = \left\{x \in \mathbf{R}^{N\times n} \mid (x + \mathcal{L}_1^{-1}\mathcal{L}_2 x_L(0))^T (I_N \otimes Q^{-1}) (x + \mathcal{L}_1^{-1}\mathcal{L}_2 x_L(0)) \leqslant 1\right\}$*, If there exist a number $\vartheta > 0$, a matrix $Q > 0$, a diagonal matrix $\Upsilon > 0$, two matrices Γ and Λ satisfying the following LMIs:*

$$\begin{bmatrix} -I_N \otimes Q & \left(\mathcal{L}_{1(i)} \otimes (\Gamma + \Lambda)_j\right)^T \\ * & -1 \end{bmatrix} < 0, \tag{50}$$

$$\begin{bmatrix} \hat{\Xi} & 0 & I_N \otimes (B\Upsilon) & I_N \otimes Q \\ * & -\beta(I_N \otimes Q) & -\mathcal{L}_1^T \otimes \Gamma^T & 0 \\ * & * & -2(I_N \otimes \Upsilon) & 0 \\ * & * & * & -\vartheta(I_N \otimes \Sigma^{-1}) \end{bmatrix} < 0, \tag{51}$$

and the sampling period $\hbar$ satisfies

$$\hbar < \frac{\alpha - \beta}{3\rho_1 + \rho_2}, \tag{52}$$

where

$$\begin{aligned} \hat{\Xi} = & I_N \otimes \left(\alpha Q + AQ + QA^T + \vartheta I_n\right) - \mathcal{L}_1 \otimes (B\Lambda) - \mathcal{L}_1^T \otimes (\Lambda^T B^T), \\ \rho_1 = & \lambda_{\max}\left(\mathcal{L}_1 \mathcal{L}_1^T\right), \\ \rho_2 = & \lambda_{\max}\left(Q\left(B\Lambda Q^{-1}A\right)^T Q^{-1} B\Lambda Q^{-1} A\right) + \lambda_{\max}\left(\left(B\Lambda Q^{-1}\right)^T Q^{-1} B\Lambda Q^{-1}\right) \lambda_{\max}(Q\Sigma) \\ & + \rho_1 \lambda_{\max}\left(Q\left(B\Lambda Q^{-1}B\right)^T Q^{-1} B\Lambda Q^{-1} B\right) \lambda_{\max}\left(\Lambda^T \Lambda Q^{-1}\right). \end{aligned}$$

In this case, the control gain K can be designed as

$$K = \Lambda Q^{-1}. \tag{53}$$

Proof. First of all, by the *Schur Complement*, (51) is equivalent to

$$\begin{bmatrix} \tilde{\Xi} & 0 & I_N \otimes (B\Upsilon) \\ * & -\beta (I_N \otimes Q) & -\mathcal{L}_1^T \otimes \Gamma^T \\ * & * & -2(I_N \otimes \Upsilon) \end{bmatrix} < 0, \tag{51'}$$

where

$$\tilde{\Xi} = I_N \otimes \left(\alpha Q + AQ + QA^T + \vartheta I_n + \vartheta^{-1} Q\Sigma Q\right) - \mathcal{L}_1 \otimes (B\Lambda) - \mathcal{L}_1^T \otimes (\Lambda^T B^T).$$

Let $P = Q^{-1}$, $G = \Gamma Q^{-1}$, $K = \Lambda Q^{-1}$, and $\Phi = \Upsilon^{-1}$, then $Q = P^{-1}$, $\Gamma = GQ = GP^{-1}$, $\Lambda = KQ = KP^{-1}$, and $\Upsilon = \Phi^{-1}$, and the inequalities (50) and (51'), respectively, turn into

$$\begin{bmatrix} -I_N \otimes P^{-1} & \left(\mathcal{L}_{1(i)} \otimes \left(GP^{-1} + KP^{-1}\right)_j\right)^T \\ * & -1 \end{bmatrix} < 0, \tag{50*}$$

and

$$\begin{bmatrix} \tilde{\Xi} & 0 & I_N \otimes (B\Phi^{-1}) \\ * & -\beta \left(I_N \otimes P^{-1}\right) & -\mathcal{L}_1^T \otimes P^{-1} G^T \\ * & * & -2\left(I_N \otimes \Phi^{-1}\right) \end{bmatrix} < 0 \tag{51*}$$

with

$$\tilde{\Xi} = I_N \otimes \left(\alpha P^{-1} + AP^{-1} + P^{-1}A^T + \vartheta I_n + \vartheta^{-1} P^{-1}\Sigma P^{-1}\right) - \mathcal{L}_1 \otimes (BKP^{-1}) - \mathcal{L}_1^T \otimes (P^{-1}K^T B^T).$$

It is also easy to see

$$\begin{aligned} \rho_2 = & \lambda_{\max}\left(P^{-1}(BKA)^T PBKA\right) + \lambda_{\max}\left((BK)^T PBK\right) \lambda_{\max}\left(P^{-1}\Sigma\right) \\ & + \rho_1 \lambda_{\max}\left(P^{-1}(BKB)^T PBKB\right) \lambda_{\max}\left(P^{-1}K^T K\right). \end{aligned}$$

Pre- and post-multiplying both sides of (50*) by $\mathrm{diag}\{I_N \otimes P, 1\}$ gives rise to the inequality (9); analogously, Pre- and post-multiplying both sides of (51*) by $\mathrm{diag}\{I_N \otimes P, I_N \otimes P, I_N \otimes \Phi\}$ yields the inequality (10).

Thus, we find that all the conditions of Theorem 1 are met. Therefore, the MAS (1) under the protocol (2) realizes the CC with the given CC region A.

4. Numerical example

This section offers a numerical example to demonstrate the effectiveness of the developed theoretical results.

Consider a MAS with two leaders and four followers. x_1, x_2, x_3, and x_4 are state vectors of followers; x_5 and x_6 are state vectors of leaders. For simplicity, suppose that the state space is two-dimensional with $x_i = [x_{i1} \ \ x_{i2}]^T$, $i = 1, 2, \cdots, 6$. Let the following parameters be given:

$$A = \frac{1}{2}\begin{bmatrix} 0.1 & -0.2 \\ 0.2 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$f(t, x_i) = 0.05 x_i \sin(t), \ i = 1, 2, \cdots, 6,$$
$$\alpha = 0.15, \ \beta = 0.1.$$

Also, the Laplacian matrix of the concerned communication topology is taken as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ 0_{2\times 4} & 0_{2\times 2} \end{bmatrix},$$

with

$$\mathcal{L}_1 = \begin{bmatrix} 2 & -0.5 & -0.5 & -0.5 \\ -0.5 & 1.5 & -0.5 & 0 \\ -0.5 & -0.5 & 2 & 0 \\ -0.5 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 0 & -0.5 \\ 0 & -0.5 \\ -1 & 0 \\ -0.5 & 0 \end{bmatrix},$$

while the topology of graph $\mathcal{G}$ is depicted in Figure 1.

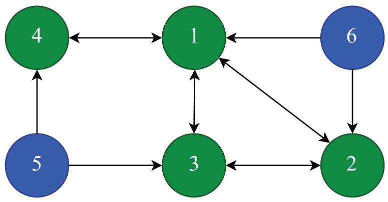

Figure 1. Network topology configuration.

It can be verified that $\Sigma = \mathrm{diag}\{0.002,\ 0.002\}$.

With the given parameters, by employing the MATLAB toolbox YALMIP to solve the inequalities (50) and (51), we find the inequalities have feasible solutions. For saving space, we here give only $Q = \begin{bmatrix} 0.0765 & 0.0702 \\ 0.0702 & 0.0646 \end{bmatrix}$, and $\Delta = \begin{bmatrix} 0.0764 & 0.0761 \end{bmatrix}$. According to Theorem 2, the MASs (1) with the protocol (2) for the given parameters can realizes the CC with a CC region; and the control gain can be taken as $K = \Lambda Q^{-1} = \begin{bmatrix} -20.4391 & 23.3774 \end{bmatrix}$. And in the light of Theorem 2, an upper bound of sampling period is calculated as 8.782×10^{-2}. Take a sampling period $\hbar = 8 \times 10^{-2}$, and randomly select a set of initial values within the corresponding CC region A. The resulting numerical results are demonstrated in Figures 2–4.

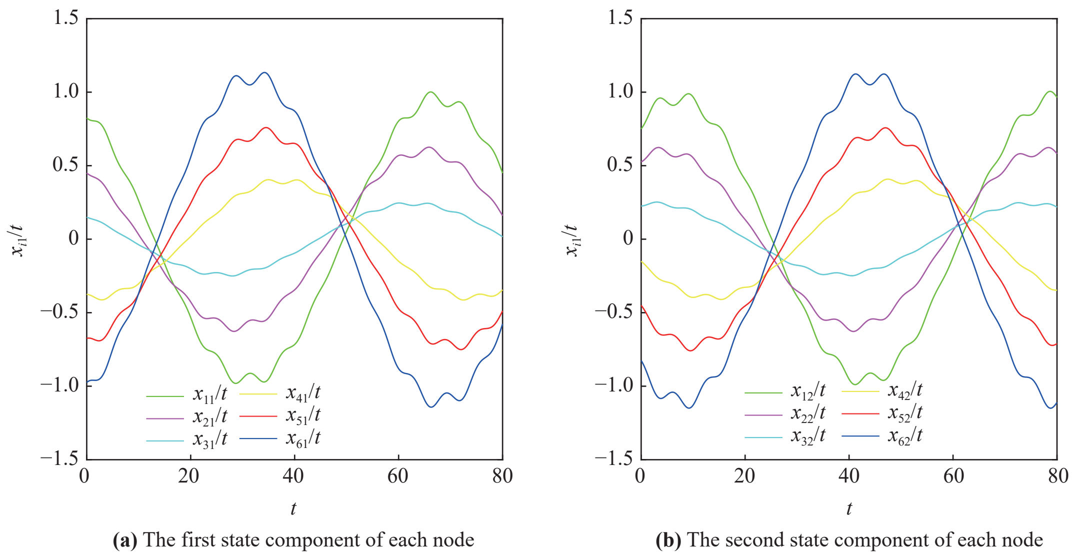

Figure 2. State trajectories of MAS without control ($u_i = 0,\ i \in \mathcal{V}_{\mathcal{F}}$.)

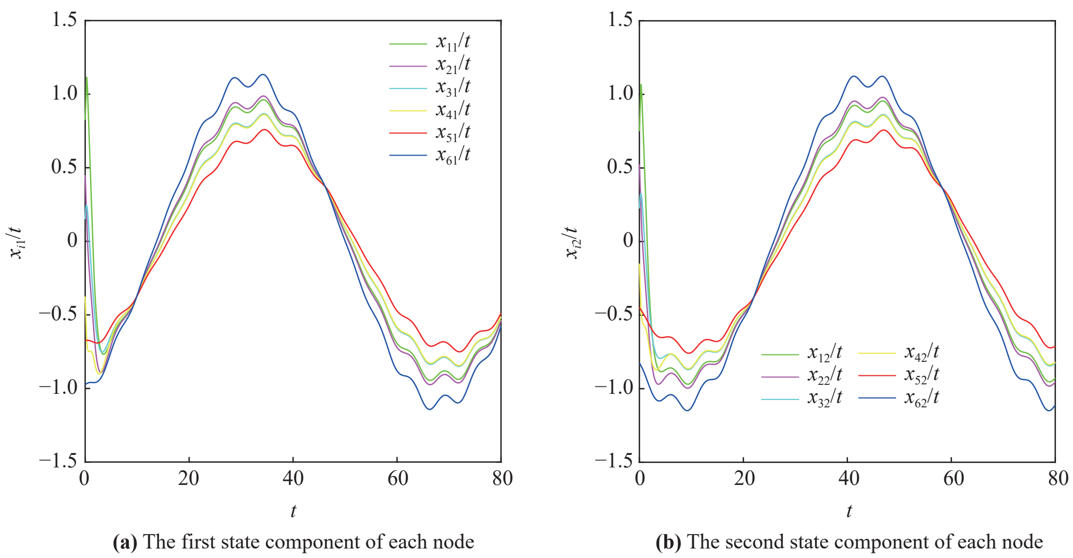

Figure 3. State trajectories of MAS with control protocol.

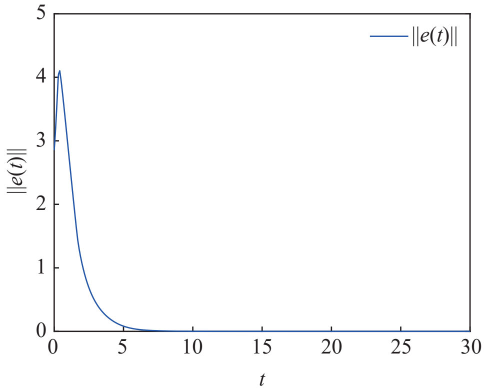

Figure 4. Evolution of CC error.

Figure 2 plots the state trajectories of the uncontrolled MAS, while Figure 3 displays the state evolution of the MAS with CC strategy (2). The evolution of the CC error $e(t)$ is shown in Figure 4, where $||\cdot||$ represents the vector norm. The simulation shows that the CC can be achieved with the given control protocol, which confirms the correctness of the theoretical findings.

5. Conclusions

In this work, we have studied the sampled-data-based CC problem for a class of nonlinear MASs with active leaders and input saturation. A framework has been established to address the CC problem for the nonlinear MASs involving sampling and input saturation. Some easy-to-verify criteria have been derived to achieve the CC relative to a CC region for the concerned MAS and the control scheme. In addition, the control gain can be obtained by solving a set of relevant matrix inequalities. Lastly, numerical simulation has further confirmed the theory developed in this paper. Future research topics would include the extension of the main results to MASs with communication constraints and/or subject to cyber attacks [44, 45].

Author Contributions: Luyang Yu: Methodology, Conceptualization, Writing-original draft; **Jiayi Ding:** Methodology, Writing-original draft; **Huihui Peng:** Methodology, Writing-original draft; **Yang Liu:** Methodology, Administration, Writing; **Yurong Liu:** Methodology, Conceptualization.

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